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Remarks on Conformal Invariance*

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Abstract. The existence of unitary representations for the special Conformal Group, is investigated for free fields in any dimension, and the connection between the correct transformation properties of the fields and weak conformal invariance pointed out.

I. Introduction

The use of conformal symmetry in quantum field theory has been advocated a decade ago by Wess [1] and Kastrup [2–4]. The renewed interest in this topic is closely connected to Wilson's [5] ideas on small distance behaviour of field operators and dynamical or anomalous dimensionality of fields. Moreover the Migdal-Polyakov construction [6–11] of conformal invariant quantum field theories offers an interesting alternative to canonical perturbation theory. It is hoped that these approaches reproduce correctly the behaviour of strong interacting systems in a particular class of high energy limits. This idea is supported by the fact that the Gell-Mann-Low limit [12, 13] of renormalizable theories is conformal invariant [14], when the coupling constant equals the Gell-Mann-Low eigenvalue.

On the other hand in "axiomatic" quantum field theory "proper conformal invariance" meets a serious difficulty, which originates from the fact, that conformal transformations can convert time – like into space – like separations and vice versa. This may spoil the fundamental concept of locality or Einstein causality. Thus it appears questionable that conformal symmetry (appart from certain limiting cases) should hold in general quantum field theory. Indeed Hortaçsu, Schroer and Seiler [15] have shown that due to the reverberation phenomenon of free fields in odd space – time dimension (i.e. the commutator is not concentrated on the light cone but spreads out into the time like region) the usual or

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canonical substitution rule,

$$\phi(x) \to (1-2bx+b^2x^2)^{-\frac{n-1}{2}} \phi\left(\frac{x-bx^2}{1-2bx+b^2x^2}\right) \tag{I.1}$$

(n = number of space dimensions, $x^2=(x^0)^2-\boldsymbol{x}^2$) for conformal transformations [1] does not lead to unitary operators in the corresponding Hilbert space. The same may happen for interacting zero mass fields in even space-time dimension as it was demonstrated by the same authors for the case of the Thirring model. Due to the absence of reverberations this can however not happen for free fields in even space time dimension [15, 16].

Moreover these authors construct hermitean, symmetric global charge operators from the conformal currents for all cases (with or without reverberations) and claim that the charges fail to possess self-adjoint extensions in the reverberating cases. However recently it has been shown by one of us [17], that conserved currents always generate global charge operators, which possess at least one self-adjoint extension if there exists an antiunitary (unitary) operator commuting (anti-commuting) with the charges. Since for the conformal currents the PCT-operator represents such an antiunitary operator, there always exist at least one set of self-adjoint infinitesimal generators, i.e. at least one unitary representation, for the special conformal transformation group.

In the present article we prove that for massless free field theories in all space-time dimensions, i.e. independent of the reverberation phenomenon, the generators of the special conformal transformations are essentially self-adjoint operators. Hence they possess one and only one self-adjoint extension.

That means in all these theories there exist a unique (up to a phase) unitary representation of the special conformal group. What breaks down due to the reverberation phenomenon in odd space-time dimensions is not the conformal invariance but the usual canonical substitution rule [1], (I.1).

In order to save the structural properties of vacuum expectation values induced by the (formal) canonical substitution rules also for the case of reverberations Hortaçsu *et al.* [15] introduced the concept of weak conformal invariance. We also show that this weak conformal invariance is a consequence of the above operator conformal invariance.

The present article is organized as follows: In section II we demonstrate our ideas and results first for the (reverberating) case of one time and one space dimension. In order to keep the calculations as simple as possible we feel free to use improper wave functions. In Section III we prove the existence of essentially self-adjoint generators of the special conformal group in the Hilbert space of massless free fields in all space dimensions $n \geqq 2$. Section IV is devoted to derive the weak conformal invariance of Hortaçsu *et al.* [15] from our conformal symmetry. In Section V we conclude with some speculations and remarks.

II. The Case of 1 Space Dimension

We shall first investigate in this section the existence of a unitary representation of the special conformal transformations,

$$x^\mu \to x_T^\mu = \frac{x^\mu - b^\mu x^2}{1 - bx + b^2 x^2} \tag{II.1}$$

on c-number solutions (wave-functions) of the D'Alembert equation in two dimensional space-time.

The transformation (II.1) leads to the following formal substitution rule for scalar wave functions:

$$f(x) \to f_b(x) = f(x_T). \tag{II.2}$$

By differentiating the transformed wave-function with respect to the group parameters, and taking b_0, b_1 and x^0 equal to zero, we obtain the formal generators in x space,

$$\begin{aligned} -i\frac{\partial}{\partial b_1} f(\boldsymbol{x},0) &= K_1 f(\boldsymbol{x},0) = i\boldsymbol{x}^2 \frac{\partial}{\partial \boldsymbol{x}} f(\boldsymbol{x},0) \\ i\frac{\partial}{\partial b_0} f(\boldsymbol{x},0) &= K_0 f(\boldsymbol{x},0) = i\boldsymbol{x}^2 \frac{\partial}{\partial x^0} f(\boldsymbol{x},0). \end{aligned} \tag{II.3}$$

In order to show that there is a unitary transformation whose generators correspond to (II.3) we shall consider positive energy solutions of the D'Alembert equation (analogous considerations will hold for negative energy solutions)

$$f(\boldsymbol{x}, x^0) = \frac{1}{(2\pi)^{1/2}} \int \frac{dk}{2|\boldsymbol{k}|} \tilde{f}(\boldsymbol{k}) e^{i(\boldsymbol{k}\boldsymbol{x} - |\boldsymbol{k}|x^0)}. \tag{II.4}$$

The $\tilde{f}(\boldsymbol{k})$ endowed with the scalar product

$$(f, g) := \int \frac{dk}{2|\boldsymbol{k}|} \overline{\tilde{f}(\boldsymbol{k})}\, \tilde{g}(\boldsymbol{k}) \tag{II.5}$$

define the Hilbert-space of positive energy solutions. From (II.3, 4) we get the generators in momentum space

$$\begin{aligned} K_1 \tilde{f}(\boldsymbol{k}) &= |\boldsymbol{k}| \frac{d^2}{d\boldsymbol{k}^2} (\varepsilon(\boldsymbol{k})\, f(\boldsymbol{k})) \\ K_0 \tilde{f}(\boldsymbol{k}) &= -|\boldsymbol{k}| \frac{d^2}{d\boldsymbol{k}^2} f(\boldsymbol{k}). \end{aligned} \tag{II.6}$$

K_0 and K_1 are from (II.6), unbounded operators, densely defined on the domain of functions of Schwartz class, which vanish with their first two derivatives at the origin.

They are furthermore symmetric on this domain and satisfy there $[K_0, K_1] = 0$. It will follow they are really self-adjoint commuting operators.

For this purpose we want to find a complete set of simultaneous (improper) eigenfunctions.

$$\begin{aligned} K_1 \tilde{f}_\lambda &= \lambda \tilde{f}_\lambda \\ K_0 \tilde{f}_\lambda &= \beta \tilde{f}_\lambda . \end{aligned} \tag{II.7}$$

The differential equations (II.7) are easily solved in terms of Bessel functions leading to a unique set of solutions which are square integrable at the origin.

$$\begin{aligned} \tilde{f}_\lambda(\boldsymbol{k}) &= (\boldsymbol{k})^{1/2} J_1(2(-\lambda \boldsymbol{k})^{1/2})\, \Theta(\boldsymbol{k}) \qquad \text{if} \quad \lambda < 0 \\ \tilde{f}_\lambda(\boldsymbol{k}) &= (-\boldsymbol{k})^{1/2} J_1(2(-\lambda \boldsymbol{k})^{1/2})\, \Theta(-\boldsymbol{k}) \quad \text{if} \quad \lambda > 0 \\ \beta &= |\lambda| . \end{aligned} \tag{II.8}$$

The completness of this eigenfunctions is reduced to the well known completness of Bessel functions. The self-adjointness of the operators K_0, K_1 is also apparent from the inexistence of normalizable solutions of the differential equations with $\lambda, \beta = \pm i$.

We can therefore build unitary operators by exponentiating K_0 and K_1 with

$$\begin{aligned} e^{-ibK} \tilde{f}(\boldsymbol{k}) &\equiv e^{-ibK} \int \frac{d\lambda}{2|\lambda|} a(\lambda)\, \tilde{f}_\lambda(\boldsymbol{k}) \\ &:= \int \frac{d\lambda}{2|\lambda|} a(\lambda) e^{-i(b_0|\lambda| - b_1\lambda)} \tilde{f}_\lambda(\boldsymbol{k}) . \end{aligned} \tag{II.9}$$

What remains to be seen is the action of this unitary representation of the special conformal group on the wave functions in x space and how is it related to the formal substitution rule (II.2).

