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Mean of the Singularities of a Gibbs Measure

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Abstract: We calculate the value of the average of the singularities of a Gibbs measure μ invariant with respect to an expansive C^2 diffeomorphism of a one-compact manifold. This is the value related to dimension that one computes numerically. We then define and study a function, known as the correlation dimension, which is related to a free energy function, and we generalize the results in higher dimension with an axiom A transformation acting on a two-compact manifold.

0. Introduction

Let μ be a measure on a compact space X. Multifractal analysis is concerned with the description of different decay rates of the measures $\mu(B(x,r))$ of balls of radius r as r goes to 0. A natural quantity to be considered is

$$M(r,\beta) = \frac{\operatorname{Log} \int \mu(B(x,r))^{\beta}\, \mu(dx)}{\operatorname{Log} r} .$$

It can be argued [P, G] that in numerical computations based on time-series associated to a dynamical system, the functions $M(r,\beta)$ are the most accessible.

We prove here the existence of the limit

$$\forall \beta \in \mathbb{R}, \quad M(\beta) = \lim_{r \to 0} M(r,\beta) , \tag{0.1}$$

and we compute $M(\beta)$ in terms of other dynamical quantities. Actually, it is known in [P] that this function M referred to as the correlation dimension, plays an important role in the numerical investigation of some models, and differs in general with other characteristic dimensions, as a Hausdorff dimension, capacity or information dimension. There exists also a numerical procedure in [G] and described in [P] which is simple and runs fast.

The aim of this paper is to compute this correlation dimension in the case when the measure μ is a Gibbs measure for an expansive smooth transformation in dimension 1, or a two dimensional hyperbolic diffeomorphism. The method used to obtain the results in dimension 2 does not allow us to generalize to higher dimension.

1. Notations and Preliminaries

Let g be a $C^{1+\gamma}$ (resp. piecewise $C^{1+\gamma}$) expansive Markovian transformation of the circle Λ (resp. the interval). Let $J = -\operatorname{Log} g'$. The function J is negative and γ-Hölder (resp. piecewise). This is a context met for example in [C].

We consider a g-invariant measure μ that is the Gibbs measure associated to a function $\varphi : \Lambda \to \mathbb{R}$ γ-Hölder. Let P_φ be the pressure of φ, defined by

$$P_\varphi = \sup_{\rho \in M_g(\Lambda)} \left[h_\rho + \int \varphi \, d\rho \right] ,$$

where $M_g(\Lambda)$ is the set of probabilities defined on Λ and g-invariant.

Multifractal analysis of the measure μ consists in analyzing the singularity sets

$$C_\alpha^+ = \left\{ x \Big/ \overline{\lim_{\substack{x \in \mathring{I} \\ x|I| \to 0}}} \frac{\operatorname{Log} \mu(I)}{\operatorname{Log} |I|} = \alpha \right\}, \qquad C_\alpha^- = \left\{ x \Big/ \underline{\lim_{\substack{x \in \mathring{I} \\ |I| \to 0}}} \frac{\operatorname{Log} \mu(I)}{\operatorname{Log} |I|} = \alpha \right\},$$

$$C_\alpha = C_\alpha^+ \cap C_\alpha^- , \tag{1.1}$$

and in estimating the Hausdorff dimension of these singularity sets. We know that on a set of full measure μ [C, SI, II], there exists a real α such that

$$\lim_{r \to 0} L^*(r) = \alpha \quad \mu \text{ a.s.} , \tag{1.2}$$

where $L^*(r) = \frac{\operatorname{Log} \mu(B(x,r))}{\operatorname{Log} r}$. This means that there exists a real α such that

$$\mu(C_\alpha^+ \cap C_\alpha^-) = 1 .$$

We know that this particular value is linked with a free energy function F which derives from the partition functions defined on $\mathbb{R}$ by

$$Z_n(\beta) = \sum_{A \in A_n} \mu(A)^\beta , \tag{1.3}$$

where A_n is a sequence of partitions of exponentially decreasing diameters and

$$F(\beta) = \lim_{n \to +\infty} -\frac{1}{n} \operatorname{Log} Z_n(\beta)$$

is obtained for any real β by a variational formula [SI, II]

$$F(\beta) = \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + \beta \int \varphi \, d\rho - \beta P_\varphi}{\int J \, d\rho} \right] . \tag{1.4}$$

This function F is in fact real analytic on $\mathbb{R}$, strictly increasing, and is either a line or strictly concave. We also have for the value of α in (1.2),

$$\alpha = F'(1) .$$

The function F also satisfies a variational principle and is actually the inverse function of a more intrinsic free energy, the dynamical free energy function G [C, SI, II]. This function G is defined in terms of the pressure P so that we have:

$$\forall \beta \in \mathbb{R}, \quad P[-F(\beta)J - \beta(\varphi - P_\varphi)] = 0 . \tag{1.5}$$

The main result of this paper is the proof of the equality

$$\forall \beta \in \mathbb{R}, \quad M(\beta) = F(\beta + 1) .$$

Let f be the Legendre–Fenchel transform of F. By [SI, II] we know that

$$f(\alpha) = \mathrm{HD}(C_\alpha) \tag{1.6}$$

for $\alpha \in [\alpha_1; \alpha_2]$, where $\alpha_1 = \inf_{\beta\in\mathbb{R}} F'(\beta)$ and $\alpha_2 = \sup_{\beta\in\mathbb{R}} F'(\beta)$, and $f \equiv -\infty$ otherwise.

