Commun. math. Phys. 35, 155—171 (1974)

From Euclidean to Relativistic Fields and on the Notion of Markoff Fields

Gerhard C. Hegerfeldt

Institut für Theoretische Physik, Universität Göttingen, Göttingen, Federal Republic of Germany

Received August 3, 1973

Abstract. Recently, Nelson [2] has constructed relativistic fields from Euclidean fields which satisfy the Markoff and reflection property as well as an additional domain assumption. In this paper we replace the Markoff and reflection property by a weaker condition, a very simple positivity condition ("T-positivity") which can be very easily expressed in terms of the expectation functional $E(f) = \langle \Omega, \exp\{i\phi(f)\}\Omega\rangle$. We show that the special role of the Markoff property in Nelson's approach is entirely due to features also shared by T-positivity. The role of Nelson's domain assumption (A′) in by-passing the difficulties with the paper of Osterwalder and Schrader [4] are made transparent, and possible ways to weaken this assumption are pointed out. If the conditions of [4] should turn out to be sufficient after all, (A′) can be replaced by a simple differentiability condition on $E(\tau f)$. Our approach can also be applied to Fermi fields. The notion of Markoff and reflection property is discussed and shown to imply T-positivity.

Introduction

The proposal of Symanzik [1] to exploit Euclidean invariance in relativistic field theory has recently found some interesting developments. These were initiated by Nelson [2, 3] who described a procedure to construct relativistic fields from Markoff fields which transform covariantly under the full Euclidean group and which satisfy an additional domain assumption. The Markoff property, a fairly restrictive condition, allows a simple Hilbert space construction. Then Osterwalder and Schrader [4] attempted to give necessary and sufficient conditions for a set of Euclidean Green's functions to lead to Wightman functions of a relativistic theory by analytic continuation. However, their crucial Lemma 8.8 is incorrect [1], and it is doubtful whether their conditions are indeed sufficient; it seems likely that one will need an extra assumption.

In this paper we construct, as Nelson, relativistic fields from Euclidean fields over $\mathscr{S}$ or $\mathscr{D}$, *without Markoff property*, however. We replace the Markoff and reflection property used in [2] by a weaker condition,

[1] This has been noted by several people. I learned this first from J. Yngvason who also gave a simple counter-example. See also [12].

namely by a very simple positivity condition ("T-positivity"), and obtain, with exactly the same domain assumption[2] as Nelson's, a simple Hilbert space construction of a relativistic field. We show that the special role played by the Markoff property in the approach of [2] is entirely due to features which are also shared by T-positivity. We also make transparent the role which the domain assumption plays in by-passing the difficulties associated with Lemma 8.8 in [4] and point out possible ways to weaken this assumption.

Let $\phi(x)$ be a commuting self-adjoint scalar field[3] over a space $\mathscr{V}$ of real-valued test functions on R^d, i.e.,

$$e^{i\phi(f_1)}e^{i\phi(f_2)} = e^{i\phi(f_1+f_2)} \tag{1.1}$$

for all $f_1, f_2 \in \mathscr{V}$, where $e^{i\phi(\tau f)}$ is a unitary operator in a Hilbert space $\mathfrak{H}$ and weakly continuous in τ so that one can recover the generator $\phi(f)$. Thus one deals with a unitary representation of an abelian group. If $\Omega \in \mathfrak{H}$ is a cyclic vector for ϕ, i.e., cyclic for $\{e^{i\phi(f)}; f \in \mathscr{V}\}$ then, by the reconstruction theorem for group representations [5], the representation is uniquely determined by the expectation functional[3]

$$E(f) = \langle \Omega, e^{i\phi(f)}\Omega \rangle . \tag{1.2}$$

We say that ϕ is a Euclidean (Bose) field if there is a unitary representation of the full Euclidean group $IO(R^d)$ in $\mathfrak{H}$ such that, for $(R, a) \in IO(R^d)$, $U(R, a)\,\phi(x)\,U^*(R, a) = \phi(Rx + a)$ and $U(R, a)\Omega = \Omega$.

Let T be the unitary time reflection operator, $T\phi(x)T = \phi(-x^0, \mathbf{X})$. Let $R^d_+ = \{x \in R^d; x^0 > 0\}$ and let E_+ be the projector onto the subspace generated by $\{e^{i\phi(f)}\Omega; supp\, f \subset R^d_+\}$. Then our positivity condition which is related to that of [4] reads

$$E_+TE_+ \geqq 0 \quad (\text{"}T\text{-positivity"}). \tag{1.3}$$

This can easily be expressed by means of the expectation functional $E(f)$. Let $(\vartheta f)(x) = f(-x^0, \mathbf{X})$. Then Eq. (1.3) is equivalent to [4]

$$\Sigma\, \lambda_i \bar{\lambda}_j E(f_i - \vartheta f_j) \geqq 0, \quad supp\, f_i \subset R^d_+,\ \lambda_i \in \mathbb{C}. \tag{1.4}$$

In Section 2 the Hamiltonian H and the Hilbert space $\mathfrak{H}_0$ of the relativistic field are constructed. $\mathfrak{H}_0$ is just the subspace of $\mathfrak{H}$ on which E_+TE_+ is nonzero. In Section 3 the domain assumption (A′) through which also continuity properties enter is explained and exploited. In Section 4 the preceding results are used to construct the relativistic

---

[2] Condition (A′) in Section 6 of [2].

[3] Cf. [7], Section 6.

[4] Indeed, Eq. (1.4) just means that, for $supp\, f_i \subset R^d_+$,

$$\langle \Sigma\, \lambda_j e^{i\phi(f_j)}\Omega,\ T\,\Sigma\, \lambda_j e^{i\phi(f_j)}\Omega \rangle \geqq 0 .$$

theory. We indicate possible ways of weakening assumption (A′) and we show how to carry over our results to Fermi fields. In Section 5 we give a formulation by means of the expectation functional $E(f)$ and present some simple examples.

In Section 6 we discuss the notion of Markoff fields. We propose a definition of localization slightly different from that of [2] and show that ϕ satisfies this Markoff property together with the reflection property if and only if $E_+ TE_+$ is a projector. It is then pointed out that also for the localization prescription of [2], and practically any other, Markoff + reflection property implies T-positivity.

We remark that T-positivity as expressed by the expectation functional is stable under limits.

2. Hamiltonian H and Hilbert Space $\mathfrak{H}_0$

Let ϕ be a Euclidean field over the real space[5] $\mathscr{S}(R^d)$ or $\mathscr{D}(R^d)$ in $\mathfrak{H}$ and assume that

$$E_+ TE_+ \geqq 0. \tag{2.1}$$

By the spectral theorem $\mathfrak{H}_+ = E_+ \mathfrak{H}$ decomposes into a direct sum,

$$\mathfrak{H}_+ = \mathfrak{H}_0 \oplus \mathfrak{H}_{\mathfrak{N}}$$

such that $E_+ TE_+$ vanishes on the null space $\mathfrak{H}_{\mathfrak{N}} = E_{\mathfrak{N}} \mathfrak{H}$ and is strictly positive on $\mathfrak{H}_0 = E_0 \mathfrak{H}$. Note that $\mathfrak{H}_{\mathfrak{N}} = \{u \in \mathfrak{H}_+ ; \langle u, Tu \rangle = 0\}$ and that $\Omega \in \mathfrak{H}_0$. We denote by T_0 the restriction of $E_0 TE_0$ to $\mathfrak{H}_0$. Then T_0^{-1} exists as a densely defined positive operator. We will use repeatedly that

$$E_+ TE_+ = E_0 TE_0 . \tag{2.2}$$

Lemma 2.1. *For $t \geqq 0$, $\mathfrak{H}_+$ and $\mathfrak{H}_{\mathfrak{N}}$ are invariant under the Euclidean time translation T_t.*

