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Analytic Continuation of Group Representations. III*

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Abstract. The connection between the ideas of "contraction" and "analytic continuation" of Lie algebras and their representations is discussed, with particular emphasis on the contraction of the Poincaré to the Galilean group.

1. Introduction

We continue the study of the relation between analytic continuation of Lie algebras, their representations and Lie algebra cohomology. The first topic we will treat will be a further development of the formalism when a Lie algebra structure is fixed, and an irreducible representation of it is analytically continued. In [4] we showed that, if the relevant first cohomology group vanishes, then the Casimir operators of the Lie algebra are constants of the deformation parameter. Here, we will study the formalism for the case where the first cohomology group does not vanish. We will obtain some insight into one of the main problems, namely, discovering when the first cohomology group is finite dimensional.

Our next topic will be to continue both the Lie algebra structure and the representation. This will provide a tie-up between Lie algebra cohomology theory and the Gell-Mann formula for the representations of Lie algebras. Again, we will find that the beautiful ideas of the Kodaira-Spencer theory of deformation of structure provide us with a deep insight into the already known situation, and should be an invaluable guide to extending the existing theory to new situations. The case of the contraction of the Poincaré to the Galilean group will be treated in some detail.

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2. The effect of continuation of representations on the universal enveloping algebra

Let G be a Lie algebra, with X, Y, . . . denoting its typical elements, $[X, Y]$ its bracket. Recall that U(G), *the universal associative enveloping algebra* of G is defined in the following way [3, 5]:

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Construct the algebra of formal products $X_1 \dots X_r$ of elements of $\mathfrak{G}$. (Technically, this is the tensor algebra of the underlying vector space of $\mathfrak{G}$ and is denoted by $T(\mathfrak{G})$.) It is an associative algebra: the product is defined in the obvious way by juxtaposition:

$$(X_1 \dots X_r)(Y_1 \dots Y_s) = (X_1 \dots X_r \quad Y_1 \dots Y_s).$$

Introduce the relations:

$$XY - YX - [X, Y] = 0.$$

The quotient (associative) algebra is defined as $U(\mathfrak{G})$. (Technically, one considers the two-sided ideal of $T(\mathfrak{G})$ generated by all elements of the form $XY - YX - [X, Y]$, for $X, Y \in \mathfrak{G}$, and defines $U(\mathfrak{G})$ as the quotient of $T(\mathfrak{G})$ by this ideal.)

U$(\mathfrak{G})$ can be made into a Lie algebra by defining the bracket as commutator:

$$[\Delta_1, \Delta_2] = \Delta_1\Delta_2 - \Delta_2\Delta_1.$$

Thus, $\mathfrak{G}$ can be considered as a Lie subalgebra of $U(\mathfrak{G})$.

If $\varphi : \mathfrak{G} \to \mathbf{L}$ is a Lie algebra homomorphism, it extends in an obvious way to an associative algebra homomorphism: $U(\mathfrak{G}) \to U(\mathbf{L})$. Explicitly, if

$$\Delta = X_1 \dots, X_r \in U(\mathfrak{G})$$

$$\varphi(\Delta) = \varphi(X_1) \dots \varphi(X_r) \in U(\mathbf{L}).$$

An element Δ of $U(\mathfrak{G})$ is a *Casimir operator* of $\mathfrak{G}$ if it belongs to the center of $U(\mathfrak{G})$: it is readily seen that this is equivalent to the condition

$$[\Delta, \mathfrak{G}] = 0.$$

Often in the applications to physics one must consider an "extended" or "complete" universal enveloping algebra, which should, roughly, consist of all "functions" of the elements of $\mathfrak{G}$, rather than simply all polynomials. One way of making this precise might be to extend $U(\mathfrak{G})$ by adding all formal power series in the elements of $\mathfrak{G}$. Very little is known about such objects: The reader might keep in mind, however, that such an extension is desirable, and that much of what we say of a purely formal nature about $U(\mathfrak{G})$ can be extended with very little effort to such an extended algebra.

Now, suppose that $\varphi_\lambda : \mathfrak{G} \to \mathbf{L}$ is a one parameter family of homomorphisms of a Lie algebra $\mathfrak{G}$ into a Lie algebra $\mathbf{L}$. As in [4] we define the map $\omega : \mathfrak{G} \to \mathbf{L}$ by the formula:

$$\omega(X) = \frac{d}{d\lambda} \varphi_\lambda(X)|_{\lambda=0}.$$

Then, as we know, ω is a 1-cocycle of $\mathfrak{G}$ with coefficients in the representation φ', where φ' assigns to each $X \in \mathfrak{G}$ the inner derivation

$$Z \to [\varphi(X), Z] \quad \text{of} \quad \mathbf{L}.$$

The cohomology class in $H^1(\varphi')$ determined by ω then "obstructs" the possibility of obtaining φ_λ from φ_0 by applying to it an inner automorphism of **L**.

Suppose that Δ is an element of $U(\mathrm{G})$. Consider the element of $U(\mathbf{L})$ given by the following formula:

$$\frac{d}{d\lambda}\varphi_\lambda(\Delta)|_{\lambda=0}\,.$$

Can it be computed in terms of ω?

Suppose, for example, that

$$\Delta = XY, \quad \text{with} \quad X, Y \in \mathrm{G}\,.$$

$$\frac{d}{d\lambda}\varphi_\lambda(\Delta)|_{\lambda=0} = \frac{d}{d\lambda}\varphi_\lambda(X)\,\varphi_\lambda(Y)|_{\lambda=0} = \omega(X)\,\varphi(Y) + \varphi(X)\,\omega(Y)\,.$$

This formula suggests that we define the right hand side as a new operation between the 1-cocycles and elements of $U(\mathrm{G})$:

$$\omega(X_1, \ldots, X_r) = \omega(X_1)\,\varphi(X_2 \ldots X_r) + \\ + \varphi(X_1)\,\omega(X_2)\,\varphi(X_3) \ldots \varphi(X_2) + \cdots + \varphi(X_1 \ldots X_{r-1})\,\omega(X_r)\,.$$

To show that it is well defined on $U(\mathrm{G})$, we must verify that:

$$\omega(XY) - \omega(YX) - \omega([X, Y] = 0\,. \tag{2.1}$$

But,

$$\begin{aligned}\omega(YX) - \omega(XY) &= \omega(Y)\,\varphi(X) + \varphi(Y)\,\omega(X) - \\ &\quad - \omega(X)\,\varphi(Y) - \varphi(X)\,\omega(Y) \\ &= [\omega(Y), \varphi(X)] - [\omega(X), \varphi(Y)] \\ &= -d\omega(X, Y) - \omega([X, Y])\,,\end{aligned}$$

whence (2.1), since $d\omega = 0$.

Let us now compute

$$\omega(\Delta_1\Delta_2), \quad \text{for} \quad \Delta_1, \Delta_2 \in U(\mathrm{G})\,.$$

In fact,

$$\begin{aligned}\omega(X_1 \ldots X_r Y_1 \ldots Y_s) &= \omega(X_1)\,\varphi(X_2 \ldots Y_s) + \\ &\quad + \varphi(X_1 \ldots Y_{s-1})\,\omega(Y_s) \\ &= \omega(X_1 \ldots X_r)\,\varphi(Y_1 \ldots Ys) + \\ &\quad + \varphi(X_1 \ldots X_r)\,\omega(Y_1 \ldots Y_s)\,.\end{aligned}$$

Thus we have:

$$\omega(\Delta_1\Delta_2) = \omega(\Delta_1)\,\varphi(\Delta_2) + \varphi(\Delta_1)\,\omega(\Delta_2) \tag{2.2}$$

for $\Delta_1, \Delta_2 \in U(\mathrm{G})$.