Let $K = (K_j)_{j=1,\ ,p}$ be a Markov partition with diameter less than the expansion constant of g [B, C, SI, II], and consider the transition matrix $A = (A_{ij})_{1\leqq i,j\leqq p}$ with

$$A_{ij} = \begin{cases} 1 & \text{if } \mathring{K}_i \cap g^{-1}(\mathring{K}_j) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} ,$$

and the subshift of finite type $\sum_A$

$$\sum_A = \{\underline{x} \in (x_n)_{n\geqq 0} \in \{1, \dots, p\}^{\mathbb{N}} / \forall i,\ A_{x_i x_{i+1}} = 1\}$$

that codes the transformation g since

$$\Pi : \sum_A \to \Lambda$$

$$\underline{x} \to \bigcap_{j\geqq 0} g^{-j}(K_j)$$

is a continuous bounded-to-one Lipschitz surjection and satisfies

$$\forall n \in \mathbb{N}, \quad \Pi \circ \sigma^n = g^n \circ \Pi ,$$

where σ is the shift on $\sum_A$.

Consider the function $\underline{\varphi}$ on $\sum_A$ defined by $\underline{\varphi} = \varphi \circ \Pi$ and the associated Gibbs measure ν_φ on $\sum_A$. We have

$$\nu_\varphi(C(n; \underline{y})) \simeq \exp\left\{ \sum_{i=0}^{n-1} \underline{\varphi}[\sigma^i(\underline{y})] - nP_\varphi \right\}, \tag{1.7}$$

where $C(n; y)$ is the cylinder of size n containing y: here and in all the paper, we denote $\simeq$ to express that the ratios of both sides are uniformly bounded by constants c and c^{-1}. The measure μ is the image of ν_φ under Π and the cylinders $C(n; y)$ are transformed by Π into intervals:

$$I(n; y) = \{x \in \Lambda / |g^i(x) - g^i(y)| \leqq \varepsilon,\ 0 \leqq i < n\} .$$

To an element U of the dynamical partition $\mathscr{P}_n = \bigvee_{i=0}^{n-1} g^{-j}(K)$, we associate $y(U) \in U$ such that

$$|g^n(U)| = |(g^n)'[y(U)]||U| \simeq 1\,, \tag{1.8}$$

or in another way

$$\exp\left\{\sum_{i=0}^{n-1} J[g^i(y(U))]\right\} \simeq |U|\,, \tag{1.9}$$

and the cylinder of size n associated to $U : C(n; y(U))$ verifies

$$\begin{aligned}\mu(U) &\simeq v_\varphi[C(n; y(U)]\\ &\simeq \exp\left\{\sum_{i=0}^{n-1} \varphi[g^i(y(U))] - nP_\varphi\right\}.\end{aligned} \tag{1.10}$$

To prove the existence of M (0.1), we follow the method of [SI, II]. We prove that the upper and lower limits of the function $M(r, \beta)$ as r goes to 0 are equal. For convenience, we consider $L(r) = -M(r, 1)$ and we observe that for any $b > 1$ we have

Proposition 1.1. *The sequence* $(L(b^{-n}))_{n \geqq 1}$ *is convergent if and only if*

$$\lim_{r \to 0} L(r) \text{ exists}\,.$$

We are going first to examine a lower bound for the limits of $L(r)$.

2. Lower Bound for $\liminf L(r)$

A lower estimate of the lower limit of $L(r)$ follows from

Theorem 2.1. *The lower limit of* $L(r)$ *verifies*

$$\varliminf_{r \to 0} L(r) \geqq \sup_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right].$$

Theorem 2.1 will follow directly from Lemmas 2.2 and 2.3.

Lemma 2.2. *The lower limit of* $L(r)$ *verifies*

$$\varliminf_{r \to 0} L(r) \geqq \sup_{\substack{\rho \in M_g(\Lambda)\\ \rho \text{ ergodic}}} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right].$$

Proof of Lemma 2.2. We consider an ergodic and g-invariant measure ρ. From the ergodic theorem, we have on a set of ρ measure 1:

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} J[g^i(y)] = \int J\, d\rho \tag{2.1}$$

and

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi[g^i(y)] = \int \varphi\, d\rho\,. \tag{2.2}$$

We are going to reduce this problem to the calculation of partition functions (1.3) [C, SI, II]. According to (2.1), let $b = \exp(-\int J\,d\rho + \varepsilon)$ for ε small enough. Indeed we have

$$L(b^{-n}) = \frac{1}{n}\operatorname{Log}_b \int \mu(B(x, b^{-n}))\,\mu(dx)$$
$$= \frac{1}{n}\operatorname{Log}_b \left\{ \sum_{U\in\mathscr{P}_n} \int_U \mu(B(x, b^{-n}))\,\mu(dx) \right\}. \tag{2.3}$$

The theorem of Shannon–McMillan [D, p. 81] leads us to consider the elements U of $\mathscr{P}_n$ of length

$$\operatorname{Log}|U| \in \left[\int J\,d\rho - \varepsilon;\ \int J\,d\rho + \varepsilon\right],$$

since for ε small enough the greatest part of the weight of the measure ρ is concentrated on those $U(> 1-\varepsilon)$ for large n. Then let A_n be the set of elements $U \in \mathscr{P}_n$ such that

$$b^{-n}\varepsilon^{2n} \leqq |U| \leqq b^{-n}.$$

For $U \in A_n$ we have for any $x \in U$,

$$\mu(U) \leqq \mu(B(x, b^{-n})),$$

therefore we get

$$\mu(U)^2 \leqq \int_U \mu(B(x, b^{-n}))\,\mu(dx),$$

and (2.3) leads to

$$L(b^{-n}) \geqq \frac{1}{n}\operatorname{Log}_b \left\{ \sum_{U\in A_n} \mu(U)^2 \right\}. \tag{2.4}$$

Consider now the right hand-side of (2.4). We have from the Shannon–McMillan theorem [D, p. 81] and [L, (2.2), SI, (2.4)] a lower estimate of $\# A_n$ since we get: for any $\varepsilon > 0$, there exists an integer N such that for any $n \geqq N$, and we have

$$\# A_n \geqq (1-\varepsilon)\exp\{n(h_\rho - \varepsilon)\}. \tag{2.5}$$

From (1.10) and (2.2) we have for $U \in A_n$,

$$2\left\{\int \varphi\,d\rho - P_\varphi\right\} - \varepsilon \leqq \frac{1}{n}\operatorname{Log}\mu^2(U) \leqq 2\left\{\int \varphi\,d\rho - P_\varphi\right\} + \varepsilon. \tag{2.6}$$

