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Non-Existence of Goldstone Bosons with Non-Zero Helicity

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Abstract. In a local relativistic quantum field theory a conserved covariant tensor current may lead to a spontaneously broken symmetry if it generates zero mass states from the vacuum (Goldstone theorem). Here it is shown that in addition it is necessary that these massless states have helicity zero if the underlaying state space has a positive metric.

1. Introduction

The well-known Goldstone-theorem in quantum field theory [1] states a connection between the symmetries of a dynamical system and its excitation spectrum. It says that a local conservation law, expressed by the condition $\partial_\mu j^\mu(x) = 0$ for some local current[1], leads inevitably to a global one, expressed by the invariance of the vacuum state under a corresponding group of transformations, if there is a gap above the vacuum in the energy spectrum of the system. Conversely, if $\partial_\mu j^\mu(x) = 0$ but the vacuum is not invariant (in this case we speak of a spontaneously broken symmetry), there must exist excitations of the system with an energy arbitrarily close to that of the vaccum state. In a relativistic theory which we want to deal with in the following, one knows more: There have to be zero-mass states in the state space of the system which can be created from the vacuum by the application of the current operator j_μ. These massless "Goldstone-particles" must have the quantum numbers carried by the current j_μ. We may ask if there is some principal restriction on these or not. Clearly there is none on the internal quantum numbers, since we can easily construct an example of a Goldstone-particle with arbitrary internal quantum numbers with the help of a corresponding massless free scalar field ϕ with $j_\mu(x) = \partial_\mu \phi(x)$ the transformation being the addition of a constant to ϕ. What remains open is the spin (i.e., helicity) of the Goldstone-particle. In order to obtain Goldstone-particles with non-zero spin, the current which creates them from the vacuum has to have additional Lorentz-indices besides its

[1] We want to emphasize that we are only dealing with translation-covariant currents, i.e., $\mathcal{U}(1, a)\, j_\mu(x)\, \mathcal{U}^{-1}(1, a) = j_\mu(x + a)$.

vector index. In the following we shall show that a higher rank tensor current cannot lead to a spontaneously broken symmetry if it can create massless Bosons with non-zero helicity from the vaccuum in a state space with positive metric.

2. Assumptions

As a mathematical framework for our investigation we assume the usual Wightman axioms for relativistic quantum field theory [2]. We think of a collection of Wightman fields defined on a domain containing $\mathfrak{R}\Omega$, where $\mathfrak{R}$ is the polynomial algebra generated by the fields smeared with test functions from $\mathscr{D}(\mathbb{R}^4)$ and Ω is the unique vacuum state. One of those fields is assumed to be a Lorentz-covariant hermitean field $t_{\mu m}$ ($\mu = 0, 1, 2, 3$; m denotes a collection of vector indices[2]) which is conserved, $\partial^\mu t_{\mu m}(x) = 0$, and normalized to $(\Omega | t_{\mu m}(x)\, \Omega) = 0$. Covariance of $t_{\mu m}$ means

$$\mathscr{U}(\Lambda, a)\, t_{\mu m}(x)\, \mathscr{U}(\Lambda, a)^{-1} = \Lambda^{-1}{}_{\mu}{}^{\mu'}\, D_m{}^{m'}(\Lambda^{-1})\, t_{\mu' m'}(\Lambda x + a) \tag{1}$$

where $\mathscr{U}$ is the unitary representation of the inhomogeneous Lorentz-group (ILG) on the Hilbert space $\mathscr{H}$ of the state vectors and $D_m{}^{m'}$ some finite dimensional one valued representation of the homogeneous Lorentz-group (HLG)[2]. Furthermore, we assume that the current operators $t_{0m}(x)$ generate one-parameter groups of automorphisms $\alpha_m(\tau)$ of $\mathfrak{R}$ by the definition

$$-i \frac{d}{d\tau} \alpha_m(\tau)\, A \,|_{\tau=0} = \lim_{r \to \infty} [Q_{rm}, A] \quad \text{for} \quad A \in \mathfrak{R} \tag{2}$$