For this purpose it is convenient to have the eigenfunctions (II.8) in x space.

From the first equation of (II.7) one gets

$$\frac{\lambda}{|\boldsymbol{k}|} \tilde{f}_\lambda(\boldsymbol{k}) = \frac{d^2}{d\boldsymbol{k}^2} (\varepsilon(\boldsymbol{k})\, \tilde{f}_\lambda(\boldsymbol{k})) + c\,\delta(\boldsymbol{k}) \tag{II.10}$$

with c a constant to be determined. Fourier transforming (II.10) we obtain the eigenvalue equation in x space

$$\lambda f_\lambda(\boldsymbol{x}) = i x^2 \frac{d}{d\boldsymbol{x}} f_\lambda(\boldsymbol{x}) + \sqrt{2\pi}\, c \tag{II.11}$$

with the solution (properly normalized)

$$f_\lambda(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}} \left\{ e^{i\frac{\lambda}{\boldsymbol{x}}} - \sqrt{2\pi}\frac{c}{\lambda} \right\}. \tag{II.12}$$

From (II.8) we see that $|\boldsymbol{k}|^{-1}\tilde{f}_\lambda$ does not have a δ singularity at the origin and therefore $f_\lambda(\boldsymbol{x}) \xrightarrow[|\boldsymbol{x}|\to\infty]{} 0$. This fixes our constant c, and gives,

$$f_\lambda(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}} \left\{ e^{i\frac{\lambda}{\boldsymbol{x}}} - 1 \right\}. \tag{II.13}$$

Notice that since $c \neq 0$ the correct x space eigenvalue equation (II.11) does not coincide with the one that would formally follow from the formal generators (II.3), differing from it by the inhomogeneous term that takes into account the proper boundary condition for $k=0$ or $|\boldsymbol{x}| \to \infty$.

From (II.9, 13)

$$e^{ib_1K_1} f_\lambda(\boldsymbol{x}) = e^{ib_1\lambda} f_\lambda(\boldsymbol{x}) = f_\lambda\left(\frac{\boldsymbol{x}}{1+b_1\boldsymbol{x}}\right) - f_\lambda\left(\frac{1}{b_1}\right). \tag{II.14}$$

Expanding

$$f(\boldsymbol{x}) = \int \frac{d\lambda}{2|\lambda|} a(\lambda) f_\lambda(\boldsymbol{x}) \tag{II.15}$$

we get

$$e^{ib_1K_1} f(\boldsymbol{x}) = f\left(\frac{\boldsymbol{x}}{1+b_1\boldsymbol{x}}\right) - f\left(\frac{1}{b_1}\right) \tag{II.16}$$

which is the correct transformation of the wave function for $x^0 = 0$, differing from the formal substitution rule (I.1) by an additional term. To get the transformation of the wave function for arbitrary times notice that from (II.4 and II.8),

$$\begin{aligned} f_\lambda(\boldsymbol{x}, x^0) &= f_\lambda(\boldsymbol{x} - x_0) \quad \text{for} \quad \lambda < 0 \\ f_\lambda(\boldsymbol{x}, x^0) &= f_\lambda(\boldsymbol{x} + x^0) \quad \text{for} \quad \lambda > 0 \end{aligned} \tag{II.17}$$

and therefore, straightforward calculation shows,

$$\begin{aligned} e^{ib_1K_1(x^0)} f_\lambda(\boldsymbol{x}, x^0) &= e^{-iHx^0} \left\{ f_\lambda\left(\frac{\boldsymbol{x}}{1+b_1\boldsymbol{x}}\right) - f_\lambda\left(\frac{1}{b_1}\right) \right\} \\ &= f_\lambda(\boldsymbol{x}_T, x_T^0) - f_\lambda\left(\frac{1}{b_1}, 0\right). \end{aligned} \tag{II.18}$$

Again using expansion (II.15) for arbitrary times

$$e^{ib_1K_1(x_0)}f(\boldsymbol{x},x^0)=f(\boldsymbol{x}_T,x_T^0)-f\left(\frac{1}{b_1},0\right). \tag{II.19}$$

Similarly,

$$\begin{aligned} e^{-ib_0K_0}f_\lambda(\boldsymbol{x}) &= e^{-ib_0|\lambda|}f_\lambda(\boldsymbol{x}) \\ &= f_\lambda\left(\frac{\boldsymbol{x}}{1-b_0^2\boldsymbol{x}^2},\frac{b_0\boldsymbol{x}^2}{1-b_0^2\boldsymbol{x}^2}\right)-f_\lambda\left(\boldsymbol{0},-\frac{1}{b_0}\right) \end{aligned} \tag{II.20}$$

and in general for arbitrary times,

$$e^{-ib_0K_0(x^0)}f_\lambda(\boldsymbol{x},x^0)=f_\lambda(\boldsymbol{x}_T,x_T^0)-f_\lambda\left(\boldsymbol{0},-\frac{1}{b_0}\right) \tag{II.21}$$

from what follows

$$e^{-ib_0K^0(x^0)}f(\boldsymbol{x},x^0)=f(\boldsymbol{x}_T,x_T^0)-f\left(\boldsymbol{0},-\frac{1}{b_0}\right). \tag{II.22}$$

Composing (II.19) with (II.22) we get the general transformation of the wave function under the special conformal group,

$$e^{-ibK}f(x)=f(x_T)-f\left(-\frac{b}{b^2}\right). \tag{II.23}$$

The discussion of the negative energy solutions follows exactly the same steps leading to a transformation law equal to (II.23).

It is the presence of the additional term in the correct transformation law for the wave-function, that allows a unitary representation of the special conformal group, without any conflict with causality, despite the fact that in two dimensional space-time, we have propagation inside the light-cone (reverberation) [15].

We proceed now to a discussion of the quantized field. In two dimensional space-time the field is an operator valued distribution with testfunction out of Schwartz class [18] with the extra proviso, necessary to avoid infrared difficulties,

$$\int d^2x\, g(x)=0\,. \tag{II.24}$$

We can formally expand the field in eigenfunctions of K_1,

$$\phi(x)=\int\frac{d\lambda}{2|\lambda|}\{a_\lambda f_\lambda(x)+a_\lambda^*\overline{f_\lambda(x)}\} \tag{II.25}$$

(with a_λ annihilation and a_λ^* creation operators), whose proper meaning is obtained by smearing both sides of (II.24) with the appropriate testfunction. From (II.24) the action of the conformal transformation on the field $\phi(x)$ is reduced to its action on the wave-functions $f_\lambda(x)$. Therefore we formally get, with $\tilde{K}_0, \tilde{K}_1$ the conformal generators in Fock-space

$$e^{ib\tilde{K}}\phi(x)e^{-ib\tilde{K}}$$

$$= \int \frac{d\lambda}{2|\lambda|}\{a_\lambda(e^{-iKb}f_\lambda(x)) + a_\lambda^*(e^{-iKb}f_\lambda(x))^*\} \tag{II.26}$$

$$= \phi(x_T) - \phi\left(-\frac{b}{b^2}\right).$$

By smearing both sides of (II.25) with a test function $g(x)$ we get the correct transformation law,

$$e^{ib\tilde{K}}\phi(g)e^{-ib\tilde{K}} = \phi(g_T) \tag{II.27}$$

with

$$g_T(x) = (1 + 2bx + b^2x^2)^{-2}\, g\left(\frac{x + bx^2}{1 + 2bx + b^2x^2}\right). \tag{II.28}$$

Eq. (II.27) no longer contains the additional term of (II.26) which disappeared after integration with the test function. However it would be erroneous to take (II.27) as fully justifying the formal substitution rule (I.1) for the field. This is realized by considering for instance the densely defined form $(\Psi_1, \phi(x)\Psi_2)$ which transforms exactly as the wave function

$$(e^{-ib\tilde{K}}\Psi_1, \phi(x)e^{-ib\tilde{K}}\Psi_2)$$

$$= (\Psi_1, \phi(x_T)\Psi_2) - \left(\Psi_1, \phi\left(-\frac{b}{b^2}\right)\Psi_2\right). \tag{II.29}$$

That the substitution rule (I.1) should not be valid for the fields defined as forms, is of course a necessity in order that the conformal transformations do not violate causality, since the formal substitution rule maps space-like separations into time-like ones, which is of course incompatible with the fact that the form

$$(\Psi_1, [\phi(x), \phi(y)]\Psi_2) \sim \varepsilon(x^0 - y^0)\,\Theta((x-y)^2)$$

does not vanish inside the light cone.