The inequality (2.4) becomes for n large enough

$$L(b^{-n}) \geqq \frac{h_\rho + 2\int \varphi\,d\rho - 2P_\varphi - 2\varepsilon}{\int -J\,d\rho + \varepsilon}$$

Since ε is arbitrary, we have then

$$\varliminf_{n\to+\infty} L(b^{-n}) \geqq \frac{h_\rho + 2\int \varphi\,d\rho - 2P_\varphi}{\int -J\,d\rho}. \tag{2.7}$$

Since the ergodic and g-invariant measure ρ is arbitrary, we proved:

$$\varliminf_{r\to 0} L(r) \geqq \sup_{\substack{\rho\in M_g(\Lambda)\\ \rho \text{ ergodic}}} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right]. \quad \square \tag{2.8}$$

We finish the proof of 2.1 with:

Lemma 2.3. *The two following expressions are equal:*

$$\sup_{\substack{\rho\in M_g(\Lambda)\\ \rho \text{ ergodic}}} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right] = \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right].$$

Proof of Lemma 2.3. Since the dynamical system expands, the map $\rho \to h_\rho$ is upper semi-continuous [D, (16.7), p. 107]. The ergodic measures are extremal and form a G_δ in $M_g(\Lambda)$ (this property comes from the specification [D, (21.9), p. 198]). The supremum on these two sets is the same, and it is achieved since $M_g(\Lambda)$ is compact. $\square$

Remark. This supremum is achieved by a unique g-invariant measure. Let the functional (a large deviation functional)

$$I(\rho) = \frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}$$

and ψ a g-invariant measure which achieves the supremum

$$I(\psi) = \sup_{\rho\in M_g(\Lambda)} I(\rho).$$

We have then for any g-invariant measure ξ

$$I(\psi)\int -J\, d\psi \geqq h_\xi + 2\int \varphi\, d\xi - 2P_\varphi,$$

or in a variational form

$$h_\xi + \int (2\varphi - 2P_\varphi + I(\psi)J)\, d\xi \leqq 0,$$

with equality for $\xi = \psi$. Since the function $\tau = 2\varphi - 2P_\varphi + I(\psi)J$ is by assumption Hölder continuous, the pressure of the function τ verifies

$$P_\tau = \sup_{\rho\in M_g(\Lambda)} \left[h_\xi + \int \tau\, d\xi\right] = 0 \tag{2.9}$$

with equality only in the case where $\psi = \mu_\tau$ the Gibbs measure of τ.

3. Upper Bound for lim sup $L(r)$

An upper estimate of the upper limit of $L(r)$ is given by

Theorem 3.1. *The upper limit of* $L(r)$ *verifies*

$$\varlimsup_{r\to 0} L(r) \leqq \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right].$$

Comparing the result with Theorem 2.1 implies the existence of the limit $M(1)(0.1)$ and we have

$$M(1) = -\sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int -J\, d\rho}\right] = \inf_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int J\, d\rho}\right].$$

Following (1.9) we have for any $U \in \mathscr{P}_n$,

$$n \inf\{J\} \leqq \operatorname{Log}|U| \leqq n \sup\{J\},$$

or equivalently

$$a_1^{-n} = (e^{\sup\{-J\}})^{-n} \leqq |U| \leqq (e^{\inf\{-J\}})^{-n} = a_2^{-n}. \tag{3.1}$$

Let b be a real such that $a_1 \leqq b \leqq a_2$ (will be made precised in (3.15)). Then Theorem 3.1 clearly follows from

Theorem 3.2. *For any cluster point S of the sequence $(L(b^{-n}))_{n\geqq 1}$ there exists a g-invariant measure ζ such that*

$$S \leqq \frac{h_\zeta + 2\int \varphi\, d\zeta - 2P_\varphi}{\int -J\, d\zeta}.$$

Proof of Theorem 3.2. We have from (2.3),

$$L(b^{-n}) = \frac{1}{n} \operatorname{Log}_b \sum_{U\in\mathscr{P}_n} \int_U \mu(B(x, b^{-n}))\, \mu(dx).$$

The proof parallels the proof in the cases of the partition functions and the free energy functions. We isolate the dominating terms in (2.3) for intervals of "same" diameter and μ-measure:

Lemma 3.3. *There exists a set $J_{k(n)}$ of intervals U of $(\mathscr{P}_n)_{n\geqq 1}$ with equal length and close μ-measure which verifies*

$$L(b^{-n}) \sim \frac{1}{n} \operatorname{Log}_b \left\{ \sum_{U\in J_{k(n)}} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \right\}.$$

Proof of Lemma 3.3. Set

$$E_i = \{U \in \mathscr{P}_n / -\operatorname{Log}|U| \in [i; i+1[\}. \tag{3.2}$$

From (3.1) the sets E_i are defined only for integers $i \in [[a_2 n]; [a_1 n] - 1]$ (linear scale). There exists an integer $i(n)$ such that for any integer i,

$$\frac{1}{n} \operatorname{Log}_b \sum_{U\in E_i} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \leqq \frac{1}{n} \operatorname{Log}_b \sum_{U\in E_{i(n)}} \int_U \mu(B(x, b^{-n}))\, \mu(dx),$$

and therefore we have

$$\frac{1}{n}\operatorname{Log}_b\left\{\sum_{U\in E_{i(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\right\}$$

$$\leqq L(b^{-n}) \leqq \frac{1}{n}\operatorname{Log}_b\left\{(a_1-a_2)n\sum_{U\in E_{i(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\right\}.$$

Hence we get:

$$L(b^{-n}) = \frac{1}{n}\operatorname{Log}_b\left\{\sum_{U\in E_{i(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\right\} + O\left(\frac{\operatorname{Log} n}{n}\right). \tag{3.3}$$