where the local generators Q_{rm} are defined as $Q_{rm} = t_{0m}(\vartheta_r \otimes \alpha)$[3] with $\vartheta_r(\vec{x}) = \vartheta(\vec{x}/r) \in \mathscr{D}(\mathbb{R}^3)$, $\vartheta(\vec{x}) = 1$ for $|\vec{x}| \leqq 1$, $\vartheta(\vec{x}) = 0$ for $|\vec{x}| \geqq 2$ and $\alpha(x^0) \in \mathscr{D}(\mathbb{R}^1)$, $\int \alpha(x^\circ)\, dx^0 = 1$. Sufficient conditions for a current to generate such automorphism groups are given in [4], hence we do not discuss this problem here. Let us only remark that the right hand side of Eq. (2) is actually independent of r for large r due to the relative locality of $t_{0\mu}(x)$ and $\mathfrak{R}$.

It is easy to see that the automorphisms $\alpha_m(\tau)$ are conserved symmetries (i.e., all vacuum expectation values are invariant) if

$$\lim_{r \to \infty} (\Omega | [Q_{rm}, A]\, \Omega) = 0$$

for all $A \in \mathfrak{R}$. On the other hand, one can prove [5]

$$\lim_{r \to \infty} (\Omega | [Q_{rm}, A]\, \Omega) = \lim_{r \to \infty} [(\Omega | Q_{rm} E_0 A \Omega) - (\Omega | A E_0 Q_{rm} \Omega)] \tag{3}$$

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[2] Here we exclude currents with spinor indices from our discussion. See however Section 5.

[3] $t_{\mu m}(f)$ denotes the value for the test function f of the distribution formally written as $\int t_{\mu m}(x) f(x)\, dx$.

where E_0 is the projection on the mass zero content of the representation of the ILG. This shows that a necessary condition for the spontaneous breakdown of the symmetry $\alpha_m(\tau)$ is that $E_0 t_{0m}(f)\,\Omega \neq 0$ for certain test functions $\in \mathscr{D}(\mathbb{R}^4)$. Let $\mathscr{H}_0$ be the part of $\mathscr{H}$ spanned by all states of the form $E_0 t_{\mu m}(f)\,\Omega$ with $f \in \mathscr{D}(\mathbb{R}^4)$. This is a non-trivial subspace of $\mathscr{H}$, invariant under the representation $\mathscr{U}$. Which helicities can these states have? To answer this question, let us for simplicity assume that the finite dimensional representation of the HLG transforming the components of $t_{\mu m}$ by $\Lambda^{-1} \otimes D(\Lambda^{-1})$ is irreducible of type (a, b)[4] (the reducible case will be dealt with in the Appendix), then it is clear that states of the form $E_0 t_{\mu m}(f)\,\Omega$ can have only certain helicities h, depending on the type (a, b) of $t_{\mu m}$. We denote the subspaces of $\mathscr{H}_0$ consisting of states of helicity h by $\mathscr{H}_{0h}$ (these are irreducible subspaces with respect to the representation of the ILG). It is known [3, 6], that the existence of a covariant basis – given in our case by the states $E_0 t_{\mu m}(f)\,\Omega$ – together with the conditions $\mathscr{H}_{0h} \neq \{0\}$ and $\mathscr{U}$ being unitary is only possible if $(a, b) = (b+h, b)$ with $b \geqq 0$ and $b+h \geqq 0$. This difference to the massive case has its reason in the non-compactness of the corresponding "little group", which is the Euclidean group $E(2)$ in two dimensions. Only these irreducible representations of $E(2)$ for which the two-dimensional translations are trivially represented are finite dimensional, i.e., lead to a finite number of independent states for fixed 4-momentum. For instance if one wants to create particles of mass zero and helicity one from the vacuum by a covariant field, one can use a field of type (1, 0) which is an antisymmetric self-dual tensor of second rank but not a vector-field which is of type (1/2, 1/2). Otherwise, one has to introduce indefinite metric [3]. In this case one may in fact have Goldstone-particles with non zero helicity as the example of the free photon field shows. There one has the spontaneously broken gauge transformations $A_\nu \to A_\nu + c_\nu$ (c_ν real numbers) with the four "photons" as Goldstone particles. (There are four Goldstone-states since one has four independent groups of transformations which are generated by the four currents $\partial_\mu A_\nu(x)$, $\nu = 0, 1, 2, 3$. However, the observables are pointwise invariant under these transformations.