III. Essentially Self-Adjoint Generators of the Conformal Group

For the construction of essentially self-adjoint generators of the special conformal group, we exploit the fact that any special conformal transformation

$$K(b)x := \frac{x - bx^2}{1 - 2bx + b^2x^2} \tag{III.1}$$

may be written as a product

$$K(b) = R\,T(b)\,R \tag{III.2}$$

of a translation

$$T(b)\,x := x + b \tag{III.3}$$

and inversions

$$Rx := -\frac{x}{x^2}, \quad x^2 = x_\mu x^\mu = (x^0)^2 - \sum_{i=1}^{n} (x^i)^2 . \tag{III.4}$$

This idea was first applied by Kastrup and Mayer [16] in four dimensional space-time. We generalize it by different methods to arbitrary space dimensions $n \geqq 2$.

Our first task is to construct a self-adjoint unitary representation $U(R)$ of the inversion in the Hilbert space $L^2(d^n\mu(\boldsymbol{p}))$; i.e. the Hilbert space of all complex functions $f(\boldsymbol{p})$; $\{\boldsymbol{p} = (p^1, \ldots, p^n), n > 1\}$ square integrable with respect to the measure $d^n\mu(\boldsymbol{p}) = (2|\boldsymbol{p}|)^{-1}\, d^n p$ with $|\boldsymbol{p}| := \left(\sum_{i=1}^{n} (p^i)^2\right)^{1/2}$. The scalar product in $L^2(d^n\mu(\boldsymbol{p}))$ is defined by:

$$(f, g) := \int d^n\mu(\boldsymbol{p})\, \overline{f(\boldsymbol{p})}\, g(\boldsymbol{p}) . \tag{III.5}$$

In general $L^r(d^n v(\boldsymbol{p}))$ denotes the usual normed L^r-space with respect to the measure $d^n v(\boldsymbol{p})$. The norm is given by:

$$\begin{aligned} \|f\|_r^v &:= \left[\int d^n v(\boldsymbol{p})\, |f(\boldsymbol{p})|^r\right]^{1/r} \\ \|f\|_r^1 &:= \left[\int d^n p\, |f(\boldsymbol{p})|^r\right]^{1/r} . \end{aligned} \tag{III.6}$$

In order to catch an idea about the structure of $U(R)$ we first construct by formal manipulations a sesquilinear form $R(g, f)$ in $L^2(d^n\mu(\boldsymbol{p}))$. In the second step we rigorously show, that $R(g, f)$ defines a self-adjoint unitary operator $U(R)$ in $L^2(d^n\mu(\boldsymbol{p}))$ for all $n > 1$.

The Fourier transform

$$\begin{aligned} \tilde{f}(x) &:= \frac{1}{(2\pi)^{n/2}} \int d^n\mu(\boldsymbol{p})\, f(\boldsymbol{p})\, e^{-i(|\boldsymbol{p}|x^0 - \boldsymbol{p}\boldsymbol{x})} \\ &= \frac{1}{(2\pi)^{n/2}} \int d^{n+1} p\, \Theta(p^0)\, \delta(p^2)\, f(\boldsymbol{p})\, e^{-ipx} \end{aligned} \tag{III.7}$$

is obviously a solution of the D'Alembert equation. Then the function $\hat{f}(x) := \tilde{f}\left(-\frac{x}{x^2}\right)$ satisfies the differential equation

$$x^2 \left[x^2 \,\square^{(n)} - 2(n-1)\, x^\lambda \frac{\partial}{\partial x^\lambda}\right] \hat{f}(x) = 0 ,$$

and finally the function

$$\tilde{f}_R(x) := (-1)^{1/2(n-1)}((x-i\varepsilon)^2)^{-1/2(n-1)}\tilde{f}\left(-\frac{x}{x^2}\right) \tag{III.8}$$

is a solution of the equation

$$((x-i\varepsilon)^2)^{\frac{n+1}{2}+2}\,\Box^{(n)}\tilde{f}_R(x) = 0 .$$

If $\tilde{f}(x)$ respectively $f(\boldsymbol{p})$ is smooth enough, so that $\tilde{f}_R(x)$ is again a solution of the D'Alembert equation, then the inner product

$$R(g,f) := i\int d^n x\,\overline{\tilde{g}(x)}\frac{\overleftrightarrow{\partial}}{\partial x^0}\tilde{f}_R(x)$$

is time independent and may be calculated at time $x^0=0$. By means of Eq. (III.8) we obtain:

$$R(g,f) = \frac{1}{(2\pi)^n}\int d^n x\int d^n\mu(\boldsymbol{p})\,\overline{g(\boldsymbol{p})}\int d^n\mu(\boldsymbol{q}) \cdot (\boldsymbol{x}^2)^{-\frac{n-1}{2}}\left(|\boldsymbol{p}| + \frac{|\boldsymbol{q}|}{\boldsymbol{x}^2}\right)e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} f(\boldsymbol{q}) . \tag{III.9}$$

If the integrals on the right hand side exist for all g, f from some linear set $\mathscr{B} \subseteqq L^2(d^n\mu(\boldsymbol{p}))$, then (III.9) obviously defines a sesquilinear form with domain $\mathscr{B}$. Moreover $R(g,f)$ is symmetric on $\mathscr{B}$

$$\overline{R(g,f)} = R(f,g) . \tag{III.10}$$

This follows immediately by means of the change of the integration variables

$$\boldsymbol{x} \to \boldsymbol{z} = \frac{\boldsymbol{x}}{\boldsymbol{x}^2} ; \; d^n x = (\boldsymbol{z}^2)^{-n}\, d^n z . \tag{III.11}$$

We want to show next, that $R(g,f)$ exists for all g, f from the dense domain

$$D^n := \Bigg\{ f(\boldsymbol{p}) \in L^2(d^n\mu(\boldsymbol{p})) \cap L^1(d^n\mu(\boldsymbol{p})) : \quad (\boldsymbol{p}^2)^r \frac{\partial^k}{\prod_{i=1}^{n}(\partial p^i)^{k_i}} f(\boldsymbol{p}) \in L^1(d^n p) \quad \text{for all} \quad 0 \leqq r \leqq n, 0 \leqq k = \sum_{i=1}^{n} k_i \leqq 2n \Bigg\} \tag{III.12}$$

and is bounded by:

$$|R(g,f)| \leqq \tfrac{2}{3}(2\pi)^{-n}\,\Omega_n[\max\{\|g\|_1^\mu;\|g\|_1^1\} + \int d^n p|\Delta^{(n)} g(\boldsymbol{p})|]\,[\max\{\|f\|_1^\mu;\|f\|_1^1\} + \int d^n q|\Delta^{(n)} f(\boldsymbol{q})|]\,. \tag{III.13}$$

Here Ω_n denotes the surface of the n-dimensional unit-sphere and $\Delta^{(n)}$ the n-dimensional Laplace-operator.

In order to assert this, consider the integral

$$\begin{aligned} I := &\left|\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}+1}} \int d^n\mu(\boldsymbol{p})\, e^{-i\boldsymbol{p}\boldsymbol{x}}\overline{g(\boldsymbol{p})} \int d^n\mu(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}} |\boldsymbol{q}|\, f(\boldsymbol{q})\right| \\ &\leqq \|g\|_1^\mu \Bigg\{ \int_{\boldsymbol{x}^2\leqq 1} d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}\right| \\ &\quad + \int_{\boldsymbol{x}^2\geqq 1} d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}\right| \\ &\leqq \|g\|_1^\mu \Bigg\{ \tfrac{1}{3}\Omega_n \|f\|_1^1 \\ &\quad + \int_{\boldsymbol{x}^2\leqq 1} d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}\right|\Bigg\}. \end{aligned}$$

By means of the variable transformation (III.11) we find

$$\begin{aligned} \hat{I} := &\int_{\boldsymbol{x}^2\leqq 1} d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}\right| \\ = &\int_{\boldsymbol{z}^2\geqq 1} d^n z (\boldsymbol{z}^2)^{-\frac{n-1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, e^{i\boldsymbol{q}\boldsymbol{z}}\right| \\ = &\int_{\boldsymbol{z}^2\geqq 1} d^n z (\boldsymbol{z}^2)^{-\frac{n+1}{2}} \left|\int d^n q\, f(\boldsymbol{q})\, \Delta^{(n)} e^{i\boldsymbol{q}\boldsymbol{z}}\right| \\ \leqq &\tfrac{1}{3}\Omega_n \int d^n q |\Delta^{(n)} f(\boldsymbol{q})|\,. \end{aligned}$$

Hence

$$\mathrm{I} \leqq \tfrac{2}{3}\Omega_n \|g\|_1^\mu \{\|f\|_1^1 + \int d^n q|\Delta^{(n)} f(\boldsymbol{q})|\}\,. \tag{III.14}$$

By exactly the same arguments it follows after the change (III.11) of the integration variables:

$$\begin{aligned} \mathrm{II} := &\left|\int d^n x (\boldsymbol{x}^2)^{-\frac{n-1}{2}} \int d^n\mu(\boldsymbol{p})\, |\boldsymbol{p}|\, \overline{g(\boldsymbol{p})}\, e^{-i\boldsymbol{p}\boldsymbol{x}} \cdot \int d^n\mu(\boldsymbol{q})\, f(\boldsymbol{q})\, e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}\right| \\ &\leqq \tfrac{2}{3}\Omega_n \|f\|_1^\mu \{\|g\|_1^1 + \int d^n q|\Delta^{(n)} g(\boldsymbol{q})|\}\,. \end{aligned} \tag{III.15}$$

However the last two inequalities together deliver the bound (III.13).