We define also for integers $k \in \mathbb{Z}$ the sets

$$J_k = \left\{U \in E_{i(n)} \Big/ \sum_{i=0}^{n-1}\varphi[g^i(y(U))] - nP_\varphi \in [k, k+1[\right\} \tag{3.4}$$

for $y(U) \in U$ (1.9). The sets J_k are defined for k varying in a linear scale:

$$a_3 n = n(\inf\varphi - P_\varphi) \leqq \sum_{i=0}^{n-1}\varphi[g^i(y(U))] \leqq n(\sup\varphi - P_\varphi) = a_4 n\,.$$

There exists an integer $k(n)$ such that for any integer $k \in [[a_3 n]; [a_4 n] - 1]$,

$$\frac{1}{n}\operatorname{Log}_b\sum_{U\in J_k}\int_U \mu(B(x,b^{-n}))\,\mu(dx) \leqq \frac{1}{n}\operatorname{Log}_b\sum_{U\in J_{k(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\,,$$

and like in (3.3) we have:

$$L(b^{-n}) = \frac{1}{n}\operatorname{Log}_b\left\{\sum_{U\in J_{k(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\right\} + O\left(\frac{\operatorname{Log} n}{n}\right). \tag{3.5}$$

All the intervals U in $J_{k(n)}$ have the "same" length $e^{-i(n)}$ and their μ-measure satisfies

$$\mu(U) \simeq \exp\{k(n)\}\,, \tag{3.6}$$

and this is the claim of Lemma 3.3. □

From (3.5) we have

$$L(b^{-n}) \sim \frac{1}{n}\operatorname{Log}_b\left\{\sum_{U\in J_{k(n)}}\int_U \mu(B(x,b^{-n}))\,\mu(dx)\right\}, \tag{3.7}$$

and to solve this problem from the point of view of partition functions, we are going to involve sums with values of type $\mu(A)^2$. Let us define like (1.3)

$$\frac{1}{n}\operatorname{Log} Z_n(2) = \frac{1}{n}\operatorname{Log}\sum_{A\in\mathscr{P}_n}\mu(A)^2 = -F_n(2)\,.$$

We can reduce this computation to intervals A of $\mathscr{P}_n$ of the "same" length and μ measure. The procedure is similar to the one in the proof of Lemma 3.3. Using Definition (3.2) there exists an integer $j(n)$ such that

$$-F_n(2) = \frac{1}{n}\,\mathrm{Log}\left\{\sum_{A\in E_{j(n)}} \mu(A)^2\right\} + O\left(\frac{\mathrm{Log}\, n}{n}\right).$$

We define now like in (3.4) the sets

$$K_p = \left\{A \in E_{j(n)} \Big/ \sum_{i=0}^{n-1} \varphi[g^i(y(A))] - nP_\varphi \in [p, p+1[\right\}. \tag{3.8}$$

Then there exists an interger $p(n)$ such that

$$-F_n(2) = \frac{1}{n}\,\mathrm{Log}\left\{\sum_{A\in K_{p(n)}} \mu(A)^2\right\} + O\left(\frac{\mathrm{Log}\, n}{n}\right). \tag{3.9}$$

Let us consider a cluster point of the sequence $(-F_n(2))_{n\geqq 1}$, for example

$$F = \lim_{j\to+\infty} -F_{n_j}(2), \quad \text{where } S = \lim_{j\to+\infty} L(b^{-n_j})\,.$$

We have then

Proposition 3.4. *There exists a g-invariant measure ξ which verifies*

$$F \leqq h_\xi + 2\textstyle\int \varphi\, d\xi - 2P_\varphi\,.$$

Proof of Proposition 3.4. Let us define the measures

$$\theta_n = \frac{1}{\# K_{p(n)}} \sum_{A\in K_{p(n)}} \delta_{y(A)} \quad \text{and} \quad \xi_n = \frac{1}{n}\sum_{i=0}^{n-1} g^i\,\theta_n\,,$$

where $y(A) \in A$ (1.9). We have

- $\xi_{n_j} \in M(\Lambda)$, the set of probability measures defined on Λ and
- $\frac{1}{n_j}\mathrm{Log}\,\# K_{p(n_j)} \in [0, a_1]$ (3.1).

Both sequences take their values in compact sets. We can suppose that

- $\xi_n \to \xi \in M_g(\Lambda)$ (observe that the weak limit is g-invariant), (3.10)
- $\frac{1}{n}\mathrm{Log}\ \# K_{p(n)} \to \gamma \in [0, a_1]$.

Let us compute

$$\int \varphi\, d\xi_n = \frac{1}{\# K_{p(n)}} \sum_{A\in K_{p(n)}} \frac{1}{n}\sum_{i=0}^{n-1} \varphi[g^i(y(A))]\,.$$

Following (1.10) we have for any $A \in K_{p(n)}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi[g^i(y(A))] - nP_\varphi = \frac{p(n)}{n} + O\left(\frac{1}{n}\right),$$

which gives when n goes to $+\infty$

$$\lim_{n\to+\infty} \frac{p(n)}{n} = \int \varphi \, d\xi - P_\varphi . \tag{3.11}$$

By the same method we prove also

$$\lim_{n\to+\infty} \frac{j(n)}{n} = \int -J \, d\xi . \tag{3.12}$$

Moreover a standard argument (due to Misiurewicz, see [SI, (2.3), II, (2.4), D, p. 145] shows that:

$$\gamma \leqq h_\xi . \tag{3.13}$$

We now claim:

$$F = \lim_{n\to+\infty} \frac{1}{n} \operatorname{Log} \sum_{A\in K_{p(n)}} \mu(A)^2 = \gamma + \int 2\,\varphi \, d\xi - 2P_\varphi . \tag{3.14}$$