Proof. Invariance of $\mathfrak{H}_+$ is obvious. Let $u \in \mathfrak{H}_{\mathfrak{N}}$. By Schwarz's inequality

$$\begin{aligned} 0 &\leqq \langle T_t u, TT_t u \rangle = \langle u, TT_{2t} u \rangle \\ &\leqq \langle u, Tu \rangle^{1/2} \langle T_{2t} u, TT_{2t} u \rangle^{1/2} = 0. \quad \text{QED.} \end{aligned} \tag{2.3}$$

Lemma 2.2. *For $t \geqq 0$, we define $\hat{P}_t u = E_0 T_t u$ for $u \in \mathfrak{H}_0$. Then $\hat{P}_t \hat{P}_s = \hat{P}_{t+s}$ for $t, s \geqq 0$.*

Proof. Since $T_t \mathfrak{H}_{\mathfrak{N}} \subset \mathfrak{H}_{\mathfrak{N}}$ and $T_t \mathfrak{H}_+ \subset \mathfrak{H}_+$ for $t \geqq 0$ and since $E_0 \mathfrak{H}_{\mathfrak{N}} = 0$, one obtains, with $u \in \mathfrak{H}_0$,

$$\begin{aligned} \hat{P}_t \hat{P}_s u &= E_0 T_t E_0 T_s u = E_0 T_t E_+ T_s u - E_0 T_t E_{\mathfrak{N}} T_s u \\ &= E_0 T_t T_s u - 0 = \hat{P}_{t+s} u. \quad \text{QED.} \end{aligned}$$

[5] Throughout we use real-valued functions.

Proposition 2.1. *We define* $P_t = T_0^{1/2} \hat{P}_t T_0^{-1/2}$ *on* $T_0^{1/2} \mathfrak{H}_0$. *Then* $\{P_t, t \geqq 0\}$ *can be extended by continuity to a self-adjoint contraction semi-group on* $\mathfrak{H}_0$. *Hence there exists a positive operator H such that*

$$P_t = e^{-tH} . \tag{2.4}$$

Proof. We first show that P_t, $t \geqq 0$, is symmetric on $T_0^{1/2} \mathfrak{H}_0$. For $u, v \in \mathfrak{H}_0$, one has

$$\begin{aligned} &\langle T_0^{1/2} u, P_t T_0^{1/2} v \rangle = \langle u, T_0 \hat{P}_t v \rangle = \langle u, T_0 E_0 T_t v \rangle \\ &= \langle u, E_+ T E_+ T_t v \rangle = \langle T_t u, T v \rangle = \langle T_t u, E_+ T E_+ v \rangle \\ &= \langle \hat{P}_t u, T_0 v \rangle = \langle P_t T_0^{1/2} u, T_0^{1/2} v \rangle . \end{aligned}$$

Using the symmetry and Schwarz's inequality one obtains by induction

$$\begin{aligned} \langle P_t T_0^{1/2} u, P_t T_0^{1/2} u \rangle &\leqq \| T_0^{1/2} u \| \, \| P_{2t} T_0^{1/2} u \| \\ &\leqq \| T_0^{1/2} u \|^{\sum_0^N 2^{-\nu}} \, \| P_{2^N t} T_0^{1/2} u \|^{2^{-N+1}} \\ &\leqq \| T_0^{1/2} u \|^2 \, \| u \|^{2^{-N+1}} \end{aligned}$$

which converges to $\| T_0^{1/2} u \|^2$ for $N \to \infty$. Hence $\| P_t \| \leqq 1$. QED.

We note that up to now no continuity or domain properties of the field have been used. Instead of P_t one could also have considered $\hat{P}_t$ which becomes a self-adjoint contraction semi-group if one introduces the norm $\| T_0^{1/2} u \|$ on $\mathfrak{H}_0$ and goes to the completion. The resulting space, however, seems to be too large for the desired continuity properties of the field operator to hold.

3. The Field Operator: Domains and Continuity

We want to define a sharp-time field on $\mathfrak{H}_0$ as a form and use for this purpose the assumption (A′) of Nelson [2]. Let $X_0, \ldots, X_{d-1}$ denote the self-adjoint generators of Euclidean time-space translations in $\mathfrak{H}$, and put

$$K = (\Sigma X_j^2)^{1/2} .$$

Let $\mathfrak{H}^k$, $-\infty \leqq k \leqq \infty$, be the associated scale. That is, for $k \geqq 0$, $\mathfrak{H}^k$ is the domain $\mathscr{D}(K^{k/2})$ with norm $\| u \|_k = \| (1+K)^{k/2} u \|$, and $\mathfrak{H}^{-k}$ is the completion of $\mathfrak{H}$ in the norm $\| \cdot \|_{-k}$. One has $\mathfrak{H}^l \supset \mathfrak{H}^k$ for $l < k$, $\mathfrak{H}^\infty = \cap \mathfrak{H}^k$, $\mathfrak{H}^{-\infty} = \cup \mathfrak{H}^k$. On $\mathfrak{H}^\infty$ a topology is defined by the basic set of neighbourhoods $N_{\varepsilon,k} = \{ u \in \mathfrak{H}^\infty ; \| u \|_k < \varepsilon \}$ of 0.

We impose the following assumption on ϕ:

(A′) $\langle u, \phi(f) v \rangle$ is defined and separately continuous on $\mathfrak{H}^\infty \times \mathscr{D}(R^d) \times \mathfrak{H}^\infty$. This is equivalent to Nelson's assumption (A′) of [2], namely that there exist finite k and l such that, for all $f \in \mathscr{D}(R^d)$, $\phi(f)$ is a bounded linear operator from $\mathfrak{H}^k$ to $\mathfrak{H}^l$ and $f \mapsto \phi(f)$ is continuous[6].

As shown in [2], (A′) implies that $\phi(f \times \delta) \equiv \phi(f, 0)$ is a bounded linear operator from $\mathfrak{H}^{k'}$ to $\mathfrak{H}^{l'}$, for $f \in \mathscr{S}(R^{d-1})$ and for some finite k' and l'. We want to interpret $\phi(f, 0)$ as a form on $\mathfrak{H}_0$. Our procedure is somewhat more complicated[7] than that of Nelson [2].

Lemma 3.1. a) *T extends to a unitary operator on each $\mathfrak{H}^k$, and it commutes with $\phi(f, 0)$.* b) *$\mathfrak{H}_0$ is invariant under space translations so that their generators are self-adjoint on $\mathfrak{H}_0$, and they commute with P_t.*

Proof. The first part follows from $[T, K] = 0$ and from $T\phi(g)T = \phi(\vartheta g)$ where $(\vartheta g)(x) = g(-x_0, \boldsymbol{x})$. For the second part we note that the space translations commute with E_+, T and T_t, hence with $E_0 T E_0$, and thus with E_0, T_0 and P_t. QED.

X_0, on the other hand, need not be defined on $\mathfrak{H}_0$. For $k \geqq 0$, we define $\mathfrak{H}_+^k = \mathfrak{H}^k \cap \mathfrak{H}_+$ while, for $k < 0$, $\mathfrak{H}_+^k$ is defined as the completion of $\mathfrak{H}_+$ in $\|\cdot\|_k$-norm.

Lemma 3.2. *$\mathfrak{H}_+^{2k}$ is dense in $\mathfrak{H}_+$.*

Proof. To show denseness for $k > 0$, it suffices to show that X_0^k is densely defined on $\mathfrak{H}_+$. Now, for $u \in \mathfrak{H}_+$ and $g \in \mathscr{S}(R)$ with support on the right half axis the vector $\int d\tau\, g(\tau)\, T_\tau u$ is in $\mathscr{D}(X_0^k) \cap \mathfrak{H}_+$, and the set of such vectors is dense in $\mathfrak{H}_+$. QED.