By now we have established the existence of a symmetric sesquilinear form with dense domain D^n in $L^2(d^n\mu(\boldsymbol{p}))$. This form $R(g,f)$ defines an operator $U(R)$ if and only if it is bounded in the $L^2(d^n\mu(\boldsymbol{p}))$-norm in g. The key for the derivation of this property is the following lemma:

Lemma 1. *For all $f(\boldsymbol{p},\boldsymbol{q})$, which together with their first two partial derivatives are from*

$$L^2(d^n\mu(\boldsymbol{p}))\otimes L^2(d^n\mu(\boldsymbol{q}))\cap L^1(d^n\mu(\boldsymbol{p}))\otimes L^1(d^n\mu(\boldsymbol{q}))\cap L^1(d^n p\otimes d^n q)$$

we have:

$$\begin{aligned}
&\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q})\, f(\boldsymbol{p},\boldsymbol{q}) \left(|\boldsymbol{p}| + \frac{|\boldsymbol{q}|}{\boldsymbol{x}^2}\right) e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \\
&= 2\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q})\, f(\boldsymbol{p},\boldsymbol{q}) \frac{|\boldsymbol{q}|}{\boldsymbol{x}^2} e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \qquad \text{(III.16)}\\
&= 2\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q})\, f(\boldsymbol{p},\boldsymbol{q})\, |\boldsymbol{p}|\, e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} .
\end{aligned}$$

Proof. From the derivation of the bound (III.13) we know that all integrals are finite. Then by means of the substitutions

$$\boldsymbol{x}\to\boldsymbol{z}=\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\,;\quad \boldsymbol{p}\leftrightarrow\boldsymbol{q}$$

we obtain at once:

$$\begin{aligned}
&\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q}) \left(|\boldsymbol{p}| + \frac{|\boldsymbol{q}|}{\boldsymbol{x}^2}\right) e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \\
&\qquad\cdot\{f(\boldsymbol{p},\boldsymbol{q})-f(\boldsymbol{q},\boldsymbol{p})\}=0 .
\end{aligned}$$

Hence we may restrict ourselves to symmetric functions $f(\boldsymbol{p},\boldsymbol{q})=f(\boldsymbol{q},\boldsymbol{p})$. However for them it follows at once by the same substitution:

$$\begin{aligned}
&\int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q})\, f(\boldsymbol{p},\boldsymbol{q})\, |\boldsymbol{p}|\, e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \\
&= \int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n-1}{2}}} \int d^n\mu(\boldsymbol{p})\, d^n\mu(\boldsymbol{q})\, f(\boldsymbol{p},\boldsymbol{q}) \frac{|\boldsymbol{q}|}{\boldsymbol{x}^2} e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} .
\end{aligned}$$

This proves Lemma 1.

The majorisation (III.14) together with Lemma I allows the sesquilinear form $R(g,f)$ to be rewritten as the following $L^2(d^n\mu(\boldsymbol{p}))$-scalar product:

$$\begin{aligned} R(g,f) &= \frac{2}{(2\pi)^n} \int d^n\mu(\boldsymbol{p})\,\overline{g(\boldsymbol{p})} \int d^n x(\boldsymbol{x}^2)^{-\frac{n+1}{2}} \\ &\cdot \int d^n q\, f(\boldsymbol{q})\, e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \\ &= \frac{2}{(2\pi)^n} \int d^n\mu(\boldsymbol{p})\,\overline{g(\boldsymbol{p})} \int d^n x(\boldsymbol{x}^2)^{-\frac{n-1}{2}} \\ &\cdot \int d^n q\, f(\boldsymbol{q})\, e^{-i\left(\frac{\boldsymbol{p}\boldsymbol{x}}{\boldsymbol{x}^2}-\boldsymbol{q}\boldsymbol{x}\right)}. \end{aligned} \tag{III.17}$$

Now in order that $R(g,f)$ is bounded in g by the $L^2(d^n\mu(\boldsymbol{p}))$-norm for all $g, f \in D^n$ it is sufficient to show, that

$$\begin{aligned} (U(R)f)(\boldsymbol{p}) := \frac{2}{(2\pi)^n} \int d^n x(\boldsymbol{x}^2)^{-\frac{n-1}{2}} \\ \cdot \int d^n q\, f(\boldsymbol{q})\, e^{-i\left(\frac{\boldsymbol{p}\boldsymbol{x}}{\boldsymbol{x}^2}-\boldsymbol{q}\boldsymbol{x}\right)} \end{aligned} \tag{III.18}$$

is from $L^2(d^n\mu(\boldsymbol{p}))$ for all $f \in D^n$, since the desired boundedness is then obtained from Schwartz' inequality.

For all $f(\boldsymbol{q}) \in D^n$ the Fourier transforms

$$\mathring{f}(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} \int d^n q\, e^{i\boldsymbol{q}\boldsymbol{x}} f(\boldsymbol{q}) \tag{III.19}$$

are from $L^2(d^n x)$ and satisfy the bounds

$$\begin{aligned} &\left| (\boldsymbol{x}^2)^k \frac{\partial^r}{\prod_{i=1}^{n} (\partial x^i)^{r_i}} \mathring{f}(\boldsymbol{x}) \right| \leqq c^{r,k}(f) < \infty \\ &r := \sum_{i=1}^{n} r_i, \qquad r_i \geqq 0, \end{aligned} \tag{III.20}$$

for all $0 \leqq r$, $k \leqq n$.

This implies that the functions $(\boldsymbol{x}^2)^{-\frac{n+1}{2}} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right)$ together with their first n derivatives are from $L^2(d^n x) \cap L^1(d^n x)$. Therefore the functions

$$\begin{aligned} (U(R)f)(\boldsymbol{p}) &= \frac{2}{(2\pi)^{n/2}} \int d^n x(\boldsymbol{x}^2)^{-\frac{n-1}{2}} \mathring{f}(\boldsymbol{x})\, e^{-i\frac{\boldsymbol{p}\boldsymbol{x}}{\boldsymbol{x}^2}} \\ &= \frac{2}{(2\pi)^{n/2}} \int d^n x(\boldsymbol{x}^2)^{-\frac{n+1}{2}} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right) e^{-i\boldsymbol{p}\boldsymbol{x}} \end{aligned} \tag{III.21}$$

and $(p^i)^r\,(U(R)f)(\boldsymbol{p})$, $\{r=0,\dots,n;\, i=1,\dots,n\}$ are bounded continuous functions from $L^2(d^n p)$. However this means that $(U(R)f)(\boldsymbol{p})$ is from $L^2(d^n\mu(\boldsymbol{p}))\cap L^1(d^n\mu(\boldsymbol{p}))\cap L^1(d^n p)$. Thus by Schwartz' inequality we get for all $g, f\in D^n$:

$$|R(g,f)|\leqq \|g\|_2^\mu\,\|U(R)f\|_2^\mu\,. \tag{III.22}$$

This bound asserts that Eq. (III.18) defines a linear operator $U(R)$ in $L^2(d^n\mu(\boldsymbol{p}))$ with dense domain D^n. Moreover $U(R)$ is symmetric on D^n since $R(g,f)$ is symmetric.

$$(U(R)f,g)=\overline{R(g,f)}=R(f,g)=(f,U(R)g)\,.$$

In the next step we show:

Theorem 1. *The symmetric operator*

$$\begin{aligned}(U(R)f)(\boldsymbol{p})=&\frac{2}{(2\pi)^n}\int d^n x(\boldsymbol{x}^2)^{-\frac{n+1}{2}}\\ &\cdot\int d^n\mu(\boldsymbol{q})\,|\boldsymbol{q}|\,e^{-i\left(\boldsymbol{p}\boldsymbol{x}-\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)}f(\boldsymbol{q})\end{aligned} \tag{III.23}$$

with $f\in D^n$ has a unique self-adjoint, unitary extension to the whole Hilbert space $L^2(d^n\mu(\boldsymbol{p}))$.