Suppose now that $\omega = dW$, for $W \in L$.

$$\begin{aligned}\omega(YZ) &= \omega(Y)\,\varphi(Z) + \varphi(Y)\,\omega(Z)\\ &= (dW)\,(Y)\,\varphi(Z) + (\varphi(Y)\,dW(Z))\\ &= [W, \varphi(Y)]\,\varphi(Z) + \varphi(Y)\,[W, \varphi(Z)]\\ &= [W, \varphi(YZ)]\,.\end{aligned}$$

The general formula is obviously:

$$(dW)\,(\Delta) = [W, \varphi(\Delta)] \quad \text{for} \quad W \in \mathbf{L}, \Delta \in U(\mathfrak{G})\,. \tag{2.3}$$

Suppose $X \in \mathfrak{G}$: let us compute

$$[\varphi(X), \omega(\Delta)]\,.$$

Suppose, for example, $\Delta = YZ$. Then,

$$\begin{aligned}[\varphi(X), \omega(YZ)] &= [\varphi(X), \omega(Y)\,\varphi(Z) + \varphi(Y)\,\omega(Z)\\ &= [\varphi(X), \omega(Y)]\,\varphi(Z) + \omega(Y)\,[\varphi(X), \varphi(Z)] +\\ &\quad + [\varphi(X), \varphi(Y)]\,\omega(Z) + \varphi(Y)\,[\varphi(X), \omega(Z)]\\ &= [\varphi(X), \omega(Y)]\,\varphi(Z) + \omega(Y)\,\varphi([X, Z]) +\\ &\quad + \varphi([X, Y]\,\omega(Z) + \varphi(Y)\,[\varphi(X), \omega(Z)]\,.\end{aligned}$$

Also,

$$\begin{aligned}\omega([X, YZ]) &= \omega([X, Y]\,Z + Y[X, Z])\\ &= \omega([X, Y]\,\varphi(Z) + \varphi([X, Y]\,\omega(Z) +\\ &\quad + \omega(Y)\,\varphi([X, Z]) + \varphi(Y)\,\omega([X, Z])\,.\end{aligned}$$

Hence,

$$\begin{aligned}[\varphi(X, \omega(YZ)] - \omega([X, YZ) &= ([\varphi(X), \omega(Y)] - \omega([X, Y]))\,\varphi(Z) +\\ &\quad + \varphi(Y)\,([\varphi(X), \omega(Z)] - \omega([X, Z]))\\ &= X(\omega)\,(YZ)\,.\end{aligned}$$

This computation obviously generalizes to higher degree elements of $U(\mathfrak{G})$, giving the formula:

$$[\varphi(X), \omega(\Delta)] = X(\omega)\,(\Delta) + \omega([X, \Delta]) \quad \text{for} \quad X \in \mathfrak{G}, \Delta \in U(\mathfrak{G})\,. \tag{2.4}$$

Thus, if Δ is a Casimir operator of $U(\mathfrak{G})$,

$$\begin{aligned}[\varphi(X), \varphi(\Delta)] &= X(\varphi)\,(\Delta)\\ &= d(X \rfloor \omega)\,(\Delta)\\ &= [\omega(X), \varphi(\Delta)]\,.\end{aligned} \tag{2.5}$$

Thus we have proved:

Theorem 2.1. *Suppose that the homomorphism satisfies the following condition: φ maps every Casimir operator of $U(\mathfrak{G})$ into a Casimir operator of $U(\mathbf{L})$. Then $\Delta \to \omega(\Delta)$ also maps a Casimir operator of $U(\mathfrak{G})$ into a Casimir operator of $U(\mathbf{L})$.*

Suppose, in particular, that $\mathbf{L}$ *is a Lie algebra of operators on a vector space that is irreducible in the sense that the only operators commuting with every element of* $\mathbf{L}$ *are the multiples of the identity operator* I.

Then,

$$\omega(\Delta) = a(\Delta, \omega) I \tag{2.6}$$

for each Casimir operator Δ *of* $U(\mathrm{G})$, *where* $a(\Delta, \omega)$ *is a bilinear, scalar-valued function defined for Casimir operators* Δ *and for* 1*-cocycles* ω. *If* ω *cobounds, it is zero, so* $a(\,,)$ *is really a bilinear function defined on Casimir operators* Δ *and cohomology classes in* $H^1(\varphi')$.

Suppose now that G is semisimple. Suppose $\Delta_1, \ldots, \Delta_l$ are the basic Casimir operators of G, i.e., all other Casimir operators are polynomials in these (l = rank, G = dimension of a Cartan subalgebra of G. Of course, it is a theorem about semisimple Lie algebras that such a basic set exists). Then, if (2.6) holds, the mapping

$$\omega \to (a(\omega, \Delta_1), \ldots, a(\omega, \Delta_l)) \tag{2.7}$$

defines a linear mapping of $H^1(\varphi')$ into C^l if, for example, C (the complex numbers) is the field of scalars for the operators $\mathbf{L}$. One conjectures that in certain circumstances this is one-one, hence, that the finite dimensionality of $H^1(\varphi')$ can be proved in this explicit way. One also can remark that the standard methods of defining unitary representations via induced representation theory do indeed provide representations depending on l parameters.

3. Cohomology modulo subgroups

Suppose G, $\mathbf{L}$, φ, and φ' are as in Section 2. In addition, let $\mathbf{K}$ be a subalgebra of G. Consider a deformation $\lambda \to \varphi_\lambda$ of φ with

$$\omega = \frac{d}{d\lambda} \varphi_\lambda|_{\lambda=0}$$

the corresponding 1-cocycle. Now, if

$$\varphi_\lambda(X) = X \quad \text{for} \quad X \in \mathbf{K},$$

we obviously have

$$\omega(X) = X \rfloor \omega = 0 .$$

Since $d\omega = 0$, we also have

$$X \rfloor d\omega = 0 .$$

Now, in general, suppose we consider the subspace of those cocycles $\omega \in C^r(\varphi')$ such that

$$0 = X \rfloor \omega = X \rfloor d\omega ,$$

which we denote by $C^r(\varphi', \mathbf{K})$. Note that

$$dC^r(\varphi', \mathbf{K}) \subset C^{r+1}(\varphi', \mathbf{K}) .$$

Thus, we can define the r-th cohomology group of φ' modulo $\mathbf{K}$ as the quotient

$$Z^r(\varphi', \mathbf{K}) | d C^{r-1}(\varphi', \mathbf{K}) .$$

Summing up, we may say that $H^1(\varphi', \mathbf{K})$ measures the possible deformations $\lambda \to \varphi_\lambda$ of the homomorphism φ such that the representation φ_λ restricted to $\mathbf{K}$ is fixed. There is obviously a homomorphism $H^r(\varphi', \mathbf{K}) \to \to H^r(\varphi')$.

This is a standard construction in cohomology theory. Let φ_K be the homomorphism φ restricted to $\mathbf{K}$. Let φ_K be the corresponding homomorphism $\mathbf{K} \to$ linear transformations on $\mathbf{L}$.

Every cochain in $C^r(\varphi')$ defines by restriction to $\mathbf{K}$ a cochain in $C^r(\varphi_K)$, hence also a linear map $H^r(\varphi') \to H^r(\varphi'_K)$. In certain dimensions, this is an exact sequence of the form:

$$H^r(\varphi') \to H^r(\varphi'_K) \to H^{r+1}(\varphi', \mathbf{K}) \to H^{r+1}(\varphi') \to \cdots . \tag{3.1}$$

("Exact sequence" means that the image of each of these homomorphisms is equal to the kernel of the succeeding one.) In practice, this is often used to compute $H^r(\varphi')$ in terms of $H^r(\varphi', \mathbf{K})$ and $H^r(\varphi'_K)$.

These constructions are of great interest for our program of computing $H^1(\varphi')$ for homomorphisms of $\mathbf{G}$ arising from unitary representations since, as we will see, $H^1(\varphi', \mathbf{K})$ is more readily computable.

For example, suppose that

$$\mathbf{G} = \mathbf{K} \oplus \mathbf{P}, \quad \text{with} \quad [\mathbf{K}, \mathbf{P}] \subset \mathbf{P}, [\mathbf{P}, \mathbf{P}] \subset \mathbf{K} ,$$

i.e., $\mathbf{K}$ is a symmetric subalgebra of $\mathbf{G}$.

Then, we have, for $\omega \in Z^1(\varphi', \mathbf{K})$, $X \in \mathbf{K}$, $X(\omega) = 0 = X \rfloor d\omega + + d(X \rfloor \omega) = 0$ hence

$$[\varphi(X), \omega(Y)] = \omega([X, Y]) \quad \text{for} \quad X \in \mathbf{K}, Y \in \mathbf{P} . \tag{3.2}$$

Thus, the set of operators $\omega(\mathbf{P})$ transforms under $\varphi(\mathbf{K})$ like the representation of $\mathrm{Ad}\,\mathbf{K}$ in $\mathbf{P}$. The cocycle condition is now

$$\omega([X, Y]) = [\varphi(X), \omega(Y)] - [\varphi(Y), \omega(X)] = 0, \quad \text{for} \quad X, Y \in \mathbf{P} . \tag{3.3}$$

As we shall see, at least for $G = SL(2, R)$, (3.2) and (3.3) seem to determine $H^1(\varphi, \mathbf{K})$ by purely algebraic means.