Lemma 3.3. *Let r be chosen so large that $\phi(f \times \delta)$ is a bounded linear operator from $\mathfrak{H}^{2r}$ to $\mathfrak{H}^{-2r}$. Then one can define a symmetric form $\phi_0(f)$ on $T_0^{1/2} E_0 \mathfrak{H}_+^{2r} \times T^{1/2} E_0 \mathfrak{H}_+^{2r}$ by*

$$\langle T_0^{1/2} E_0 u, \phi_0(f) T_0^{1/2} E_0 v \rangle = \langle u, T\phi(f \times \delta) v \rangle \qquad (3.1)$$

where $u, v \in \mathfrak{H}_+^{2r}$.

Proof. First note that $\phi(f \times \delta) v \in \mathfrak{H}_+^{-2r}$ for $v \in \mathfrak{H}_+^{2r}$. Indeed, let $f_n \in \mathscr{D}(R_+^d)$ such that $\phi(f_n) \to \phi(f \times \delta)$ on $\mathfrak{H}^{2r}$. Then $\phi(f_n) v \in \mathfrak{H}_+^{-2r}$, and $\lim \phi(f_n) v = \phi(f \times \delta) v \in \mathfrak{H}_+^{-2r}$. Now we intend to show that the r.h.s. of Eq. (3.1) depends only on $T_0^{1/2} E_0 u$ and on $T_0^{1/2} E_0 v$. Let $w_n \in \mathfrak{H}_+$ such that $w_n \to \phi(f \times \delta) v$ in $\|\cdot\|_{-2r}$-norm. Then, by Eq. (2.2), $\langle u, T w_n \rangle = \langle E_0 u, T w_n \rangle$ so that also the limit depends on $E_0 u$ only. Approximating $\phi(f \times \delta) u$ in a similar way yields dependence on $E_0 v$ only; here Lemma 3.1 a) has been used. Now the correspondence between $E_0 u$ and $T_0^{1/2} E_0 u$ as well as $E_0 v$ and $T_0^{1/2} E_0 v$ is one-to-one. Hence the r.h.s. of

[6] (A′) implies that there are norms $\|\cdot\|_l$, $|\cdot|_{(m)}$, $\|\cdot\|_k$ on $\mathfrak{H}^\infty$, $\mathscr{S}(R^d)$ and $\mathfrak{H}^\infty$, respectively, such that $|\langle u, \phi(f) v \rangle| \leqq \|u\|_l \, |f|_{(m)} \, \|v\|_k$. Hence, in particular, $\|\phi(f)\|_{k,l} \leqq |f|_{(m)}$ so that the former is a continuous semi-norm.

[7] In [2], T is the unit operator on the space corresponding to our $\mathfrak{H}_0$, and the Markoff property holds.

Eq. (3.1) depends on the latter vectors only. The symmetry of ϕ_0 follows from that of the r.h.s. QED.

Lemma 3.4. *$T_0^{1/2}E_0\mathfrak{H}_+^{2k}$ is dense in $\mathscr{D}(H^k)\cap\bigcap_{i>0}\mathscr{D}(X_i^k)$, $k=1,2,\ldots$, and for $u\in\mathfrak{H}_+^{2k}$ one has*

$$T_0^{1/2}E_0X_0^ku=(iH)^kT_0^{1/2}E_0u\,. \tag{3.2}$$

Proof. Since $T_0^{1/2}$ and E_0 commute with X_i, the above subspace is contained in $\bigcap_{i>0}\mathscr{D}(X_i^k)$. For the rest it suffices to prove Eq. (3.2), by Lemma 3.2. Let $u\in\mathscr{D}(X_0^k)\cap\mathfrak{H}_+$. Then

$$X_0^ku=\operatorname*{s\text{-}lim}_{\tau_j\to+0}(-i)^k\left(\prod_{j=1}^{k}\frac{T_{\tau_j}-1}{\tau_j}\right)u \tag{3.3}$$

which, incidentally, lies again in $\mathfrak{H}_+$. Applying $T_0^{1/2}E_0$ to the r.h.s. without the *lim* and using $u=E_0u+E_{\mathfrak{N}}u$ together with Lemma 2.2, we obtain

$$(-i)^k\left\{\left(\prod_j\frac{P_{\tau_j}-1}{\tau_j}\right)T_0^{1/2}E_0u+T_0^{1/2}E_0\left(\prod_j\frac{T_{\tau_j}-1}{\tau_j}\right)E_{\mathfrak{N}}u\right\}. \tag{3.4}$$

By Lemma 2.1, the second term vanishes. Since, by Eq. (3.3), the norm limit of Eq. (3.4) exists, the first term converges strongly for $\tau_j\to+0$ yielding Eq. (3.2). QED.

Now we are going to prove continuity properties of $\phi_0(f)$ which are decisive for the following.

Proposition 3.1. *Let $E_+TE_+\geqq 0$ and let (A′) hold. Define $\phi_0(f)$ as in Eq. (3.1). Let $\tilde{\mathfrak{H}}_0$ be the Hilbert space $\tilde{\mathfrak{H}}_0=\mathfrak{H}_0\cap\mathscr{D}(X_1^r)\cap\ldots$ with the norm*

$$\|u\|^{\sim}=\left\|\left(1+\sum_{j>0}|X_i|\right)^ru\right\|. \tag{3.5}$$

Let $\tilde{H}$ be the restriction of H to $\tilde{\mathfrak{H}}_0$ [8]*, and let $\{\tilde{\mathfrak{H}}_0{}^k\}$ be the associated scale spaces. Then $\phi_0(f)$ can be extended by continuity to a bounded operator from $\tilde{\mathfrak{H}}_0^{2r}$ to $\tilde{\mathfrak{H}}_0^{-2r}$ which is continuous in f.*

Proof. We first show boundedness. Let $u^0\in T_0^{1/2}E_0\mathfrak{H}_+^\infty$, and let $u\in\mathfrak{H}_+^\infty$ with $T_0^{1/2}E_0u=u^0$. If one puts $X\equiv(1-iX_0)\left(1+\sum_{j>0}|X_j|\right)$, then $A\equiv X^{-r}\phi(f\times\delta)X^{-r}$ is a bounded operator on $\mathfrak{H}$ and $TX=X^*T$. Thus, by Eq. (3.1),

$$\begin{aligned}\langle u^0,\phi_0(f)u^0\rangle&=\langle u,T\phi(f\times\delta)u\rangle\\&=\langle X^ru,TAX^ru\rangle\,.\end{aligned}$$

[8] $\tilde{H}$ is self-adjoint on $\tilde{\mathfrak{H}}_0$, by Lemma 3.1.

For $u \in \mathfrak{H}_+$, one has $X^r u \in \mathfrak{H}_+$ and also $Au \in \mathfrak{H}_+$ since $(1 - iX_0)^{-1} = \int_0^\infty dt e^{-t} T_t$. Hence one deals with the positive operator $E_+ T E_+$ and can repeatedly apply Schwars's inequality,

$$\begin{aligned}\langle X^r u, TAX^r u\rangle &\leqq \langle X^r u, TX^r u\rangle^{1/2} \langle X^r u, TA^2 X^r u\rangle^{1/2} \\ &\leqq \langle X^r u, TX^r u\rangle^{\sum_1^N 2^{-\nu}} \langle X^r u, TA^{2^N} X^r u\rangle^{2^{-N}}.\end{aligned}$$

The last term is bounded by $\|A\| \, \|X^r u\|^{2^{-N+1}}$ which tends to $\|A\|$. By Eqs. (2.2) and (3.2) and Lemma 3.1, we obtain

$$\langle X^r u, TX^r u\rangle = \left\|(1+H)^r \left(1 + \sum_{j>0} |X_j|\right)^r u^0\right\|^2.$$

This proves the boundedness. Since the r.h.s. of Eq. (3.1) defines a continuous linear functional on $\mathscr{S}(R^{d-1})$ so does any limit, by the weak completeness of $\mathscr{S}$, and the remark in footnote 6 applies. QED.

