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A Study of Metastability in the Ising Model

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Abstract. We consider a two-dimensional Ising ferromagnet with (+) boundary conditions and negative external field, where a Markovian time evolution is assumed.

We construct, suitably restricting the allowed configurations at $t=0$, a non equilibrium state with positive magnetization such that:

1) only one phase is present,

2) the relaxation time for unit volume is finite and can be made very large.

These results are obtained following a general method for describing metastable states proposed by Lebowitz and Penrose and exploiting the analysis of the Ising-spin-configurations in terms of contours given by Minlos and Sinai.

Introduction

In 1971 Penrose and Lebowitz [1] proposed a general method for describing metastable states in statistical mechanics, where for the first time the dynamical and static aspects of the problem were coupled in a precise way.

Given a finite system K and called S the set of its possible configurations (s) and $G(s)$ the associated Gibbs distribution, the program is to find a subset R of S such that, if at $t=0$ we consider the state described by the probability distribution:

$$P(s) \propto G(s) \qquad s \in R$$
$$P(s) = 0 \qquad s \notin R$$

then, for suitable values of the thermodynamical parameters:

i) only one thermodynamic phase is present;

ii) the conditional probability $p(t)$ that the system, being in R at $t=0$, has escaped from R by the time t is very small.

iii) the relative weight of the configurations contained in R is negligible at equilibrium.

These properties translate in quantitative and precise terms what is expected on physical grounds from a metastable state. Lebowitz and Penrose were able to successfully study in this framework the Kac potentials in the so called Van der Waals limit [2]. In this case the smooth analytic behaviour exhibited at the first order phase transition gives a natural way to identify uniquely, via an analytic continuation, a class of non equilibrium states, whose relaxation time to the stable equilibrium states is found to go to infinity in the above mentioned limit.

The aim of this paper is to study, with the general method proposed in [1], a system with short range interactions. In contrast with the generalized Van der Waals case, in this case there are weighty arguments [3–5] for the existence of a singularity forbidding any real analytic continuation beyond transition point. This fact leads us to expect some ambiguity in the definition of metastable states.

Also on a purely physical ground one cannot expect a blind extension of the results on metastability of the Van der Waals systems to the short range ones.

We will consider an Ising spin system with n.n. interactions, where the dynamics of a Markovian process is assumed [6–8]. In Section I, we briefly recall the known equilibrium results for this system and derive, by the use of a master equation, expressions of the dynamical quantities of interest.

In Section II the subset R is defined and the physical ideas leading to this choice are illustrated.

In Section III, studying the static properties of our state we prove that it is actually a pure phase. We also show, that the relative weight of configurations contained in R at equilibrium decreases exponentially with the size of the system.

In Section IV we explicitly work out rigorous lower and upper bounds for the relaxation time and discuss the choice of R that maximizes this quantity. We further show that, for suitable values of the thermodynamical parameters, it can be made very large.

In Section V we summarize the results of the previous sections in two theorems. The reader is referred to this last section for a full picture of our results.

For the purpose of geometrical visualization all the calculations refer to the two dimensional case; generalization of the results to higher dimensions does not present any particular difficulty.

We remark that the contours technique employed in this paper can be generalized to a continuous system [9] (the Widom and Rowlinson model). This should allow to apply our analysis to a system with a natural time evolution.

Section I

a) Equilibrium Properties of the Ising Model. We consider an Ising ferromagnet, with n. n. interaction only, enclosed in a finite box Λ on a two-dimensional square lattice Z^2. Boundary conditions are specified by the choice of a spin configuration τ outside Λ.

The energy of a spin configuration $\boldsymbol{\sigma}$ inside Λ is given by:

$$E_{\Lambda,\tau}(\boldsymbol{\sigma}) = -\frac{J}{2} \sum_{\langle k,k'\rangle \in \Lambda} \sigma_k \sigma_{k'} - \frac{J}{2} \sum_{\substack{\langle k,k'\rangle \\ k \in \Lambda \\ k' \notin \Lambda}} \sigma_k \tau_{k'} - \frac{h}{2} \sum_{k \in \Lambda} \sigma_k \tag{I.1}$$

where $J > 0$ and $\frac{h}{2}$ is the external magnetic field. The spin correlation functions, for any $A \subseteqq \Lambda$, are defined through:

$$\langle \sigma_A \rangle_{\Lambda,\tau,h} = \frac{\sum_{\boldsymbol{\sigma}} \prod_{k \in A} \sigma_k e^{-\beta E_{\Lambda,\tau}(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}} e^{-\beta E_{\Lambda,\tau}(\boldsymbol{\sigma})}} \tag{I.2}$$

and the free energy, (multiplied by $-\beta$), is given by:

$$F(\beta, h, \Lambda, \tau) = \frac{1}{|\Lambda|} \ln \sum_{\boldsymbol{\sigma}} e^{-\beta E_{\Lambda,\tau}(\boldsymbol{\sigma})} = \frac{1}{|\Lambda|} \ln Z^{\tau}(\Lambda, h). \tag{I.3}$$

The equilibrium properties of the Ising model are described in terms of the spin correlation functions $\langle \sigma_A \rangle_{\tau,h}$ and the free energy $F(\beta, h)$ of the infinite volume system, which are obtained as limits of $\langle \sigma_A \rangle_{\Lambda,\tau,h}$ and $F(\beta, h, \Lambda, \tau)$ with a proper choice of τ for any Λ as $\Lambda \to \infty$ [10]. The main results on equilibrium properties may be summarized as follows:

For $\beta < \beta_c$ (β_c is the reciprocal of the critical temperature) the system is always in a pure phase. The free energy $F(\beta, h)$ is an analytic function of β and h, even in h. The spin correlation functions do not depend on boundary conditions and have cluster properties, that is: $\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \to 0$ as the distance between A and B, $d(A, B) \to \infty$.

Similar results hold when $\beta > \beta_c$, if $h \neq 0$. For $\beta > \beta_c$, at $h = 0$, the system undergoes a phase transition: The free energy is not differentiable in h, at $h = 0$, the spin correlation functions depend on boundary conditions, and do not describe in general a pure phase.

From now on, we only consider the situation where spins outside Λ are all positive: in this case for $\beta \geqq \beta'$, with a proper $\beta' > \beta_c$, the following results hold [11].

i) The spin correlation functions decrease, as $\Lambda \to \infty$, to limits $\langle \sigma_A \rangle_{+,h}$ satisfying, for any $h \geqq 0$

$$|\langle \sigma_A \sigma_B \rangle_{+,h} - \langle \sigma_A \rangle_{+,h} \langle \sigma_B \rangle_{+,h}| \leqq \text{const.}\, e^{-(\beta-\beta')d(A,B)},$$

ii) the central limit theorem holds for quantities like the total magnetization or the total energy of the system.

These properties, which characterize a pure phase, may be obtained when the temperature is low enough by the contour description of the Ising model.

In general, for fixed boundary conditions, a spin configuration is uniquely specified by the set of contours, that is the set of non intersecting polygonals separating opposite spins [10].

With the boundary conditions chosen, all the contours are closed and in a given configuration the outer contours are the ones not embraced by other contours.

