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Irreducible Multiplier Corepresentations and Generalized Inducing

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Abstract. Wigner's classification of irreducible corepresentations into three types is generalised to irreducible multiplier corepresentations. Representations of Types I, II, and III have commutants isomorphic to $\boldsymbol{R}, \boldsymbol{H}$, and $\boldsymbol{C}$, respectively. The more general problem of relating irreducible multiplier corepresentations of a group to those of an invariant subgroup is considered, and some algebraic aspects of "generalized inducing" are described. The Wigner classification is then re-obtained as a very simple instance of the general theory.

§ 1. Introduction

1.1. Consider an irreducible PUA-representation (see Parthasarathy [6]) $\boldsymbol{U}$ of a group G with respect to a fixed UA-decomposition $G=G^+\cup G^-$, with G^- non-empty. (Thus G must possess an invariant subgroup G^+ of index 2.) If U is a version of $\boldsymbol{U}$, then $U(g)$ is a unitary ($g\in G^+$) or antiunitary ($g\in G^-$) operator on a complex Hilbert space $\mathscr{H}$ which satisfies

$$U(g_1)\,U(g_2)=\sigma(g_1,g_2)\,U(g_1g_2)\,, \tag{1.1}$$

where the (generalised) multiplier (with respect to the fixed UA-decomposition $G=G^+\cup G^-$) σ is a function $G\times G\to\boldsymbol{T}$ (=complex numbers of unit modulus) which satisfies, for all $g_1,g_2,g_3\in G$, the equation

$$\sigma(g_1,g_2)\,\sigma(g_1g_2,g_3)=\sigma(g_1,g_2g_3)\,\sigma(g_2,g_3)^{g_1}\,. \tag{1.2}$$

Here $\lambda^g(\lambda\in\boldsymbol{C}, g\in G)$ is defined to be λ if $g\in G^+$ and $\bar{\lambda}$ if $g\in G^-$. The map $g\mapsto U(g)$, from G into the unitary antiunitary group of $\mathscr{H}$, is called a multiplier corepresentation of G with multiplier σ, or simply a σ-corepresentation. The multiplier σ' of any other version U' of $\boldsymbol{U}$ will be equivalent to the multiplier σ, satisfying that is

$$\sigma'(g_1,g_2)=[\lambda(g_1)\,\lambda(g_2)^{g_1}/\lambda(g_1g_2)]\,\sigma(g_1,g_2) \tag{1.3}$$

for some $\lambda: G\to\boldsymbol{T}$.

As is usual, we will always adjust the overall phase of the $U(g)$ so that

$$U(e) = I, \qquad (e = id_G), \tag{1.4}$$

whence the multiplier σ satisfies

$$\sigma(e, g) = \sigma(g, e) = 1, \qquad \text{for all} \quad g \in G. \tag{1.5}$$

It will also prove convenient to make some arbitrary, but fixed, choice of element $a \in G^-$. We also write $G^+ = H$, and so the fixed UA-decomposition of G with which we are working is

$$G = H \cup aH. \tag{1.6}$$

The restriction U_H of U to the subgroup H is a σ_H-representation (whose operators are all unitary) of H, where σ_H denotes the restriction of σ to $H \times H$ and so is a multiplier for H in the more usual sense – i.e. satisfies (1.2) except that no complex conjugation is involved in the r.h.s.

1.2. Now Wigner [9] showed that ordinary (i.e. $\sigma \equiv 1$) irreducible corepresentations can be classified into three distinct types, according as U_H is irreducible (Type I) or decomposes into a direct sum $D \oplus C$ of two irreducible representations D, C, with C linearly equivalent (Type II) or linearly inequivalent (Type III) to D. In Theorems A and B below we generalise Wigner's results to the case when the multiplier σ is non-trivial. In this connection, observe that the direct sum $D \oplus C$ is defined as a multiplier representation only when D, C have the same multiplier. Our proofs are essentially basis-free versions of Wigner's proofs, with multipliers judiciously inserted. Actually – see § 4 – the mathematical ideas which are involved are special instances (in which the invariant subgroup is of index 2) of those employed by Clifford [3]; this last author in turn traces their earlier history, going back to the work of Frobenius. We also show (Theorem C) that the three different types of irreducible multiplier corepresentations may be distinguished by their having different commutants – see also Ascoli and Teppati [1] in this connection.

1.3. It is perhaps worth stressing that in our generalization of Wigner's results the relevant notion of equivalence of two multiplier representations D_1, D_2 of H is still ordinary equivalence:

$$PD_1(h)\,P^{-1} = D_2(h), \qquad \text{for all } h \in H, \tag{1.7}$$

and *not* "projective equivalence":

$$PD_1(h)\,P^{-1} = \lambda(h)\,D_2(h). \tag{1.8}$$

In the case of linear equivalence [P in Eq. (1.7) a linear isomorphism] it follows that D_1 and D_2 must have the same multiplier: $\sigma_1 = \sigma_2$; in the case of antilinear equivalence (P an antiisomorphism) it follows that $\sigma_1 = \overline{\sigma_2}$.

Actually, as is well known, linear (antilinear) equivalence of D_1 with D_2 implies unitary (antiunitary) equivalence; in particular when D_1, D_2 in (1.7) are irreducible, P is necessarily proportional to a unitary (antiunitary) mapping $\mathscr{H}_1 \to \mathscr{H}_2$ ($\mathscr{H}_i$ = carrier space of $D_i, i = 1, 2$) – for (1.7) implies, on using $D(h)^\dagger = D(h^{-1})$, that $P^\dagger P$ commutes with every $D_1(h)$ and so is proportional to the identity operator on $\mathscr{H}_1$.

The reason why ordinary equivalence is still important when dealing with multiplier representations is as follows. While one wishes, at any rate in quantum mechanics, to classify multiplier representations of a group H only up to *projective* unitary equivalence, nevertheless for a *decomposable* representation (such as U_H in Theorem A) this involves knowing its irreducible constituents up to *ordinary* unitary equivalence. For example, suppose C, D are σ-representations of H and λ is a non-trivial 1-dimensional unitary representation (character) of H. Then the σ-representation D' of H defined by $D'(h) = \lambda(h) D(h)$ is clearly [take $P = I$ in Eq. (1.8)] projectively unitarily equivalent to D. Nevertheless it will often be the case that D' and D are unitarily inequivalent – *in which case the σ-representations $U = C \oplus D$ and $U' = C \oplus D'$ are unitarily inequivalent, even projectively speaking.* A similar example for σ-corepresentations – with λ a *generalized character*, satisfying

$$\lambda(g)\,\lambda(g')^g = \lambda(gg') \tag{1.9}$$

– shows that we need to know the irreducible constituents of a decomposable σ-corepresentation up to ordinary unitary equivalence. Perhaps we have laboured this point too much – everyone knows in quantum mechanics that overall phases are unimportant, but that relative phases do carry physical content (in the absence of superselection rules).

1.4. The authors' interest in the general problem which is answered by Theorem B – namely that of constructing the irreducible σ-corepresentations of G out of the irreducible σ_H-representations of H – arose in connection with the extended Poincare group $\mathscr{P}$, where the physically relevant UA-decomposition is of course $\mathscr{P} = \mathscr{P}^{\uparrow} \cup \mathscr{P}^{\downarrow}$ (i.e. space inversion S represented linearly and time reversal T represented antilinearly). In [10], Wigner determined the physically relevant representations of $\mathscr{P}$ by (in effect) proving Theorem B in the case $G = \mathscr{P}$, $H = \mathscr{P}^{\uparrow}$. Another way of finding the representations of $\mathscr{P}$ is to proceed at the little group level, as in a forthcoming paper by us [7]. In the case of the positive mass representations the relevant little groups, for representations of $\mathscr{P}$ and $\mathscr{P}^{\uparrow}$, are[1] $G = \mathrm{SU}(2) \times F_4$ and $H = \mathrm{SU}(2) \times F_2$, respectively, where F_2 is the group of order 2 generated by S and F_4 is the group of order 4 generated by S and T. Whichever approach is adopted, it helps to have to hand the general Theorem B. In the second approach, the theory of Clifford [3] and Mackey [5] needs to be generalised[2] so as to deal with cases where some of the group elements are represented antiunitarily.