Now, $e^{-t\tilde{H}}$ maps each $\tilde{\mathfrak{H}}_0^k$ into $\tilde{\mathfrak{H}}_0^\infty$. This has a simple but interesting consequence.

Corollary 3.1. *Let $g \in \mathscr{S}(R_+)$ and define $\check{g}$ by*

$$\check{g}(q) = \int d\tau \, e^{-q\tau} g(\tau) \upharpoonright q \geqq 0 .$$

Then, for $g_i \in \mathscr{S}(R_+)$ and $f_0, f_i \in \mathscr{S}(R^{d-1})$, $i = 1, 2, \ldots$,

$$\begin{aligned}&\left|\int \prod_{i=1}^n \{d\tau_i g_i(\tau_i)\} \langle \Omega, \phi_0(f_0) e^{-\tau_1 \tilde{H}} \phi_0(f_1) \ldots e^{-\tau_n \tilde{H}} \phi_0(f_n) \Omega\rangle\right| \\ &\qquad \leqq \|\phi_0(f_0)\Omega\|_{-2r} \prod_{i=1}^n \{\|\phi_0(f_i)\|_{2r,-2r} \sup_q |(1+q)^{2r} \check{g}_i(q)|\} .\end{aligned} \tag{3.6}$$

Proof. Let $u \in \tilde{\mathfrak{H}}_0^{2r}$. Then

$$\begin{aligned}&\|\int d\tau \, g(\tau) e^{-\tau\tilde{H}} \phi_0(f) u\|_{2r} \\ &\qquad \leqq \|(1+\tilde{H})^r \int d\tau \, g(\tau) e^{-\tau\tilde{H}} (1+\tilde{H})^r\| \; \|\phi_0(f)\|_{2r,-2r} \|u\|_{2r},\end{aligned}$$

and for the first factor on the r.h.s. one obtains $\sup_q |(1+q)^{2r} \check{g}(q)|$, by the spectral theorem[9]. From this Eq. (3.6) results by Schwarz's inequality. QED.

From this we obtain a result which can be used for analytic continuation.

---

[9] Note that if E_λ is the spectral decomposition of a self-adjoint operator then $\|\int f(\lambda) \, dE_\lambda\| = \sup |f(\lambda)|$.

Corollary 3.2. *Let $\xi_i = (\tau_i, \boldsymbol{x}_i - \boldsymbol{x}_{i-1})$, $i = 1, 2, \ldots, n$. Then, considered as a distribution on $\mathscr{S}(R^{d-1}) \times \mathscr{S}(R_+) \times \cdots \times \mathscr{S}(R^{d-1})$,*

$$\begin{aligned}\langle \Omega, \phi_0(\boldsymbol{x}_0) e^{-\tau_1 \tilde{H}} \ldots e^{-\tau_n \tilde{H}} \phi_0(\boldsymbol{x}_n) \Omega \rangle \\ = \int e^{-\sum_k (\xi_k^0 q_k^0 - i \xi_k \boldsymbol{q}_k)} \tilde{W}_n(q_1, \ldots, q_n) d^{nd} q\end{aligned} \tag{3.7}$$

where $q_k = (q_k^0, \boldsymbol{q})$ and where $\tilde{W}_n$ is a distribution on $\mathscr{S}(R^{dn})$ with support in $\{(q_1, \ldots, q_n); q_k^0 \geqq 0, k = 1, \ldots, n\}$.

Proof. By Lemma 8.2 of [4], the map $g \mapsto \check{g}$ is a continuous map of $\mathscr{S}(R_+)$ onto a dense subset of $\mathscr{S}(\bar{R}_+)$, where the latter consists of functions $f_+(q) = f(q) \upharpoonright R_+$, $f \in \mathscr{S}(R)$, with norms

$$|f_+|_m = \sup_{q \geqq 0, \alpha \leqq m} (1 + q^2)^{m/2} |f^{(\alpha)}(q)|,$$

$m = 0, 1, \ldots$. Eq. (3.6) shows that the l.h.s. of Eq. (3.7), when smeared in $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n$, can be extended to a distribution on $\mathscr{S}(\bar{R}_+) \ldots \mathscr{S}(\bar{R}_+)$. The rest is translation invariance and continuity in the f_i's. QED.

Now we want to show how $e^{-t\tilde{H}} \phi_0(f)$ acts as an operator.

Lemma 3.5. *Let the assumptions be as in Proposition 3.1, and let g $\mathscr{S}(R_+)$. Let $v \in \mathfrak{H}_+^\infty$. Then*

$$\int d\tau\, g(\tau) e^{-\tau \tilde{H}} \phi_0(f) T_0^{1/2} E_0 v = T_0^{1/2} E_0 \int d\tau\, g(\tau) T_\tau \phi(f \times \delta) v \tag{3.8}$$

where the expression following E_0 on the r.h.s. lies in $\mathfrak{H}_+^\infty$.

Proof. The last statement follows easily from the spectral theorem and the transformation properties of $\phi(f \times \delta)$ under space translations [10]. Now let $u \in \mathfrak{H}_+^{2r}$ and put $g(P_\tau) = \int d\tau\, g(\tau) P_\tau$ and similarly for $\hat{P}_\tau$ and T_τ. Then, by Proposition 2.1, Lemmas 2.2 and 3.3 and Eq. (2.2),

$$\begin{aligned}&\langle T_0^{1/2} E_0 u, g(P_\tau) \phi_0(f) T_0^{1/2} E_0 v \rangle \\ &= \langle T_0^{1/2} \bar{g}(\hat{P}_\tau) E_0 u, \phi_0(f) T_0^{1/2} E_0 v \rangle \\ &= \langle T_0^{1/2} E_0 \bar{g}(T_\tau) u, \phi_0(f) T_0^{1/2} E_0 v \rangle \\ &= \langle \bar{g}(T_\tau) u, T \phi(f \times \delta) v \rangle = \langle u, T g(T_\tau) \phi(f \times \delta) v \rangle \\ &= \langle T_0^{1/2} E_0 u, T_0^{1/2} E_0 g(T_\tau) \phi(f \times \delta) v \rangle .\end{aligned}$$

Since $T_0^{1/2} E_0 \mathfrak{H}_+^{2r}$ is dense in $\mathfrak{H}_0$, Eq. (3.8) follows. QED.

From this we have a corollary which is crucial for the next section.

Corollary 3.3. *Let $\mathscr{S}_<(R^{dn})$ denote the subspace of functions $f \in \mathscr{S}(R^{dn})$ such that $f^{(\alpha)}(x_1, \ldots, x_n) = 0$ for all α unless $x_1^0 < \cdots < x_n^0$. Then, under the*

[10] See also the remark at the end of this section.

assumptions of Proposition 3.1,

$$\begin{aligned}&\langle \Omega, \phi_0(\boldsymbol{x}_1) e^{-(x_2^0-x_1^0)\tilde{H}} \phi_0(\boldsymbol{x}_2) \ldots e^{-(x_n^0-x_{n-1}^0)\tilde{H}} \phi_0(\boldsymbol{x}_n)\Omega\rangle \\ &\qquad = \langle \Omega, \phi(x_1) \ldots \phi(x_n)\Omega\rangle\end{aligned} \tag{3.9}$$

as distributions on $\mathscr{S}_<(R^{dn})$.

Proof. We smear the l.h.s. of Eq. (3.9) in all $\boldsymbol{x}_i$ with $f_i \in \mathscr{S}(R^{d-1})$ and in $\xi_i^0 = x_{i+1}^0 - x_i^0$ with $g_i \in \mathscr{S}(R_+)$. By repeated application of Lemma 3.5 and by Eq. (3.1) together with $T_0\Omega = T\Omega = \Omega$ we obtain

$$\left\langle \Omega, \phi(f_1, 0) \int \left\{ \prod_{i=2}^{n} d\xi_i^0\, g_i(\xi_i^0) \right\} \phi(f_2, \xi_2^0) \ldots \phi(f_n, \xi_2^0 + \cdots + \xi_n^0)\Omega \right\rangle . \tag{3.10}$$

From this the lemma results by translation invariance. QED.