As we have seen, an $\omega \in Z^1(\varphi')$ induces a mapping of $U(\mathbf{G}) \to U(\mathbf{L})$. Suppose that $\mathbf{L}$ is a Lie algebra of skew-Hermitian operators on a Hilbert space H, and that $\varphi(\mathbf{G})$ is an irreducible family of operators on H. We constructed a homomorphism $H^1(\varphi') \to R^l$ by choosing a basis $\Delta_1, \ldots, \Delta_l$ of Casimir operators for $\mathbf{G}$, proving that for $\omega \in Z^1(\varphi')$, $\omega(\Delta_1), \ldots, \omega(\Delta_l)$ are multiples $ib_1, \ldots, ib_l$ of the identity operator in H, and mapping the cohomology class determined by ω into $(b_1, \ldots, b_l) \in R^l$. We would like to get some idea of how to compute the image in R^l of $H^1(\varphi', \mathbf{K})$.

Suppose that H_0 is a subspace of H invariant and irreducible under the action of $\varphi(\mathbf{K})$, such that all of H can be built up by applying to H_0 polynomials in the operators $\varphi(\mathbf{P})$. We may ask: when do two Casimir operators Δ_1 and Δ_2 of $\mathfrak{G}$ give the same multiple of the identity in H? Obviously, this is so if and only if $\varphi(\Delta_1)$ and $\varphi(\Delta_2)$ have the same value in H_0. Let us look at the simplest case, namely we suppose that

$$\varphi(\mathbf{K}) H_0 = 0 . \tag{3.4}$$

(If $\mathbf{K}$ is a maximal compact subalgebra of a semisimple $\mathfrak{G}$, these are what HELGASON [2] calls *representations of class* 1. They are also sometimes known as *spherical representations*, since they are the representations whose matrix elements are the "spherical functions" in the sense of CARTAN and GELFAND.) Then, clearly, $\varphi(\Delta_1)$ and $\varphi(\Delta_2)$ will be the same if $\Delta_1 - \Delta_2$ belongs to the left ideal $U(\mathfrak{G})\mathbf{K}$ generated by K. Notice, however, that if a Casimir operator Δ in $U(\mathfrak{G})$ belongs to $U(\mathfrak{G})\mathbf{K}$, and if $\omega \in Z^1(\varphi', \mathbf{K})$ then

$$\omega(\Delta) = 0 .$$

For, we know that $\omega(\Delta)$ is a multiple of the identity on H. To prove it is zero, it suffices to show that it is zero in H_0. However, if

$$\Delta = \Delta_1' \Delta_2', \quad \text{with} \quad \Delta_1' \in U(\mathfrak{G}), \Delta_2' \in U(\mathbf{K}) ,$$

then

$$\omega = \omega(\Delta_1')\, \varphi(\Delta_2') + \varphi(\Delta_1')\, \omega(\Delta_2') = \omega(\Delta')\, \varphi(\Delta_2') \quad \text{since} \quad \omega(\Delta_2') = 0 .$$

Then,

$$\omega(\Delta)\,(H_0) = 0, \quad \text{since} \quad \varphi(\Delta_2') H_0 = 0 .$$

This suggests that we choose (if possible) our basic Casimir operators $\Delta_1, \ldots, \Delta_l$ so that $\Delta_{m+1}, \ldots, \Delta_l$ is a basis for the Casimir operators in $U(\mathfrak{G})K$. Then, we might conjecture that the mapping $H^1(\varphi', \mathbf{K}) \to R^m$ that assigns $(b_1, \ldots, b_m) \in R^m$ to $\omega \in Z^1(\varphi', \mathbf{K})$ is one-one. (At least for certain symmetric spaces, one can prove that m = rank of the symmetric space G/K = dimension of maximal Abelian subalgebra of $\mathbf{P}$.)

4. Computation for $SL(2, R)$

Now suppose that $\mathfrak{G}$ is the Lie algebra of $SL(2, R)$, i.e., $\mathfrak{G}$ is generated by elements X, Y, Z with:

$$[Z, X] = Y; \ [Z, Y] = -X$$
$$[X, Y] = -Z .$$

Suppose that φ is an irreducible representation of $\mathfrak{G}$ by skew-Hermitian operators on a Hilbert space H, and that $\mathbf{L}$ is the Lie algebra of skew-Hermitian operators on H. Define

$$X^+ = X - iY; \ X^- = -(X + iY) .$$

Then

$$[Z, X^+] = iX^+; \ [Z, X^-] = -iX^- . \tag{4.1}$$

Let **K** be spanned by Z, **P** by X and Y. Notice that $\varphi(X^+)$ and $\varphi(X^-)$ are Hermitian adjoints of each other.

Suppose that $\omega \in Z^1(\varphi', \mathbf{K})$. We will compute $\omega(X)$ and $\omega(Y)$ using (3.2) and (3.3). Notice that it suffices to compute $\omega(X^+)$ and $\omega(X^-)$, which are Hermitian adjoints of each other.

We will proceed, as customary, by diagonalizing the operator $\varphi(Z)$. (Since Z generates a compact subgroup of $SL(2, R)$, $\varphi(Z)$ has discrete eigenvalues. It can be proved that these eigenvalues are simple and, with proper normalization, are integers.) There are two cases: either the eigenvalues of $\varphi(Z)$ from go $-\infty$ to ∞, or they are bounded in one direction. We will work with the first case for the moment. Suppose then that H is written as a direct sum $\sum_{r=-\infty}^{\infty} H^r$ of one-dimensional subspaces, each H^r generated by a single element ψ_r of norm one, with

$$\varphi(Z)\psi_r = ir\psi_r .$$

(4.1) shows that $\varphi(X^+)$ and $\varphi(X^-)$ are creation and annihilation operators, sending H^r into H^{r+1} and H^{r-1}, respectively. Suppose, say, that:

$$\varphi(X^+)\psi_r = \alpha_r \psi_{r+1}$$
$$\varphi(X^-)\psi_r = \alpha^*_{r-1} \psi_{r-1} .$$

(α^* denotes the complex conjugate of α.) Similarly,

$$\omega(X^+)\psi_r = \beta_r \psi_{r+1}$$
$$\omega(X^-)\psi_r = \beta^*_{r-1} \psi_{r-1} .$$

The cocycle condition is:

$$[\varphi(X^+), \omega(X^-)] = [\varphi X^-), \omega(X^+)] ,$$

or

$$\alpha_{r-1}\beta^*_{r-1} - \beta^*_r \alpha_r = \alpha^*_r \beta_r - \beta_{r-1}\alpha^*_{r-1} . \tag{4.2}$$

Now

$$X^+X^- = -X^2 - Y^2 + iYX - iXY = -\Delta + iZ + Z^2 ,$$

where $\Delta = X^2 + Y^2 - Z^2$ is the Casimir operator of **G**. Then,

$$-\omega(\Delta) = \omega(X^+X^-) = \omega(X^+)\,\varphi(X^-) + \varphi(X^+)\,\omega(X^-) .$$

We want to show that $\omega(\Delta)$ determines the cohomology class in $H^1(\varphi', \mathbf{K})$ to which ω belongs. Suppose then that $\omega(\Delta) = 0$. Then

$$0 = \beta_{r-1}\alpha^*_{r-1} + \alpha_{r-1}\beta^*_{r-1} . \tag{4.3}$$

We want to show that

$$\omega = dA ,$$

where A is a skew-Hermitian operator on H. The condition $\omega(Z) = 0$ forces

$$[\varphi(Z), A] = 0$$

hence A maps H^r into H^r. Say, that $A\psi_r = i a_r \psi_r$.

$$\omega(X^+) = [\varphi(X^+), A]$$
$$\omega(X^-) = [\varphi(X^-), A] .$$

Since the second of these relations follows from the first on taking adjoints, we can solve the first, which takes the form:

$$\beta_r = i a_r \alpha_r - i a_{r+1} \alpha_r \tag{4.4}$$

which can be looked on now as a set of *equations* for a_r. Notice first that (4.4) implies (4.3):

$$\alpha_r^* \beta_r = |\alpha_r|^2 (i a_r - i a_{r-1}) \tag{4.5}$$

which implies (4.3). Conversely, (4.5) can be solved for a_r by recurrence. (4.3) then guarantees that the a_r are real numbers.