1.5. In § 4 we deal with some of the chief algebraic aspects of such a generalization – see Parthasarathy [6] and Lever [4] for some relevant measure-theoretic and analytical details. The Wigner classification can then be seen (§ 4.5) as a particularly simple instance of the general theory; in particular a corepresentation of Type III is a simple instance of a corepresentation obtained by "generalized inducing".

§ 2. Classification of Irreducible Multiplier Corepresentations

Throughout this section h, h_1, h_2 will denote arbitrary elements of the subgroup H of index 2 of G, while a will denote some fixed choice of element such that $G = H \cup aH$. We will repeatedly use such elementary facts as $a^2 \in H$, $aha^{-1} \in H$, $ah_1ah_2 \in H$, etc. Before coming to Theorems A–C it will be convenient to first state and prove a preliminary lemma.

Lemma. *Suppose we are given*

(i) *a multiplier σ for the group G with respect to the decomposition $G = H \cup aH$;*

(ii) *a σ_H-represention U of H with carrier space $\mathscr{H}$:*

$$U(h_1)\,U(h_2) = \sigma(h_1, h_2)\,U(h_1 h_2),\ h_1, h_2 \in H\,; \tag{2.1}$$

(iii), (iv) *an antiunitary operator $U(a)$ on $\mathscr{H}$ which satisfies*

$$\text{(iii)}\qquad U(h)\,U(a) = [\sigma(h, a)/\sigma(a, a^{-1}ha)]\,U(a)\,U(a^{-1}ha)\,; \tag{2.2}$$

$$\text{(iv)}\qquad U(a)^2 = \sigma(a, a)\,U(a^2)\,. \tag{2.3}$$

If we now extend U to the whole of G by defining $U(g)$ for $g \in aH$ by

$$\text{(v)}\qquad U(ah) = \sigma(a, h)^{-1}\,U(a)\,U(h)\,, \tag{2.4}$$

then U is a σ-corepresentation of G.

Proof. We need to check that the representation property (1.1) holds for the four kinds of choice $g_1 = h_1$ or ah_1 and $g_2 = h_2$ or ah_2.

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[1] In [6] Parthasarathy incorrectly (§ 5, Case 3) takes the little group for $\mathscr{P}$ to be $\mathrm{SU}(2) \times F_2$. This error results from omitting the complex conjugation in our Eq. (4.1).

[2] But see § 6 of [3], where such a generalization is already embarked upon. See also [11], [12] and [13].

The choice $g_1 = ah_1$, $g_2 = ah_2$ involves the lengthiest check, and we give it here, supposing first of all that the multiplier σ is identically equal to 1:

$$\begin{aligned} U(ah_1)\,U(ah_2) &= U(a)\,U(h_1)\,U(a)\,U(h_2)\,, \quad \text{by (v)}\,,\\ &= U(a)^2\,U(a^{-1}h_1 a)\,U(h_2)\,, \quad \text{by (iii)}\,,\\ &= U(a^2)\,U(a^{-1}h_1 a h_2)\,, \quad \text{by (iv) and (ii)}\,,\\ &= U(ah_1 a h_2)\,, \quad \text{by (ii)}\,. \end{aligned}$$

This check carries through for a general multiplier σ upon making repeated use of Eq. (1.2). (See also § 3.1.)

Theorem A. *For any irreducible σ-corepresentation U of G (with respect to the decomposition $G = H \cup aH$), one of the following three mutually exclusive possibilities must hold for the restriction U_H of U to the subgroup H:*

Type I: U_H is irreducible.

Types II and III: U_H decomposes into a direct sum $U_H = D \oplus C$ of irreducible σ_H-representations C, D, which are either linearly equivalent (Type II) or linearly inequivalent (Type III).

A σ-corepresentations U of Type II can (upon identifying the carrier space $\mathscr{H}_C$ of C with that $\mathscr{H}_D$ of D) be cast in the form

$$U(h) = \begin{pmatrix} D(h) & 0 \\ 0 & D(h) \end{pmatrix}, \quad U(a) = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}, \tag{2.5}$$
$$U(ah) = \sigma(a,h)^{-1}\,U(a)\,U(h)\,,$$

where K is an antiunitary operator satisfying

$$K D(a^{-1}ha)\,K^{-1} = [\sigma(a, a^{-1}ha)/\sigma(h,a)]\,D(h)\,, \tag{2.6}$$

for all $h \in H$, and also satisfying

$$K^2 = -\sigma(a,a)\,D(a^2)\,. \tag{2.7}$$

A σ-corepresentation U of Type III can be cast in the form

$$U(h) = \begin{pmatrix} D(h) & 0 \\ 0 & C(h) \end{pmatrix}, \quad U(a) = \begin{pmatrix} 0 & \sigma(a,a)\,D(a^2)\,K^{-1} \\ K & 0 \end{pmatrix},$$
$$U(ah) = \sigma(a,h)^{-1}\,U(a)\,U(h)\,, \tag{2.8}$$

where the irreducible σ_H-representations C, D of H are linearly inequivalent, and where $K: \mathscr{H}_D \to \mathscr{H}_C$ is antiunitary and satisfies

$$K D(a^{-1}ha)\,K^{-1} = [\sigma(a, a^{-1}ha)/\sigma(h,a)]\,C(h) \tag{2.9}$$

for all $h \in H$.

Proof. Let $\mathscr{H}$ denote the carrier space of U (and hence of U_H) and let $\mathscr{H}_D \subset \mathscr{H}$ denote the carrier space of some irreducible σ_H-representation D which is contained in the σ_H-representation U_H. Let $\mathscr{H}_C$ denote the image of $\mathscr{H}_D$ under the action of $U(a)$, and let K denote the restriction of $U(a)$ to $\mathscr{H}_D, \mathscr{H}_C$; thus $K: \mathscr{H}_D \to \mathscr{H}_C$ is antiunitary, and $K^{-1}: \mathscr{H}_C \to \mathscr{H}_D$ exists.

On restricting the following identity [an immediate consequence of Eq. (1.1)]

$$U(a)\,U(a^{-1}ha)\,U(a)^{-1} = [\sigma(a, a^{-1}ha)/\sigma(h, a)]\,U(h) \tag{2.10}$$

to the subspace $\mathscr{H}_C$, we see that the invariance of $\mathscr{H}_D$ under U_H implies the invariance of $\mathscr{H}_C$ under U_H, and that the restriction C of U_H to $\mathscr{H}_C$ is given as in Eq. (2.9). Clearly the irreducibility of the σ_H-representation D implies the irreducibility of the σ_H-representation C.

The irreducibility of D implies that $\mathscr{H}_C \cap \mathscr{H}_D$ can not be a proper subspace of $\mathscr{H}_D$. Moreover, since G is generated by H and a, the irreducibility of U implies that $\mathscr{H}_D + \mathscr{H}_C = \mathscr{H}$. Consequently either $\mathscr{H} = \mathscr{H}_D = \mathscr{H}_C$ or $\mathscr{H} = \mathscr{H}_D \oplus \mathscr{H}_C$; hence the three possibilities listed in the first part of the theorem are indeed the only ones to consider.