Proof. It is sufficient to show, that $U(R)$ is isometric (and thus bounded) on D^n, i.e.

$$\|U(R)f\|_2^\mu=\|f\|_2^\mu \quad \text{for all} \quad f\in D^n\,.$$

$$\begin{aligned}(\|U(R)f\|_2^\mu)^2=&(2\pi)^{-2n}\int d^n\mu(\boldsymbol{p})\int d^n y(\boldsymbol{y}^2)^{-\frac{n+1}{2}}\\ &\cdot e^{i\boldsymbol{p}\boldsymbol{y}}\int d^n q\, e^{-i\frac{\boldsymbol{q}\boldsymbol{y}}{\boldsymbol{y}^2}}\,\overline{f(\boldsymbol{q})}\int d^n x(\boldsymbol{x}^2)^{-\frac{n+1}{2}}\\ &\cdot e^{-i\boldsymbol{p}\boldsymbol{x}}\int d^n k\, e^{i\frac{\boldsymbol{k}\boldsymbol{x}}{\boldsymbol{x}^2}}\, f(\boldsymbol{k})\,.\end{aligned}$$

Since the y- and x-integrals converge absolutely we may interchange them with the p-integration. This leads to the integral [18].

$$\int d^n p|\boldsymbol{p}|^{-1}e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}=\left(2^{n-1}\pi^{n/2}\Gamma\left(\frac{n-1}{2}\right)\Gamma(1/2)^{-1}\right)[(\boldsymbol{x}-\boldsymbol{y})^2]^{-\frac{n-1}{2}}$$

Applying in the resulting expression again the variable transformation

(III.11) we end up with:

$$(\| U(R) f \|_2^{\mu})^2 = (2\pi)^{-2n} \int d^n y \int d^n x \int d^n \mu(\boldsymbol{p}) e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}$$

$$\cdot \int d^n q e^{-i\boldsymbol{q}\boldsymbol{x}} \overline{f(\boldsymbol{q})} \int d^n k e^{i\boldsymbol{k}\boldsymbol{x}} f(\boldsymbol{k})$$

$$= \int d^n \mu(\boldsymbol{p}) |f(\boldsymbol{p})|^2 = (\| f \|_2^{\mu})^2 .$$

This proves Theorem 1.

In the final step we construct by means of the unitary operators $U(R)$ essentially self-adjoint generators K^{μ}, $\{\mu = 0, 1, \ldots, n\}$ of the special conformal group.

Let $\Delta^n(P)$ be the dense domain of all functions from $L^2(d^n \mu(\boldsymbol{p}))$ vanishing at infinity faster than any inverse polynomial in p^k.

$$\Delta^n(P) = \{ f \in L^2(d^n \mu(\boldsymbol{p})) : |(\boldsymbol{p}^2)^r f(\boldsymbol{p})| \leqq c_r(f) < \infty ; \quad r = 0, 1, 2, \ldots \} . \tag{III.24}$$

$\Delta^n(P)$ is contained in the domain of the essentially self-adjoint generators $P^{\mu} \{\mu = 0, \ldots, n\}$ of the translation group:

$$(P^{\mu} f)(\boldsymbol{p}) = p^{\mu} f(\boldsymbol{p}), \qquad p^0 := |\boldsymbol{p}| . \tag{III.25}$$

Moreover $\Delta^n(P)$ is stable under the application of P^{μ}:

$$P^{\mu} \Delta^n(P) \subseteqq \Delta^n(P) . \tag{III.26}$$

Let $\Delta^n(R)$ be the image of $\Delta^n(P)$ under the unitary operators $U(R)$:

$$\Delta^n(R) := U(R) \Delta^n(P) . \tag{III.27}$$

By the symmetry of $U(R)$ it follows

$$\Delta^n(P) = U(R) \Delta^n(R) . \tag{III.28}$$

Lemma 2. *$\Delta^n(R)$ and $\Gamma^n := \Delta^n(P) \cap \Delta^n(R)$ are both dense in $L^2(d^n \mu(\boldsymbol{p}))$ and Γ^n is stable under $U(R)$:*

$$U(R) \Gamma^n \subseteqq \Gamma^n . \tag{III.29}$$

Proof. First observe that the Schwartz space $\mathscr{S}_n$ of strongly decreasing c^{∞}-functions [18] is dense in $L^2(d^n \mu(\boldsymbol{p}))$ and obviously contained in $\Delta^n(P)$. It remains to be shown that $\mathscr{S}_n$ is cobtained in $\Delta^n(R)$. In view of (III.28) it suffices to prove that

$$(U(R) f)(\boldsymbol{p}) = (2\pi)^{-\frac{n}{2}} \int d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right) e^{-i\boldsymbol{p}\boldsymbol{x}}$$

$$= \frac{1}{(2\pi)^n} \int d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} e^{-i\boldsymbol{p}\boldsymbol{x}} \int d^n q f(\boldsymbol{q}) e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}}$$

vanishes faster than any inverse polynomial at infinity for all $f(\boldsymbol{q}) \in \mathscr{S}_n$. This however follows by repeated applications of the following three relations and the fact that the Fourier transformation maps $\mathscr{S}_n$ onto $\mathscr{S}_n$.

$$\frac{\boldsymbol{x}^k}{\boldsymbol{x}^2} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right) = i \int d^n q \left[\frac{\partial}{\partial q^k} f(\boldsymbol{q})\right] e^{i\frac{\boldsymbol{qx}}{\boldsymbol{x}^2}}$$

$$\mathring{f}^{(k)}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right) := \frac{\partial}{\partial \frac{\boldsymbol{x}^k}{\boldsymbol{x}^2}} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right) = i \int d^n q \, e^{i\frac{\boldsymbol{qx}}{\boldsymbol{x}^2}} q^k f(\boldsymbol{q}) \tag{III.30}$$

$$p^k(U(R)f)(\boldsymbol{p}) = -\frac{2i}{(2\pi)^n} \int d^n x (\boldsymbol{x}^2)^{-\frac{n+1}{2}} \cdot e^{-i\boldsymbol{px}} \left\{\frac{\partial}{\partial x^k} - (n+1)\frac{x^k}{\boldsymbol{x}^2}\right\} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right)$$

$$= -\frac{2i}{(2\pi)^n} \int d^n x (\boldsymbol{x}^2)^{-\frac{n-1}{2}-2} e^{-i\boldsymbol{px}} \cdot \left\{\sum_{r=1}^{n} \left(\delta^{rk} - 2\frac{x^r x^k}{\boldsymbol{x}^2}\right) \frac{\partial}{\partial \frac{x^r}{\boldsymbol{x}^2}} - (n+1) x^k\right\} \mathring{f}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}^2}\right).$$

Finally the relation (III.29) is a trivial consequence of the definition of the domains $\Delta^n(P)$, $\Delta^n(R)$ and Γ^n

This proves Lemma 2.

On the dense domain $\Delta^n(R)$ we define in accordance with the Eq. (III.2) the operators K^μ, $\{\mu = 0, 1, \ldots, n\}$ by:

$$K^\mu := U(R)\, P^\mu \, U(R). \tag{III.31}$$

Since P^μ is essentially self-adjoint and $U(R)$ unitary and self-adjoint also K^μ is essentially self-adjoint. Moreover P^μ and K^μ have the same spectrum. Finally from the definition of $\Delta^n(R)$ it obviously follows:

$$K^\mu \Delta^n(R) \subseteqq \Delta^n(R). \tag{III.32}$$

In a last step we have to establish, that the operators K^μ above are identical with the generators $\hat{K}^\mu$ of the special conformal group defined by:

$$\tilde{f}(x) \to \tilde{f}_b(x) := (1 - 2bx + b^2 x^2)^{-\frac{n-1}{2}} \tilde{f}\left(\frac{x - bx^2}{1 - 2bx + b^2 x^2}\right) \tag{III.33}$$

$$= \tilde{f}(x) - i b_\mu (\hat{K}^\mu \tilde{f})(x) + O((b^\mu)^2)$$

or what is the same:

$$
\begin{aligned}
(\hat{K}^\mu \tilde{f})(x) &= i \frac{\partial}{\partial b_\mu} \tilde{f}_b(x)|_{b^\mu=0} \\
&= i\left[(n-1)x^\mu + 2x^\mu x^\nu \frac{\partial}{\partial x^\nu} - x^2 \frac{\partial}{\partial x_\mu}\right] \tilde{f}(x).
\end{aligned} \tag{III.34}
$$