Clearly, the same argument applies in case the spectrum of $\varphi(Z)$ is bounded below. We have an additional fact here however: $\omega(\Delta)$ is always zero. For, there is then an element ψ of H which is annihilated by $\varphi(X^-)$ and $\omega(X^-)$. Thus,

$$\varphi(\Delta)(\psi) = 0 .$$

Since $\varphi(\Delta)$ is a scalar operator, it is zero. This is the cohomological version of the fact that the representations of this form are part of the "discrete series", and cannot be deformed continuously. This should give us a way of making precise what is meant by "discrete series" in the case of more complicated groups.

5. Analytically continuing Lie algebra structures and representations together

So far, we have been considering continuation of representations and Lie algebra structures separately. An understanding of the combination of the two ideas is essential for an understanding of the Gell-Mann formula, which is just a particular case. Let us show this in the case of $SL(2, R)$, based on the treatment in [4].

Let us start off with the Lie algebra of the group of rigid motions in the plane, generated by elements Z, X', Y' satisfying:

$$\begin{gathered}[Z, X'] = Y';\ [Z, Y'] = -X' \\ [X', Y'] = 0 .\end{gathered} \tag{5.1}$$

Suppose an irreducible representation of this algebra by skew-Hermitian operators on a Hilbert space H is given. The operator $X'^2 + Y'^2$ is a Casimer operator of this algebra: let us normalize so that $X'^2 + Y'^2 = I$.

Form operators

$$X_\lambda = 1/2i[Z^2, X'] + \lambda X'$$
$$Y_\lambda = 1/2i[Z^2, Y^+] + \lambda Y$$
$$Z_\lambda = Z .$$

Then, as was shown in [4],

$$[X_\lambda, Y_\lambda] = -Z$$
$$[Z, X_\lambda] = Y_\lambda$$
$$[Z, Y_\lambda] = -X_\lambda$$

these operators $(X_\lambda, Y_\lambda, Z)$ form a representation of the Lie algebra of $SL(2, R)$.

We now want to investigate more precisely what happens as $\lambda \to \infty$. Let us set $\varepsilon = 1/\gamma$. Define

$$\begin{aligned} \varphi_\varepsilon(X') &= 1/2\varepsilon i[Z^2, X'] + X' = \varepsilon X_\lambda \\ \varphi_\varepsilon(Y') &= \varepsilon Y_\lambda \\ \varphi_\varepsilon(Z') &= Z . \end{aligned} \tag{5.2}$$

Then,

$$[\varphi_\varepsilon(X'), \varphi_\varepsilon(Y')] = -\varepsilon^2 Z$$
$$[Z, \varphi_\varepsilon(X')] = \varepsilon Y_\lambda = \varphi_\varepsilon(Y')$$
$$[Z, \varphi_\varepsilon(Y')] = -\varphi_\varepsilon(X') .$$

These formulas can be interpreted as follows:

Let G be the *vector space* spanned by the elements X', Y', Z. For each ε, define a Lie algebra structure as $[\,,]_\varepsilon$ on G by the following formulas:

$$\begin{aligned} &[X', Y']_\varepsilon = -\varepsilon^2 Z \\ &[Z, X']_\varepsilon = Y', \; [Z, Y']_\varepsilon = X' . \end{aligned} \tag{5.3}$$

Define φ_ε as above. Then, for each ε, the above formulas define φ_ε as a linear representation of the $[\,,]_\varepsilon$ Lie algebra. There is no longer any singularity at $\varepsilon = 0$ or $\lambda = \infty$. Thus, passing from the "Inonu-Wigner" picture with which we began (where the Lie algebra structure remains fixed, and the representation is continued and the basis of the algebra is changed simultaneously) to the "Kodaira-Spencer" picture (where the Lie algebra and representation are continued simultaneously) is an enormous aid to a proper mathematical understanding of the situation.

Thus, we can look at the Gell-Mann formula (5.1) in the following way: start off with the Lie algebra defined by (5.1), which is the Lie algebra of the group of rigid motions of the plane. Define an analytic continuation of the Lie algebra structure by the formulas (5.3). This continuation is nonrigid in the Kodaira-Spencer sense, since for $\varepsilon > 0$

the algebra is not isomorphic to the one with which we started at $\varepsilon = 0$. The Gell-Mann formula itself, i.e., (5.2), now provides an analytic continuation of the representation of the $[\,,]_0$ structure that is given, each representation for ε being a representation of the $[\,,]_\varepsilon$ structure.

Let us now look for the interpretation of this in terms of cohomology. Let us change notations to conform with our earlier work. Suppose $\mathbf{G}$ and $\mathbf{L}$ are Lie algebras, with the bracket in $\mathbf{G}$ given by $[X, Y]$, and suppose φ is a homomorphism $\mathbf{G} \to \mathbf{L}$. Again, let φ' be the homomorphism for $\mathbf{G}$ into the linear transformations on $\mathbf{L}$ given by:

$$\varphi'(X)(Z) = [\varphi(X), Z] \quad \text{for} \quad X \in \mathbf{G}, Z \in \mathbf{L}\,.$$

Suppose a one-parameter family

$$(X, Y) \to [X, Y]_\lambda$$

of Lie algebra structures is given on $\mathbf{G}$, reducing to the given one for $\lambda = 0$. Let $\gamma : \mathbf{G} \to$ (linear maps on $\mathbf{G}$) be the adjoint representation of the $\lambda = 0$ Lie algebra on $\mathbf{G}$, i.e.,

$$\gamma(X)(Y) = [X, Y] \quad \text{for} \quad X, Y \in \mathbf{G}\,.$$

Then, we know that the formula:

$$\omega(X, Y) = \frac{d}{d\lambda}[X, Y]_\lambda|_{\lambda=0}$$

defines ω as a two-cocycle relative to γ, i.e., on element in $Z^2(\gamma)$, whose cohomology class in $H^2(\gamma)$ measures the "nonisomorphism" of the structure at $\gamma = 0$ and that for small, but nonzero γ.

Suppose further that, for each λ, φ_λ is a linear mapping of $\mathbf{G} \to \mathbf{L}$ reducing to φ for $\lambda = 0$, such that:

$$\varphi_\lambda([X, Y]_\lambda) = [\varphi_\lambda(X), \varphi_\lambda(Y)] \quad \text{for} \quad X, Y \in \mathbf{G}\,. \tag{5.4}$$

Define $\varphi : \mathbf{G} \to \mathbf{L}$ by the formula

$$\theta(X) = \frac{d}{d\lambda}\varphi_\lambda(X)|_{\lambda=0}$$

θ is a one-cochain in $C^1(\varphi')$. However, it is not a cocycle. In fact, let us differentiate (5.4) and set $\lambda = 0$:

$$\theta([X, Y]) + \varphi(\omega(X, Y)) = [\theta(X), \varphi(Y)] + [\varphi(X), \theta(Y)]\,.$$

This gives the formula:

$$\varphi(\omega) = d\theta \tag{5.5}$$

where $\varphi(\omega)$ is the two-chain in $C^2(\varphi')$ given by

$$\varphi(\omega)(X, Y) = \varphi(\omega(X, Y))\,.$$

Thus, ω considered as a cocycle in $C^2(\gamma)$ is not necessarily a coboundary, but its image under φ, $\varphi(\omega)$, is a coboundary, and the element φ in $C^1(\gamma)$ is the first term in the analytic continuation of φ.