In the cases $\mathscr{H} = \mathscr{H}_D \oplus \mathscr{H}_C$ (Types II and III) we have

$$U(h) = \begin{pmatrix} D(h) & 0 \\ 0 & C(h) \end{pmatrix}, \qquad U(a) = \begin{pmatrix} 0 & L \\ K & 0 \end{pmatrix}, \tag{2.11}$$

where, in view of Eq. (2.10) the antiunitary operators $K: \mathscr{H}_D \to \mathscr{H}_C$ and $L: \mathscr{H}_C \to \mathscr{H}_D$ must satisfy

$$\begin{aligned} KD(a^{-1}ha)\,K^{-1} &= [\sigma(a, a^{-1}ha)/\sigma(h, a)]\,C(h) \\ LC(a^{-1}ha)\,L^{-1} &= [\sigma(a, a^{-1}ha)/\sigma(h, a)]\,D(h) \end{aligned} \tag{2.12}$$

and, since $U(a)^2 = \sigma(a, a)\,U(a^2)$, must also satisfy

$$KL = \sigma(a, a)\,C(a^2)\,, \qquad LK = \sigma(a, a)\,D(a^2)\,. \tag{2.13}$$

According to this last equation, we have $L = \sigma(a, a)\,D(a^2)\,K^{-1}$; hence we have obtained the "canonical form" (2.8) in the last part of the theorem.

When (Type II) C is linearly equivalent to D, it is convenient to identify $\mathscr{H}_C$ with $\mathscr{H}_D$ and to take C actually equal to D:

$$\mathscr{H} = \mathscr{H}_D \oplus \mathscr{H}_D\,, \qquad U(h) = D(h) \oplus D(h)\,; \tag{2.14}$$

in which case we deduce from Eq. (2.12) that LK^{-1} commutes with each $D(h)$ and hence must equal ωI, where (since LK^{-1} is unitary) $|\omega| = 1$. Moreover, when $C = D$, Eq. (2.13) yields $LK = KL$, i.e.

$\omega K^2 = K\omega K = \bar{\omega} K^2$, and so ω must be real; thus $\omega = \pm 1$. Actually the value $\omega = +1$ can not arise, given, as we are, that U is irreducible: for when $L = +K$ the proper subspace of $\mathscr{H}$ consisting of those vectors of the form $\phi \oplus \phi$, $\phi \in \mathscr{H}_D$, is clearly invariant under the $U(h)$ and under $U(a)$, and hence under $U(g)$ for all $g \in G$. We are thus left with the possibility $\omega = -1$, i.e. $L = -K$, in agreement with the "canonical form" (2.5) in the second part of the theorem. This possibility does give rise to an irreducible U – see the next remark (also Theorem C, Remark). □

Remark. The preceding derivation of the canonical form of a Type II representation can be carried through in terms of a carrier space $\mathscr{H} = \mathscr{H}_D \otimes \mathscr{H}_2$, where $\mathscr{H}_2$ is a 2-dimensional Hilbert space. Instead of Eq. (2.5), we arrive at

$$U(h) = D(h) \otimes I\,, \qquad U(a) = K \otimes k\,, \tag{2.5'}$$

where the antiunitary operator K satisfies Eq. (2.6) and where, since $U(a)^2 = \sigma(a,a)\, U(a^2)$, we must have $K^2 \otimes k^2 = \sigma(a,a)\, D(a^2) \otimes I$. Hence

$$K^2 = \varepsilon\sigma(a,a)\, D(a^2)\,, \qquad k^2 = \varepsilon I\,, \qquad (\varepsilon = \pm 1)\,.$$

Whether or not U is reducible depends on whether or not the antiunitary operator $\boldsymbol{k}$ on $\mathscr{H}_2$ possesses invariant rays. We thus rule out $\varepsilon = +1$ ($\boldsymbol{k}$ a conjugation), but allow $\varepsilon = -1$ (when $v(\neq 0)$ and kv are always linearly independent). Of course a *linear* operator k on $\mathscr{H}_2$ always has an invariant ray, and this entails that in the corresponding case (see § 3.4) of an irreducible σ-representation there is no analogue of Type II. See § 4.5 in this connection.

Theorem B. *Given a multiplier σ for G (with respect to the UA-decomposition $G = H \cup aH$), then each irreducible σ_H-representation D of H determines (up to unitary equivalence) a unique irreducible σ-corepresentation U of G whose restriction to H contains D as a subrepresentation. The details are as follows.*

Given σ and D, define, for each $h \in H$,

$$E(h) = [\bar{\sigma}(h,a)/\bar{\sigma}(a, a^{-1}ha)]\, D(a^{-1}ha)\,; \tag{2.15}$$

then E is an irreducible $\bar{\sigma}_H$-representation of H. The type of U may then be determined as follows:

Type I: there exists an antiunitary operator K satisfying $KE(h)K^{-1} = D(h)$ and $K^2 = +\sigma(a,a)\, D(a^2)$;

Type II: there exists an antiunitary operator K satisfying $KE(h)K^{-1} = D(h)$ and $K^2 = -\sigma(a,a)\, D(a^2)$;

Type III: the representations D and E are antilinearly inequivalent— i.e. no antilinear operator K exists satisfying $KE(h)K^{-1} = D(h)$ for all $h \in H$.

Given σ and D, the σ-corepresentation U of G is given (up to unitary equivalence) in the above three cases as follows:

$$\text{Type I} \qquad U(h)=D(h)\,, \qquad U(a)=K\,; \tag{2.16}$$

$$\text{Type II} \quad U(h)=\begin{pmatrix} D(h) & 0 \\ 0 & D(h) \end{pmatrix}, \quad U(a)=\begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix}; \tag{2.17}$$

$$\text{Type III} \quad U(h)=\begin{pmatrix} D(h) & 0 \\ 0 & KE(h)K^{-1} \end{pmatrix}, \quad U(a)=\begin{pmatrix} 0 & \sigma(a,a)\,D(a^2)\,K^{-1} \\ K & 0 \end{pmatrix}, \tag{2.18}$$

where K in the first two cases is as in the second part of the theorem and in the last case denotes any antiunitary mapping, and where, in all three cases, U(ah) is defined by

$$U(ah)=\sigma(a,h)^{-1}\,U(a)\,U(h)\,. \tag{2.19}$$

Proof. A straightforward check [using Eq. (1.2)] shows that E is indeed a (irreducible) $\bar{\sigma}_H$-representation of H, and (using also the lemma) shows that U, as defined in the last part of the theorem, is indeed a σ-corepresentation of G. Different choices K_1, K_2 of K lead to unitarily equivalent σ-representations U_1, U_2 of G:

$$PU_1(g)\,P^{-1}=U_2(g)\,, \qquad (P \text{ unitary})\,.$$

For in the first two cases we must have (by Schur's lemma $K_2=\omega K_1$, $|\omega|=1$, and so we can take $P=vI$, where $v^2=\omega$, while in the third case we can take $P=I\oplus K_2K_1^{-1}$. The rest of the theorem follows quickly from Theorem A. □

Theorem C. *The irreducible σ-corepresentation U of G is of Types I, II or III according as its commutant [U] is isomorphic to the algebra of the real numbers* ***R****, quaternions* ***H*** *or complex numbers* ***C****.*

Proof. By definition the commutant $[U]$ consists of all the *linear* operators on the carrier space $\mathscr{H}$ of U which commute with $U(g)$ for every $g\in G$. If X, Y belong to $[U]$, then so do $X+Y, XY$, and λX, for $\lambda\in \boldsymbol{R}$; however iX does not belong to $[U]$, since $(iX)\,U(g)=-U(g)\,(iX)$ whenever $g\in aH$. Thus $[U]$ is an algebra over $\boldsymbol{R}$ but not over $\boldsymbol{C}$. (Actually – see § 3.3 – the irreducibility of U implies that $[U]$ is necessarily a division algebra over $\boldsymbol{R}$.)