For any $A \in K_{p(n)}$, we have by (1.10) and (3.8),

$$\mu(A) \simeq \exp\{p(n)\} .$$

We obtain therefore

$$\sum_{A\in K_{p(n)}} \mu(A)^2 \simeq \# K_{p(n)} \exp\{2p(n)\}$$

which leads to

$$\frac{1}{n} \operatorname{Log} \sum_{A\in K_{p(n)}} \mu(A)^2 \sim \frac{1}{n} \operatorname{Log}_b \# K_{p(n)} + 2\frac{p(n)}{n} .$$

Going to the limit and using (3.9) and (3.10) we get (3.14). Using (3.13) and (3.14) we get Proposition 3.4. □

Now and for the following we take $\operatorname{Log} b = \int -J \, d\xi$. We have then

$$\frac{F}{\operatorname{Log} b} = \lim_{n\to+\infty} \frac{1}{n} \operatorname{Log}_b \sum_{A\in K_{p(n)}} \mu(A)^2 \leqq \frac{h_\xi + 2\int \varphi \, d\xi - 2P_\varphi}{\int -J \, d\xi} . \tag{3.15}$$

Remember with (3.7) that

$$L(b^{-n}) \sim \frac{1}{n} \operatorname{Log}_b \left\{ \sum_{U\in J_{k(n)}} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \right\} .$$

There are therefore two cases which depend on the values $e^{-i(n)}$ and b^{-n}. But it seems that the weights of the sums, which are maximum for the values of type $\mu(A)^2$ with $|A| \simeq e^{-j(n)} \simeq b^{-n}$ (3.12), are also maximum for the values of type $\int_U \mu(B(x, b^{-n}))\, \mu(dx)$ with $|U| \simeq e^{-i(n)} \simeq b^{-n}$ (means "$i(n) = n \operatorname{Log} b$").

** *First case.* $e^{-i(n)} > b^{-n}$. We have then for a certain constant C,

$$\sum_{U\in J_{k(n)}} \int_U \mu(B(x, b^{-n}))\, \mu(dx) \leqq C \sum_{A\in \mathscr{P}_n} \mu(A)^2 . \tag{3.16}$$

Cut an interval $U \in J_{k(n)}$ in three pieces: $[c,d],[d,e]$ and $[e,f]$ with $|d-c| = |f-e| = b^{-n} \ll |e-d| \simeq |U| \simeq e^{-i(n)}$ (see Fig. 1).

We have then:

* $\forall x \in [c,d],\ B(x,b^{-n}) \subset [h,d] \cup U$ (where $h = a - b^{-n}$) and therefore

$$\mu(B(x,b^{-n})) \leqq \mu([h,d]) + \mu(U)\,,$$

* $\forall x \in [e,f],\ \mu(B(x,b^{-n})) \leqq \mu([e,p]) + \mu(U)$ (where $p = f + b^{-n}$), and
* $\forall x \in [d,e],\ \mu(B(x,b^{-n})) \leqq \mu(U)\,.$

For the points of $[d,e]$ we can compare $\mu(B(x,b^{-n}))$ and $\mu(U)$, otherwise it may happen that the weights $\mu([h,d])$ and $\mu([e,p])$ are much bigger than $\mu(U)$, and we want to control these subset distortions. Here is described the general situation:

From Fig. 2, we see that the interval U has two neighbours V and W; four cases may occur according to whether V and W belong to $J_{k(n)}$. Let's study first the simple case:

* $V \in J_{k(n)}$: the intervals U and V are in $J_{k(n)}$ which contains intervals of similar lengths and μ-measures,

$$\mu(V) \leqq c\,e\,\mu(U)\,,$$

where c comes from (1.7) and e from (3.4) and (3.6). We have then

$$\mu([h,d]) \leqq \mu(V) + \mu(U) \leqq (1+ec)\,\mu(U)\,,$$

and therefore for any x of $[c,d]$,

$$\mu(B(x,b^{-n})) \leqq (2+ec)\,\mu(U)\,.$$

In this case we can also compare $\mu(B(x,b^{-n}))$ and $\mu(U)$.

* $V \notin J_{k(n)}$: we have two possibilities:

– $\mu(V) \leqq \mu(U)$ and $\mu(B(x,b^{-n})) \leqq 2\mu(U)\,,$

– $\mu(U) \leqq \mu(V)$ and $\mu(B(x,b^{-n})) \leqq 2\mu(V)\,.$

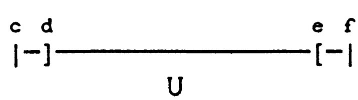


Fig. 1.

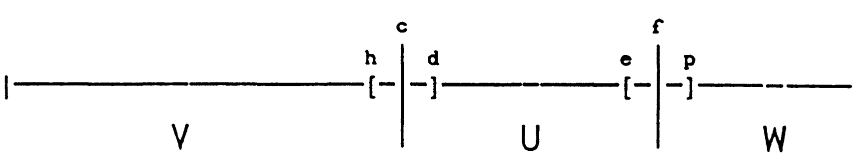


Fig. 2.

We make a similar operation for W and we get

$$\int_U \mu(B(x,b^{-n}))\,\mu(dx) \leqq C'(\mu(U)^2 + \mu(V)^2 + \mu(W)^2),$$

and this leads to (3.16).