In order to see this, we first translate $\hat{K}^\mu$ to an operation in $L^2(d^n\mu(\boldsymbol{p}))$ by means of the Fourier transform (III.7). Since it is easily seen, that $\square^{(n)}\tilde{f}(x)=0$ implies $\square^{(n)}(\hat{K}^\mu \tilde{f})(x)=0$ we may calculate the action of $\hat{K}^\mu$ in $L^2(d^n\mu(\boldsymbol{p}))$ from:

$$
\begin{aligned}
(g, \hat{K}^\mu f) &= i \int_{x^0=0} d^n x\, \overline{\tilde{g}(x)} \overleftrightarrow{\frac{\partial}{\partial x^0}} (\hat{K}^\mu f)(x) \\
&= (2\pi)^{-n} \int_{x^0=0} d^n x \int d^n\mu(\boldsymbol{p})\, \overline{g(\boldsymbol{p})} \int d^n\mu(\boldsymbol{q})\, f(\boldsymbol{q})\, e^{-ipx} \\
&\quad \cdot \left\{\left[i(|\boldsymbol{p}|+|\boldsymbol{q}|) - \frac{\partial}{\partial x^0}\right] [(n-1)x^\mu - 2i x^\mu x^\nu q_\nu + i x^2 q^\mu]\, e^{iqx}\right\}.
\end{aligned} \tag{III.35}
$$

By straightforward calculations we find for all f from the dense subset $\mathscr{S}_n \subset L^2(d^n\mu(\boldsymbol{p}))$ and all $g \in L^2(d^n\mu(\boldsymbol{p}))$

$$
\begin{aligned}
(g, \hat{K}^0 f) &= -\int d^n\mu(\boldsymbol{p})\, \overline{g(\boldsymbol{p})}\, |\boldsymbol{p}|\, \Delta^{(n)} f(\boldsymbol{p}) \\
(g, \hat{K}^k f) &= \int d^n\mu(\boldsymbol{p})\, \overline{g(\boldsymbol{p})} \\
&\quad \cdot \left\{(n-1)\frac{\partial}{\partial p^k} + 2\boldsymbol{p}\nabla \frac{\partial}{\partial p^k} - p^k \Delta^{(n)}\right\} f(\boldsymbol{p}).
\end{aligned} \tag{III.36}
$$

On the other hand we obtain for all g from the dense subset D^n and all $f \in \mathscr{S}_n$ by means of the symmetry properties of $U(R)$

$$
\begin{aligned}
(g, K^\mu f) &= (U(R)g, P^\mu U(R) f) \\
&= \frac{4}{(2\pi)^{2n}} \int d^n\mu(\boldsymbol{p}) \int \frac{d^n x}{(\boldsymbol{x}^2)^{\frac{n+1}{2}}} \int d^n q\, e^{i\left(\boldsymbol{p}\boldsymbol{x} - \frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \overline{g(\boldsymbol{q})} \\
&\quad \cdot p^\mu \int \frac{d^n y}{(\boldsymbol{y}^2)^{\frac{n+1}{2}}} \int d^n k\, e^{-i\left(\boldsymbol{p}\boldsymbol{x} - \frac{\boldsymbol{k}\boldsymbol{y}}{\boldsymbol{y}^2}\right)} f(k). \\
p^0 &= |\boldsymbol{p}|.
\end{aligned} \tag{III.37}
$$

Since the x- and y-integrations converge absolutely and give rise to functions from $L^2(d^n\mu(\boldsymbol{p})) \cap L^1(d^n\mu(\boldsymbol{p})) \cap L^1(d^n p)$ for $g \in D^n$ and $f \in \mathscr{S}_n$, we may interchange the p-integration with the x- and y-integrals.

Case 1. K^0: Applying the Eq. (III.30) after the p-integration has been performed we get:

$$(g, K^0 f) = \frac{2}{(2\pi)^n} \int d^n x (\boldsymbol{x}^2)^{-(n+1)} \int d^n q \int d^n k \, \overline{g(\boldsymbol{q})} \, f(k) e^{i(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{x}\cdot(\boldsymbol{x}^2)^{-1}}$$

$$= -\frac{2}{(2\pi)^n} \int \frac{d^n x}{(\boldsymbol{x}^2)^n} \int d^n q \int d^n k \, e^{i(\boldsymbol{k}-\boldsymbol{q})\frac{\boldsymbol{x}}{\boldsymbol{x}^2}} \, \overline{g(\boldsymbol{q})} \, (\Delta^{(n)} f(\boldsymbol{k})) .$$

Thus

$$(g, K^0 f) = \int d^n \mu(k) \, \overline{g(\boldsymbol{k})} \, (-|\boldsymbol{k}| \, \Delta^{(n)} f(\boldsymbol{k})) . \tag{III.38}$$

Case 2. K^r: We first rewrite the q-integral by means of Lemma 1 and then proceed in the same way as in the case of K^0:

$$\begin{aligned}(g, K^r f) = {} & \frac{4}{(2\pi)^{2n}} \int d^n \mu(\boldsymbol{p}) \int d^n x (\boldsymbol{x}^2)^{-\frac{n-1}{2}} \\ & \cdot \int d^n \mu(\boldsymbol{q}) e^{i\left(\boldsymbol{p}\boldsymbol{x} - \frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}\right)} \overline{g(\boldsymbol{q})} p^k |\boldsymbol{p}| \int d^n y (\boldsymbol{y}^2)^{-\frac{n+1}{2}} \\ & \cdot \int d^n \mu(\boldsymbol{k}) e^{-i\left(\boldsymbol{p}\boldsymbol{y} - \frac{\boldsymbol{k}\boldsymbol{y}}{\boldsymbol{y}^2}\right)} f(\boldsymbol{k}) \, |\boldsymbol{k}| \\ = {} & \frac{i}{(2\pi)^n} \int d^n x (\boldsymbol{x}^2)^{-(n+1)} \int d^n \mu(\boldsymbol{q}) \, \overline{g(\boldsymbol{q})} \int d^n k \, f(\boldsymbol{k}) \\ & \cdot \left\{ (n+1) x^r - i k^r + 2i \boldsymbol{k}\boldsymbol{x} \frac{x^r}{\boldsymbol{x}^2} \right\} e^{i(\boldsymbol{k}-\boldsymbol{q})\frac{\boldsymbol{x}}{\boldsymbol{x}^2}} .\end{aligned}$$

From this it follows by the familiar change of integration variables (III.11):

$$\begin{aligned}(g, K^r f) = {} & \int d^n \mu(\boldsymbol{k}) \, \overline{g(\boldsymbol{k})} \\ & \cdot \left\{ (n-1) \frac{\partial}{\partial k^r} + 2 \boldsymbol{k} \boldsymbol{\nabla} \frac{\partial}{\partial k^r} - k^r \Delta^{(n)} \right\} f(\boldsymbol{k}) .\end{aligned} \tag{III.39}$$

Since D^n is dense in $L^2(d^n \mu(\boldsymbol{p}))$ we obtain the desired identity from a comparison of (III.36) with (III.38) and (III.39):

$$(K^\mu f)(\boldsymbol{p}) = (\hat{K}^\mu f)(\boldsymbol{p}) \quad \text{for all} \quad f \in \mathcal{S}_n . \tag{III.40}$$

Once we have established the existence of essentially self-adjoint generators K^μ for the special conformal group, we obtain its unitary representation $U(b)$ by exponentiation and the fact that K^μ and P^μ have the same spectrum contained in the closed forward cone

$$\overline{V_0^+} := \{p^\mu : p^2 \geqq 0, \; p^0 \geqq 0\} \tag{III.41}$$

$$U(b) = e^{i b_\mu K^\mu} := \int_{\overline{V_+^0}} e^{i b_\mu P^\mu} \, dE_k(p) . \tag{III.42}$$

Moreover in order to keep the considerations of the next section as elementary as possible, we may feel free to take the limits to improper functions. However every step can easily be rewritten in terms of wave-packets ($\in L^2(d^n\mu(\boldsymbol{p}))$).