Remarks. (i) The existence of the n-point functions as distributions on $\mathscr{S}(R^{dn})$ can be directly inferred from assumption (A′). The latter implies [2] that $\phi(\delta^{(d)}) = \phi(\mathbf{0}, 0)$ is a bounded operator from $\mathfrak{H}^{2k}$ to $\mathfrak{H}^{-2k}$ for some $k > 0$. Since, for $f \in \mathscr{S}(R^d)$, $\int dx\, f(x) T_{x^0} T_{\boldsymbol{x}}$ maps $\mathfrak{H}^{-\infty}$ into $\mathfrak{H}^{\infty}$, by the spectral theorem, one has

$$\int d\xi_1\, f_1(\xi_1)\, \phi(\xi_1) \ldots \int d\xi_n\, f_n(\xi_n)\, \phi(\xi_1 + \cdots + \xi_n)\Omega \in \mathfrak{H}^{\infty} .$$

Hence any product of field operators can be applied to Ω, mapping it into $\mathfrak{H}^{\infty}$. One can also show that $\phi(f)$ maps $\mathfrak{H}^{\infty}$ into $\mathfrak{H}^{\infty}$ so that if (A′) holds field operators can be applied arbitrarily often on $\mathfrak{H}^{\infty}$, not only on Ω.

(ii) The r.h.s. of Eq. (3.9) is symmetric in $x_1, \ldots, x_n$ since ϕ is abelian.

4. Transition to Relativistic Theory

We are now able to go over to a relativistic field by analytic continuation. Further below we will also show that the Hilbert space on the relativistic theory is $\mathfrak{H}_0$, too, and write down the relativistic field operators.

Theorem 4.1. *Let* ϕ *be a commutative field over* $\mathscr{S}(R^d)$ *or* $\mathscr{D}(R^d)$ *in a Hilbert space* $\mathfrak{H}$ *with cyclic vector* Ω. *Let* ϕ *be covariant under the full Euclidean group* $\{(R, a); R \in O(d), a \in R^d\}$ *with* Ω *as unique translation invariant state. Let* T *be the unitary time reflection operator, and let* E_+ *be the projector onto the subspace* $\mathfrak{H}_+$ *generated by vectors of the form* $e^{i\phi(f)}\Omega$, *supp* $f \subset \{x; x^0 > 0\}$. *Then, if*

(i) $E_+ T E_+ \geqq 0$ *and*

(ii) *condition* (A′) *of Section 3 holds,*

the n-point functions $\langle \Omega, \phi(x_1) \ldots \phi(x_n)\Omega\rangle$, $x_1^0 < \cdots < x_n^0$, *can be analytically continued in* $\xi_j^0 = x_j^0 - x_{j-1}^0$, $j = 2, \ldots, n$, *to* $\operatorname{Re} \xi_j^0 > 0$. *For* $\xi_j^0 \to -i(t_j - t_{j-1})$ *they yield the Wightman distributions of a relativistic* *theory, satisfying relativistic invariance, positivity, causality, spectral condition and cluster property.*

Proof. Eq. (3.7) together with Eq. (3.9) shows that one can analytically continue in ξ_j^0. The proof of relativistic invariance etc. is then identical to that of Theorem 2 and 3 in [2] or to the similar procedure in [4]. Here Euclidean invariance and the commutativity of ϕ enter crucially. QED.

Realization of Relativistic Field Operators in $\mathfrak{H}_0$

By Proposition 3.1, ϕ_0 maps $\tilde{\mathfrak{H}}_0^{2r}$ into $\tilde{\mathfrak{H}}_0^{-2r}$. Let $f(t, \boldsymbol{x}) \in \mathscr{S}(R^d)$ and put $f_t(\boldsymbol{x}) = f(t, \boldsymbol{x})$. Then, by [6],

$$A(f) = \int dt\, e^{i\tilde{H}t} \phi_0(f_t) e^{-i\tilde{H}t} \tag{4.1}$$

maps $\tilde{\mathfrak{H}}^{\infty}$ into $\tilde{\mathfrak{H}}_0^{\infty}$. Comparison with Eq. (3.9) shows that

$$\langle \Omega, A(f_1) \ldots A(f_n)\Omega \rangle$$

is just the analytically continued n-point function at imaginary times smeared with $f_1 \times \cdots \times f_n$. Hence instead of using Corollary 3.2 one could have performed the analytic continuation by inserting operators $e^{it_j\tilde{H}}$ on the l.h.s. of Eq. (3.9), quite analogous to [2]. The operator in Eq. (4.1) is the relativistic field. By the spectral condition it is seen [11] that $\tilde{\mathfrak{H}}_0^{2k} = \mathscr{D}(H^k)$ so that $\tilde{H} = H$.

Remarks. (i) It may seem surprising, that, instead of $\hat{P}_t = E_0 T_t$, we have used $P_t = T_0^{1/2} \hat{P}_t T_0^{-1/2}$ for the construction of the Hamiltonian and that this works. However, this is not quite unexpected. Since T and $\phi(f, 0)$ commute it is not unreasonable that in

$$\langle \Omega, \phi_0(f_1) T_0^{1/2} \hat{P}_{\tau_2} T_0^{-1/2} \phi_0(f_2) T_0^{1/2} \hat{P}_{\tau_3} T_0^{-1/2} \ldots \Omega \rangle$$

the operators $T_0^{1/2}$ and $T_0^{-1/2}$ drop out in some sense, reflecting some sort of commutativity of T_0 and ϕ_0.

(ii) The operators $T_0^{1/2}$ appearing in the definition Eq. (3.1) of ϕ_0 are important. If they are dropped one *does* get an analog of Proposition 3.1 if one introduces the new norm $\|T_0^{1/2} u\|$ and uses the generator for P_t, but then Eq. (3.9) does not hold.

(iii) The role of assumption (A′) is seen to be twofold. It assures the existence of the Euclidean n-point functions and, together with the T-positivity, it allows their analytic continuation, thus by-passing the difficulties associated with the incorrect Lemma 8.8 of [4]. Corollary 3.1

---

[11] By the spectral condition, the representation of the Poincaré group decomposes into a direct integral of irreducible ones with $m^2 \geqq 0$. For the latter one has $P_0 = \sqrt{\boldsymbol{P}^2 + m^2}$ so that $\mathscr{D}(P_0) \subset \mathscr{D}(\boldsymbol{P})$.

shows possible ways to weaken (A′). One could try to obtain, instead of Eq. (3.6), an estimate which, firstly, contains derivatives of $\check{g}$ and which, secondly, contains an explicit n-dependence of the exponent $2r$ and of the order of the derivative.

(iv) All previous results can be extended to *Fermi fields* and higher *spin Bose fields* in an obvious way. If one assumes Euclidean Fermi fields to anticommute for all points (not only for non-coinciding ones), a doubling of the fields as in the case of the free field seems unavoidable. It is simplest to assume that Ω, the unique translation invariant state, is in the domain of all field products and cyclic. $\mathfrak{H}_+$ can be defined similarly as before as the subspace generated by

$$\{\Omega, \psi_{k_1}(f_1)\dots\psi_{k_n}(f_n)\,\Omega;\ f_i\in\mathscr{S}(R_+^{dn}),\quad i=1,\dots,n;\ n=1,2,\dots\}.$$

With T-positivity, $E_+TE_+\geqq 0$, and assumption (A′) for $\psi_1,\dots,\psi_\sigma$ all results of Section 2 to 4 go through as before, with the only modification that in the proof of Theorem 4.1 the anti-symmetry of the n-point functions has to be used as in [4].