** *Second case.* $e^{-i(n)} \leqq b^{-n}$. We use here a new partition (we shall use cylinders of size $l(n)$). Let $l(n)$ be the greatest integer k such that

$$\forall A \in \mathscr{P}_k, \quad |A| \geqq b^{-n}.$$

Following Proposition 3.4, since $S = \lim_{j\to+\infty} L(b^{-n_j})$, there exists a g-invariant measure χ such that (we write n instead of n_j)

$$\lim_{n\to+\infty} \frac{1}{l(n)} \operatorname{Log}\left\{\sum_{A\in\mathscr{P}_{l(n)}} \mu(A)^2\right\} \leqq h_\chi + 2\int \varphi\, d\chi - 2P_\varphi, \tag{3.17}$$

with

$$\frac{1}{l(n)} \operatorname{Log}\left\{\sum_{A\in\mathscr{P}_{l(n)}} \mu(A)^2\right\} \sim \frac{1}{l(n)} \operatorname{Log}\left\{\sum_{A\in K_{q(n)}} \mu(A)^2\right\},$$

where for intervals $A \in K_{q(n)}$ we have

$$|A| \simeq e^{-q(n)} \geqq b^{-n} \ (q(n) \leqq n \operatorname{Log} b) \ (3.9), \tag{3.18}$$

$$\frac{q(n)}{l(n)} \sim \int -J\, d\chi \ (3.12),$$

$$\frac{1}{l(n)} \operatorname{Log} \mu(A) \sim \int \varphi\, d\chi - P_\varphi \ (3.11),$$

where the measure χ is defined by

$$\rho_n = \frac{1}{\# K_{q(n)}} \sum_{A\in K_{q(n)}} \delta_{y(A)} \quad \text{and} \quad \chi_n = \frac{1}{l(n)} \sum_{i=0}^{l(n)-1} g^i \rho_n \to \chi \in M_g(\Lambda).$$

We get therefore

$$\frac{1}{n} \operatorname{Log}_b \left\{\sum_{A\in\mathscr{P}_{l(n)}} \mu(A)^2\right\} \leqq \frac{h_\chi + 2\int \varphi\, d\chi - 2P_\varphi}{\int -J\, d\chi}, \tag{3.19}$$

since using (3.18),

$$\frac{1}{n\operatorname{Log} b} = \frac{l(n)}{n \operatorname{Log} b}\,\frac{1}{l(n)} \quad \text{and} \quad \frac{l(n)}{n \operatorname{Log} b} = \frac{l(n)}{q(n)}\,\frac{q(n)}{n \operatorname{Log} b} \leqq \frac{1}{\int -J\, d\chi}.$$

Like in the first case and (3.16) we get

$$\sum_{U\in J_{k(n)}} \int_U \mu(B(x,b^{-n}))\,\mu(dx) \leqq C \sum_{A\in\mathscr{P}_{l(n)}} \mu(A)^2. \tag{3.20}$$

Comparing the expression (3.7) with the results (3.4) and (3.16), (3.19) and (3.20), we obtain

$$S \leqq \sup(I(\xi); I(\chi)) = I(\zeta) = \frac{h_\zeta + 2\int \varphi\, d\zeta - 2P_\varphi}{\int -J\, d\zeta},$$

and this achieves the proof of Theorem 3.2. □

4. Study of the Correlation Function *M*

From Sects. 2 and 3 follows the existence of the limit

$$M(1) = \lim_{r \to 0} M(r, 1)$$

and the expression

$$M(1) = \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + 2\int \varphi\, d\rho - 2P_\varphi}{\int J\, d\rho}\right].$$

For any positive β, the same analysis applies to the quantities $M(r, \beta)$ defined in (0.1). We get

Proposition 4.1. *We have for any positive* β,

$$M(\beta) = \lim_{r \to 0} M(r, \beta) = \frac{\operatorname{Log} \int \mu(B(x, r))^\beta\, \mu(dx)}{\operatorname{Log} r}$$

$$= \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + (\beta + 1)\int \varphi\, d\rho - (\beta + 1)P_\varphi}{\int J\, d\rho}\right] = F(\beta + 1). \quad (4.1)$$

Recall that F was defined in [C, SI, II]. Observe also that there is nothing to prove for $\beta = 0$, and that for $\beta < 0$ the proofs are also analogous. The minimum in (4.1) is achieved since the functional is lower semicontinuous and $M_g(\Lambda)$ is compact. Proposition 4.1 defines the real function $M(\beta)$ that we are going to analyze.

Define G as the dynamical free energy function for any pair (x, y) of $\mathbb{R}^2$ by

$$G(x, y) = P[(x + 1)(\varphi - P_\varphi) + yJ]. \quad (4.2)$$

Since the function φ is Hölder continuous, the function G is real analytic in both variables [R]. Observe that

Proposition 4.2. *We have for any real* β,

$$G(\beta, -M(\beta)) = 0.$$

Proof of Proposition 4.2. Let $\beta \in \mathbb{R}$ and consider the Hölder continuous function ξ_β,

$$\xi_\beta = (\beta + 1)(\varphi - P_\varphi) - M(\beta)J.$$

Its Gibbs measure μ_β is the measure for which the minimum in (4.1) is achieved and this means that

$$\sup_{\psi \in M_g(\Lambda)} \left[h_\psi + \int [(\beta + 1)(\varphi - P_\varphi) - M(\beta)J]\, d\psi\right] = 0.$$

The statement of 4.2 follows. □

As a consequence we get

Proposition 4.3. *The function M is real analytic and is strictly increasing on $\mathbb{R}$, and we have for any real β,*

$$M'(\beta) = \frac{\int \varphi \, d\mu_\beta - P_\varphi}{\int J \, d\mu_\beta} .$$

Proof of Proposition 4.3 (See [M, Ma, R, SI, II]). We have for any real β,

$$\left(\frac{\partial G}{\partial x}\right)(\beta, -M(\beta)) = \int \varphi \, d\mu_\beta - P_\varphi < 0 , \tag{4.3}$$

and

$$\left(\frac{\partial G}{\partial y}\right)(\beta, -M(\beta)) = \int J \, d\mu_\beta < 0 . \tag{4.4}$$

The expression (4.4) is never 0 so by the implicit function theorem and (4.2) the function M is real analytic on $\mathbb{R}$. When differentiating (4.2) we get

$$\left(\frac{\partial G}{\partial x} - M'(\beta)\frac{\partial G}{\partial y}\right)(\beta, -M(\beta)) = 0 ,$$

hence

$$\forall \beta \in \mathbb{R}, \qquad M'(\beta) = \frac{\int \varphi \, d\mu_\beta - P_\varphi}{\int J \, d\mu_\beta} > 0 . \qquad \square$$