Our intention now is to show that a current of type $(b+h, b)$ with $h \neq 0$ cannot generate a spontaneously broken symmetry.

3. Statements and Proofs

We shall give different proofs for currents of type $(b+h, b)$ with $2b+h \geqq 2$ and the remaining cases (1, 0) and (0, 1).

Lemma 1. *Let $t_{\mu m}(x)$ be a covariant Wightman field of type $(b+h, b)$ with $2b+h \geqq 2$ then $\lim_{r\to\infty} \|E_0 Q_{rm}\Omega\| < \infty$ ($\partial^\mu t_{\mu m}(x) = 0$ is not used for this conclusion).*

[4] a and b can acquire non-negative integer or half-integer values.

Proof. According to Ref. [3] (compare also [6]) the scalar product of two covariant states[5] of type $(b+h, b)$ describing zero-mass particles of helicity h has the following form:

$$(\Omega|t_{\mu m}(f)\,E_0\,t_{\nu n}(g)\,\Omega)=\int \bar{\tilde{f}}(p)\,\tilde{g}(p)\,P_{\mu\nu mn}(p)\,\theta(p^0)\,\delta(p^2)\,d^4p \tag{5}$$

where $P_{\mu\nu mn}(p)$ is a homogeneous polynomial in p of degree $2h+4b$:

$$P_{\mu\nu mn}(p)=c\,p_0^{2h+4b}\sum_{k=0}^{2b+h}\gamma_k\left(\frac{\vec{p}\cdot\vec{\Pi}}{p_0}\right)^k_{\mu\nu mn}. \tag{6}$$

$\vec{\Pi}_{\mu\nu mn}$ are certain constant matrices representing infinitesimal rotations in the representation $(b+h, b)$ of the HLG. The degree $2h+4b$ of the polynomial $P_{\mu\nu mn}$ corresponds to the maximum value of the quantum number n of Ref. [3], the only case compatible with $\mathscr{H}_{0h}\neq\{0\}$ and positive metric.

This leads to

$$\begin{aligned}\|E_0\,Q_{rm}\Omega\|^2&=c\int|\tilde{\vartheta}_r(\vec{p})|^2\,|\tilde{\alpha}(p^0)|^2\,P_{00mn}(p)\,\delta(p^2)\,\Theta(p^0)\,d^4p\\&=c\,r^{4-2h-4b}\int|\tilde{\vartheta}(\vec{p})|^2\,|\tilde{\alpha}(|\vec{p}|)|^2\,P_{00mn}(|\vec{p}|,\vec{p})\,\frac{d^3\vec{p}}{2|\vec{p}|}\end{aligned} \tag{7}$$

using $\tilde{\vartheta}_r(\vec{p})=r^3\,\tilde{\vartheta}(r\vec{p})$ and $P_{00mm}\left(\frac{|\vec{p}|}{r},\frac{\vec{p}}{r}\right)=r^{-2h-4b}\,P_{00mm}(|\vec{p}|,\vec{p})$. Since $\tilde{\vartheta}(\vec{p})$ is rapidly decreasing, we see that for $h+4b\geqq 2$ we get

$$\lim_{r\to\infty}\|E_0\,Q_{rm}\Omega\|^2<\infty\,.$$

Lemma 2. *If* $\lim_{r\to\infty}\|E_0\,Q_{rm}\Omega\|<\infty$, *then the automorphism group generated by* Q_{rm} *leaves* Ω *invariant.*

Proof. As remarked earlier, we have to show that $\lim_{r\to\infty}(\Omega|[Q_{rm}, A]\Omega)=0$ for all $A\in\mathfrak{R}$.