Now, this does not quite reflect the situation in the case developed above; ω defined as the first derivative is zero, since the parameter λ occurs to different order in the continuation of the representation and the Lie algebra structure. Suppose then that

$$\frac{d}{d\lambda}[X, Y]_\lambda|_{\lambda=0} = 0 \quad \text{for} \quad X, Y \in \mathbf{G}.$$

Define now

$$\omega_2(X, Y) = \frac{d^2}{d\lambda^2}[X, Y]_\lambda|_{\lambda=0} \quad \text{for} \quad X, Y \in \mathbf{G}.$$

Since the first derivations are zero, it is readily seen that ω_2 so defined also satisfies the cocycle condition. Then,

$$d\theta = 0,$$

i.e., θ itself is a cocycle. Let

$$\theta_2(X) = \frac{d^2}{d\lambda^2}\varphi_\lambda(X)|_{\lambda=0}.$$

Differentiating (5.4) twice gives now:

$$\theta_2([X, Y]) + \varphi\omega(X, Y)$$
$$= [\theta_2(X), \varphi(Y)] + [\varphi(X), \theta_2(Y)] + 2[\theta_1(X), \theta_1(Y)].$$

This can be rewritten as

$$-d\theta_2(X, Y) + \varphi\omega(X, Y) = 2[\theta_1(X), \theta_1(Y)].$$

Now, the right-hand side obviously is a two-cocycle in $C^2(\varphi')$ since the left-hand side is such a cocycle. Let us denote this cocycle by

$$[\theta_1, \theta_1].$$

(This operation is discussed in the review article by NIJENHUIS and RICHARDSON [6]. It turns out to depend only on the cohomology class determined by θ_1 in $H^1(\varphi')$). Then, we can write the relation as:

$$\varphi\omega = d\theta_2 + 2[\theta_1, \theta_1]$$

i.e., the cohomology class determined by $\varphi\omega$ in $H^1(\varphi')$ can be written as a "square" of an element of $H^1(\varphi')$.

In summary, we have shown that there are interesting relations between the deformation theory and the analytic continuation problems that are of importance for the application of group-theoretical ideas to elementary particle physics. Before proceeding further with the general theory (in a later paper) it is appropriate to work out a further example that is of the greatest importance for physics.

6. Contraction of the Poincaré group into the Galilean group

Let $\mathbf{T}$ be a vector space over the real numbers, considered as an Abelian Lie algebra. (One might think of $\mathbf{T}$ as the Lie algebra of the group of space-time translations.) Denote elements of $\mathbf{T}$ by such letters as X, Y, etc. Suppose a $(X, Y) \to Q(X, Y)$ is a nondegenerate, symmetric bilinear form in $\mathbf{T}$. Let $\mathbf{K}(Q)$ be the Lie algebra (under commutator) of all linear transformations $A : \mathbf{T} \to \mathbf{T}$ that satisfy:

$$Q(AX, Y) + Q(X, AY) = 0 .$$

Thus, each such A is the infinitesimal generator of a one-parameter group of linear transformations on $\mathbf{T}$ that preserve the form $Q(,)$. Form the Lie algebra $\mathfrak{G}(Q)$ as the semidirect sum of $\mathbf{K}(Q)$ and $\mathbf{T}$, i.e., as a vector space $\mathfrak{G}(Q)$ is the direct sum of $\mathbf{K}(Q)$ and $\mathbf{T}$ with the bracket defined as follows:

$$\begin{aligned} &[X, Y] = 0 \quad \text{for} \quad X, Y \in \mathbf{T} \\ &[A_1, A_2] = A_1A_2 - A_2A_1 \quad \text{for} \quad A_1, A_2 \in \mathbf{K}(Q) \\ &[A, X] = A(X) \quad \text{for} \quad A \in \mathbf{K}(Q), X \in \mathbf{T} . \end{aligned}$$

Now suppose Q_λ is a one-parameter family of such bilinear forms, reducing to the given one at $\lambda = 0$. We can, of course, form $\mathfrak{G}(Q_\lambda)$ for every value of λ. In what sense can this be considered an analytic continuation of $\mathfrak{G}(Q)$, and how can we investigate the limit as $\lambda \to \infty$?

Since Q_λ is nondegenerate, for each λ there is a linear transformation $B_\lambda : \mathbf{T} \to \mathbf{T}$ with nonzero determinant such that

$$Q_\lambda(X, Y) = Q(B_\lambda X, Y) \quad \text{for} \quad X, Y \in \mathbf{T} .$$

Thus,

$$Q_\lambda(X, Y) = Q_\lambda(Y, X) \quad \text{forces} \quad Q(BX, Y) = Q(B_\lambda Y, X) = Q(Y, B_\lambda X)$$

i.e., $B_\lambda^* = B_\lambda$, where B_λ^* denotes the adjoint of B_λ *with respect to the form* Q.

Suppose $A \in \mathbf{K}(Q_\lambda)$:

$$Q_\lambda(AX, Y) + Q_\lambda(X, AY) = 0 ,$$

or

$$0 = Q(B_\lambda AX, Y) + Q(B_\lambda X, AY) = Q(B_\lambda AX, Y) + Q(X, B_\lambda AY)$$

Hence,

$$B_\lambda A \in \mathbf{K}(Q) .$$

Thus, there is a map $A \to B_\lambda A = \alpha_\lambda(A)$ from $\mathbf{K}(Q_\lambda)$ to $\mathbf{K}(Q)$ that is *not* a Lie algebra isomorphism. Thus, we can define a one-parameter family $[,]_\lambda$ of Lie algebra structures on $\mathfrak{G}(Q)$ by carrying over the Lie algebra structure on $\mathfrak{G}(Q_\lambda)$ via this isomorphism:

$$\begin{aligned} &[X, Y]_\lambda = 0 \quad \text{for} \quad X, Y \in \mathbf{T} \\ &[A, Y]_\lambda = \varphi_\lambda^{-1}A, Y = B_\lambda^{-1}AY \quad \text{for} \quad A \in \mathbf{K}(Q), X \in \mathbf{T} \\ &[A_1, A_2]_\lambda = \alpha_\lambda[\alpha_\lambda^{-1}A_1, \alpha_\lambda^{-1}A_2] \\ &\quad = B_\lambda(B_\lambda^{-1}A_1B_\lambda^{-1}A_2 - B_\lambda^{-1}A_2B_\lambda^{-1}A_1) \\ &\quad = A_1B_\lambda^{-1}A_2 - A_2B_\lambda^{-1}A_1 . \end{aligned} \tag{6.1}$$

Now, we can pass to the limit as $\lambda \to \infty$: If

$$B = \lim_{\lambda \to \infty} B_\lambda^{-1} ,$$

the limiting algebra has the structure

$$\begin{aligned} &[\mathrm{A}, \mathbf{T}]_\infty = 0 \\ &[A, Y]_\infty = BAY \quad \text{for} \quad A \in \mathbf{K}(Q),\ Y \in \mathbf{T} \\ &[A_1, A_2]_\infty = A_1 B A_2 - A_2 B A_1 \quad \text{for} \quad A_1, A_2 \in \mathbf{K}(Q) . \end{aligned} \tag{6.2}$$

Further, if B_λ^{-1} is analytic Y_λ in the neighborhood of infinite, then the formulas (6.1) show that the algebra for large λ is a perfectly smooth deformation in the Kodaira-Spencer sense of the ∞-algebra, which we denote by G_∞.

The structure of G_∞ can be exhibited quite nicely if B is a projection operator $B^2 = B$ as it is for the case where $\mathrm{G}(Q)$ is the Poincaré group, and G_∞ is the Galilean group. (There, B_λ is the diagonal matrix

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\lambda = c^2$; $c =$ velocity of light) and B is the matrix

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} .$$

Then

$$\mathbf{T} = B\mathbf{T} \oplus (I - B)\mathbf{T} ,$$

$$Q(B\mathbf{T}, (I - B)\mathbf{T}) = Q(\mathbf{T}, B(I - B)\mathbf{T}) = 0$$

(since $B^2 = B$, and $B = B^*$).

Let

$$A = I - 2B .$$

Then, $s^2 = I + 4B^2 - 4B = I$.

$$Q(sX, sY) = Q(X, s^2 Y) = Q(X, Y) .$$

Thus, s is an automorphism of $\mathbf{T}$ whose square is the identity which preserves the form Q: s defines a symmetric automorphism of $\mathbf{K}(Q)$ by the formula:

$$s(A) = sAs \quad \text{for} \quad A \in \mathbf{K}(Q) .$$

Let $\mathbf{L}$ be the set of all $A \in \mathbf{K}(Q)$ such that

$$s(A) = A .$$

Let $\mathbf{P}$ be the set of all $A \in \mathbf{K}(Q)$ such that

$$s(A) = A .$$

Then

$$\mathbf{K}(Q) = \mathbf{L} \oplus \mathbf{P}, \ [\mathbf{L}, \mathbf{L}] \subset \mathbf{L}, \ [\mathbf{L}, \mathbf{P}] \subset \mathbf{P}, \ [\mathbf{P}, \mathbf{P}] \subset \mathbf{L}.$$

i.e., $\mathbf{L}$ is a symmetric subalgebra of $\mathbf{K}(Q)$.