We get also

Proposition 4.4. *The function M is concave. It is strictly concave unless J and φ are homologous, i.e. there is $K \in C^\gamma(\Lambda)$ such that $J = \varphi + K \circ g - K$.*

Proof of Proposition 4.4. When differentiating the above formula, we get for any real β,

$$M''(\beta) = \frac{\partial}{\partial x}\left(\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}\right)(\beta, -M(\beta)) - M'(\beta)\frac{\partial}{\partial y}\left(\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}\right)(\beta, -M(\beta)) ,$$

and finally

$$M''(\beta) = \left\{ \frac{\left(\frac{\partial^2 G}{\partial x^2}\right)\left(\frac{\partial G}{\partial y}\right)^2 - 2\left(\frac{\partial G}{\partial x}\right)\left(\frac{\partial G}{\partial y}\right)\left(\frac{\partial^2 G}{\partial x \partial y}\right) + \left(\frac{\partial^2 G}{\partial y^2}\right)\left(\frac{\partial G}{\partial x}\right)^2}{\left(\frac{\partial G}{\partial y}\right)^3} \right\}(\beta, -M(\beta)) .$$

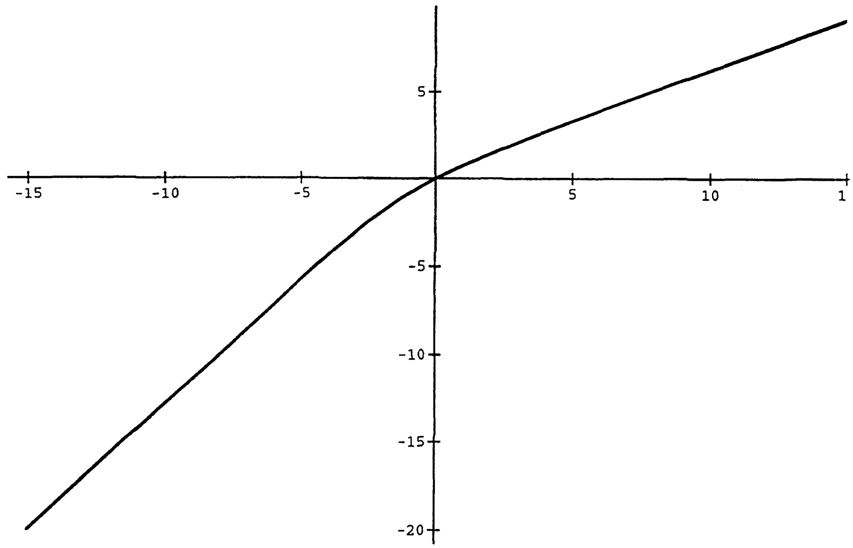
From [R, SI, II] we have for any real β the following equalities:

$$\left(\frac{\partial^2 G}{\partial x^2}\right)(\beta, -M(\beta)) = \sum_{k \in \mathbb{Z}} \left\{ \int \varphi \, \varphi \circ g^k \, d\mu_\beta - \left(\int \varphi \, d\mu_\beta\right)^2 \right\} , \tag{4.5}$$

$$\left(\frac{\partial^2 G}{\partial x \partial y}\right)(\beta, -M(\beta)) = \sum_{k \in \mathbb{Z}} \left\{ \int \varphi J \circ g^k \, d\mu_\beta - \int \varphi \, d\mu_\beta \int J \, d\mu_\beta \right\} , \tag{4.6}$$

and

$$\left(\frac{\partial^2 G}{\partial y^2}\right)(\beta, -M(\beta)) = \sum_{k \in \mathbb{Z}} \left\{ \int J J \circ g^k \, d\mu_\beta - \left(\int \varphi \, d\mu_\beta\right)^2 \right\} . \tag{4.7}$$



Graph 1.: Graph of the correlation dimension function
$M : \mathbb{R} \to \mathbb{R}$

And we have

$$\left(\frac{\partial^2 G}{\partial x \partial y}\right)^2 (\beta, -M(\beta)) < \left(\frac{\partial^2 G}{\partial x^2}\right)(\beta, -M(\beta)) \left(\frac{\partial^2 G}{\partial y^2}\right)(\beta, -M(\beta)), \tag{4.8}$$

which becomes an equality only when J and φ are homologous (see [SII, (2.3.1)]). The conclusion follows. □

Here we describe in the general case the behaviours of the correlation dimension function M and its derivative M'. We can prove that

* $a = \lim_{\beta \to +\infty} M'(\beta) = \lim_{\beta \to +\infty} \frac{M(\beta)}{\beta}$;
* $b = \lim_{\beta \to -\infty} M'(\beta) = \lim_{\beta \to -\infty} \frac{M(\beta)}{\beta}$;
* there exist positive reals δ_1 and δ_2 such that

$$\lim_{\beta \to +\infty} [M(\beta) - a(\beta + 1) + \delta_1] = 0 \quad \text{and} \quad \lim_{\beta \to -\infty} [M(\beta) - b(\beta + 1) + \delta_2] = 0,$$

where the numbers δ_1 and δ_2 are the Hausdorff dimensions of g-invariant measures ρ_1 and ρ_2 (where $HD(\rho) = \inf\{HD(A)/\rho(A) = 1\}$).

5. Extension of the Results in Dimension 2

Consider a compact manifold X of dimension 2, for example the torus, on which acts an Axiom A C^2 diffeomorphism g. We associate to this dynamical system a g-invariant measure μ, in the first case the Bowen–Margulis measure and in the second case a Gibbs measure.

We introduce canonical coordinates [B, R, SI, II] and a local product structure using local stable manifolds $W^s_{\text{loc}}(x)$ (where g contracts) and local unstable manifolds $W^u_{\text{loc}}(x)$ (where g expands).