The probability to find a family $\gamma_1, \ldots, \gamma_n$ of outer contours is expressed by [12]

$$\varrho_{\Lambda,h}(\gamma_1, \ldots, \gamma_n) = \frac{\prod_1^n{}_i \, \mu(\gamma_i, h) \sum'_{\{\tilde\gamma\} \subset \Lambda} \prod_{\tilde\gamma \in \{\tilde\gamma\}} \mu(\tilde\gamma, h)}{\Xi_\Lambda(\beta, h)} \tag{I.4}$$

with $\mu(\gamma, h) = e^{-\beta J|\gamma| - \beta \frac{h}{2}|\Theta(\gamma)|} \zeta^-(\gamma, h)$

$$\Xi_\Lambda(\beta, h) = \sum_{\{\gamma\} \subset \Lambda} \prod_{\gamma \in \{\gamma\}} \mu(\gamma, h)$$

where $|\gamma|$ is the lenght of the contour γ, $\zeta^{-(+)}(\gamma, h)$ is the partition function for the region $\Theta(\gamma)$ enclosed in γ, where all the internal spins adjacent to the boundary are negative (positive). The primed sum runs over the families $\{\tilde\gamma\}$ of outer contours compatible with $\gamma_1, \ldots, \gamma_n$, that is no one of the $\tilde\gamma$ intersects of embraces any of the γ_i, or is contained in any of the regions $\Theta(\gamma_i)$. If we consider in the denominator of (I.4), only the contribution of the set of configurations obtained by "erasing" $\gamma_1, \ldots \gamma_n$, that is by flipping all the spins inside each of the $\Theta(\gamma_i)$, in the configurations where $\gamma_1, \ldots \gamma_n$ are present, we get the inequalities

$$\varrho_{\Lambda,h}(\gamma_1, \ldots, \gamma_n) \leqq \prod_1^n{}_i \, e^{-\beta J|\gamma_i|} \frac{\zeta^-(\gamma_i, h)}{\zeta^+(\gamma_i, h)}. \tag{I.5}$$

Observing that:

$$\frac{\zeta^-(\gamma, h)}{\zeta^+(\gamma, h)} = \frac{\zeta^-(\gamma, h)}{\zeta^-(\gamma, -h)} = \exp\left\{+\frac{\beta}{2} \int_{-h}^{+h} \sum_{k \in \Theta(\gamma)} \langle \sigma_k \rangle_{\Theta(\gamma), -, x} dx\right\} \tag{I.6}$$

whe see that, when $h \geqq 0$, $\zeta^-(\gamma, h)/\zeta^+(\gamma, h) \leqq 1$ and $\varrho_{\Lambda,h}(\gamma_1, \dots \gamma_n) \leqq \exp\left\{-\beta J \sum_{i\,1}^{n} |\gamma_i|\right\}$ that is the upper bound (I.5) decreases exponentially with the total lenght of the contours.

The interest of this description lies in the fact that the outer contours correlation functions obey a set of integral equations a la Kirkwood and Saltzbourg. These equations were derived and studied in detail by Minlos and Sinai in a fundamental series of papers [12–14]. We will summarize in the sequel that part of their analysis which will concern us.

For any $h \geqq 0$, $\beta > \beta^{||}$, we can uniquely define a function over the families of compatible outer contours, satisfying:

i) $\varrho_h(\gamma_1 \dots \gamma_n)$ is analytic in β and h if $h \neq 0$,

ii) $\varrho_n(\gamma_1 \dots \gamma_n) \leqq \exp\left\{-\beta J \sum_{i\,1}^{n} |\gamma_i|\right\}$,

iii) $\left|\varrho_{\Lambda,h}(\gamma_1 \dots \gamma_n) - \prod_{i\,1}^{n} \chi_\Lambda(\gamma_i)\, \varrho_h(\gamma_1 \dots \gamma_n)\right| \leqq e^{-(\beta-\beta'')\frac{J}{h}\sum_{1}^{n}|\gamma_i|}\, \Phi(\Lambda, \beta)$

where $\chi_\Lambda(\gamma_i) = 1$ if $\Theta(\gamma_i) \subseteqq \Lambda$, is zero otherwise, and $\Phi(\Lambda, \beta) \xrightarrow[\Lambda \to \infty]{} 0$.

The deduction of these results relies essentially on the exponential behaviour of the upper bound (I.5), for $h \geqq 0$. This allows one to prove that the spin correlation functions (that can be expressed as sums of contours correlation functions with coefficients not depending on Λ) converge uniformly in Λ, and to derive by standard methods their analyticity and cluster property.

b) Master Equation. Following a standard procedure [6–8], we introduce a time evolution in the Ising model by assigning the probability $W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})$, of a transition from the configuration $\boldsymbol{\sigma}$ to the configuration $\boldsymbol{\sigma}'$ in the unit time, satisfying the detailed balance condition:

$$W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})\, e^{-\beta E_\Lambda(\boldsymbol{\sigma})} = W_\Lambda(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\, e^{-\beta E_\Lambda(\boldsymbol{\sigma}')} \tag{I.7}$$

we further require that, for any pair of configurations $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$, there is a sequence $\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \dots, \boldsymbol{\sigma}^n$ of configurations s.t. $\boldsymbol{\sigma}^1 \equiv \boldsymbol{\sigma}$, $\boldsymbol{\sigma}^n \equiv \boldsymbol{\sigma}'$, and $W_\Lambda(\boldsymbol{\sigma}^{i+1}, \boldsymbol{\sigma}^i) \neq 0$, $i = 1, \dots, n-1$.

If $p_\Lambda(\boldsymbol{\sigma}; t)$ is the probability distribution over the configurations in Λ at the time t, we can write the master equation as:

$$\frac{\partial}{\partial t} p_\Lambda(\boldsymbol{\sigma}; t) = \sum_{\boldsymbol{\sigma}'} \{W_\Lambda(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\, p_\Lambda(\boldsymbol{\sigma}', t) - W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})\, p_\Lambda(\boldsymbol{\sigma}, t)\}\,. \tag{I.8}$$

Let now R_Λ be a subset of the set of spin configurations in Λ, and consider the initial condition:

$$p_\Lambda(\boldsymbol{\sigma}, 0) = \begin{cases} 0 & \boldsymbol{\sigma} \notin R_\Lambda \\ K e^{-\beta E_\Lambda(\boldsymbol{\sigma})} & \boldsymbol{\sigma} \in R_\Lambda \end{cases} \tag{I.9}$$

where K is the normalization constant.

The solution of the Eq. (I.8) for this initial condition is such that the quantity $p(R_\Lambda; t) = \sum_{\boldsymbol{\sigma} \in R_\Lambda} p_\Lambda(\boldsymbol{\sigma}; t)$ never increases with time, while its derivative never decreases. This is proven in Appendix A.

If we consider now the quantity $1 - p(R_\Lambda, t)$ (i.e. the conditional probability that the system, being in R_Λ at $t = 0$, has escaped by the time t) its rate of increase is maximum at $t = 0$; following [1] we can define the "escape rate" from R_Λ, v_{R_Λ}, as:

$$v_{R_\Lambda} = -\frac{d}{dt} p(R_\Lambda; t)\Big|_{t=0}. \tag{I.10}$$

From (I.8) and (I.9), taking into account condition (I.7) we have:

$$v_{R_\Lambda} = K \sum_{\boldsymbol{\sigma} \in R_\Lambda} \sum_{\boldsymbol{\sigma}' \notin R_\Lambda} W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})\, e^{-\beta E_\Lambda(\boldsymbol{\sigma})}. \tag{I.11}$$

The assumptions on $W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})$ guarantee that the Gibbs distribution is the unique equilibrium distribution and then the probability of return to R_Λ is given, as in [1], by the probability to be in R_Λ at equilibrium, that is

$$P_{R_\Lambda} = \frac{\sum_{\boldsymbol{\sigma} \in R_\Lambda} e^{-\beta E_\Lambda(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}} e^{-\beta E_\Lambda(\boldsymbol{\sigma})}}. \tag{I.12}$$

Section II

In Section I we have briefly sketched a geometrical description of the spin configuration for the Ising model. The main physical implication of this analysis can be formulated in the following way (see for instance [15]): among all configurations in the gran canonical ensemble with + boundary conditions and $h \geqq 0$, the most probable will be those with a large majority of positive spins and small and rare "islands", uniformely distributed, of negative spins.

This picture can be considered a rigorous version of the phenomenological theory [5] and allows one to study in a quantitative way approximations and limits of the dropled model [3]. For instance in the case of finite Λ the bounds for the probability of subclasses of configurations given by Eq. (I.5), when those configuration contain "large contours" (i.e. $|\gamma| \geqq \ln|\Lambda|$) give a rigorous and physically transparent estimate of the relevance of the fluctuations that could give rise to the opposite phase. In particular, the bounds on the temperature that guarantee the validity of Minlos and Sinai analysis (cf. Section I) can be interpreted as a sufficient condition to keep these fluctuations under control in the thermodynamic limit.

With this in mind it is easy to understand how, both in the phenomenological and rigorous theories, when $h<0$ the positive volume effect [see (I.5)] due to the external field, enhancing the weight of large area contours, naturally leads to a break down when $\Lambda \to \infty$. Nevertheless it is worth noticing that in this case, where the powerful tool of the integral equations become useless, the description of the configurations in terms of contours still makes sense and we can attempt, as in the phenomenological theories, to define a metastable state by eliminating the configurations containing "large droplets". If c is an integer let us define a subset R_Λ of spin configurations in Λ in the following way.