We now make use of the canonical forms I, II, III of Theorem A. In view of Schur's lemma, the equation $XU(h)=U(h)X$, for all $h\in H$, already forces X to be of the form

$$\text{Type I: } \alpha I\,; \quad \text{Type II: } \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix}; \quad \text{Type III: } \begin{pmatrix} \alpha I & 0 \\ 0 & \delta I \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in \boldsymbol{C}$. The further property $XU(a) = U(a)X$ then forces X to be of the form

$$\text{Type I: } \lambda I\,; \quad \text{Type II: } \begin{pmatrix} \alpha I & \beta I \\ -\bar{\beta} I & \bar{\alpha} I \end{pmatrix}; \quad \text{Type III: } \begin{pmatrix} \alpha I & 0 \\ 0 & \bar{\alpha} I \end{pmatrix}, \tag{2.20}$$

where $\lambda \in \boldsymbol{R}$, $\alpha, \beta \in \boldsymbol{C}$. The theorem follows. □

Remark. It follows from Eq. (2.20) that a non-zero $X \in [U]$ is always invertible, and hence can not be a proper projection. Thus each Type I, II, III of corepresentation U is indeed irreducible.

§ 3. Remarks

3.1. The above proof of Theorem B involves (partly via the proof of the preliminary lemma) repeated and tedious appeal to Eq. (1.2). This can be avoided by making use of the following obvious generalization to corepresentation theory of a well-known resource in representation theory (see, for example, Theorems 10.8, 10.16 in [8]).

Given a multiplier σ for G, make $G \times \boldsymbol{T}$ into a group G^σ by defining

$$(g_1, \omega_1)(g_2, \omega_2) = (g_1 g_2, \sigma(g_1, g_2)\, \omega_1 (\omega_2)^{g_1})\,. \tag{3.1}$$

Then instead of dealing with the multiplier corepresentations U of G (all with σ as multiplier), we may equivalently deal with those ordinary corepresentations U^σ of G^σ which satisfy

$$U^\sigma(e, \omega) = \omega I\,, \tag{3.2}$$

the passage from the one kind of representation to the other being effected by

$$U^\sigma(g, \omega) = \omega U(g)\,. \tag{3.3}$$

Theorems A and B now follow from the corresponding theorems for ordinary corepresentations. For example in the case of a Type III corepresentation the canonical form

$$U^\sigma(a, 1) = \begin{pmatrix} 0 & D^\sigma[(a, 1)^2]\, K^{-1} \\ K & 0 \end{pmatrix}$$

implies that of Eq. (2.8):

$$U(a) = \begin{pmatrix} 0 & \sigma(a, a)\, D(a^2)\, K^{-1} \\ K & 0 \end{pmatrix},$$

since $(a, 1)^2 = (a^2, \sigma(a, a))$. Similarly the definition

$$E^\sigma(h, 1) = D^\sigma[(a, 1)^{-1}\, (h, 1)\, (a, 1)]$$

implies that of Eq. (2.15), since we find

$$(a, 1)^{-1}(h, 1)(a, 1) = (a^{-1}ha, \bar{\sigma}(h, a)/\bar{\sigma}(a, a^{-1}ha))$$

after using

$$(a, 1)^{-1} = (a^{-1}, \sigma(a^{-1}, a)^{-1})$$

and [set $g_1 = a^{-1}, g_2 = a, g_3 = a^{-1}ha$ in Eq. (1.2)]

$$\sigma(a^{-1}, a) = \sigma(a^{-1}, ha)\,\bar{\sigma}(a, a^{-1}ha).$$

3.2. In stating the above lemma and theorems we took care not to use the fact that $\bar{\sigma} = \sigma^{-1}$, i.e. that $\sigma(g_1, g_2) \in \boldsymbol{T}$. Also no essential use was made of the unitarity or antiunitarity of the operators involved. Thus, on replacing unitary and antiunitary by linear isomorphism and anti-isomorphism, the theorems apply to general multiplier corepresentations. (In the above proof of Theorem A we showed that $L = \omega K$ with ω real and used the unitarity of LK^{-1} to *deduce* that $\omega = \pm 1$; in the more general situation we can *arrange* for ω to be ± 1 by means of a similarity transformation $U(g) \to PU(g)P^{-1}$ with $P = \lambda I \oplus \lambda^{-1} I$, for suitable $\lambda \in \boldsymbol{R}$.) In particular the representation U_H subduced by U is fully reducible; this applies even when (as in § 4) the invariant subgroup H is not of index 2.

3.3. When U is irreducible, Schur's lemma entails that every non-zero $X \in [U]$ possesses an inverse; thus $[U]$ is a division algebra over $\boldsymbol{R}$. By a well-known theorem of Frobenius, we thus know from the outset that $[U]$ is isomorphic to $\boldsymbol{R}$, $\boldsymbol{H}$ or $\boldsymbol{C}$. It is tempting therefore to use Theorem C as a *definition* of the three possible types of irreducible multiplier corepresentations, and to carry on to use the knowledge of $[U]$ to derive the "canonical forms" of Theorem A. However nothing appears to be gained thereby, in that one still needs most of the ideas of the above proof of Theorem A, combined with further considerations.

3.4. While on the topic of the representations of a group G possessing a subgroup H of index 2, let us mention the obvious generalization to multiplier representations of "Clifford's theorem" (see, for example, Boerner [2], Theorem 13.3).

If U is an irreducible σ-representation of G, then either U_H is irreducible (Type "I") or else decomposes into a direct sum $D \oplus C$ of two inequivalent irreducible σ_H-representations (Type "III"), where C, D are related as in Eq. (2.9), except that $K: \mathscr{H}_D \to \mathscr{H}_C$ is now a *linear* isomorphism. *(Note that there is no Type "II"*—the analogue of Type II being reducible, as we previously noted in the remark after Theorem A. See also § 4.5.)

Given σ and D (an irreducible σ_H-representation of H), define the (irreducible) σ_H-representation E of H by

$$E(h) = [\sigma(h, a)/\sigma(a, a^{-1}ha)]\,D(a^{-1}ha). \tag{3.4}$$

(E is called the *conjugate* of D by $a \in G$—see Eq. (4.1) for the generalization of this when G^- is non-empty.) There are two possibilities. Either E and D are (linearly) equivalent or they are inequivalent. In the former case there exists a pair K, $-K$ of linear automorphisms satisfying

$$K E(h) K^{-1} = D(h), \quad \text{and} \quad K^2 = \sigma(a, a) D(a^2), \tag{3.5}$$

and D can be extended *in two*[3] *inequivalent ways* (contrast this with the unique extension in the corepresentation case) to yield the σ-representations $U^\pm$ of G (of Type "$I\pm$"):

$$\textit{Type "}I\pm\textit{"} \qquad U^\pm(h) = D(h), \quad U^\pm(a) = \pm K. \tag{3.6}$$

(Of course U^+ is projectively equivalent to U^-.)

In the latter case D determines a unique (up to equivalence) irreducible σ-representation U of G (of Type "III") such that U_H contains D:

$$\textit{Type "III"} \quad U(h) = \begin{pmatrix} D(h) & 0 \\ 0 & E(h) \end{pmatrix}, \quad U(a) = \begin{pmatrix} 0 & \sigma(a, a) D(a^2) \\ I & 0 \end{pmatrix}. \tag{3.7}$$

In this last "canonical form" we could insert an arbitrary *linear* isomorphism K to make it correspond with Eq. (2.18)—the choice $K = I$ which we have made amounts to making a (non-natural) identification of two different carrier spaces.