For later use we want to derive the (improper) eigenfunctions of K^μ in $\boldsymbol{x}$-space (at time $x^0=0$). They are most easily obtained from (III.31) and a complete set of eigenfunctions $\tilde{f}_{\boldsymbol{p}}(\boldsymbol{x})$ of P^μ:

$$\begin{aligned}&\tilde{f}_{\boldsymbol{p}}(\boldsymbol{x}) := \lim_{\lambda\to\boldsymbol{p}} (2\pi)^{-\frac{n}{2}} \int d^n\mu(\boldsymbol{q}) e^{i\boldsymbol{q}\boldsymbol{x}} f_\lambda(\boldsymbol{q})\\ &f_\lambda \in L^2(d^n\mu(\boldsymbol{q}))\,; \qquad f_{\boldsymbol{p}}(\boldsymbol{q}) = 2|\boldsymbol{p}|\,\delta(\boldsymbol{q}-\boldsymbol{p})\,.\end{aligned} \tag{III.43}$$

Eq. (III.31) implies:

$$K^\mu \widetilde{(U(R)f)}_{\boldsymbol{p}}(\boldsymbol{x}) = p^\mu \widetilde{(U(R)f)}_{\boldsymbol{p}}(\boldsymbol{x})\,. \tag{III.44}$$

All we have to do is to rewrite Eq. (III.23) by means of Lemma 1 and take the Fourier transform. This leads to:

$$\widetilde{(U(R)f)}_\lambda(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} (\boldsymbol{x}^2)^{-\frac{n-1}{2}} \int d^n\mu(\boldsymbol{q}) e^{i\frac{\boldsymbol{q}\boldsymbol{x}}{\boldsymbol{x}^2}} f_\lambda(\boldsymbol{q})\,.$$

Hence we get in the limit for the complete set of eigenfunctions of K^μ:

$$\begin{aligned}\widetilde{(U(R)f)}_{\boldsymbol{p}}(\boldsymbol{x}) &= \lim_{\lambda\to\boldsymbol{p}} \widetilde{(U(R)f)}_\lambda(\boldsymbol{x})\\ &= (2\pi)^{-\frac{n}{2}} (\boldsymbol{x}^2)^{-\frac{n-1}{2}} e^{i\frac{\boldsymbol{p}\boldsymbol{x}}{\boldsymbol{x}^2}}\end{aligned} \tag{III.45}$$

IV. Conformal Transformations for $n>1$ Space Dimensions

From the previous sections we have seen that the conformal generators are self-adjoint for any space dimension, the one dimensional case having been treated separately in Section II, and the case of $n>1$ space dimensions in Section III.

From (III.45) the (improper) eigenfunctions of the conformal generators in $\boldsymbol{x}$ space, for $x^0=0$, are ($n>1$, positive energy solutions)

$$\tilde{f}_\lambda(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} (\boldsymbol{x}^2)^{-\frac{n-1}{2}} e^{i\frac{\lambda\boldsymbol{x}}{\boldsymbol{x}^2}} \tag{IV.1}$$

with

$$\begin{aligned}&\boldsymbol{K}\tilde{f}_\lambda(\boldsymbol{x}) = \boldsymbol{\lambda}\tilde{f}_\lambda(\boldsymbol{x})\\ &K^0\tilde{f}_\lambda(\boldsymbol{x}) = \sqrt{\boldsymbol{\lambda}^2}\,\tilde{f}_\lambda(\boldsymbol{x})\,.\end{aligned} \tag{IV.2}$$

Notice that the $\tilde{f}_\lambda$ are solutions of the formal eigenvalue equations in $\boldsymbol{x}$ space

$$i\left\{-\boldsymbol{x}^2\frac{\partial}{\partial x^k}+2x_k x_j\frac{\partial}{\partial x_j}+(n-1)x_k\right\}\tilde{f}_\lambda(\boldsymbol{x})=\lambda_k\tilde{f}_\lambda(\boldsymbol{x}) \quad \text{(IV.3)}$$

the absence of a inhomogeneous term analogous to (II.11), being due to the fact, for $n>1$ there are normalizable functions which do not vanish for $\boldsymbol{k}=0$. So from

$$(g, K_i f_\lambda)=\lambda_i(g, f_\lambda)$$

we conclude

$$\frac{1}{|\boldsymbol{k}|}K_i f_\lambda(\boldsymbol{k})=\lambda_i\frac{1}{|\boldsymbol{k}|}f_\lambda(\boldsymbol{k})$$

without a $\delta(\boldsymbol{k})$-term analogous to (II.10).

From (IV.1, 2) we conclude

$$\begin{aligned} e^{i\boldsymbol{b}\boldsymbol{K}}\tilde{f}_\lambda(\boldsymbol{x}) &= e^{i\boldsymbol{b}\lambda}\tilde{f}_\lambda(\boldsymbol{x}) \\ &= (1+2\boldsymbol{b}\boldsymbol{x}+\boldsymbol{b}^2\boldsymbol{x}^2)^{-\frac{n-1}{2}}\tilde{f}_\lambda(\mathrm{x}_T) \end{aligned} \quad \text{(IV.4)}$$

and expanding

$$\tilde{f}(\boldsymbol{x},0)=\int\frac{d^n\lambda}{2|\boldsymbol{\lambda}|}a_\lambda\tilde{f}_\lambda(\boldsymbol{x}) \quad \text{(IV.5)}$$

we obtain the transformation law of an arbitrary wave function for $x^0=0$:

$$e^{i\boldsymbol{b}\boldsymbol{K}}\tilde{f}(\boldsymbol{x},0)=\sigma(\boldsymbol{x})^{-\frac{n-1}{2}}\tilde{f}(\boldsymbol{x}_T,0) \quad \text{(IV.6)}$$

with

$$\sigma(x)=1-2bx+b^2x^2\,. \quad \text{(IV.7)}$$

For time zero the transformation law (IV.6) coincides with the formal substitution rule (I.1). To obtain the action of the finite conformal transformations for arbitrary times we have to use the time evolution operator

$$e^{i\boldsymbol{b}\boldsymbol{K}(x^0)}\tilde{f}(\boldsymbol{x},x^0)=e^{-iHx^0}\left\{\sigma(\boldsymbol{x})^{\frac{1-n}{2}}\tilde{f}(\boldsymbol{x}_T,0)\right\}. \quad \text{(IV.8)}$$

The positive energy time evolution kernel e^{-iHx^0} is given by

$$\begin{aligned}(e^{-iHx^0})_{\boldsymbol{x},\boldsymbol{y}} &= \frac{1}{(2\pi)^n}\int d^n k\, e^{-i|\boldsymbol{k}|x^0}e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} \\ &= c_n x^0[(\boldsymbol{x}-\boldsymbol{y})^2-(x^0-i\varepsilon)^2]^{-\frac{n+1}{2}}\end{aligned} \quad \text{(IV.9)}$$

with

$$c_n = (2\pi)^{-n} i(-1)^{n-1} (n-1)!\, \Omega_n ,$$

Ω_n = surface of unit sphere in n dimensions.

From (IV.6, 8, 9) we find

$$e^{i\boldsymbol{b}\boldsymbol{K}(x^0)} \tilde{f}(\boldsymbol{x}, x^0) = c_n x^0 \cdot \int d^n y [(\boldsymbol{x}-\boldsymbol{y})^2 - (x^0 - i\varepsilon)^2]^{-\frac{n+1}{2}} \sigma(\boldsymbol{y})^{\frac{1-n}{2}} \tilde{f}(\boldsymbol{y}_T, 0) . \tag{IV.10}$$

Introducing $\boldsymbol{y}_T$ as new integration variable, with the Jacobian

$$\frac{\partial(\boldsymbol{y}_T)}{\partial(\boldsymbol{y})} = \sigma(\boldsymbol{y})^{-n} \tag{IV.11}$$

and using the identity

$$(x-y)^2 = \sigma(x)\, \sigma(y)\, (x_T - y_T)^2 \tag{IV.12}$$

we arrive at

$$e^{i\boldsymbol{b}\boldsymbol{K}(x^0)} \tilde{f}(\boldsymbol{x}, x^0) = c_n x_T^0 \int d^n y_T\, \sigma(x)\, \sigma(\boldsymbol{y})^{\frac{n+1}{2}} \cdot [\sigma(x)\, \sigma(\boldsymbol{y}) \{\boldsymbol{x}_T - \boldsymbol{y}_T)^2 - (x_T^0)^2 + i\varepsilon x_T^0\}]^{-\frac{n+1}{2}} \tilde{f}(\boldsymbol{y}_T, 0) . \tag{IV.13}$$

If it were not for the ε prescription in (IV.13) one could pull out the factors $\sigma(x)\,\sigma(\boldsymbol{y})$ from the root and therefore justify the substitution rule (I.1). This can indeed be done for odd space dimensions when the denominator in (IV.13) has integral powers in agreement with the results of [15]. For even space dimensions however, although we have a unitary representation of the space-part of the special conformal transformations, as seen from (IV.6), the substitution rule (I.1), is not valid for times different from zero. On the other hand, the positive energy solutions of the D'Alembert equation, can be analytically continued to negative imaginary values of the time variable. We have therefore, for $x^0 = -ix_4$, $x_4 > 0$:

$$e^{i\boldsymbol{b}\boldsymbol{K}(-ix_4)} \tilde{f}(\boldsymbol{x}, -ix_4) = -c_n i x_{4T} \cdot \int d^n y_T\, \sigma(x)\, \sigma(\boldsymbol{y})^{\frac{n+1}{2}} \tilde{f}(\boldsymbol{y}_T, 0) \cdot [\sigma(x)\, \sigma(\boldsymbol{y}) \{(\boldsymbol{x}_T - \boldsymbol{y}_T)^2 + (x_{4T})^2\}]^{-\frac{n+1}{2}} . \tag{IV.14}$$

In (IV.14) all three factors under the root are positive and therefore can be separately pulled-out leading to

$$e^{ib\boldsymbol{K}(-ix_4)}\tilde{f}(\boldsymbol{x}, -ix_4) = \sigma(x)^{-\frac{n-1}{2}}\tilde{f}(\boldsymbol{x}_T, -ix_{4T}) \tag{IV.15}$$

which is the formal rule (I.1) for negative imaginary times.