5. Functional Formulation and Examples

The conditions of Theorem 4.1 can be very simply expressed by means of the expectation functional

$$E(f)=\langle\Omega, e^{i\phi(f)}\Omega\rangle,$$

except for condition (A′) which has to be checked separately.

The following conditions (F), (E), (C) are the well-known conditions on $E(f)$ to determine an abelian self-adjoint field over $\mathscr{S}$ (or $\mathscr{D}$) which is Euclidean covariant and has a unique invariant cyclic vector Ω.

(F) For fixed f, $E(\tau f)$ is continuous in τ, and for all $f_i\in\mathscr{S}(R^d)$, $\lambda_i\in\mathbb{C}$, $\Sigma\,\lambda_i\bar{\lambda}_j E(f_i-f_j)\geqq 0$.

(E) For all $f\in\mathscr{S}(R^d)$ and $(R,a)\in IO(d)$, $E(f_{(R,a)})=E(f)$.

(C) $\lim_{|g|\to\infty} E(f_a+g)=E(f)\,E(g)$ for all $f,g\in\mathscr{S}(R^d)$.

As already remarked in the Introduction, T-positivity is equivalent to

(T) Let $(\vartheta f)(f)=f(-x^0,\boldsymbol{x})$. Then for all $f_i\in\mathscr{S}(R_+^d)$ and all $\lambda_i\in\mathbb{C}$

$$\Sigma\,\lambda_i\bar{\lambda}_j E(f_i-\vartheta f_j)\geqq 0\,. \tag{4.1}$$

In case the conditions of [4] should turn out to be sufficient after all one can replace (A′) by

(D) For each $f\in\mathscr{S}(R^d)$, $E(\tau f)$ is a C^∞-function of τ at $\tau=0$, with the second derivative being continuous in f.

This last condition can be shown to be necessary and sufficient for Ω to be in the domain of all field operator products and for the n-point functions to be distributions on $\mathcal{S}(R^{dn})$. The latter can then be obtained by differentiating $E(\Sigma \tau_i f_i)$ at $\tau_i = 0$.

Examples

1. The Constant Field. We consider the functional

$$E(f) \equiv \langle \Omega, e^{i\phi(f)}\Omega \rangle = \exp\{i\alpha \int d^s x\, f(x) - \beta \int d^s x\, |f(x)|^2 + \int d^s x \int d\sigma(\lambda)\,[e^{i\lambda f(x)} - 1 - i\lambda f(x)/(1+\lambda^2)]\} \tag{5.1}$$

where α, β are real constants with $\beta \geqq 0$, and where σ is a positive measure on the real line satisfying

$$\int d\sigma(\lambda)\, \lambda^2/(1+\lambda^2) < \infty, \quad \sigma\{0\} = 0. \tag{5.2}$$

These functionals are well-known [7]. For $\alpha = \beta = 0$ and σ even they belong to the fields underlying the ultra-local models [8, 9]. Properties (F), (C) and (E) are satisfied. If $d\sigma$ has a sufficiently rapid decrease at infinity, then also (D) is fulfilled, since one can then differentiate under the integral. In view of the *exp* and of the integrals over x one sees immediately that $E(f_1 + f_2) = E(f_1)\, E(f_2)$ whenever f_1 and f_2 have disjoint supports. Hence for $f_i \in \mathcal{S}(R^s_+)$, $i = 1, \ldots, n$, $E(f_i - \vartheta f_j) = E(f_i)\, \overline{E(\vartheta f_j)}$. Since by (E) one has $E(\vartheta f) = E(f)$, condition (T) is trivially satisfied. The n-point functions of ϕ can be easily computed by differentiations, but this is not necessary here. It follows directly from the multiplicative property of $E(f)$ that for *noncoinciding points* one has

$$\langle \Omega, \phi(x_1) \ldots \phi(x_n)\Omega \rangle = \prod_i \langle \Omega, \phi(x_i)\Omega \rangle = c^n \tag{5.3}$$

where $\langle \Omega, \phi(x)\Omega \rangle = \text{const} = c$, by translation invariance[12]. Hence analytic continuation yields constant Wightman functions also.

2. The Free Field. In Eq. (5.1) we put $\alpha = 0$ and $\sigma = 0$, and replace f by $(-\Delta + m^2)^{-1/2} f$, where Δ is the s-dimensional (Euclidean invariant) Laplacian and $m \geqq 0$ for $s \geqq 3$ and $m > 0$ otherwise. Thus we consider

$$E(f) = \exp\{-\|(-\Delta + m^2)^{-1/2} f\|^2\}. \tag{5.4}$$

Conditions (F), (E), (C), (D) are automatically satisfied. To check T-positivity, the following lemma is useful.

Lemma 5.1. *The $n \times n$-matrix $A(\tau) = (e^{\tau\alpha_{ij}})$ is positive-definite for every $\tau > 0$ if and only if the matrix (α_{ij}) is conditionally positive-definite, i.e., if and only if, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $\Sigma \lambda_i = 0$,*

$$\Sigma \lambda_i \bar{\lambda}_j \alpha_{ij} \geqq 0.$$

[12] It is zero if σ is even, i.e., if $d\sigma(\lambda) = d\sigma(-\lambda)$. Hence these fields can be added to any Euclidean field without altering the associated Wightman field (if there is any).

The proof is a transcription of the proof of Theorem 4 in Chapter III, §4 of [7] and is therefore omitted. Replacing $f_i(x)$ by $f_i(x/\tau)$, $\tau>0$, the lemma shows that $E(f)$ in Eq. (5.4) satisfies (T) *if and only if, for* $\Sigma\,\lambda_i=0$ and $f_i\in\mathscr{S}(R^s_+)$,

$$0\leqq \Sigma\,\lambda_i\bar{\lambda}_j\langle f_i-\vartheta f_j,(-\Delta+m^2)^{-1}\quad(f_i-\vartheta f_j)\rangle$$

or

$$0\leqq \Sigma\,\lambda_i\bar{\lambda}_j\langle(-\Delta+m^2)^{-1/2}f_j,\vartheta(-\Delta+m^2)^{-1/2}f_i\rangle\,. \tag{5.5}$$

We will show that Eq. (5.5) holds without restriction on the λ_i's, for all $f_i\in\mathscr{S}(R^s_+)$[13]. To this end we note that the r.h.s. is not only defined for $f_i\in\mathscr{S}(\mathbb{R}^s)$ but also for f_i in the Sobolev space $\mathscr{H}^{-1}(\mathbb{R}^s)$, in particular for "sharp time" functions of the form

$$f_i(x)=g_i(\mathbf{x})\,\delta(x^0-t_i),\qquad g_i\in\mathscr{S}(\mathbb{R}^{s-1})\,. \tag{5.6}$$

For $t_i>0$ these functions are total in the subspace of functions having support in $\{x;x^0>0\}$, and hence it suffices to prove Eq. (5.5) for f_i given by Eq. (5.6). Using Fourier transforms one finds

$$\begin{aligned}&\langle f_j,(-\Delta+m^2)^{-1}\,\vartheta f_i\rangle\\&=\pi\langle g_j,g_i\rangle\,(\mathbf{p}^2+m^2)^{-1/2}\exp\{-|t_i+t_j|\sqrt{\mathbf{p}^2+m^2}\}\,,\end{aligned} \tag{5.7}$$

and so Eq. (5.5) is satisfied if $t_i>0$, $i=1,\dots,n$. The relativistic field associated with this Euclidean field is the free field of mass m.