§ 4. Irreducible Multiplier Corepresentations of a Group G Possessing an Invariant Subgroup H

4.1. As previously, let G be a group having a fixed UA-decomposition $G = G^+ \cup G^-$, with G^- non-empty. Each σ-corepresentation U of G then subduces a σ_H-corepresentation $U_H = U \downarrow H$ of a subgroup H of G; all such subduced multiplier corepresentations of H are of course with respect to the fixed UA-decomposition $H = H^+ \cup H^-$, where $H^\pm = H \cap G^\pm$ (and where H^- may or may not be empty). Following Clifford ([3]), one is interested in the general problem of the structure of U_H in the case when U is irreducible and H is an invariant subgroup of G, and in the reverse problem of constructing U from an irreducible σ_H-corepresentation D of H.

In this section we outline some aspects of this general problem; the previous classification of the irreducible σ-corepresentations of G into three Wigner types then fits into place as the special case $H = G^+$. In particular a Type III σ-corepresentation U of G will be seen to be a very simple instance of an *induced σ-corepresentation.* However we wish here to *concentrate entirely upon the algebraic aspects of the problem,*

[3] Clifford [3] calls U^+ and U^- *associates.*

and so will assume that the representations are finite-dimensional and that the subgroups are of finite index. (See [4, 6] for relevant measure – theoretic and analytical material needed to generalize the full-blown Mackey theory [5] to multiplier corepresentations of l.c. groups.)

4.2. The Relevant Action of G upon $\hat{H}^\sigma$. Let σ be a multiplier for G and let D be a σ_H-corepresentation of the invariant subgroup H of G. For fixed $g \in G$ the *conjugate* representation of D, by $g \in G$, is that σ_H-corepresentation[4] gD of H defined (up to unitary equivalence) by

$$(gD)(h) = [\sigma(h, g)/\sigma(g, g^{-1}hg)]\, D(g^{-1}hg)^g . \tag{4.1}$$

(More generally, this definition applies even when H is not invariant, gD then being a $\sigma_{gHg^{-1}}$—corepresentation of the conjugate subgroup gHg^{-1}.) See Eq. (4.3) for the origin of this definition. Here, for any linear or antilinear operator A on the carrier space V of D, we define

$$A^g = \begin{cases} A, & \text{if} \quad g \in G^+ , \\ \bar{A}, & \text{if} \quad g \in G^- , \end{cases} \tag{4.2}$$

where $\bar{A} \equiv KAK^{-1}$ denotes the complex conjugate of A with respect to some *fixed* antiunitary mapping $K: V \to \bar{V}$. Thus when $g \in G^-$, the carrier space of gD is an anti-space $\bar{V}$ of V. (If we like, we can – in a non-canonical way – take $\bar{V}$ to be V and K to be a conjugation: $K^2 = I$.)

If D is irreducible, then clearly so is gD. If $\hat{H}^\sigma$ denotes the set of all the irreducible σ_H-corepresentations of H (with unitarily equivalent representations being identified) then we use Eq. (4.1) to make $\hat{H}^\sigma$ into a G-space. *This action of G upon $\hat{H}^\sigma$ is forced upon us, whenever we deal with σ-corepresentations with respect to the given UA-decomposition $G^+ \cup G^-$ of G, as witness the following.*

Let U be an irreducible σ-corepresentation of G with carrier space $\mathscr{H}$, and let V denote the carrier space of some irreducible subrepresentation D of the subduced σ_H-corepresentation U_H. Upon considering the restriction to the subspace $U(g)\,V$ of the identity

$$U(h) = [\sigma(h, g)/\sigma(g, g^{-1}hg)]\, U(g)\, U(g^{-1}hg)\, U(g)^{-1} , \tag{4.3}$$

we find (see [3], the argument being a slight generalization of that in the proof of Theorem A above) that *the elements of $\hat{H}^\sigma$ which occur in the decomposition of U_H are precisely all the conjugates gD, $g \in G$, of D; thus U is associated with a single orbit in $\hat{H}^\sigma$.*

The *(generalized) isotropy subgroup* $G_D \subset G$ of the irreducible σ_H-corepresentation D of the last paragraph is defined by

$$G_D = \{g : g \in G, gD \sim D\} , \tag{4.4}$$

[4] Here, and in the ensuing, D and gD will merely be σ_H-representations in the case $H \subset G^+$.

where "$\sim$" denotes unitary equivalence. In more detail $G_D = G_D^+ \cup G_D^-$, where G_D^+ consists of all $g \in G^+$ such that the σ_H-corepresentation

$$h \mapsto [\sigma(h, g)/\sigma(g, g^{-1}hg)]\, D(g^{-1}hg) \tag{4.5}$$

is unitarily equivalent to D, and G_D^- consists of all $g \in G^-$ such that the $\bar{\sigma}_H$-corepresentation

$$h \mapsto [\bar{\sigma}(h, g)/\bar{\sigma}(g, g^{-1}hg)]\, D(g^{-1}hg) \tag{4.6}$$

is antiunitarily equivalent to D. Of course G_D contains H as an invariant subgroup.

For convenience we also write $G' = G_D$. Let $P = G/G'$ denote the space of left cosets of G', and for each $p \in P$ choose a coset representative $r_p \in G: p = r_p G'$. We choose $r_{p_0} = e$, where p_0 denotes the coset G'. Let $p \mapsto gp$ denote the canonical left action of G upon $P: g r_p G' = r_{gp} G'$; thus $r_{gp}^{-1} g r_p \in G'$ for each $p \in P, g \in G$.

Define the subspace $\mathscr{H}' \subset \mathscr{H}$ to be the sum (not usually direct) of the subspaces $U(g)\, V$ for $g \in G'$; then $\mathscr{H}'$ carries a $\sigma_{G'}$ – corepresentation U' of G', the operator $U'(g')$, $g' \in G'$, being the restriction to $\mathscr{H}'$ of $U(g')$. (Equivalently $\mathscr{H}'$ is the sum of all the equivalent irreducible invariant subspaces of $\mathscr{H}$ belonging to D.) Let $\mathscr{H}_p$ denote the image of $\mathscr{H}'$ under $U(r_p)$, and denote by $K_p: \mathscr{H}'(=\mathscr{H}_{p_0}) \to \mathscr{H}_p$ the corresponding restriction of $U(r_p)$. Corresponding to the coset decomposition $G = \bigcup_p (r_p G')$ we then have (arguing as in [3]) the direct sum decomposition

$$\mathscr{H} = \bigoplus_p \mathscr{H}_p, \qquad \mathscr{H}_p = K_p \mathscr{H}', \tag{4.7}$$

where $\mathscr{H}_p$ is the carrier space of a multiple $n = n_p$ of the irreducible σ_H-corepresentation $r_p D$, the multiplicity n being in fact uniform, i.e. the same for each $p \in P$. Since all conjugates gD of D occur in U_H, the index of G' in G equals the number of (U-equivalence classes of) representations $\in \hat{H}^\sigma$ which are conjugate to D with respect to G. (Of course a different choice of coset representative r_p would not affect the space $\mathscr{H}_p$, but would change the choice of isomorphism or anti-isomorphism $K_p: \mathscr{H}' \to \mathscr{H}_p$.)

The representation U of G is now seen to be an imprimitive one, based upon the transitive G-space $P = G/G'$. Upon restricting the identity

$$U(g) = [\sigma(g, r_p)/\sigma(r_{gp}, r_{gp}^{-1} g r_p)]\, U(r_{gp})\, U(r_{gp}^{-1} g r_p)\, U(r_p)^{-1} \tag{4.8}$$

to the subspace $\mathscr{H}_p$, we immediately obtain Eq. (4.10) below, and so see that the σ-corepresentation U of G is that representation $U' \uparrow G$ obtained from the $\sigma_{G'}$-corepresentation U' of the subgroup G' by "*generalized inducing*", provided the latter term is defined as in the next subsection.