Calling $\{\gamma\}_{\boldsymbol{\sigma}}$ the collection of outer contours $\gamma_1, \dots \gamma_n$ associated to the spin configuration $\boldsymbol{\sigma}$

$$R_\Lambda = \{\boldsymbol{\sigma} : |\Theta(\gamma)| \leqq c^2 \, \forall \gamma \in \{\gamma\}_{\boldsymbol{\sigma}}\}$$

and define a state by the following probability distribution:

$$p_\Lambda(\boldsymbol{\sigma}) = \begin{matrix} K e^{-\beta E_\Lambda(\boldsymbol{\sigma})} & \boldsymbol{\sigma} \in R_\Lambda \\ 0 & \boldsymbol{\sigma} \notin R_\Lambda \end{matrix} \tag{II.1}$$

where $E_\Lambda(\boldsymbol{\sigma})$ is defined in Section I and K is a normalization constant.

At this stage c^2 is completely arbitrary but we expect the metastability requirements listed in the introduction to pin down a critical value or at least a range of critical values for the size of the droplets (i.e. for the area of our contours).

The next two sections will be devoted to a detailed analysis of the static and dynamical properties of a state defined by Eq. (II.1). At the end of Section IV our analysis will be completed fixing a range of allowed values for c^2 such that the metastability requirements are best satisfied. On a purely static ground, without going through all the details of the next three sections, it is possible to get an idea of the critical size and the difficulties related to its determination.

If we introduce $\lambda(\gamma, h) = \varrho_{\Lambda,h}(\gamma)/\tilde{\varrho}_{\Lambda,h}(\gamma)$ where $\varrho_{\Lambda,h}(\gamma)$ is the probability to find a configuration with the outer contour γ and $\tilde{\varrho}_{\Lambda,h}(\gamma)$ the probability to find a configuration where the contour γ is absent and the region $\Theta(\gamma)$ is neither crossed nor surrounded by any other contour, we get from Eq. (I.4)

$$\lambda(\gamma, h) = e^{-\beta J|\gamma|} \zeta^-(\gamma, h)/\zeta^+(\gamma, h) \,. \tag{II.2}$$

It is worth noticing that this quantity is Λ-independent (i.e. it is not affected by the break down of the droplet picture for large Λ's when $h<0$).

From the definition of $\varrho_{\Lambda,h}(\gamma)$ and $\tilde{\varrho}_{\Lambda,h}(\gamma)$ it is easy to convince ourselves that $\ln\lambda(\gamma,h)$ is the free energy variation associated to the droplet γ. Its behaviour can be analized in order to discriminate the outer contours regarding to their ability to take or give free energy for their growth.

We remark that, when $h<0$

$$\ln\lambda(\gamma,h) = -\beta J|\gamma| + \frac{\beta}{2}\int_{-|h|}^{+|h|}\sum_{k\in\Theta(\gamma)}\langle\sigma_k\rangle_{\Theta(\gamma),+,x}dx\,. \tag{II.3}$$

The next step, the determination of the critical size, should come out from some sort of stationarity for $\ln\lambda(\gamma,h)$. This task is very hard, because the shape of the contour has to be taken into account.

But when β is very large, it is reasonable to assume that square shapes play a dominant role. If we consider square shapes only, $\ln\lambda(\gamma,h)$ is bounded by two quadratic forms:

$$-4\beta Jl + m^*(\beta)\,\beta|h|\,l^2 \leqq \ln\lambda(q,h) \leqq -4\beta Jl + \beta|h|\,l^2 \tag{II.4}$$

where l is the side of the square contour q and $m^*(\beta)$ is the magnetization in the infinite volume limit for $h=0^+$. When $\beta\to\infty$ we get an absolute minimum at $l=\frac{2J}{|h|}$ (i.e. the estimate for the critical size given by the droplet model [3]).

Section III

To investigate the static properties of the state described by (II.1) we introduce the outer contour correlation functions $\varrho^c_{\Lambda,h}$: for any set $\gamma_1,\ldots,\gamma_n$ of compatible outer contours in Λ, s.t. $|\Theta(\gamma_i)|\leqq c^2$

$$\varrho^c_{\Lambda,h}(\gamma_1,\ldots,\gamma_n) = \frac{\prod_1^n{}_i\,\mu(\gamma_i,h)\sum'_{\{\tilde{\gamma}\}\subset\Lambda,\ |\Theta(\tilde{\gamma})|\leqq c^2}\prod_{\tilde{\gamma}\in\{\tilde{\gamma}\}}\mu(\tilde{\gamma},h)}{\Xi^c_\Lambda(\beta,h)} \tag{III.1}$$

where

$$\Xi^c_\Lambda(\beta,h) = \sum_{\{\gamma\}\subset\Lambda,\ |\Theta(\gamma)|\leqq c^2}\ \prod_{\gamma\in\{\gamma\}}\mu(\gamma,h)\,.$$

It can be easily shown (see Appendix B) that the set of the $\varrho^c_{\Lambda,h}$ so defined satisfy the inequalities:

$$\varrho^c_{\Lambda,h}(\gamma_1,\ldots\gamma_n) \leqq e^{-\beta\sum_1^n{}_i|\gamma_i|\left(J-\frac{|h|\,c}{4}\right)} \leqq e^{-2\alpha\sum_1^n{}_i|\gamma_i|} \tag{III.2}$$

provided that

$$c \leqq \frac{4(\beta J - 2\alpha)}{\beta |h|}. \tag{III.3}$$

We show in Appendix B that, as in the equilibrium case, the Minlos-Sinai equations hold, and it is possible to define a function ϱ_h^c over the families of compatible outer contours, such that, in the region of β, $h<0$ where (III.3) is satisfied with $\alpha \geqq \ln 6$:

$$\begin{aligned} &\varrho_h^c(\gamma_1, \dots \gamma_n) = 0 \qquad \text{if any } \gamma_i \text{ is s.t. } |\Theta(\gamma_i)| > c^2 \\ &\varrho_h^c(\gamma_1, \dots \gamma_n) \leqq e^{-2\alpha \sum_1^n i |\gamma_i|} \\ &\left| \varrho_{\Lambda,h}^c(\gamma_1, \dots \gamma_n) - \prod_1^h {}_i X_\Lambda(\gamma_i)\, \varrho_h^c(\gamma_1, \dots \gamma_n) \right| \leqq e^{-\alpha \sum_1^h i |\gamma_i|} G(\beta, \Lambda) \end{aligned} \tag{III.4}$$

where χ_Λ is the same of Section I, and $G(\beta, \Lambda) \xrightarrow[\Lambda \to \infty]{} 0$. These properties are sufficient to guarantee that, in this region of β and h:

i) the van Hove limits $\langle \sigma_A \rangle_h^c$ of the spin correlation functions defined as:

$$\langle \sigma_A \rangle_{\Lambda,h}^c = \frac{\sum\limits_{\sigma \in R_\Lambda} \sigma_A e^{-\beta E_\Lambda(\sigma)}}{\sum\limits_{\sigma \in R_\Lambda} e^{-\beta E_\Lambda(\sigma)}} \tag{III.5}$$

exist, and have cluster properties, like the equilibrium ones. In particular, the limit of the magnetization, that can be expressed as:

$$\langle \sigma_0 \rangle_h^c = 1 + \sum_{\gamma : 0 \in \Theta(\gamma)} \varrho_h^c(\gamma) \langle \sigma_0 - 1 \rangle_{\Theta(\gamma), -, h} \tag{III.6}$$

is positive,

ii) the limit of the free energy

$$F_\Lambda^c(\beta, h) = \frac{1}{|\Lambda|} \ln \sum_{\sigma \in R_\Lambda} e^{-\beta E_\Lambda(\sigma)} = \frac{1}{|\Lambda|} \ln Z_c^+(\Lambda, h) \tag{III.7}$$

exists, and it is infinitely differentiable in β and h.

iii) The central limit theorem holds, for quantities like the total magnetization and the total energy.

Let us consider the case of $|h|$ very small. We remark that the smaller $|h|$ is, the looser becomes the constraint (III.3) on c, and for $h \to 0^-$ the constraint becomes trivial.