3. Some Generalized Free Fields. We consider the functional

$$E(f)=\exp\{-\textstyle\int d\varrho(m^2)\,\|(-\Delta+m^2)^{-1/2}f\|^2\} \tag{5.8}$$

where $d\varrho$ is a positive measure on $[a,\infty)$, $a>0$, satisfying $\int d\varrho(m^2)\cdot(1+m^2)^{-1}<\infty$. Then $E(f)$ is defined and continuous on $\mathscr{S}(R^s)$. Lemma 5.1 shows that $E(f)$ satisfies condition (F), and (E), (C), (D) are also seen to hold. Eq. (5.5) together with Lemma 5.1 shows that (T) holds. The associated relativistic field is clearly a generalized free field.

Considering the usual momentum Fock space and making an automorphism of the test function space, the fields of the last two examples can be written as a sum of creation and annihilation operators. It is then seen that, on the n-particle space, one has $(1+K)^{-1}\leqq 1/n$, and from this one infers that the smeared creation operators are bounded operators from $\mathfrak{H}$ to $\mathfrak{H}^{-1}$. Hence the annihilation operators, which are their adjoints, are bounded operators from the dual of $\mathfrak{H}^{-1}$ to the dual of $\mathfrak{H}$, i.e., from $\mathfrak{H}^1$ to $\mathfrak{H}$. Therefore the field is a bounded operator from $\mathfrak{H}^1$ to $\mathfrak{H}^{-1}$. Hence (A′) holds. This procedure is similar to that at the end of [3].

[13] This means that $\hat{E}_+\,\vartheta\hat{E}_+\geqq 0$, where $\hat{E}_+$ is the projector onto the closure of the space $(-\Delta+m^2)^{-1/2}\,\mathscr{S}(R^s_+)$.

Remarks. 1. A suitable choice of ϱ yields examples like[14]

$$E(f)=\exp\{-\|(-\Delta+m_0^2)^{-1/4}f\|^2\},$$

while functionals like

$$E(f)=\exp\{-\|(-\Delta+m^2)^{-k/2}f\|^2\}, \quad 2\leqq k\in\mathbb{N},$$

can be shown to violate (T).

2. From the way in which the free fields was obtained from the "ultralocal" functionals of Eq. (5.1) it is tempting also to consider the functionals

$$E(f)=E_{UL}((-\Delta+m^2)^{-1/2}f)$$

where E_{UL} is obtained from Eq. (5.1) by putting $\alpha=\beta=0$. Conditions (F) and (E) are trivially satisfied since one has just made a transformation of the test function space, corresponding to an automorphism of the unitary group. With suitable behaviour of σ at infinity (D) is fulfilled too and (C) follows since Δ commutes with translations. (T) may or may not be verifiable. This question which reduces to the positivity of ϑ for a subspace of $\mathscr{S}(R^s)$ is under study.

6. On the Notion of Markoff Fields and Its Connection with *T*-Positivity

In this section we introduce a notion of Markoff field which slightly differs in the localization from that of Nelson [2] and which is just as useful for the construction of relativistic fields. With its help the reflection property which is so extensively used in [2] can be very simply expressed. We then show that T-positivity is a natural generalization of Markoff + reflection property.

Nelson defines a Markoff field over $\mathscr{D}(R^d)$ in the following way. Let ϕ be an abelian field over $\mathscr{D}(R^d)$ in $\mathfrak{H}$ with cyclic vector Ω, let $U\subset R^d$ be an open set, let $\mathfrak{H}_U$ be the subspace of $\mathfrak{H}$ generated by $\{e^{i\phi(f)};\ supp\, f\subset U\}$ and let E_U be the projector onto $\mathfrak{H}_U$. Let $F\subset R^d$ be a closed set, and define

$$\mathfrak{H}_F=\bigcap_{U\supset F}\mathfrak{H}_U\,. \tag{6.1}$$

Let E_F be the projector onto $\mathfrak{H}_F$. We denote by U' the complement of U and by ∂U its boundary. Then in [2] ϕ is said to satisfy the *Markoff property* if, for all open sets U,

$$E_{U'}E_U=E_{\partial U}E_U\,. \tag{6.2}$$

[14] The exponent equals $2/\pi\int dm'\langle f,(-\Delta+m_0^2+m'^2)^{-1}f\rangle$ which can be brought to the form of Eq. (5.8).

This can easily be expressed in probabilistic language in terms of conditional expectations[15].

In the construction of relativistic fields the Markoff property is needed for open half spaces $\{x; x^0 \gtrless s\}$ only, no other open sets appear. This suggests a slightly different Markoff notion which is more symmetric in the localization.

Definition 6.1. Let $\mathscr{V}$ be a space of functions on R^d, and let ϕ be an abelian field over $\mathscr{V}$ in $\mathfrak{H}$ with cyclic vector Ω. Let $U_{>s}$ and $U_{<s}$ be the half spaces $\{x \in R^d;\ x^0 \gtrless s\}$, respectively. Let $\mathfrak{H}_{<s}$ be the Hilbert space generated by $\{e^{i\phi(f)}\Omega;\ \operatorname{supp} f \subset U_{<s}\}$, and similarly for $\mathfrak{H}_{>s}$. Put $\mathfrak{H}_{\leqq s} = \mathfrak{H}_{<s}$ and $\mathfrak{H}_s = \mathfrak{H}_{<s} \cap \mathfrak{H}_{>s}$, and denote the corresponding projectors by $E_{>s}$, $E_{<s} = E_{\leqq s}$, E_s. Then ϕ is said to satisfy the *Markoff property of second kind*[16] if

$$E_{\leqq s} E_{>s} = E_s E_{>s}. \tag{6.3}$$

Remark. In our definition the Hilbert space associated with a *closed* half space in R^d equals that for the open half space contained in it. This is a more symmetric localization and yields as a trivial consequence

Corollary 6.1. *ϕ has the Markoff property of second kind if and only if $E_{<s}E_{>s}$ is a projector,*

$$E_{<s} E_{>s} = E_s. \tag{6.4}$$

Proof. Since $\mathfrak{H}_s \subset \mathfrak{H}_{>s}$, Eq. (6.3) is equivalent to Eq. (6.4).

When one has a unitary representation T_t of the translations in 0-direction under which ϕ transforms covariantly and which leave Ω fixed, then Eq. (6.3) can be further simplified.

---

[15] One can realize $\mathfrak{H}$ as a function space $L^2(X, \Sigma, \mu)$ with Ω corresponding to 1. (X, Σ, μ) is a probability space, i.e., X is a space, Σ a σ-algebra of subsets of X and μ a measure on Σ with $\mu(X) = 1$. One can further realize $\phi(f)$ as multiplication by a function so that $\phi(f)$ can be regarded as a linear stochastic process over $\mathscr{D}(R^d)$. Let $\mathscr{B}(U) \subset \Sigma$ be the smallest σ-algebra with respect to which all functions in $\mathfrak{H}_U$ are measurable, and denote by $\mathscr{F}(U)$ the set of all functions which are measurable with respect to $\mathscr{B}(U)$. The conditional expectation with respect to a sub-σ-algebra $\mathscr{B}$ of Σ of a random variable (i.e., measurable function) u is defined by a Radon-Nikodym derivative as

$$E\{u \mid \mathscr{B}\} = \frac{(u\, d\mu)\restriction_{\mathscr{B}}}{d\mu\restriction_{\mathscr{B}}}$$

Then Eq. (5.1) can be extended to read

$$E\{u \mid \mathscr{B}(U')\} = E\{u \mid \mathscr{B}(\partial U)\}$$

for all $u \in \mathscr{F}(U)$ and for all open sets $U \subset R^d$.

[16] This can again be expressed in terms of conditional expectations and linear stochastic processes over $\mathscr{V}$, just as in the preceding footnote.