Now, $s(A) = A$ if $(I - 2B)A = A(I - 2B)$, i.e., $BA = AB$. $s(A) = -A$ if $(I - 2B)A = A(2B - I)$, or $A - 2BA = 2AB - A$, or $BA + AB = A$. Suppose, now that, as for the case where $\mathbf{G}(Q)$ is the Poincaré group,

$$\dim \mathbf{T} = \dim B\mathbf{T} + 1 .$$

Then, if Y spans $(I - B)\mathbf{T}$, $A \in \mathbf{L}$, $AY = aY$, and $Q(AY, Y) = 0$, forcing $a = 0$. (Otherwise, $Q(Y, Y) = 0$, and, since $Q(Y, B\mathbf{T}) = 0$, $Q(Y, \mathbf{T}) = 0$ forcing $Y = 0$ since the form Q is nondegenerate.)

Thus, $BA = A = AB$ for $A \in \mathbf{L}$. Hence

$$[A, Y]_\infty = AY = [A, Y] \quad \text{for} \quad A \in L, Y \in \mathbf{T}$$

$$\begin{aligned} [A_1, A_2]_\infty &= A_1 B A_2 - A_2 B A_1 = A_1 A_2 - A_2 A_1 \\ &= [A_1, A_2] \quad \text{for} \quad A_1 \in \mathbf{L}, A_2 \in \mathbf{K}(Q) . \end{aligned}$$

Thus the adjoint action of $\mathbf{L}$ on the $[\,,]_\infty$ algebra is precisely the same as the adjoint action of $\mathbf{L}$ on $\mathbf{G}(Q)$.

Let us continue to work out the rest of the structure of the $[\,,]_\infty$ algebra:

$$[A, Y]_\infty = BAY \in B\mathbf{T} \quad \text{if} \quad A \in \mathbf{L}, Y \in \mathbf{T}.$$

For $A_1, A_2 \in \mathbf{P}$,

$$\begin{aligned} [A_1, A_2]_\infty &= A_1 B A_2 - A_2 B A , \\ &= A_1 (0\, A_2 B + A_2) - A_2(-A_1 B + A_1) \\ &= -A_1 A_2 B + A_1 A_2 + A_2 A_1 B - A_2 A_1 \\ &= [A_1, A_2]\,(I - B) = 0 , \end{aligned}$$

since $[A_1, A_2] \in \mathbf{L}$.

We can sum this up as follows:

Theorem 6.1. *The Lie algebra* $[\,,]_\infty$ *has the following structure: it is the semidirect sum of the semisimple subalgebra* $\mathbf{L}$ *and the solvable ideal* $\mathbf{P} + \mathbf{T}$. $\mathbf{P} + \mathbf{T}$ *in turn is the semidirect sum of the Abelian ideal* $\mathbf{T}$ *and the Abelian subalgebra* $\mathbf{P}$. *Its commutator algebra,* $[\mathbf{P} + \mathbf{T}, \mathbf{P} + \mathbf{T}]$ *is just* $B\mathbf{T}$.

All this applies to the case where $\mathbf{G}(Q)$ is the Lie algebra of the Poincaré group; and where the $[\,,]_\infty$ algebra is that of the Galilean group. Then, $\mathbf{L}$ is just $SO(3, R)$. As L. Michel has pointed out, the fact that the complement of $\mathbf{L}$ in the Galilean algebra is an ideal, while it is not in the Poincaré algebra, is the group theoretical fact responsible for making the $SU(6)$ theory of Gursey, Radicati and Sakita, Galilean invariant while making the explanation, if any, of its relativistic invariance more complicated and, up to the present, unsolved.

7. Analytic continuation of the representations of the Poincaré group into those of the Galilean group

We have just seen that the Lie algebra structure of the Poincaré group can be deformed smoothly (in the Kodaira-Spencer sense) into the Lie algebra of the Galilean group. The next step on the program should be to see if the unitary representations can be so deformed. This involves an interesting new point, since, as is well known [1], the physically interesting representations of the Galilean group are only representations *up to a factor*, i.e., are true representations of a central extension of the Galilean group. To understand this well, before we consider more general situations, let us consider the simplest possible case, namely, of the Poincaré group in one-space dimension, x, and one-time dimension, t. The homogenerous part of the group, i.e., the Lorentz subgroup, is then that which leaves invariant the form

$$c^2 t^2 - x^2 . \tag{7.1}$$

The Lorentz subgroup is then parameterized by a coordinate θ, with transformations given by

$$\begin{aligned} x &\to x \cosh\theta + ct \sinh\theta \\ t &\to \frac{x}{c} \sinh\theta + t\cosh\theta . \end{aligned} \tag{7.2}$$

Thus, we have, for each value of c, a group of transformations G^c acting on (x, t) space. In this picture, the Galilean group is defined as the "limit" (as explained in [3]) of these groups as $c \to \infty$.

Let us try to construct linear representations of each of these groups. This can be done most readily by letting all these homogeneous groups act on the "velocity space"[1] corresponding to the space-time space (x, t).

Let $v = \frac{x}{t}$. Then, by the transformations (7.2) v is transformed according to the rule.

$$v \to \frac{x\cosh\theta + ct\sinh\theta}{\frac{x}{c}\sinh\theta + t\cosh\theta} = \frac{v\cosh\theta + c\sinh\theta}{\frac{v}{c}\sinh\theta + \cosh\theta} . \tag{7.3}$$

Let us deduce the infinitesimal transformation X on v-space obtained by differentiating (7.3) with respect to θ at $\theta = 0$:

$$v \to X(v) = c\frac{v^2}{c} ,$$

or

$$X = \left(c - \frac{v^2}{c}\right)\frac{d}{dv} .$$

[1] From a more general point of view, the "velocity space" is the projective space associated with the vector space (x, t).

By passing to this v quotient space, the translation part of the Poincaré group has been lost. It can be regained by constructing functions on v-space that transform under the one-parameter group (7.3) by a linear representation of the Lorentz group; as is well-known, the relativistic expressions for momentum and energy do precisely this:

$$E(v) = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \frac{mc^3}{\sqrt{c^2 - v^2}} \tag{7.4}$$

$$p(v) = \frac{mv}{\sqrt{1 - v^2/c^2}} = \frac{mvc}{\sqrt{c^2 - v^2}} = \frac{v}{c^2} E .$$

Now,

$$X(E) = mc^2(c^2 - v^2)\left(\frac{v}{(c^2 - v^2)^{3/2}}\right) = \frac{mc^2 v}{\sqrt{c^2 - v^2}} = cp$$

$$\begin{aligned} X(p) &= \frac{1}{c^2}\left((X(v)\, E + vX(E))\right) \\ &= \frac{1}{c^2}\left(\frac{c^2 - v^2}{c} E + \frac{mc^2v^2}{\sqrt{c^2 - v^2}}\right) \\ &= \frac{1}{c^2} mc^2(c^2 - v^2) + \frac{mc^2v^2}{c^2 - v^2} \\ &= \frac{1}{c^2}\left(\frac{mc^4}{\sqrt{c^2 - v^2}}\right) = \frac{mc^2}{\sqrt{c^2 - v^2}} = \frac{1}{c} E . \end{aligned}$$

Let H be now the space of square-integrable complex-valued functions $v \to \psi(v)$ on v-space, define X, E and p as skew-Hermitian operators on H as follows:

$$X(\psi) = \frac{1}{c}(c^2 - v^2)\frac{d}{dv}\psi - \frac{iv}{c}\psi$$

$$E(\psi) = iE\psi$$

$$p(\psi) = ip\psi$$

Thus, the commutation relations for these operators are

$$[X, E] = cp$$

$$[X, p] = \frac{1}{c} E$$

$$[E, p] = 0 .$$

Then

$$\left[\frac{x}{c}, E\right] = p$$

$$\left[\frac{x}{c}, p\right] = \frac{1}{c^2} E \tag{7.5}$$

$$[E, p] = 0 .$$

We recognize that (7.5) gives us a one-parameter family of Lie algebras that depends analytically on $\frac{1}{c}$. The limiting algebra at $c = \infty$ is just that of the Galilean group. $\frac{X}{c}$ is analytic in $\frac{1}{c}$, and converges to the operator $\frac{d}{dv}$, which is just the operator of constant acceleration with *respect to the Galilean group*. However, the operator E is not analytic in $\frac{1}{c}$. The physical interpretation suggests a way to proceed. Let us "re-normalize" E by subtracting off a constant that "becomes infinite" at $c = \infty$. We interpret this in the following group-theoretic way: enlarge the Lie algebra defined by (7.5) by adding an element 1 that commutes with all the other operators, i.e., the enlarged algebra is the direct sum with a one-dimensional Abelian subalgebra. Define:

$$E' = E - mc^2 1 .$$

E' is now analytic in $\frac{1}{c}$ at $c = \infty$. In terms of the basis $(X/c, E', p, 1)$ this algebra becomes:

$$\left[\frac{X}{c}, E'\right] = p$$

$$\left[\frac{X}{c}, p\right] = \frac{1}{c} 2E = \frac{1}{c^2}(E' + mc^2) = \frac{E'}{c^2} + m . \tag{7.6}$$