It then follows easily that the irreducibility of U implies that of U'. Conversely, just as in § 4 of [3], *if U' is any irreducible $\sigma_{G'}$-corepresentation of the isotropy group G' of $D \in \hat{H}^\sigma$, such that U'_H contains D as a subrepresentation, then the induced σ-corepresentation $U' \uparrow G$ of G is irreducible.*

4.3. Generalized Inducing. Let σ be a multiplier of G, let G' be any subgroup (of finite index) of G, and let U' be any $\sigma_{G'}$-corepresentation[5] of G' with carrier space $\mathscr{H}'$. For each coset $p \in P = G/G'$, choose (i) a Hilbert space $\mathscr{H}_p$ isomorphic to $\mathscr{H}' = \mathscr{H}_{p_0}$, (ii) a coset representative r_p, with $r_{p_0} = e$, (iii) a fixed unitary ($r_p \in G^+$) or antiunitary ($r_p \in G^-$) mapping $K_p : \mathscr{H}' \to \mathscr{H}_p$, with $K_{p_0} = I$. Then the induced σ-corepresentation $U' \uparrow G$ of G is that representation U, having carrier space $\mathscr{H} = \bigoplus_p \mathscr{H}_p$, defined by

$$U(g) = \bigoplus_p U_p(g) , \tag{4.9}$$

where $U_p(g) : \mathscr{H}_p \to \mathscr{H}_{gp}$ is [taking our cue from Eq. (4.8)] defined by

$$U_p(g) = [\sigma(g, r_p)/\sigma(r_{gp}, r_{gp}^{-1} g r_p)] \, K_{gp} U'(r_{gp}^{-1} g r_p) \, K_p^{-1} . \tag{4.10}$$

If $\sigma \equiv 1$, then one sees immediately that U is indeed a corepresentation. For an arbitrary multiplier σ on G (with respect to the given UA-decomposition $G^+ \cup G^-$) our lead-in via Eq. (4.8) strongly suggests that U is indeed a σ-corepresentation. A direct check, using Eq. (1.2), that the multipliers do work out, is possible but somewhat tedious and it is simpler to proceed as in § 3.1.

It should be noted that a different choice $\hat{K}_p$ of the mappings K_p merely results in a σ-corepresentation $\hat{U}$ which is unitarily equivalent to U, for clearly $\hat{U}(g) = T U(g) T^{-1}$, where $T = \bigoplus_p \hat{K}_p K_p{}^{-1}$ is unitary. Of course, in a matrix realization of U, we can arrange for each K_p to be the identity matrix – if $\{e_i\}$ is a orthonormal basis for $\mathscr{H}'$, choose $\{K_p e_i\}$ as orthonormal basis for $\mathscr{H}_p$.

Also note that different choices of coset representatives lead to unitary equivalent representations. For, in the case $\sigma = 1$ of ordinary corepresentations, U is clearly unchanged under the simultaneous replacements $r_p \mapsto r_p h_p$, $K_p \to K_p U'(h_p)$, for any $h_p \in G'$. Hence, using the method of § 3.1, and noting that $(r_p h_p, 1) = (r_p, 1)(h_p, \sigma(r_p, h_p)^{-1})$, we deduce that the induced σ-corepresentation U is unchanged under the replacements $r \mapsto r_p h_p$, $K_p \mapsto K_p \sigma(r_p, h_p)^{-1} U'(h_p)$, $h_p \in G'$.

Two cases can be distinguished, as in [6], according as we induce from a subgroup G' which is or is not contained in G^+.

Case (a): $G' \cap G^- \neq \phi$. In this case $P = G/G'$ can be identified also with $G^+/(G')^+$, and for convenience we may choose r_p to lie in G^+, and hence K_p to be unitary, for every $p \in P$. (Even if we do not make this

[5] In Case (b), (see below), U' is in fact a representation, not a corepresentation.

choice, the induced representation U above has the correct property of being antiunitary upon the coset G^- – for, in the present Case (a), U' is a genuine corepresentation, being antiunitary upon $G' \cap G^-$.) Observe then *that the restriction U_+ of U to G^+ is precisely the representation which arises by ordinary inducing (see for example § 13a of* [2], *with appropriate insertion of multipliers) from the restriction U'_+ of U' to $(G')^+$:*

$$U_+ = U'_+ \uparrow G^+ . \tag{4.11}$$

Case (b): $G' \subset G^+$. In this case $P = G/G'$ splits into two G^+-orbits:

$$P = P^+ \cup P^- , \qquad P^+ = G^+/G' , \qquad P^- = aP^+ , \tag{4.12}$$

where a denotes any element of G^- (but which will be kept fixed throughout the following). The direct sum decomposition

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- , \qquad \text{where} \qquad \mathscr{H}_\pm = \bigoplus_{p \in P^\pm} \mathscr{H}_p , \tag{4.13}$$

clearly decomposes the representation U_+ of G^+ into two subrepresentations, say X_+ and X_-, where

$$X_\pm(g) = \bigoplus_{p \in P^\pm} U_p(g) , \qquad g \in G^+ . \tag{4.14}$$

Note that X_+ is that representation of G^+ induced from the representation U' of G'. Since $X_- = \bigoplus_{p \in P^+} U_{ap}$, we soon see, on choosing $r_{ap} = ar_p$ for $p \in P^+$, that X_- is unitarily equivalent to the conjugate representation aX_+ of X_+ by $a \in G^-$. *Thus, in the present case, U_+ decomposes at least to the extent:*

$$U_+ \sim (U' \uparrow G^+) \oplus a(U' \uparrow G^+) . \tag{4.15}$$

We can also check that $a(U' \uparrow G^+) \sim (aU') \uparrow G^+$, where [see the remark in parenthesis after Eq. (4.1)] aU' is a representation of the subgroup $aG'a^{-1}$.

4.4. The Determination of the Irreducible Multiplier Corepresentations of a Group G Containing an Invariant Subgroup $H \subset G^+$. We return now to the set-up in § 4.2, with H invariant in G. *However we will now assume that $H \subset G^+$.* We lose almost nothing by making this assumption, since if it does not apply we can always switch our attention to $H^+ = H \cap G^+$ which is also invariant in G. The plan of campaign (cf. [3, 5]) for determining all the irreducible multiplier corepresentations U of G (with respect to the given UA-decomposition) thus runs as follows:

(1) determine the equivalence classes of multipliers of G;

(2) for each multiplier class, fix upon one multiplier σ and determine the G-orbits in $\hat{H}^\sigma$;

(3) for each orbit, fix upon one point D (i.e. an irreducible σ_H-representation of H) and determine its (generalized) isotropy group $G_D = G'$;

(4) determine (up to unitary equivalence) all the irreducible $\sigma_{G'}$-(co)representations U' of G' such that U'_H is a multiple of D;

(5) for each U' construct $U = U' \uparrow G$.

We will say nothing here about Step (1). Instead we concentrate on another weak link, namely Step (4). If the operators $U'(g')$ are all unitary [i.e. Case (b) of § 4.3] then it is well known that the U' are fairly amenably related to D and to the irreducible multiplier representations of G'/H – see Theorem 3 of [3] and Theorem 8.3 of [5]. We now show that the situation is entirely analogous even when the U' are genuine corepresentations [i.e. Case (a) of § 4.3].