Corollary 6.2. *Let $T_t\phi(x)T_t^* = \phi(x^0+t, \boldsymbol{x})$, $T_t\Omega = \Omega$. Let $E_- = E_{<0}$ and $E_+ = E_{>0}$. Then ϕ has the Markoff property of second kind if and only if $E_- E_+$ is a projector,*

$$E_- E_+ = E_0 . \tag{6.5}$$

Proof. This follows from $E_{\gtrless s} = T_s E_{\lessgtr 0} T_s^*$ and $E_s = T_s E_0 T_s^*$. QED.

With this Markoff notion one can work just as with that introduced in [2]. The *reflection property* needed in [2] means that there is a unitary operator T in $\mathfrak{H}$ satisfying $T\phi(x^0, \boldsymbol{x})T = \phi(-x^0, \boldsymbol{x})$ and $T\Omega = \Omega$ such that T is the identity on the Hilbert space belonging to the hyperplane $\{x \in R^d;\ x^0 = 0\}$. Correspondingly we will say that a Markoff field of the second kind satisfies the reflection property if T is the identity on $\mathfrak{H}_0$. The next necessary and sufficient condition is an easy consequence of the definitions.

Theorem 6.1. *Let ϕ be a field in $\mathfrak{H}$ with cyclic vector Ω over a space $\mathscr{V}$ of functions on R^d. Let $\{T_t;\ t \in R\}$ and T be unitary operators such that $T_t\phi(x)T_t^* = \phi(x^0+t, \boldsymbol{x})$, $T\phi(x)T = \phi(-x^0, \boldsymbol{x})$, and $T_t\Omega = T\Omega = \Omega$. Then ϕ is a Markoff field of the second kind satisfying the reflection property if and only if $E_+ T E_+$ is a projector.*

Proof. The necessity follows from $E_0 = TE_0 = TE_- E_+ = E_+ TE_+$. Conversely, if $E_+ TE_+$ is a projector, E say, then $E^3 = TE_- E_+ E_- E_+ = E = TE_- E_+$. Hence $(E_- E_+)^2 = E_- E_+$, and since $\|E_- E_+\| \leqq 1$ it follows that $E_- E_+$ is a projector [10]. From this and from $E^2 = E_+ E_- E_+ = E_- E_+ = E = TE_- E_+$ it then follows that T is 1 on $E_- E_+ \mathfrak{H}$. QED.

We note that if $E_+ TE_+$ is a projector, it projects onto $\mathfrak{H}_0 = \mathfrak{H}_+ \cap \mathfrak{H}_-$, and T is 1 on $\mathfrak{H}_0$. In particular $E_+ TE_+ \geqq 0$. The next corollary shows that T-positivity is the natural generalization of Markoff + reflection property for practically any localization prescription.

Corollary 6.3. *Let ϕ, $\mathscr{V}$, $\mathfrak{H}$, Ω, T be as in Theorem 6.1. If $U \subset R^d$ is an open set, let $\mathfrak{H}_U$ be the Hilbert space generated by $\{e^{i\phi(f)}\Omega;\ \mathrm{supp}\, f \subset U\}$. If $F \subset R^d$ is a closed set define $\mathfrak{H}_F$ in such a way that $\mathfrak{H}_F \supset \mathfrak{H}_A$ whenever $F \supset A$, but otherwise arbitrary*[17]. *If, with this localization, ϕ is a Markoff field which satisfies the reflection property, then $E_+ TE_+ \geqq 0$.*

Proof. Let $\bar{E}_\pm$ and $\bar{E}_0$ be the projectors onto the subspaces associated with $\{x;\ x^0 \gtreqqless 0\}$ and $\{x;\ x^0 = 0\}$ so that $\bar{E}_- \bar{E}_+ = \bar{E}_0 \bar{E}_+$ and $T\bar{E}_0 = \bar{E}_0$. Then multiplication by $E_+ T$ yields $E_+ TE_+ = E_+ \bar{E}_0 E_+ \geqq 0$. QED.

Remarks. (i) Under additional continuity properties different localizations will give the same result. Thus in the case of a field over the Sobolev space $\mathscr{H}^{-1}(R^d)$ Nelson's definition of the Markoff property satisfies also Definition 6.1 since, in view of the assumed continuity of $e^{i\phi(f)}$, the Hilbert spaces belonging to open and closed half spaces and hyperplane are the same in the two localizations.

---

[17] It suffices to consider open and closed half spaces and hyperplanes as in Definition 6.1.

(ii) The localization in Definition 6.1 may be changed slightly without spoiling the symmetry. One still associates to both open and closed half spaces again the same Hilbert space, but one now takes $\bar{\mathfrak{H}}_{>s} = \bigcap_{r<s} \mathfrak{H}_{>s}$, etc., where $\mathfrak{H}_{>s}$ is as in Definition 6.1, and puts $\bar{\mathfrak{H}}_s = \bar{\mathfrak{H}}_{<s} \cap \bar{\mathfrak{H}}_{>s}$. Eq. (5.3) is then replaced, in obvious notation, by $\bar{E}_{<s} \bar{E}_{>s} = \bar{E}_s \bar{E}_{>s}$. Then Corollaries 6.1 and 6.2 as well as Theorem 6.1 hold with the obvious changes. Although $\bar{\mathfrak{H}}_0 \supset \mathfrak{H}_0$, for the relativistic field no different Hilbert space results [cf. Eq. (3.9)].

(iii) One may generalize the above considerations in an obvious way to Fermi fields. With localization as in Definition 6.1 it would be natural to say that a Fermi field possesses the Markoff and reflection property if and only if $E_+ T E_+$ is a projector (cf. Remark (iv) in Section 4). Markoff Fermion field have also been discussed in [11].

Acknowledgments. I would like to thank J. Yngvason for stimulating discussions in connection with assumption (A′) of [2].

Note Added in Proof. In a forthcoming paper we are going to give a functional characterization of Markoff fields (which satisfy the reflection property) and show with its help that T-positivity is indeed more general. It will turn out that the generalized free fields of Section 5 are not Markoff fields, neither over $\mathscr{H}^{-1}$ nor over $\mathscr{S}$, although they satisfy T-positivity. It can also be shown that reflection invariance is not needed for the construction of a relativistic field, i.e., the existence of the operator T is not needed. T-positivity has then to be expressed by the functional $E(f)$ as in Eq. (1.4). The derivation uses bilinear forms instead of the operator T.

References

1. Symanzik, K.: Euclidean quantum field theory. Proceedings of the International School of Physics "Enrico Fermi", Varenna Course XLV 1968. Edited by R. Jost. New York: Academic Press 1969
2. Nelson, E.: J. Funct. Anal. **12**, 97—112 (1973)
3. Nelson, E.: J. Funct. Anal. **12**, 211—227 (1973)
4. Osterwalder, K., Schrader, R.: Commun. math. Phys. **31**, 83—112 (1973)
5. Neumark, M. A.: Normierte Algebren, p. 400. Berlin: VEB Deutscher Verlag der Wissenschaften 1959
6. Nelson, E.: J. Funct. Anal **11**, 211—219 (1972)
7. Gelfand, I. M., Vilenkin, N. Y.: Generalized functions, Vol. 4, Chapter III, § 4, Theorem 5. New York: Academic Press 1964
8. Klauder, J. R.: Commun. math. Phys. **18**, 307—318 (1970)
9. Hegerfeldt, G. C., Klauder, J. R.: Field product renormalization and Wilson-Zimmermann expansion in a class of Euclidean invariant models. Il Nuovo Cim. (in press)
10. Yosida, K.: Functional analysis, 2nd Edition, p. 84. Berlin-Heidelberg-New York: Springer 1968
11. Wilde, I. F.: The free Fermion field as a Markov field. Preprint E.T.H. Zürich 1973
12. Glaser, V.: On the equivalence of the Euclidean and Wightman formulation of field theory. Cern preprint TH 1706

Communicated by A. S. Wightman

G. C. Hegerfeldt
Institut für Theoretische Physik der Universität
D-3400 Göttingen, Bunsenstr. 9
Federal Republic of Germany