$\lim_{r\to\infty}\|E_0\,Q_{rm}\Omega\|^2<\infty$ implies that there exists a weakly convergent subsequence $E_0\,Q_{r_nm}\Omega\xrightarrow{w}\chi$. Then from (3) we get

$$\lim_{n\to\infty}(\Omega|[Q_{r_nm}, A]\,\Omega)=(\chi|A\Omega)-(A^*\Omega|\chi)\,.$$

On the other hand, this is equal to

$$\begin{aligned}&\lim_{n\to\infty}(\Omega|[Q_{r_nm}, \mathscr{U}(\xi)\,A\,\mathscr{U}^{-1}(\xi)]\,\Omega)\\&\quad=(\chi|\mathscr{U}(\xi)\,A\Omega)-(\mathscr{U}(\xi)\,A^*\Omega|\chi)\end{aligned}$$

[5] A state $\psi_m\in\mathscr{H}$ is called covariant of type (a, b) if it transforms like $t_\mu(x)\,\Omega$ under $\mathscr{U}(\Lambda, a)$ where $t_\mu(x)$ is a covariant field of type (a, b) defined in Eq. (1).

and this holds for all $\xi \in \mathbb{R}^4$ ($\mathscr{U}(\xi) \equiv \mathscr{U}(1, \xi)$). Choose now $\xi^0 = 0$, $\vec{\xi} = \lambda \vec{e}$, $\lambda > 0$. Then the cluster theorem yields $w - \lim_{\lambda \to \infty} \mathscr{U}(\xi) = P_\Omega$, the projector on Ω. Hence we get for the above

$$= (\chi | \Omega)(\Omega | A \Omega) - (\Omega | A^* \Omega)(\Omega | \chi),$$

but

$$(\Omega | \chi) = \lim_{n \to \infty} (\Omega | Q_{r_n m} \Omega) = 0.$$

Combining Lemma 1 and 2 we get:

Proposition 1. *Let* $t_{\mu m}(x)$ *be a covariant Wightman-field of type* $(b+h, b)$ *with* $2b + h \geqq 2$ *and* $\partial^\mu t_{\mu m}(x) = 0$, *then the group of automorphisms of* $\mathfrak{R}$ *generated by* $Q_{rm} = t_{0m}(\vartheta_r \otimes \alpha)$ *is not spontaneously broken.*

It remains to show that also currents of type (1, 0) or (0, 1) cannot produce Goldstone-particles. The covariant fields of these types are antisymmetric self-dual resp. anti-self-dual tensor fields of second rank: $t_{\mu m}(x) = F_{\mu\nu}(x)$ with

$$F_{\mu\nu}(x) = -F_{\nu\mu}(x)$$

and

$$2F_{\mu\nu}(x) = \pm i \varepsilon_{\mu\nu\varrho\sigma} F^{\varrho\sigma}(x). \tag{8}$$

Proposition 2. *Let* $F_{\mu\nu}(x)$ *be an antisymmetric conserved tensor current of rank* 2, *then* $\lim_{r \to \infty} (\Omega | [Q_{r\nu}, A] \Omega) = 0$ *for all* $A \in \mathfrak{R}$, $\nu = 0, 1, 2, 3$.

Proof. Since $F_{00} = 0$, we need only consider $\nu = i = 1, 2, 3$. For $A \in \mathfrak{R}$ the C^∞-functions

$$h_i(\vec{x}) = \int [(\Omega | A E_0 F_{0i}(x^0, \vec{x}) \Omega) - (\Omega | F_{0i}(x^0, \vec{x}) E_0 A \Omega)] \alpha(x^0) \, dx^0 \tag{9}$$

have compact support ([5], Lemma 4.4.2). Therefore, their Fourier transforms, $\tilde{h}_i(\vec{p})$, are analytic functions of rapid decrease. From $\partial^\mu F_{\mu\nu}(x) = -\partial^\nu F_{\mu\nu}(x) = 0$ and $F_{00}(x) = 0$ we get $\partial^i F_{0i}(x) = 0$ ($i = 1, 2, 3$) and hence $p^i h_i(\vec{p}) = 0$. If we take the derivative of this equation with respect to p_j at $\vec{p} = 0$, we find $\tilde{h}_j(0) = 0$. But this is all we need because

$$\begin{aligned} &\lim_{r \to \infty} (\Omega | [A, Q_{ri}] \Omega) \\ &= \lim_{r \to \infty} [(\Omega | A E_0 Q_{ri} \Omega) - (\Omega | Q_{ri} E_0 A \Omega)] \\ &= \lim_{r \to \infty} \int \tilde{h}_i(\vec{p}) \, \tilde{\vartheta}_r(\vec{p}) \, d^3\vec{p} = \lim_{r \to \infty} \int \tilde{h}_i(\vec{p}/r) \, \tilde{\vartheta}(\vec{p}) \, d^3\vec{p} = \tilde{h}_i(0) = 0. \end{aligned} \tag{10}$$