The limiting algebra as $c \to \infty$ now exists, and is by its construction, the representation depending on c is analytic in $\frac{1}{c}$ at $c = 0$. However, the limiting algebra at $c = 0$ is not that of the Galilean group, but a central extension of it. This explains why the "interesting" physical representations of the Galilean group are not true representations but representations only up to a factor.

8. The Gell-Mann formula for contraction of the Poincaré to the Galilean group

Now, let us ask whether there is a formula representing the Lie algebra of the Poincaré group as functions of the generators of the Lie algebra of the central extension of the Galilean group constructed in the last section. (For simplicity, we continue to work with the groups corresponding to one-space dimension.) We suppose then that X'', E'', p'' and 1 are the generators of a Lie algebra, with the structure relations.

$$[X'', E''] = p'', [X'', p''] = 1$$

$$[1, X''] = 0 = [E'', p''] = [1, E''] = [1, p''] . \tag{8.1}$$

(For simplicity, we also suppose $m = 1$). Define

$$X = cX''$$

$$E = \frac{c^3}{\sqrt{c^2 - 2E''}} \tag{8.2}$$

$$p = \frac{cp''}{\sqrt{c^2 - 2E''}} .$$

Suppose we are given an irreducible representation of the algebra elements defined by the $(X'', E'', p'', 1)$ satisfying (8.1). Looking at the computations in Section 7, we see that the operators defined by (8.2) will satisfy the structure relations of the Poincaré algebra providing that

$$E'' = 1/2\,p''^2 .$$

However, $[X'', E'' - 1/2\,p''^2] = p'' - p'' \cdot 1$, and 1 is in the center of the algebra defined by (8.1); if the representations is irreducible, it is a multiple of the identity, which we can normalize to be the identity operator. Thus, $E'' - 1/2\,p''^2$ is a multiple of the identity, since it commutes with all generators of the algebra. Notice that a scalar multiple of the identity can be added to E'' without affecting the structure relations (8.1). Thus, we can normalize so that

$$E'' - \frac{1}{2}\,p''^2 = 0 ,$$

at which point we see that (8.2) is a "Gell-Mann formula" which "expands" a representation of this central extension of the Galilean group to a representation of a central extension of the Poincaré algebra which, however, is isomorphic to a direct sum of the "true" Poincaré algebra and a one-dimensional center, since the Poincaré algebra has no other kind of central extensions. (This property of the Poincaré algebra is well known to the experts, although it is hard to find a direct, simple proof in the literature. Since we have developed in [4] Lie algebra cohomology theory independently of the much more complicated and general literature on homological algebra, we will now, for the reader's convenience, give an exposition of the cohomology theory of Abelian extensions of Lie algebras.)

9. The connection between Lie algebra cohomology and extensions by an Abelian ideal

Let φ be a homomorphism of a Lie algebra $\mathbf{G}$ onto a Lie algebra $\mathbf{P}$, with kernel $\mathbf{K}$, which is, of course, an ideal of $\mathbf{G}$. In addition, we suppose that $\mathbf{K}$ is Abelian. Let π be any *linear* map $\mathbf{P} \to \mathbf{G}$ such that

$$\varphi\pi(X) = X \quad \text{for} \quad X \in \mathbf{P}, \quad \text{i.e.,} \quad \varphi\pi = 1 , \qquad (9.1)$$

where 1 is interpreted as the identity map.

For $X, Y \in \mathbf{P}$, define

$$\omega_\pi(X, Y) = \pi[X, Y] - [\pi X, \pi Y] . \qquad (9.2)$$

Thus, ω_π is identically zero if and only if π is a homomorphism. If this is the case, then $\mathbf{G}$ is the semidirect product of $\mathbf{K}$ and the subalgebra $\pi(\mathbf{P})$. The aim of the cohomology theory is to show when a new π' can be chosen so that $\omega_{\pi'} = 0$ by modifying π in a certain way. Note first that

$$\varphi\,\omega_\pi(X, Y) = 0, \quad \text{i.e.,} \quad \omega_\pi(P, P) \subset K . \tag{9.3}$$

Using the Jacobi identity gives:

$$\begin{aligned}\omega_\pi(X, [Y, Z]) &= \pi[X, [Y, Z]] - [\pi X, \pi[Y, Z]] \\ &= \pi[[X, Y], Z] + \pi[Y, [X, Z]] - [\pi X, \pi[Y, Z]] \\ &= \omega_\pi([X, Y], Z]) + [\pi[X, Y], \pi Z] \\ &\quad + \omega_\pi(Y, [X, Z] + [\pi Y, \pi[X, Z]] - [\pi X, \pi[Y, Z]] .\end{aligned} \tag{9.4}$$

Further,

$$[\pi[X, Y], \pi Z] = [[\pi X, \pi Y], \pi Z] + [\omega_\pi(X, Y), \pi Z] .$$

Hence

$$\begin{aligned}&\omega_\pi(X, [Y, Z]) - \omega_\pi([X, Y], Z) - \omega_\pi(Y, [X, Z]) \\ &\quad = [\pi[X, Y], \pi Z] + [\pi Y, \pi[X, Z]] - [\pi X, \pi[Y, Z]] \\ &\quad = [\omega_\pi(X, Y), \pi Z] - [\omega_\pi(X, Z), \pi Y] + [\omega_\pi(Y, Z), \pi X] .\end{aligned} \tag{9.5}$$

We would like to interpret this as a condition $d\omega_\pi = 0$, where ω_π is taken as a two-cochain of **P** defined by some representation of φ' of P by linear transformations on **K**. (9.5) suggests that we try to do this by defining

$$\varphi'(X)\,(W) = [\pi(X), W] \quad \text{for} \quad W \in \mathbf{K}, X \in \mathbf{P} .$$

Let us see under what conditions this is successful.

$$\begin{aligned}\varphi'([X, Y]) &= [\pi([X, Y)], W] \\ &= [\omega_\pi(X, Y) + [\pi X, \pi Y], W] \\ &= [\omega_\pi(X, Y)\,W] + [\pi X, [\pi, (Y)], W] \\ &\quad - [\pi Y, [\pi X, W]] .\end{aligned}$$

Thus, $\varphi'([X, Y])$ will equal $[\varphi'(X), \varphi'(Y)]$ if and only if

$$[\omega_\pi(X, Y), \mathbf{K}] = 0 .$$

ince $\omega_\pi(X, Y) \in \mathbf{K}$, the simplest hypothesis that assures this is that

$$[\mathbf{K}, \mathbf{K}] = 0 . \tag{9.6}$$

Let us then assume (9.6). Further, we see that as our notation indicates, $\varphi' : \mathbf{P} \to$ (linear transformations on **K**) is independent of π.