The real time analogue of (IV.15) obtained by analytic continuation from (IV.15) requires that the time variable in $\sigma(x)$ be given a small negative imaginary part. This will imply, since the negative energy solutions require the opposite ε prescription a non-local transformation law for the quantized field in even space dimensions, i.e. one which can not be represented by a change of coordinates, hence no conflict with causality.

Only in the non-reverberating situation will (I.1) give the correct transformation of the field.

In order to obtain the action of the finite K_0 transformations, let us compute first the time evolved (for negative imaginary times) of the eigenfunctions $\tilde{f}_\lambda(\boldsymbol{x})$

$$\begin{aligned}\tilde{f}_\lambda(\boldsymbol{x}, -ix_4) &= c_n(2\pi)^{-\frac{n}{2}}(-ix_4)\int d^n y(\boldsymbol{y}^2)^{\frac{1-n}{2}} \\ &\cdot [(\boldsymbol{x}-\boldsymbol{y})^2 + x_4^2]^{-\frac{n+1}{2}} e^{-i\frac{\boldsymbol{\lambda y}}{\boldsymbol{y}^2}}.\end{aligned} \tag{IV.16}$$

Making the variable transformation $\boldsymbol{y}' = -\frac{\boldsymbol{y}}{\boldsymbol{y}^2}$, $x' = -\frac{\boldsymbol{x}}{\boldsymbol{x}^2}$ we easily arrive at

$$\begin{aligned}\tilde{f}_\lambda(\boldsymbol{x}, -ix_4) &= (2\pi)^{-\frac{n}{2}}\left(\frac{2}{|\boldsymbol{\lambda}|}\right)^{\frac{1}{2}}(\boldsymbol{x}^2 + x_4^2)^{-\frac{n-1}{2}} \\ &\cdot \exp\left\{\frac{-\sqrt{\lambda^2}\,x_4 + i\boldsymbol{\lambda x}}{\boldsymbol{x}^2 + x_4^2}\right\}.\end{aligned} \tag{IV.17}$$

From

$$\begin{aligned}e^{-ib_0K_0(-ix_4)}\tilde{f}_\lambda(\boldsymbol{x}, -ix_4) &= e^{-Hx_4 - ib_0K_0}\tilde{f}_\lambda(\boldsymbol{x}, 0) \\ &= e^{-ib_0\sqrt{\lambda^2}}\tilde{f}_\lambda(\boldsymbol{x}, -ix_4)\end{aligned} \tag{IV.18}$$

and since K_0 is a positive definite operator, which allows to extend the unitary operator to a bounded one for negative imaginary values of the parameter $b_0 = -ib_4$, $b_4 > 0$, we get using (IV.17, 18)

$$e^{-b_4K_0(-ix_4)}\tilde{f}_\lambda(\boldsymbol{x}, -ix_4) = \sigma(x)^{\frac{1-n}{2}}\tilde{f}_\lambda(\boldsymbol{x}_T, -ix_{4T}) \tag{IV.19}$$

and therefore for an arbitrary wave function,

$$e^{-b_4K_0(-ix_4)}\tilde{f}(\boldsymbol{x}, -ix_4) = \sigma(x)^{\frac{1-n}{2}}\tilde{f}(\boldsymbol{x}_T, -ix_{4T}) \tag{IV.20}$$

which is again the formal rule (I.1), for negative imaginary times and values of the b_0 group parameter. Again for real values of the time and the group parameter the analogue of (IV.20) requires that both the time and b_0 in $\sigma(x)$ be given a small negative imaginary part.

Composing (IV.17) with (IV.20) we get the transformation law

$$e^{-ibK(x^0)}\tilde{f}(x)=\sigma(x)^{\frac{1-n}{2}}\tilde{f}(x_T)$$
$$b_0=-ib_4,\quad x^0=-ix_4\,. \tag{IV.21}$$

One has therefore justified the substitution rule (I.1) for the analytically continued special conformal transformations acting on analytically continued solutions of the D'Alembert equation for any space dimension. This is closely related to the concept of weak conformal invariance of Hortaçsu, Seiler and Schroer [15].

Similar considerations hold for the negative energy solutions, with the important difference that analytic continuations have to be made to positive imaginary values of both the time and the group parameter b_0. This means that in the reverberating case we do not have a simple substitution rule valid for a general solution containing both positive and negative energy parts.

The transformation properties of the quantized field follow immediately from (IV.21) by means of the expansion

$$\phi^{(-)}(x)=\int\frac{d^n\lambda}{2|\lambda|}a_\lambda\tilde{f}_\lambda(x);\;\phi^{(+)}(x)=[\phi^{(-)}(x)]^* \tag{IV.22}$$

in terms of annihilation and creation operators. As in the one dimensional example the action of the conformal group on the field ϕ is reduced to its action on the wave functions $\tilde{f}_\lambda$. As for wave functions, the annihilation part of the field can be continued from a distribution valued operator to an operator for negative imaginary times and the (K_0) part of the conformal group analytically continued to negative imaginary values of the group parameter. Therefore one finally obtains,

$$e^{ib\tilde{K}}\phi^{(-)}(x)e^{-ib\tilde{K}}=\sigma(x)^{\frac{1-n}{2}}\phi^{(-)}(x_T) \tag{IV.23}$$
$$b_0=-ib_4,\quad x^0=-ix_4$$

and the conjugate expression for $\phi^{(+)}(x)$. Eq. (IV.23) provides (for free fields) the operator realization of weak conformal invariance [15], which now follows from the existence of a unitary representation of the special conformal group.

V. Conclusion

The fact that for free fields weak conformal invariance is a consequence of a unitary representation of the special conformal group, coupled with the existence of at least one self-adjoint extension of the conformal generators, makes one suspect, that weak conformal invariance would always be a manifestation of a unitary representation of the special conformal group.

In the interacting case, explicit transformation properties for the field, might be very hard to get, due to the impossibility of a covariant splitting in positive and negative energy parts and merit a separate investigation.

We conclude with a few remarks on the following problem: Do the separate unitary operators of the Poincaré, dilatation and special conformal group generate a representation of the whole conformal group?

For $n=1$, it is clear from (II.27) that we have a unitary representation of the whole group.

For $n>1$, we get from (IV.23),

$$e^{i\tilde{K}b}\phi(0)e^{-i\tilde{K}b}=\phi(0)\,. \tag{V.1}$$

On the other hand if we had a unitary representation of the whole conformal group then,

$$e^{i\tilde{K}b}e^{i\tilde{P}a}=TU\Lambda V \tag{V.2}$$

where we have used the canonical decomposition of any group element in terms of translations (T), special conformal transformations (U), special Lorentz transformations (Λ) and dilatations (V), [15]. In (V.2), it would follow from the existence of a representation,

$$\begin{aligned} T&=e^{i\tilde{P}a_T}\\ V&=e^{i(\log|\sigma(a)|)D} \end{aligned} \tag{V.3}$$

and U, Λ act trivially on the field at the origin.

With (V.2, 3) we would find,

$$\begin{aligned} e^{i\tilde{K}b}\phi(x)e^{-i\tilde{K}b}&=e^{i\tilde{K}b}e^{i\tilde{P}x}\phi(0)e^{-i\tilde{P}x}e^{-i\tilde{K}b}\\ &=(|\sigma(x)|)^{\frac{1-n}{2}}\phi(x_T)\,. \end{aligned} \tag{V.4}$$

The transformation law (V.4) is only compatible with the correct transformation law of the field under special conformal transformations if $n=4l+1$. In all other cases we probably get representations of the covering of the conformal group. Those questions will be dealt with in more detail in a future publication.

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