Therefore, the infinite volume contour correlation functions exist, for any fixed c, also in the limit $h \to 0^-$. Denoting by ϱ the set of outer contour correlation functions of the equilibrium state, for $h = 0^+$, it can be shown (see Appendix B) that for any set $\gamma_1, \dots, \gamma_n$ of compatible outer contours, s.t. $|\Theta(\gamma_i)| \leqq c^2$

$$\left|\varrho_h^c(\gamma_1, \ldots, \gamma_n) - \prod_1^n {}_i X^c(\gamma_i)\, \varrho(\gamma_1, \ldots, \gamma_n)\right| \leqq e^{-\alpha \sum_1^n |\gamma_i|} B(\beta, h, c) \quad \text{(III.8)}$$

where $X^c(\gamma) = 1$ if $|\Theta(\gamma)| \leqq c^2$, is zero otherwise, and

$$B(\beta, h, c) \to 0 \quad \text{as} \quad h \to 0^-, c \to \infty \quad \text{if} \quad c|h| \leqq 4(\beta J - 2\ln 6).$$

If we consider then any sequence of c, such that $c \to \infty$ as $h \to 0^-$, we get $\varrho_h^c \to \varrho$ as $h \to 0^-$, provided that for any c in the sequence the contraint (III.3) is satisfied, with $\alpha \geqq 2\ln 6$.

It is therefore impossible, on a purely static ground, to define uniquely a sequence of states joining the equilibrium state as $h \to 0^-$ (i.e. an extrapolation of the "equation of state" from positive to negative h) but a large class of sequences that exhibits this property does exist.

Consider now the relative weight of the configurations in R_Λ at equilibrium P_{R_Λ}. It may be written as:

$$P_{R_\Lambda} = \frac{\sum_{\sigma \in R_\Lambda} e^{-\beta E_\Lambda(\sigma)}}{\sum_\sigma e^{-\beta E_\Lambda(\sigma)}} = \frac{Z_c^+(\Lambda, h)}{Z^+(\Lambda, h)}. \quad \text{(III.9)}$$

Using then the inequalities:

$$e^{+\beta J|\partial\Lambda|} Z^+(\Lambda, -|h|) \geqq Z^-(\Lambda, -|h|) \geqq Z^-(\Lambda, 0)\, e^{\frac{\beta}{2}|h| m^*(\beta)|\Lambda|} \quad \text{(III.10)}$$

and remembering that the magnetization in R_Λ, for c satisfying (III.3) is positive, then $Z_c^+(\Lambda, -|h|) \leqq Z_c^+(\Lambda, 0)$, we get:

$$P_{R_\Lambda} \leqq \frac{Z_c^+(\Lambda, -|h|)}{Z_c^+(\Lambda, 0)} \frac{Z^-(\Lambda, 0)}{Z^+(\Lambda, -|h|)} \leqq e^{\beta J|\partial\Lambda| - \frac{\beta}{2}|h| m_i^*(\beta)|\Lambda|}. \quad \text{(III.11)}$$

Then $P_{R_\Lambda} \to 0$, in the infinite volume limit.

Section IV

In this section we investigate the dynamical properties of the state whose probability distribution at $t = 0$ is given by (II.1): we assume that the time evolution of this state is described by (I.8) where $W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma})$ satisfies:

$$W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma}) \neq 0 \quad \text{iff} \quad \exists k \in \Lambda \text{ s.t. } \begin{array}{l} \sigma_k' = -\sigma_k \\ \sigma_i' = \sigma_i \quad i \neq k \end{array} \quad \text{(IV.1)}$$

$$W_m \leqq W_\Lambda(\boldsymbol{\sigma}', \boldsymbol{\sigma}) \leqq W_M$$

where W_m, W_M are positive constants independent of Λ. For any $\boldsymbol{\sigma}$ in Λ, $k \in \Lambda$, call $(\boldsymbol{\sigma}, k)$ the spin configuration defined by $\sigma'_i = \sigma_i (i \neq k)$, $\sigma'_k = -\sigma_k$ and define for any $k \in \Lambda$ the subset R^k_Λ of R_Λ as

$$R^k_\Lambda = \{\boldsymbol{\sigma} \in R_\Lambda : (\boldsymbol{\sigma}, k) \notin R_\Lambda\}\,. \tag{IV.2}$$

By definition R^k_Λ contains all the configurations $\boldsymbol{\sigma} \in R_\Lambda$ such that k is external to all the outer contours and is adjacent[1] to a number of outer contours (at most four) that give rise, when $\sigma_k \to -\sigma_k$, to a single outer contour with area greater than c^2.

The expression (I.11) for the escape rate may now be written in the form:

$$v_{R_\Lambda} = \sum_{k \in \Lambda} \sum_{\boldsymbol{\sigma} \in R^k_\Lambda} W_\Lambda((\boldsymbol{\sigma}, k), \boldsymbol{\sigma}) \frac{e^{-\beta E_\Lambda(\boldsymbol{\sigma})}}{\sum\limits_{\boldsymbol{\sigma} \in R_\Lambda} e^{-\beta E_\Lambda(\boldsymbol{\sigma})}}\,. \tag{IV.3}$$

To derive from (IV.3) an upper bound for v_{R_Λ} observe that in any $\boldsymbol{\sigma} \in R^k_\Lambda$, the total length of outer contours adjacent to k is $\geqq 4c$. From (IV.1), (III.1), and (III.2) it follows then

$$v_{R_\Lambda} \leqq W_M \sum_{k \in \Lambda} \sum_{1}^{4} {}_{\nu} \sum_{(\gamma_1, \ldots \gamma_\nu)} \varrho^c_{\Lambda,h}(\gamma_1, \ldots, \gamma_\nu)\,. \tag{IV.4'}$$

Where $\gamma_1, \ldots, \gamma_\nu$ are adjacent to k, and $|\gamma_1| + \cdots + |\gamma_\mu| \geqq 4c$

$$v_{R_\Lambda} \leqq W_M |\Lambda|\, 4 \sum_{l \geqq 2c} 3^{2l+3} e^{-\beta(J - |h| \frac{c}{4}) 2l}\,. \tag{IV.4''}$$

The last inequality is obtained observing that, for any $k \in \Lambda$, the number of events caracterized by the presence of ν outer contours $(1 \leqq \nu \leqq 4)$ with total lenght $2l$, all adjacent to k, is less than or equal to $4 \cdot 3^{2l+4-1}$. Then for

$$c \leqq \frac{4(\beta J - \ln 3)}{\beta |h|}$$

we have

$$v_{R_\Lambda} \leqq W_M |\Lambda|\, e^{-(\beta J - \ln 3) hc + \beta |h| c^2} F_2(\beta, h, c) \tag{IV.4}$$

with

$$F_2 = 108/(1 - e^{-2(\beta J - \ln 3 - |h| \frac{c}{4})})\,.$$

We next evaluate a lower bound for v_{R_Λ}.

Let Λ_1 be the set of lattice points in Λ whose distance from the lattice points external to Λ is greater than $\sqrt{2}c$, and consider, for any $k \in \Lambda_1$ the set Q_k of square outer contours with side c adjacent to k and containing one of its n.n. (the one on the left, for example). For any $k \in \Lambda_1$ there are c of these contours and any $\boldsymbol{\sigma} \in R^k_\Lambda$ contains at most one of the elements of Q_k.

[1] We say that $k \in \Lambda$ is adjacent to γ if k is external to γ, and at least one of its n.n. or n.n.n. is contained in $\Theta(\gamma)$.

We get then from (IV.3), (IV.1), and (III.1)

$$\begin{aligned} v_{R_\Lambda} &\geqq W_m \sum_{k\in\Lambda_1}\sum_{q\in Q_k} \varrho^c_{\Lambda,h}(q) \\ &\geqq W_m(|\Lambda| - \sqrt{2}c|\partial\Lambda|)\, c e^{-4\beta J c + \beta m^*(\beta)|h|c^2} F_1(\beta, h, c) \end{aligned} \tag{IV.5}$$

where $F_1(\beta, h, c)$ is a positive function (cf. Appendix C). The last inequality follows from

$$\varrho^c_{\Lambda,h}(q) \geqq e^{-4\beta J c + \beta m^*(\beta)|h|c^2} F_1(\beta, h, c) \tag{IV.6}$$

which is proven in Appendix C.

The bounds (IV.4) and (IV.5) nicely fit what is predicted by the phenomenological theories.

The existence of a lower and an upper bound both proportional to the volume guarantees that the relaxation time decreases linearly with the volume and, furthermore, the argument of the exponential in the upper (lower) bound is nothing other than a lower (upper) bound for the free energy of a square droplet of area c^2 as expected from the theory of spontaneous nucleation cf. (II.4).