On the one hand, suppose that U' is given, with U'_H equal to a multiple n of D. Then we may take the carrier space $\mathscr{H}'$ of U' in the form

$$\mathscr{H}' = V \otimes \mathscr{H}_n ; \tag{4.16}$$

where $\mathscr{H}_n$ denotes an n-dimensional Hilbert space and where V carries D:

$$U'(h) = D(h) \otimes I , \qquad h \in H . \tag{4.17}$$

For convenience, let us temporarily drop the primes from $\mathscr{H}'$, G', U'. In other words we are in the special case of § 4.2 which arises when $G' = G$, i.e. when the representation D is selfconjugate: $gD \sim D$, for all $g \in G$. Thus for each $g \in G^+(G^-)$ there exists a unitary (antiunitary) operator $T(g)$ such that

$$T(g)\, D(g^{-1}hg)\, T(g)^{-1} = [\sigma(g, g^{-1}hg)/\sigma(h, g)]\, D(h) \tag{4.18}$$

holds for all $h \in H$.

For each $g \in G^+(G^-)$, choose any unitary (antiunitary) operator $\Sigma(g)$ on $\mathscr{H}_n$. Then, upon using Eqs. (4.17), (4.18) in Eq. (4.3), it follows that for each $g \in G$ the unitary operator $U(g) \circ (T(g) \otimes \Sigma(g))^{-1}$ commutes with $D(h) \otimes I$ for all $h \in H$, and hence (Schur's lemma holding, since G is an irreducible σ_H*-representation*) is of the form $I \otimes L(g)$ for some unitary operator $L(g)$ on $\mathscr{H}_n$. On writing $\Omega(g) = L(g)\, \Sigma(g)$ we see that U is of the form

$$U(g) = T(g) \otimes \Omega(g) , \qquad g \in G . \tag{4.19}$$

Of course the irreducibility of U implies that of Ω.

Thus the PUA-representation $\boldsymbol{U}$ of G under consideration, with carrier space $\mathscr{H} = V \otimes \mathscr{H}_n$, determines uniquely two other PUA-representations $\boldsymbol{T}$ and $\boldsymbol{\Omega}$ of G, with carrier spaces V and $\mathscr{H}_n$. For any versions U, T, Ω of $\boldsymbol{U}$, $\boldsymbol{T}$, $\boldsymbol{\Omega}$ such that Eq. (4.19) holds, the multipliers

σ, τ, ω are of course related by

$$\sigma = \tau\omega\,, \tag{4.20}$$

and T satisfies Eq. (4.18). The PUA-representation $\boldsymbol{\Omega}$ of G is effectively a PUA-representation of the quotient group G/H, since from Eq. (4.17) we have $\boldsymbol{\Omega}(h) = \boldsymbol{I}$, and hence $\boldsymbol{\Omega}(gh) = \boldsymbol{\Omega}(g)$ for all $g \in G$, $h \in H$; i.e. $\boldsymbol{\Omega} = \boldsymbol{\Omega}_0 \circ f$, where $\boldsymbol{\Omega}_0$ is some PUA-representation of G/H and f is the canonical homomorphism $G \to G/H$. Thus we can choose versions [satisfying Eqs. (4.19), (4.20)] such that

$$T(h) = D(h)\,, \qquad \Omega(h) = I\,, \qquad g \in G\,, \tag{4.21}$$

and such that the multiplier ω satisfies

$$\omega(gh, g'h') = \omega(g, g')\,, \qquad g, g' \in G\,, \qquad h, h' \in H\,; \tag{4.22}$$

i.e. ω is of the form $\omega = \omega_0 \circ (f \times f)$ for some multiplier ω_0 for G/H.

On the other hand, starting from a given $D \in \hat{H}^\sigma$ which is self-conjugate with respect to G (where we are still dropping the primes!), then there exists for each $g \in G^+$ (G^-) a unitary (antiunitary) operator $T(g)$ satisfying Eq. (4.18). Using Schur's lemma, it follows quickly that $\boldsymbol{T}(g)$ is uniquely determined by g and that $T(g)\,T(g')\,T(gg')^{-1}$ commutes with $D(h)$ and hence is a scalar multiple of the identity operator. Thus D, together with σ, determines uniquely a PUA-representation $g \to \boldsymbol{T}(g)$ of G, having the same carrier space V as D.

Any version of $\boldsymbol{T}$ according gives rise to a τ-corepresentation T of G where the multiplier τ is determined up to equivalence by σ and D. Defining the multiplier ω for G by Eq. (4.20), we wish to prove that ω is equivalent to a multiplier which satisfies Eq. (4.22). To this end, note first of all that according to Eq. (4.18) we may choose T so that

$$T(h) = D(h)\,, \qquad \text{for all } h \in H\,. \tag{4.23}$$

This choice entails that σ and τ agree on $H \times H$:

$$\omega(h, h') = 1\,, \qquad \text{for all } h, h' \in H\,. \tag{4.24}$$

Given that $D(h) = T(h)$, and hence $D(g^{-1}hg) = T(g^{-1}hg)$, we see that Eq. (4.18) holds also when σ is replaced by τ; thus ω satisfies

$$\omega(g, g^{-1}hg) = \omega(h, g)\,, \qquad g \in G\,, \qquad h \in H\,. \tag{4.25}$$

Next we show that we can arrange for ω to satisfy

$$\omega(g, h) = 1 = \omega(h, g)\,, \qquad g \in G\,, \qquad h \in H\,. \tag{4.26}$$

in other words the version T can be chosen so as to satisfy

$$\begin{aligned} T(gh) &= \sigma(g, h)^{-1}\, T(g)\, D(h)\,, \\ T(hg) &= \sigma(h, g)^{-1}\, D(h)\, T(g)\,, \qquad g \in G\,, \qquad h \in H\,. \end{aligned} \tag{4.27}$$

Choosing coset representatives s_i, we can certainly arrange for $\omega(s_i, h) = 1$, by taking $\lambda(s_i h) = \omega(s_i, h)$ in Eq. (1.3). Setting $g_1 = s_i$, $g_2 = h'$, $g_3 = h$ in Eq. (1.2) then yields $\omega(g, h) = 1$, (with $g = s_i h'$ = a general element of G). Equation (4.25) now yields $\omega(h, g) = 1$ also. Finally the desired property (4.22) follows quickly from Eq. (1.2) upon using Eq. (4.26): setting $g_1 = g, g_2 = g'$, $g_3 = h$ we obtain $\omega(g, g') = \omega(g, g'h)$ and setting $g_1 = g$, $g_2 = h$, $g_3 = g'$ we obtain $\omega(gh, g') = \omega(g, hg') = \omega(g, g')$. (Incidentally, setting $g_1 = h$, $g_2 = g, g_3 = g'$ we also obtain $\omega(hg, g') = \omega(g, g')^h$, whence we deduce that Eqs. (4.23), (4.24) imply that $H \subset G^+$; this should not surprise us, since we have been leaning heavily upon the assumption $H \subset G^+$ in our appeal to Schur's lemma.)

Thus, given σ and a self-conjugate $D \in \hat{H}^\sigma$, we have shown that there exists a multiplier τ for G and a τ-corepresentation T of G which satisfies Eqs. (4.18), (4.27) [and in particular Eq. (4.23)]. The multiplier $\omega = \sigma/\tau$ thus satisfies Eq. (4.22) (and in particular Eqs. (4.24), (4.26)) and so determines a multiplier ω_0 for the quotient group G/H. Since $\boldsymbol{T}$ is uniquely determined by D and σ, the multiplier τ, and hence ω and ω_0, is determined up to equivalence.

If Ω_0 is any ω_0-corepresentation of G/H, then $U = T \otimes \Omega$, with $\Omega = \Omega_0 \circ f$, is a σ-corepresentation of G such that U_H is a multiple of D. Since every linear operator belonging to the commutant $[D \otimes I]$ is of the form $I \otimes X$, we see that the map $X \mapsto I \otimes X$ sets up an isomorphism between the commutants $[\Omega_0]$ and $[T \otimes \Omega]$. Since X is a projection if and only if $I \otimes X$ is a projection, it follows that $T \otimes \Omega$ is irreducible if and only if Ω_0 is irreducible[6]. Since similarly a linear mapping X intertwines $\Omega_0^{(1)}, \Omega_0^{(2)}$ if and only if $I \otimes X$ intertwines $T \otimes \Omega^{(1)}$, $T \otimes \Omega^{(2)}$, we see[7] also that $\Omega_0^{(1)}, \Omega_0^{(2)}$ are unitarily inequivalent if and only if $T \otimes \Omega^{(1)}$, $T \otimes \Omega^{(2)}$ are unitarily inequivalent.