(Proof: if π' is another map $\mathbf{P} \to \mathbf{G}$ with $\varphi\pi' = 1$, then

$$\varphi(\pi - \pi') = 0, \quad \text{i.e.,} \quad (\pi - \pi')\,(\mathbf{P}) \subset K ,$$

hence

$$\varphi'(X)\,(W) = [\pi(X), W] = [\pi'(X), W] \quad \text{for} \quad W \in \mathbf{K}, X \in \mathbf{P}) .$$

Having interpreted ω_π as an element of $Z^2(\varphi')$, i.e., as a two-cocycle relative to the representation φ', let us look at the condition that it be a coboundary. Suppose that

$$\omega_\pi = d\theta \quad \text{where } \theta \text{ is a map: } \mathbf{P} \to \mathbf{K} .$$

Then,

$$\omega_\pi(X, Y) = \varphi'(X)\,\varphi(Y) - \varphi'(Y)\,\varphi(X) - \varphi([X, Y]\,,$$

$$\pi[X, Y] - [\pi X, \pi Y] = [\pi(X), \varphi(Y)] - [\pi(Y), \varphi(X)] - \theta([X, Y])\,,$$

or

$$(\pi + \theta)\,([X, Y]) = [\pi + \varphi)\,(X), (\pi + \varphi)\,(Y)] \quad \text{for} \quad X, Y \in \mathbf{P}\,,$$

i.e., $\pi + \varphi$ is a *homomorphism* $\mathbf{P} \to \mathbf{G}$. Reversing the steps proves that

Theorem 9.1. *If* $\mathbf{K}$ *is abelian, the algebra* $\mathbf{G}$ *is a semidirect product of the ideal* $\mathbf{K}$ *and a subalgebra isomorphic to* $\mathbf{P}$ *if and only if the cohomology class determined by* ω_π *in* $H^2(\varphi')$ *is zero.*

Suppose now that conversely we are given a Lie algebra $\mathbf{P}$ a representation φ' of $\mathbf{P}$ by linear transformations on an Abelian Lie algebra $\mathbf{K}$ and an element $\omega \in Z^2(\varphi')$. We construct an extension of $\mathbf{G}$ whose kernel is $\mathbf{K}$ in the following way.

As a vector space $\mathbf{G}$ is isomorphic to the direct sum

$$\mathbf{K} \oplus \mathbf{P}:$$

The bracket within $\mathbf{K}$ is given by their given Lie algebra structure

$$[X, Y] = \varphi'(X)\,(Y) \quad \text{for} \quad X \in \mathbf{P},\ Y \in \mathbf{K}$$

$$[X, Y] \text{ as computed in } \mathbf{G}$$

$$= [X, Y] \text{ as computed in } \mathbf{P} + \omega(X, Y)\,.$$

In this way one proves the well-known result that the extensions of $\mathbf{P}$ with kernel $\mathbf{K}$ are in one-one correspondence with $H^2(\varphi')$.

Suppose now that we consider extensions for which $[\mathbf{K}, \mathbf{G}] = 0$, i.e., $\mathbf{K}$ is in the center of $\mathbf{G}$. They are called *central extensions*. Clearly, then, the representation φ' is the representation which assigns the zero operator to each element of $\mathbf{P}$. Let us now compute several of these cohomology groups in the case where $\mathbf{K}$ is one-dimensional, i.e., let us classify in certain cases of $\mathbf{P}$ the possible central extensions with a one-dimensional center. (This is the case of most importance for applications to quantum mechanics for there one is interested in projective unitary representations of the Lie groups P whose Lie algebra is $\mathbf{P}$, i.e., assignments of unitary operators to elements of P that are not true representations, but such that the actions of P on the "probabilities" (which are the absolute value squared of the "amplitudes") is a true representation).

Suppose first that $\mathbf{P}$ is, as for the Poincaré group, a semidirect sum $\mathbf{L} \oplus \mathbf{T}$ of an Abelian ideal $\mathbf{T}$ and a semisimple subalgebra $\mathbf{L}$ such that the action of $\mathbf{L}$ on $\mathbf{T}$ is irreducible. Let φ be the trivial representation of $\mathbf{P}$ on the vector space of the real numbers R. For $r = 1, 2, \ldots$, there is then a subspace $W(\varphi) \in Z(\varphi)$ such that

$$Z^r(\varphi) = W^r(\varphi) \oplus d\,C^{r+1}(\varphi) \tag{9.7}$$

$$X(W^r(\varphi)) \subset W^r(\varphi) \quad \text{for all} \quad X \in \mathbf{L}\,.$$

Suppose $r = 2$, and $\omega \in W^2(\varphi)$, $X \in \mathbf{L}$. Then
$$0 = X(\omega) = d(X \rfloor \omega)\,,$$
hence:
$$X \rfloor \omega = \theta_X + d\theta'_X$$
where $\theta_X \in W^1(\varphi)$, and $\varphi'_X \in C^0(\varphi)$.

But, $d(C^0(\varphi)) = 0$, since φ is the trivial representation of $\mathbf{P}$. Thus for $X, Y \in \mathbf{L}$,
$$X(Y \rfloor \omega) = [X, Y] \rfloor \omega\,,$$
or
$$0 = X(\theta_Y) = \theta_{[X,Y]}$$
hence
$$\theta_{[L,L]} = \theta_L = 0\,,$$
or
$$X \rfloor \omega = 0 \quad \text{for} \quad X \in \mathbf{L}\,.$$
Thus, ω is determined by its restriction to $\mathbf{T}$. But, $\omega(\mathbf{T}, \mathbf{T})$ is then a real-valued skew-symmetric form on $\mathbf{T}$ that is invariant under the action of $\mathbf{L}$. If, for example, $\mathbf{P}$ is the Poincaré algebra, there is no such nonzero form. Hence, $H^2(\varphi) = 0$ and we have proved: *The Poincaré Lie algebra has no nontrivial central extensions.*

Now, let us turn to the situation that includes the Galilean group. Suppose again that $\mathbf{P} = \mathbf{L} \oplus \mathbf{T}$, with
$$[\mathbf{T}, \mathbf{T}] = 0,\ [\mathbf{L}, \mathbf{T}] \subset \mathbf{T}\,.$$
However, suppose that
$$\mathbf{L} = \mathbf{L}' \oplus \mathbf{L}'', \mathbf{T} = \mathbf{T}' \oplus \mathbf{T}'',$$
with
$$[\mathbf{L}', \mathbf{L}'] \subset \mathbf{L}',\ [\mathbf{L}', \mathbf{L}''] \subset \mathbf{L}'',\ [\mathbf{L}', \mathbf{T}'] = 0$$
$$[\mathbf{L}', \mathbf{T}''] \subset \mathbf{T}'',\ \dim \mathbf{T}' + 1$$
$$[\mathbf{L}'', \mathbf{L}''] = 0$$
$$[\mathbf{L}'', \mathbf{T}'] \subset \mathbf{T}''$$
$$[\mathbf{L}'', \mathbf{T}''] \subset \mathbf{T}''$$
$$\mathbf{L}' \text{ is semisimple.}$$

Since $\mathbf{L}'$ is semisimple, subspaces $W^r(\varphi)$ exist satisfying (9.7) such that
$$X(\omega) = 0 \quad \text{for} \quad \omega \in W^r(\varphi), X \in \mathbf{L}'\,.$$
Just as for the Poincaré group, one then proves that
$$X \rfloor \omega = 0 \quad \text{for} \quad X \in \mathbf{L}'\,.$$
Suppose further that $\operatorname{Ad}\mathbf{L}'$ *acting on* $\mathbf{L}''$ *and* T'' *is irreducible and the representations are equivalent, i.e., there is a vector space isomorphism* $\alpha : \mathbf{L}'' \to \mathbf{T}''$ *such that*
$$[X, \alpha(Y)] = \alpha([X, Y]) \quad \text{for} \quad X \in \mathbf{L}', Y \in \mathbf{L}''\,.$$

Further, suppose that $\mathbf{L}''$ and $\mathbf{T}''$ admit no nonzero real-valued skew-symmetric bilinear forms that are invariant under $\mathrm{Ad}\,\mathbf{L}'$, but that they do admit an invariant symmetric bilinear form $B(\ ,\ )$. Then, we have

$$\begin{aligned} \omega(\mathbf{L}'', \mathbf{L}'') = 0 = \omega(\mathbf{T}'', \mathbf{T}'') \\ 0 = \omega(\mathbf{T}', \mathbf{P}) = 0\,. \end{aligned} \tag{9.8}$$

Then ω must satisfy

$$\omega(X, Y) = a\,B(\alpha(X), Y) \quad \text{for} \quad X \in \mathbf{L}'', Y \in \mathbf{T}'' \tag{9.9}$$

where a is a real constant.

This shows that $\dim H^2(\varphi) \leqq 1$. Conversely, we must show that (9.8) and (9.9) define a nonzero element of $H^2(\varphi)$. This can be done by a straightforward calculation that we leave to the reader. The result is then:

Up to a normalization, there is but one nontrivial central extension of the Galilean algebra.

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