We notice that all the estimates giving rise to the above bounds are independent of the volume (except for the surface term in the lower bound) and therefore we can consider the excape rate per unit volume v as a Λ-independent quantity. To get a step further in the determination of the class of states that satisfy best metastability requirements, it would be useful to explicitly evaluate the value or the values of c for which the excape rate v has an absolute minimum.

We are not able to do this explicitly but, at least for suitable values of β and h, the structure of our bounds allows us to define a range of values of c in which v will certainly reach a minimum.

For instance, when $J > |h| \geqq AJe^{-\beta J}$. ($A$ is a numerical constant) and β very large; v reaches a minimum for c lying in the range[2]:

$$\frac{2J}{|h|} - \sqrt{\frac{2(J-|h|)}{|h|}}\left(1 + 0\left(\frac{1}{\beta}\right)\right) \leqq c \leqq \frac{2J}{|h|} + \sqrt{\frac{2(J-|h|)}{|h|}}\left(1 + 0\left(\frac{1}{\beta}\right)\right) \tag{IV.8}$$

and its actual value is bound by:

$$B_1 W_m e^{-4\beta J^2/m^*(\beta)\beta|h|} \leqq \min_c v \leqq B_2 W_M e^{-4(\beta J - \ln 3)^2/\beta|h|} \tag{IV.9}$$

where B_1 and B_2 are numerical constants.

[2] In the evalutation of (IV.8) we have assumed that

$$W_m/W_M \geqq 1 - \mathrm{th}(\beta J - \beta|h|)$$

in agreement with the generally accepted form of the transition probability for single spin flip processes [8].

It is worth noticing that Eq. (IV.8) gives a statistically very well defined "critical size" centered around the value suggested at the end of Section II.

To conclude this section we remark that (IV.9) not only shows that the escape rate per unit volume can be made very small compared with the rate of the Markovian process but also exhibits a dependence on the thermodynamical parameters of the same nature of that suggested by the phenomenological theories. In particular when $\beta \to \infty$ both exponentials go as $4\beta J^2/|h|$ and this value is identical to that obtained by Langer [16] in the framework of the droplet model.

Conclusions

The results of the previous sections can be summarized by the following two theorems.

Theorem 1. *Given an Ising spin system with n.n. interaction of strength* $-\frac{J}{2} < 0$ *and external magnetic field* $\frac{h}{2} < 0$, *contained in a box* Λ *with boundary conditions* $+$, *call* R_Λ *the subset of all spin configurations* $\boldsymbol{\sigma}$ *such that any region bounded by a contour has area less or equal to* c^2 *where* c *is a fixed integer number and* $S_{c,\Lambda}$ *the state defined by the probability distribution*

$$p_\Lambda(\boldsymbol{\sigma}) = K e^{-\beta E_\Lambda(\boldsymbol{\sigma})} \qquad \text{if } \boldsymbol{\sigma} \in R_\Lambda$$

$$p_\Lambda(\boldsymbol{\sigma}) = 0 \qquad \text{otherwise}$$

where K *is a normalization constant and* $E_\Lambda(\boldsymbol{\sigma})$ *is the energy of the spin configuration* $\boldsymbol{\sigma}$.

If $Z_c^+(\Lambda, h)$ *is the partition sum over* R_Λ, *and* $\langle \sigma_A \rangle_{\Lambda,h}^c$, $A \subseteq \Lambda$, *the related spin correlation function, when* $\beta > 2\ln 6/J$, $c < 4(\beta J - 2\ln 6)/\beta|h|$.

a) *The following limits exist and are finite:*

$$F^c(\beta, h) = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ln Z_c^+(\Lambda, h)$$

$$\langle \sigma_A \rangle_h^c = \lim_{\Lambda \to \infty} \langle \sigma_A \rangle_{\Lambda,h}^c .$$

b) *The infinite volume correlation functions have cluster properties.*

c) *The central limit theorem holds for the magnetization and the energy.*

d) *Consider a sequence of* c*'s such that* $c \to \infty$ *as* $h \to 0^-$ *then*

$$\lim_{h \to 0^-} \langle \sigma_A \rangle_h^c = \langle \sigma_A \rangle_{+,0}$$

$$\lim_{h \to 0^-} F^c(\beta, h) = F(\beta, 0)$$

where $\langle \sigma_A \rangle_{+,0}$ *and* $F(\beta, 0)$ *are the corresponding quantities at equilibrium.*

e) *If* $Z^+(\Lambda, h)$ *is the partition sum over the full equilibrium ensemble, the following limit holds:*

$$\lim_{\Lambda \to \infty} Z_c^+(\Lambda, h)/Z^+(\Lambda, h) = 0 .$$

If we define the time evolution of our system as described by a Markovian master equation, Theorem 2 holds.

Theorem 2. *Assuming that the system is in the state* $S_{c,\Lambda}$ *at* $t=0$ *and calling* $p(R_\Lambda; t)$ *the probability to find it at time t in a configuration belonging to* R_Λ.

α) *For any* $\Lambda, p(R_\Lambda; t)$ *is monotonic decreasing and* $\frac{d}{dt} p(t; R_\Lambda)$ *is monotonic increasing function of t.*

β) *Define for any* Λ *the escape rate* v_{R_Λ} *from* R_Λ *as*

$$v_{R_\Lambda} = -\frac{d}{dt} p(R_\Lambda; t)\Big|_{t=0} .$$

Then, when $\beta > 2\ln 6/J$, $c \leqq 4(\beta J - 2\ln 6)/\beta|h|$

$$F_1(\beta, h, c)\, e^{-4\beta J c + \beta m^*(\beta)|h|c^2}\left(1 + o\left(\frac{|\partial\Lambda|}{|\Lambda|}\right)\right)$$

$$\leqq \frac{v_{R_\Lambda}}{|\Lambda|} \leqq F_2(\beta, h, c)\, e^{-(\beta J - \ln 3)4c + \beta|h|c^2}$$

where $m^*(\beta)$ *is the equilibrium magnetization at* $h=0^+$, *and* F_1 *and* F_2 *are bounded, positive functions.*

The above two theorems give a series of properties that nicely fit what is expected on a physical ground from a metastable state.

$S_{c,\Lambda}$ is a state to which thermodynamics applies (Point a)) and it is in a pure phase (Points b) and c)). Furthermore the limit of the free energy for $h \to 0^-$ over a suitable sequence of such states joins very smoothly to equilibrium value at $h=0^+$ (Point d). Configurations belonging to R_Λ are very unlikely in the full equilibrium ensemble (Point e).

The escape rate, in full agreement with the phenomenological theories of nucleation, is asymptotically proportional to the volume and in a suitable range of c that depends on β and h, the escape rate per unit volume can be made very small compared with the rate of the single spin flip process (cf. end of Section IV).

Comparing our results with what was previously known for long range forces systems [1] we see that in contrast with the Van der Waals theory where the isotherm in the metastable region is simply the analytic continuation of the equilibrium isotherm, we actually get a less defined prescription. In particular we are not able to discriminate between a large class of sequences of subensembles that give rise to smooth extensions of the isotherm for negative magnetic fields.

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Appendix A

In this appendix, we show that the quantity $-\frac{d}{dt}p(R_\Lambda;t)$ defined in Section I, is a monotonic decreasing function of t.

To simplify our notations, we label the spin configurations in Λ with an integer n, $n=1,\ldots,2^{|\Lambda|}$, and define:

$$W_{mn}=W_\Lambda(\boldsymbol{\sigma}',\boldsymbol{\sigma}) \qquad p_n(t)=p_\Lambda(\boldsymbol{\sigma},t) \qquad g_n=e^{-\beta E_\Lambda(\boldsymbol{\sigma})} \tag{A1}$$

where n and m are the indices corresponding to $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ respectively. The Λ dependence is omitted.