(We can interchange the limit $r \to \infty$ with the integral because the $\tilde{h}_i(\vec{p})$ are bdd. functions and $\tilde{\vartheta}(\vec{p})$ is of rapid decrease.)

Remark. The same kind of proof can be applied to all conserved currents of type $(h, 0)$ or $(0, h)$ since these are antisymmetric tensors of rank $2h$.

4. Result

Taking together Propositions 1 and 2 we may state the following:

Theorem. *Let $t_{\mu m}$ be Wightman fields transforming covariantly according to some irreducible one valued representation of the* HLG *with* $\partial^\mu t_{\mu m}(x) = 0$. *Assume furthermore that the corresponding operators Q_{rm}* (Eq. (2)) *generate transformation groups on $\mathfrak{R}$ which let the set of observables invariant. If the states $E_0 t_{\mu m}(f)\,\Omega$ have non-vanishing helicity, then the $\alpha_m(\tau)$ are conserved symmetries (the vacuum functional is invariant).*

Remark 1. The finite-dimensional representation of the HLG transforming the components of $t_{\mu m}(x)$ with $\Lambda^{-1} \otimes D(\Lambda^{-1})$ (Eq. (1)) may also be reducible as will be shown in the Appendix.

Remark 2. The result of the theorem can also be stated as follows: there are no Goldstone-Bosons of helicity different from zero in a Lorentz-covariant field theory on a state space with positive metric.

5. Half-Integer Spin

The case that $D(\Lambda^{-1})$ belongs to a two-valued representation of the HLG has been excluded from the preceding discussion. However, we want to add two remarks concerning this case.

i) Lemma 1 stays true in case of halfinteger spin $h > \frac{3}{2}$, hence by Lemma 2 one has also in this case

$$(\Omega \,|\, [Q_{rm}, A]\, \Omega) = 0$$

for sufficiently large r.

ii) A symmetry transformation α_τ should have the property that it maps observables into observables. It is easy to see that this can not be the case for all τ if $t_{\mu m}$ transforms according to a two valued representation of the HLG.

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Appendix

If the finite dimensional representation of the HLG acting on the indices of $t_{\mu m}(x)$ according to Eq. (1) is not irreducible we can decompose $t_{\mu m}$ in a unique way into a sum of irreducible parts and then apply Lemma 1 to the irreducible parts. The only case causing trouble is when $t_{\mu m}$ contains parts of type $(1, 0)$ or $(0, 1)$, since then Lemma 1 does not apply. On the other hand we cannot apply Prop. 2, since the antisymmetric tensor current corresponding to the (1, 0) or (0, 1) part of $t_{\mu m}$ is not known to be conserved any more.

Let $t_{\mu\nu}$ be a conserved tensor current of rank two. Decompose it into its symmetric and antisymmetric parts $S_{\mu\nu} + F_{\mu\nu}$. $E_0 S_{\mu\nu}$ generates from Ω states with helicity ± 2 and $E_0 F_{\mu\nu}$ states with helicity ± 1. To apply Prop. 2 we need to show that $\partial^\mu E_0 F_{\mu\nu}(x)\Omega = 0$. Denote by $P_{\pm 1}$ the projection on the states of helicity ± 1. Then

$$\begin{aligned}\partial^\mu E_0 F_{\mu\nu}(x)\Omega &= \partial^\mu E_0 (P_{+1} + P_{-1})\, t_{\mu\nu}(x)\,\Omega \\ &= E_0 (P_{+1} + P_{-1})\, \partial^\mu t_{\mu\nu}(x)\,\Omega = 0\,.\end{aligned}$$

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