By our previous discussion leading up to Eq. (4.19), every irreducible σ-corepresentation U of G, such that U_H is a multiple of D, is of the kind $U = T \otimes \Omega$ just considered. *Thus the mapping $\Omega_0 \mapsto T \otimes \Omega$ sets up a $1-1$ correspondence (unitarily equivalent representations being identified) between the irreducible ω_0-corepresentations of G/H and those irreducible σ-corepresentations of G which reduce on H to a multiple of D.*

So much (after restoring the primes!) for step (4). As described in §4.2, we can now combine together Steps (4) and (5). *Given σ and $D \in \hat{H}^\sigma$, and hence also $G' \equiv G_D$ and $\boldsymbol{O} \equiv$ the G-orbit of D, the mapping*

$$\Omega_0 \mapsto U = (T \otimes \Omega) \uparrow G$$

[6] By a modification of the foregoing proof, this conclusion holds more generally, for corepresentations which are not UA (and which therefore may reduce without decomposing).

[7] If W intertwines $T \otimes \Omega^{(1)}$, $T \otimes \Omega^{(2)}$, then $W \in [D \otimes I]$ and so W must be of the form $I \otimes X$.

sets up a $1-1$ correspondence (unitarily equivalent representations being identified) between the irreducible ω_0-corepresentations[8] Ω_0 *of G'/H and those irreducible σ-corepresentations U of G having **O** as orbit.* Of course $D_1, D_2 \in \hat{H}^\sigma$ can give rise thereby to unitarily equivalent representations if and only if D_1, D_2 lie on the same G-orbit.

Following § 4.3 we can usefully distinguish between two cases.

Case (a): $G' \cap G^- \neq \phi$; i.e. $D \sim aD$, for some $a \in G^-$. In this case Ω_0, $U' = T \otimes \Omega$ and U are all genuine corepresentations; *moreover they must necessarily be all of the same Wigner type*[9]. For, since $(V_1 \oplus V_2)\uparrow G^+ = (V_1 \uparrow G^+) \oplus (V_2 \uparrow G^+)$, the fact that U is of the same type as U' follows quickly from Eq. (4.11) upon applying the ordinary theory (e.g. § 8 of [5]) to the chain $H \lhd (G')^+ \subset G^+$. The further fact that U' is of the same Wigner type[10] as Ω_0 follows from Theorem C since, as previously noted, $[T \otimes \Omega]$ is isomorphic to $[\Omega_0]$. (Incidentally all three Wigner types can arise—take $H = \{e\}$, for example.) One way of determining all the irreducible ω_0-corepresentations Ω_0 of G'/H is to first of all determine all the irreducible ω_0-representations Γ_0 of $(G')^+/H$ and then apply Theorem B. As noted in the second part of that theorem, the Wigner type of Ω_0 (and hence that of U) is determined entirely by the properties of Γ_0.

Case (b): $G' \subset G^+$; i.e. D and aD are unitarily inequivalent for all $a \in G^-$, i.e. *D and aD belong to distinct G^+-orbits in $\hat{H}^\sigma$.* The σ_{G^+}-representations $U' \uparrow G^+$ and $a(U' \uparrow G^+) = (aU') \uparrow G^+$ in Eq. (4.15) are thus unitarily inequivalent, since their orbits contain D and aD, respectively, and so are distinct. Thus, in the present case, *U is always of Wigner Type III.*

4.5. Classification of Irreducible Multiplier Corepresentations. If G is any group having a UA-decomposition $G = G^+ \cup G^-$, with G^- non-empty, then its irreducible σ-corepresentations can be determined, given a certain knowledge of the irreducible σ_{G^+}-representations of G^+, by applying the plan of attack of § 4.4 to the simple special case $H = G^+$. Three distinct possibilities are then seen to arise, and we thereby rederive the previous classification (Theorems A and B) into three Wigner types.

Given then a multiplier σ for $G = H \cup aH$, and given an irreducible σ_H-representation D of H, the details are as follows.

Type III: $G' = H$, i.e. D and aD are unitarily inequivalent, i.e. D and E [in Eq. (2.15)] are antiunitarily inequivalent. Thus, as in § 4.4, U is obtained from $U' = D$ by [Case (*b*) of] generalized inducing: $U = D \uparrow G$, and we reobtain Eq. (2.18) as a special case of Eq (4.9).

Types I, II: $G' = G$, i.e. $D \sim aD$. Thus $G'/H \simeq Z_2 = \{e, e'\}$, say. Bearing in mind that Z_2 here inherits the UA-decomposition $\{e\} \cup \{e'\}$, we see that there are just two inequivalent multipliers for Z_2, ω_0^+ and ω_0^-,

[8] In Case (b), Ω_0 is merely a ω_0-representation.

[9] Not of course to be confused with Von Neumann-Murray type referred to in [5].

[10] Of course, by Eq. (4.23), T is of Wigner Type I.

where $\omega_0^{\pm}(e', e') = \pm 1$. Up to unitary equivalence we easily see that there is just one irreducible $\omega_0^{\pm}$-corepresentation $\Omega_0^{\pm}$ of Z_2, given as follows: the corepresentation Ω_0^{+} is 1-dimensional, with e' represented by a conjugation, while the ω_0^{-}-corepresentation Ω_0^{-} is 2-dimensional, with e' represented by an antiunitary operator k satisfying $k^2 = -I$, and hence such that $ke_1 = e_2$, $ke_2 = -e_1$ for some orthonormal basis $\{e_1, e_2\}$.

Setting $K = T(a)$ in Eq. (4.18), the two possibilities $\omega^{\pm}(a, a) = \pm 1$ correspond to D being such that $K^2 = \pm\sigma(a, a)\, D(a^2)$, and give rise to the Wigner Types I and II for U:

Type I: $D \sim aD$, $K^2 = +\sigma(a, a)\, D(a^2)$; hence $U = U' = T \otimes \Omega^{+} \cong T$ is given by Eqs. (2.16) and (2.19).

Type II: $D \sim aD$, $K^2 = -\sigma(a, a)\, D(a^2)$; hence $U = U' = T \otimes \Omega^{-}$ is given by Eq. (2.5′) [equivalently by Eq. (2.17)] and Eq. (2.19).

Remark. Following on from § 3.4, let us compare the above results with the "Clifford's theorem" results, where $G^{+} = H$ and $G^{-} = aH$ are both represented unitarily:

Type "III": $G' = H, D \nsim aD$; hence $U = D \uparrow G$, and we obtain Eq. (3.7) as a special case of Eq. (4.9) (the K's now being unitary).

Type "I ±": $G' = G$, $D \sim aD$, $U^{\pm} = T \otimes \Gamma^{\pm}$. Here $G'/H \simeq Z_2$ has, up to equivalence, only the *one* multiplier (in the ordinary sense) $\omega_0 \equiv 1$; however there are *two* inequivalent irreducible representations $\Gamma^{\pm}$ of Z_2, *both* 1-dimensional, given by $\Gamma^{\pm}(e') = \pm 1$. (In the corepresentation case, there were *two* multipliers for Z_2, each giving rise to *one* irreducible corepresentation, the dimensions being 1 and 2.) It follows that $U^{+}(g) = \pm U^{-}(g)$ for $g \in G^{\pm}$.

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