The detailed balance condition is then written:

$$W_{mn}g_n=W_{nm}g_m \tag{A2}$$

and the master equation is

$$\begin{aligned}\frac{d}{dt}p_n(t)&=\sum_{m=1}^{2^{|\Lambda|}}(W_{nm}p_m(t)-W_{nn}p_n(t))\\&=\sum_{m=1}^{2^{|\Lambda|}}\Big(W_{nm}-\sum_{l=1}^{2^{|\Lambda|}}W_{lm}\delta_{mn}\Big)p_n(t)\,.\end{aligned} \tag{A3}$$

Following the procedure introduced in [17], we define the $2^{|\Lambda|}\times 2^{|\Lambda|}$ matrix T as:

$$T_{nm}=g_n^{-1/2}g_m^{1/2}\left(W_{nm}-\sum_{l=1}^{2^{|\Lambda|}}W_{lm}\delta_{mn}\right) \tag{A4}$$

T is a real symmetric matrix, and all its eigenvalues λ^i, $i=1,\ldots,2^{|\Lambda|}$ are non positive [17]. This can be easily checked using (A2) and the fact that $W_{nm}\geqq 0$. The null eigenvalue is not degenerate if, for any n and m, there is a sequel $n_1,\ldots,n_r$ such that $n_1=n$, $n_r=m$, and $W_{n_i,n_{i+1}}\neq 0$.

Denoting with $\{C_n^i\}$, $n=1,\ldots,2^{|\Lambda|}$ the normalized eigenvector of T corresponding to λ^i, we easily find:

$$\begin{aligned}\frac{d}{dt}g_n^{-1/2}p_n(t)&=\sum_{m=1}^{2^{|\Lambda|}}T_{nm}g_m^{-1/2}p_m(t)\\&=\sum_{i=1}^{2^{|\Lambda|}}\lambda^iC_n^i\sum_{m=1}^{2^{|\Lambda|}}g_m^{-1/2}p_m(t)\,C_m^i\\p_n(t)&=g_n^{1/2}\sum_{i=1}^{2^{|\Lambda|}}e^{\lambda^it}C_n^i\sum_{m=1}^{2^{|\Lambda|}}g_m^{-1/2}p_m(0)\,C_m^i\,.\end{aligned} \tag{A5}$$

Let now M be the subset of indices corresponding to the subset R_Λ of spin configurations in Λ. Inserting in (A5) the initial condition:

$$\begin{aligned} p_m(0) &= g_m \sum_{1}^{2^{|\Lambda|}} {}_k\, g_k \Big/ \sum_{k\in M} g_n = K_\Lambda g_m \qquad m \in M \\ p_m(0) &= 0 \qquad\qquad\qquad\qquad\qquad\quad m \notin M \end{aligned} \tag{A6}$$

we find eventually:

$$\begin{aligned} p(R_\Lambda; t) &= \sum_{m\in M} p_m(t) \\ &= K_\Lambda \sum_{1}^{2^{|\Lambda|}} {}_i\, e^{\lambda_i t} \Big(\sum_{m\in M} p_m^{1/2} C_m^i \Big)^2 \end{aligned} \tag{A7}$$

That proves the initial assertion.

Appendix B

In the first part of this appendix, we show that the deduction of (III.4), that guarantee the existence of the thermodynamic limit for the $\varrho^c_{\Lambda,h}$, may be carried out by a mere transcription of the procedure described in [11], introduced by Minlos and Sinai to prove the existence of the thermodynamic limit of the $\varrho_{\Lambda,h}$, for $h \geqq 0$.

It is sufficient to check that, when $h \leqq 0$ and β is high enough, the probability distribution (II.1) satisfyies the following conditions.

1) The probability to find a family $\gamma_1, \ldots, \gamma_n$ of compatible outer contours in Λ, such that $|\Theta(\gamma_i)| \leqq c^2$, has a bound that decreases exponentially with the total lenght of the contours, that is:

$$\varrho^c_{\Lambda,h}(\gamma_1, \ldots, \gamma_n) \leqq e^{-2\alpha \sum_1^h {}_i |\gamma_i|} \qquad \alpha > 0\,. \tag{B1}$$

2) The ratio between $\varrho^c_{\Lambda,h}(\gamma_1, \ldots, \gamma_n)$ and the probability $\varrho^{c;\gamma_1}_{\Lambda,h}(\gamma_2, \ldots, \gamma_h)$ to find a configuration $\boldsymbol{\sigma}$ containing the outer contours $\gamma_2, \ldots, \gamma_n$, and such that no one of the outer contours surrounds $\Theta(\gamma_1)$ or intersects its boundary, does not depend on $\gamma_2, \ldots, \gamma_n$, and has a bound that exponentially decreases with $|\gamma_1|$, that is:

$$\varrho^c_{\Lambda,h}(\gamma_1, \ldots \gamma_n)/\varrho^{c\,\gamma_1}_{\Lambda,h}(\gamma_2, \ldots \gamma_n) = \lambda(\gamma_1, h) \leqq e^{-2\alpha|\gamma_1|}\ \alpha > 0\,. \tag{B2}$$

We remark that the set of spin configuration obtained by erasing any of the contours $\gamma_1, \ldots \gamma_n$, that is by flipping all the spins inside $\Theta(\gamma_i)$, in all the configurations in R_Λ where $\gamma_1, \ldots, \gamma_n$ are present, is still a subset of R_Λ.

The argument already used in Section I gives then:

$$\varrho^c_{\Lambda,h}(\gamma_1,\dots\gamma_n)=\frac{\prod_1^n{}_i\,\mu(\gamma_i,h)\sum'_{\substack{\{\tilde\gamma\}:\,|\Theta(\tilde\gamma)|\leqq c^2\\ \Theta(\tilde\gamma)\subseteqq\Lambda}}\prod_{\tilde\gamma\in\{\tilde\gamma\}}\mu(\tilde\gamma,h)}{\sum_{\substack{\{\tilde\gamma\}:\,|\Theta(\tilde\gamma)|\leqq c^2\\ \Theta(\tilde\gamma)\subseteqq\Lambda}}\prod_{\tilde\gamma\in\{\tilde\gamma\}}\mu(\tilde\gamma,h)} \tag{B3}$$

$$\leqq\prod_1^n{}_i\,e^{-\beta J|\gamma_i|}\zeta^-(\gamma_i,h)/\zeta^+(\gamma_i,h)\,.$$

$$\varrho^c_{\Lambda,h}(\gamma_1,\dots,\gamma_n)/\varrho^{c,\gamma_1}_{\Lambda,h}(\gamma_2,\dots,\gamma_n)=e^{\beta J|\gamma_1|}\zeta^-(\gamma_1,h)/\zeta^+(\gamma_1,h)\,. \tag{B4}$$

When $h\leqq 0$, $|\Theta(\gamma)|\leqq c^2$, $\zeta^-(\gamma,h)/\zeta^+(\gamma,h)\leqq e^{\beta|h|\,|\Theta(\gamma)|}\leqq e^{\beta|h|\frac{c}{4}|\gamma|}$ the above conditions (B1) and (B2) are then satisfied, provided that $\beta J-\beta|h|c/4\geqq 2\alpha$, $\alpha>0$.

A set of integral equations for the $\varrho^c_{\Lambda,h}$ may be now derived as in [13]: it is sufficient to remark that the probability $\varrho^{c,\gamma_1}_{\Lambda,h}(\gamma_1,\dots,\gamma_n)$ may be expressed as a linear combination of the $\varrho^c_{\Lambda,h}(\tilde\gamma_1,\dots,\tilde\gamma_k,\gamma_2,\dots,\gamma_n)$, $k\geqq 1$.

If we define the linear operators X_Λ, X_c and A^c_Λ on the linear space of the functions φ defined over the sets of compatible outer contours as:

$$(X_\Lambda\varphi)(\gamma_1,\dots\gamma_n)=\prod_1^n{}_i\,X_\Lambda(\gamma_i)\,\varphi(\gamma_1,\dots\gamma_n) \tag{B5}$$

where

$$X_\Lambda(\gamma)=1\quad\text{if}\quad\Theta(\gamma)\subseteqq\Lambda\,,$$
$$X_\Lambda(\gamma)=0\quad\text{otherwise}$$

$$(X_c\varphi)(\gamma_1,\dots\gamma_n)=\prod_1^n{}_i\,X_c(\gamma_i)\,\varphi(\gamma_1,\dots\gamma_n) \tag{B6}$$

where

$$X_c(\gamma)=1\quad\text{if}\quad|\Theta(\gamma)|\leqq c^2$$
$$X_c(\gamma)=0\quad\text{otherwise}$$

$$\begin{aligned}&(A^c_\Lambda\varphi)(\gamma_1)\\&=X_c(\gamma_1)X_\Lambda(\gamma_1)\,\lambda(\gamma_1,h)\\&\cdot\Big\{\sum_{k\geqq 1}(-1)^k\sum_{\{\tilde\gamma_1\dots\tilde\gamma_k\}:\,\tilde\gamma_i\cap\gamma_1\neq\varphi}(X_cX_\Lambda\varphi)(\tilde\gamma_1,\dots\tilde\gamma_k)-\sum_{\tilde\gamma:\,\Theta(\tilde\gamma)\supseteqq\Theta(\gamma_1)}(\chi_cX_\Lambda\varphi)(\tilde\gamma)\Big\}\end{aligned} \tag{B7}$$

and for $n\geqq 2$

$$\begin{aligned}(A^c_\Lambda\varphi)(\gamma_1,\dots\gamma_n)&=X_c(\gamma_1)X_\Lambda(\gamma_1)\,\lambda(\gamma_1,h)\Big\{\varphi(\gamma_2,\dots\gamma_n)\\&\quad+\sum_{k\geqq 1}(-1)^k\sum_{\{\tilde\gamma_1,\dots\tilde\gamma_k\}:\,\tilde\gamma_i\cap\gamma_1\neq\varphi}(X_cX_\Lambda\varphi)(\tilde\gamma_1,\dots,\tilde\gamma_k,\gamma_2,\dots\gamma_n)\\&-\sum_{\tilde\gamma:\,\Theta(\tilde\gamma)\geqq\Theta(\gamma_1)}(X_cX_\Lambda\varphi)(\tilde\gamma,\gamma_2,\dots\gamma_n)\Big\}\end{aligned}$$

where the sum are over sets of compatible outer contours. The set of integral equations satisfied by the $\varrho^c_{\Lambda,h}$ may be written in the form:

$$\varrho^c_{\Lambda,h} = X_c X_\Lambda \lambda_h + A^c_\Lambda \varrho^c_{\Lambda,h} \tag{B8}$$

where $\lambda_h(\gamma_1, \dots \gamma_n) = 0$ for $n > 1$, and $\lambda_h(\gamma_1) = \lambda(\gamma_1, h)$.

We remark that the Minlos Sinai equations, satisfied by the $\varrho_{\Lambda,h}$, may be obtained from (B8) substituting X_c with the unity in the inhomogeneous terms and in the definition of the kernel.

Consider now the Banach space $\mathscr{B}$ of the function φ, defined over the sets of compatible outer contours, with the norm:

$$\|\varphi\| = \sup_{n,(\gamma_1,\dots\gamma_n)} \frac{|\varphi(\gamma_1, \dots \gamma_n)|}{e^{-\alpha(|\gamma_1| + \dots + |\gamma_n|)}}. \tag{B9}$$

An upper bound for the norm of A^c_Λ is given by:

$$\|A^c_\Lambda\| = \sup_\varphi \frac{\|A^c_\Lambda \varphi\|}{\|\varphi\|} \leqq \sup_{\gamma_1} \left(\frac{\zeta}{3}\right)^{|\gamma_1|} \left\{\left(1 + \frac{\zeta^4}{1-\zeta^2}\right)^{|\gamma_1|} + \frac{\zeta^4}{(1-\zeta^2)^2}\right\} \tag{B10}$$

where $\zeta = 3e^{-\alpha}$. This bound is obtained by the inequality:

$$\begin{aligned}|(A^c_\Lambda \varphi)(\gamma_1, \dots \gamma_n)| \leqq e^{-\alpha \sum_1^n{}_i |\gamma_i|} \|\varphi\| \, e^{-\alpha|\gamma_1|} \Bigg\{ & \sum_0^{|\gamma_1|}{}_k \binom{|\gamma_1|}{k} \left(\sum_2^\infty{}_l \zeta^{2l}\right)^k \\ & + \sum_1^\infty{}_x \sum_{l \geqq x+1} \zeta^{2l} \Bigg\}.\end{aligned} \tag{B11}$$

That is deduced with standard majorization techniques [13]. The bound (B10) does not depend on Λ, and is less than one when $\zeta \leqq 1/2$, that is $\alpha \geqq \ln 6$.

This result, together with the condition (B1), is sufficient to guarantee the existence of the thermodynamic limit of the $\varrho^c_{\Lambda,h}$.

The existence of this limit is sufficient for the existence of the limit of the spin correlation functions $\langle \sigma_A \rangle^c_{\Lambda,h}$, as it is easily checked.

The Minlos Sinai equation, when $\|A^c_\Lambda\| < 1$, also guarantee the validity of cluster properties for the outer contours correlation functions [13] that, as in the general case, allow to prove exponential cluster properties for the spin correlation functions. These properties are then also sufficient to prove the validity of the central limit theorem [10].

Consider now $\eta_{\Lambda,h} = \varrho^c_{\Lambda,h} - X_c \varrho_{\Lambda,0}$ (where $\varrho_{\Lambda,0}$ are the outer contour correlation functions defined in Section I for $h = 0$). It is easy to check that, when $\beta J - \beta|h| \, c/4 > 2\ln 6$, $\eta_{\Lambda,h}$ is the unique solution in $\mathscr{B}$ of the equation:

$$\eta_{\Lambda,h} = \xi^c_{\Lambda,h} + A^c_\Lambda \eta_{\Lambda,h} \tag{B12}$$

where

$$\xi^c_{\Lambda,h} = X_c(\lambda_h - \lambda_0) + A^c_\Lambda X_c \varrho_{\Lambda,0} - X_c A_\Lambda \varrho_{\Lambda,0}$$

and A_Λ is the kernel appearing in the M.S. equation for $\varrho_{\Lambda,0}$. To prove (III.8), it is then sufficient to show that:

$$|\xi^c_{\Lambda,h}(\gamma_1, \dots \gamma_n)| \leqq e^{-\alpha \sum_1^n |\gamma_i|} B(\beta, h, c) \tag{B13}$$

where $B(\beta, h, c) \xrightarrow[c \to \infty]{h \to 0^-} 0$.

It follows from the definition of $\xi^c_{\Lambda,h}$:

$$\begin{aligned}
&|\xi^c_{\Lambda,h}(\gamma_1, \dots \gamma_n)| \\
&\leqq |\lambda(\gamma_1, h) - \lambda(\gamma_1, 0)| \Bigg[\sum_{k \geqq 0} \sum_{\substack{(\tilde\gamma_1, \dots \tilde\gamma_k):\, |\Theta(\tilde\gamma_i)| \leqq c^2 \\ \tilde\gamma_i \cap \gamma_1 \neq \phi}} |\varrho_{\Lambda,0}(\tilde\gamma_1, \dots \tilde\gamma_k \gamma_2, \dots \gamma_n)| \\
&\quad + \sum_{\substack{\tilde\gamma:\, |\Theta(\tilde\gamma)| \leqq c^2 \\ \Theta(\tilde\gamma) \supseteqq \Theta(\gamma_1)}} |\varrho_{\Lambda,0}(\tilde\gamma, \gamma_2 \dots \gamma_n)| \Bigg] \\
&\quad + \lambda(\gamma_1, 0) \Bigg[\sum_{k \geqq 1} \sum_{\substack{\{\tilde\gamma_1, \dots \tilde\gamma_k\}:\, \tilde\gamma_i \cap \gamma_1 \neq \varphi \\ \exists \tilde\gamma_i:\, |\Theta(\tilde\gamma_i)| > c^2}} |\varrho_{\Lambda,0}(\tilde\gamma_1, \dots \tilde\gamma_k, \gamma_2, \dots \gamma_n)| \\
&\quad + \sum_{\substack{\tilde\gamma:\, |\Theta(\tilde\gamma)| \supset c^2 \\ \Theta(\tilde\gamma) \supset \Theta(\gamma_1)}} |\varrho_{\Lambda,0}(\tilde\gamma, \gamma_2, \dots, \gamma_n)| \Bigg]
\end{aligned} \tag{B14}$$

where $\gamma_1, \dots \gamma_n$ are s.t. $|\Theta(\gamma_i)| \leqq c^2$.

Recalling that $\varrho_{\Lambda,0}(\gamma_1, \dots \gamma_n) \leqq e^{-\beta J \sum_1^n |\gamma_i|}$, $\lambda(\gamma, 0) = e^{-\beta J}$, $\beta J > 2\alpha$, and using standard techniques of majorization, we can get upper bounds for the terms in the square brackets of the form:

$$C_1 e^{-\alpha \sum_1^n |\gamma_i| + 2\alpha|\gamma_1| - \delta|\gamma_1|} \quad ; \quad C_2 e^{-\alpha \sum_1^n |\gamma_i|} (3e^{-\alpha})^c \, e^{\frac{\beta}{2} J |\gamma_1|}$$

respectively, where we have assumed $\alpha \geqq \ln 6$, and C_1, C_2, δ are positive constant. We find then:

$$\begin{aligned}
&|\xi^c_{\Lambda,h}(\gamma_1, \dots \gamma_n)| \\
&\leqq e^{-\alpha \sum_1^n |\gamma_i|} \left[C_1 e^{-(\beta J - 2\alpha)|\gamma_1| - \delta|\gamma_1|} (e^{\beta |h| \, |\Theta(\gamma_1)|} - 1) + C_2 (3e^{-\alpha})^c \right].
\end{aligned} \tag{B15}$$

It is now easy to check that the square bracket term goes to zero, as $c \to \infty$, $|h| \to 0$, provided that for any c and $|h|: \beta J - 2\alpha \geqq |h| \, c/4 \geqq |h| \, |\Theta(\gamma_1)|/|\gamma_1|$.

Appendix C

In this appendix we give a lower bound for $\varrho^c_{\Lambda,h}(q)$ as the maximum of two independent estimate. We recall that q (see Section IV) is a square contour of maximal area.

a) First Estimate. Remarking that in any $\boldsymbol{\sigma} \in R_\Lambda$ there are no contours embracing q, the probability to find q, according to the probability distribution (II.1) may be written as:

$$\varrho^c_{\Lambda,h}(q) = \lambda(q,h) \frac{\left\langle \prod_{i \in \partial^+ q \cup \partial^- q} \left(\frac{1+\sigma_i}{2}\right) \Theta(R_\Lambda, \boldsymbol{\sigma}) \right\rangle_{\Lambda,+,h}}{\langle \Theta(R_\Lambda, \boldsymbol{\sigma}) \rangle_{\Lambda,+,h}} \tag{C1}$$

where $\lambda(q,h)$ is given by (II.2), $\Theta(R_\Lambda, \boldsymbol{\sigma})$ is such that

$$\Theta(R_\Lambda, \boldsymbol{\sigma}) = 1 \quad \text{if} \quad \boldsymbol{\sigma} \in R_\Lambda$$
$$\Theta(R_\Lambda, \boldsymbol{\sigma}) = 0 \quad \text{otherwise}$$

and $\partial^{+(-)} q$ is the set of sites facing the contour q from outside (inside). Then, noticing that $\Theta(R_\Lambda, \boldsymbol{\sigma})$ is an increasing function over the complete set of configurations $\boldsymbol{\sigma}$, by F.K.G. inequalities [18], we get:

$$\begin{aligned} &\left\langle \prod_{i \in \partial^+ q \cup \partial^- q} \left(\frac{1+\sigma_i}{2}\right) \Theta(R_\Lambda, \boldsymbol{\sigma}) \right\rangle_{\Lambda,+,h} \\ &\qquad \geqq \left\langle \prod_{i \in \partial^+ q \cup \partial^- q} \left(\frac{1+\sigma_i}{2}\right) \right\rangle_{\Lambda,+,h} \langle \Theta(R_\Lambda, \sigma) \rangle_{\Lambda,+,h} \end{aligned} \tag{C2}$$

$$\begin{aligned} \left\langle \prod_{i \in \partial^+ q \cup \partial^- q} \left(\frac{1+\sigma_i}{2}\right) \right\rangle_{\Lambda,+,h} &\geqq \prod_{i \in \partial^+ q \cup \partial^- q} \left(\frac{1+\langle \sigma_i \rangle_{\Lambda,+,h}}{2}\right) \\ &\geqq \left(\frac{1+m(\beta,h)}{2}\right)^{8c}, \end{aligned} \tag{C3}$$

where $m(\beta, h)$ is the equilibrium magnetization and therefore, for $h < 0$ we get:

$$\varrho^c_{\Lambda,h}(q) \geqq \lambda(q,h) \left(\frac{1-m(\beta,|h|)}{2}\right)^{8c}. \tag{C4}$$

b) Second Estimate. Equation (B7) for $\varrho^c_{\Lambda,h}(q)$ reads:

$$\varrho^c_{\Lambda,h}(q) = \lambda(q,h) \left[1 + \sum_{k \geqq 1} (-1)^k \sum_{(\tilde{\gamma}_1, \ldots \tilde{\gamma}_k):\, \tilde{\gamma}_i \cap q \neq \phi} \varrho^c_{\Lambda,h}(\tilde{\gamma}_1, \ldots \tilde{\gamma}_k)\right] \tag{C5}$$

where $\tilde{\gamma}_1, \ldots \tilde{\gamma}_k$ are compatible contours intersecting q.

If we remark that the square bracket term in (C5) is the probability to find a configuration $\boldsymbol{\sigma} \in R_\Lambda$ such that no contour intersect the square q, we get:

$$\varrho^c_{\Lambda,h}(q) \geqq \lambda(q,h) \left[1 - \sum_{\gamma:\, \gamma \cap q \neq \phi} \varrho^c_{\Lambda,h}(\gamma)\right] \tag{C6}$$

that, by the use of standard majorization techniques, becomes

$$\varrho^c_{\Lambda,h}(q) \geqq \lambda(q,h)\left[1-4c\frac{e^{-(\beta J-\ln 3)4+\beta|h|c}}{1-e^{-2(\beta J-\ln 3-\beta|h|c/4)}}\right]. \tag{C7}$$

We remark that, for fixed β and $|h|$ the square bracket term gets negative for large c but, in a suitable range of β, $|h|$ and c (C7) is better that (C4). If we recall now Eqs. (II.9) and (II.4) and put

$$F_1(\beta,h,c) = \max\left\{\left(\frac{1-m(\beta,|h|)}{2}\right)^{8c}, \quad 1-4c\frac{e^{-4(\beta J-\ln 3)+\beta|h|c}}{1-e^{-2(\beta J-\ln 3-\beta|h|c/4)}}\right\} \tag{C8}$$

we get (IV.6).

References

1. Lebowitz, J. L., Penrose, O.: J. Stat. Phys. **3**, 211 (1971)
2. Lebowitz, J. L., Penrose, O.: J. Math. Phys. **7**, 98 (1966)
3. Fisher, M. E.: Physics **3**, 255 (1967)
4. Andreev, A. F.: Soviet Phys. JEPT **18**, 1415 (1964)
5. Frenkel, J.: Kinetic Theory of Liquids, Chapter VII. New York: Dover Publications 1955
6. Glauber, R. J.: J. Math. Phys. **4**, 294 (1963)
7. Holley, K.: Commun. math. Phys. **23**, 87 (1971)
8. Kawasaki, K.: In: Domb, C., Green, M. S. (Eds.): Phase Transitions and Critical Phenomena, New York: Academic Press 1972
9. Cassandro, M., Da Fano, A.: Commun. math. Phys. **36**, 277—286 (1974)
10. Gallavotti, G., Martin Löf, A., Miracle Solé, F.: In: Battelle Seattle 1971. Rencontres, Berlin-Heidelberg-New York: Springer 1971
11. Martin Löf, A.: Commun. math. Phys. **32**, 75 (1973)
12. Minlos, K. A., Sinai, Ya. G.: Math. Sb. **73**, 375 (1967) (English Translation: Math. USSR-Sbornik **2**, 335)
13. Minlos, R. A., Sinai, Ya. G.: Trudy Moskov Mat. Pbs. **17**, 213 (1967) (English Translation: Trans. Moscow Math. Soc. **17**, 237)
14. Minlos, K. A., Sinai, Ya. G.: Trudy Moskov Mat. Obs. **19**, 113 (1968) (English Translation: Trans. Moscow Math. Soc. **19**, 121)
15. Gallavotti, G.: Rivista Nuovo Cim. **2**, 133 (1972)
16. Langer, J.: Ann. Phys. **41**, 108 (1967)
17. Kadanoff, L. P., Swift, J.: Phys. Rev. **165**, 310 (1968)
18. Lebowitz, J. L., Monroe, J. L.: Commun. math. Phys. **25**, 276 (1972)

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