Define stable Markov partitions $(\mathscr{P}^s_n)_{n\geqq 1}$ and unstable Markov partitions $(\mathscr{P}^u_n)_{n\geqq 1}$. Consider the "product" partition $(\mathscr{P}_n)_{n\geqq 1}$ whose elements verify

$$A = [U, V]$$

with $(U, V) \in \mathscr{P}^s_n \times \mathscr{P}^u_n$, [SI, II]

Consider also the functions

$$J^s(x) = \text{Log Jacobian}\,(Dg : E^s_x \to E^s_{gx})$$

and

$$J^u(x) = -\text{Log Jacobian}\,(Dg : E^u_x \to E^u_{gx})\,.$$

Since g is C^2, Dg is C^1 and the functions J^s and J^u are negative and Hölder continuous functions. We get a basic set Λ which contains the supports of the measures of interest.

Firstly consider the measure μ of maximal entropy

$$h_\mu = h = \sup_{\psi\in M_g(\Lambda)} h_\psi\,.$$

We obtain

Theorem 5.1. *For any real β we have the following limit*:

$$M(\beta) = \lim_{r\to 0} \frac{\text{Log} \int \mu(B(x,r))^\beta\, \mu(dx)}{\text{Log}\, r}\,.$$

In fact $M(\beta)$ can be decomposed into $M^s(\beta) + M^u(\beta)$, where

$$M^s(\beta) = \inf_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - (\beta+1)h}{\int J^s\, d\rho}\right] \quad \text{and} \quad M^u(\beta) = \inf_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - (\beta+1)h}{\int J^u\, d\rho}\right]\,.$$

Proof of Theorem 5.1. We have seen in [SI] that the measure μ verifies locally

$$\mu = \mu^s \times \mu^u\,, \tag{5.1}$$

where the measures μ^s and μ^u are defined respectively on the stable and unstable manifolds. For example, there exists for each interval U of $\mathscr{P}^s_n$ an element $y(U) \in U$ (1.9) such that

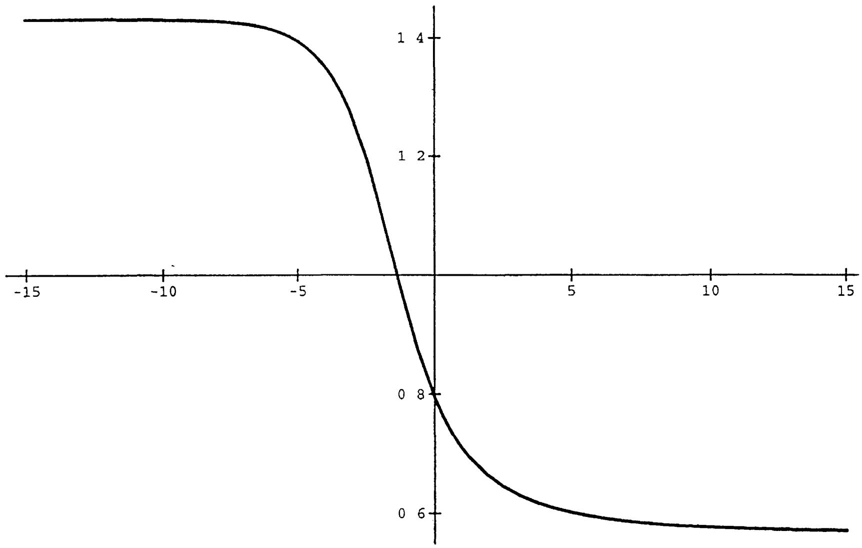
$$\exp\left\{\sum_{i=0}^{n-1} J^s[g^i(y(U))]\right\} \simeq |U| \tag{5.2}$$

and

$$\mu^s(U) \simeq e^{-nh}\,. \tag{5.3}$$

Symmetrically, there exists for each interval $V \in \mathscr{P}^u_n$ an element $y(V) \in V$ such that

$$\exp\left\{\sum_{i=0}^{n-1} J^u[g^i(y(V))]\right\} \simeq |V| \tag{5.4}$$



Graph 2.: Graph of the derivative of the correlation dimension function $M' : \mathbb{R} \to]a; b[\subset \mathbb{R}^{+*}$

and

$$\mu^u(V) \simeq e^{-nh} . \tag{5.5}$$

Following Sects. 2 and 3 we first take ρ a g-invariant and ergodic measure. We consider A_n^s the set of elements $U \in \mathscr{P}_n^s$ such that

$$\mathrm{Log}\,|U| \in \left[\int J^s\, d\rho - \varepsilon;\ \int J^s\, d\rho + \varepsilon\right] .$$

Identically we consider the set A_n^u of elements $V \in \mathscr{P}_n^u$ such that

$$\mathrm{Log}\,|V| \in \left[\int J^u\, d\rho - \varepsilon;\ \int J^u\, d\rho + \varepsilon\right] .$$

Let $A_n = [A_n^s; A_n^u]$ and define the real $b = \inf(c; d)$, where

$$c = \exp\left(\int -J^s\, d\rho + \varepsilon\right) \quad \text{and} \quad d = \exp\left(\int -J^u\, d\rho + \varepsilon\right) .$$

We have then

$$\begin{aligned} L(b^{-n}) &= \frac{1}{n} \mathrm{Log}_b \int \mu(B(x, b^{-n}))\, \mu(dx) \\ &= \frac{1}{n} \mathrm{Log}_b \sum_{A \in \mathscr{P}_n} \int_A \mu(B(x, b^{-n}))\, \mu(dx) \\ &\geqq \frac{1}{n} \mathrm{Log}_b \sum_{A \in A_n} \mu(A)^2 \\ &= \frac{1}{n} \mathrm{Log} \sum_{\substack{A \in A_n \\ A = [U, V] \\ (U, V) \in A_n^s \times A_n^u}} \mu([U, V]) . \end{aligned} \tag{5.6}$$

From (5.1), (5.6) becomes

$$\frac{1}{n}\operatorname{Log}_b \sum_{A\in A_n} \mu(A)^2 \sim \frac{1}{n}\operatorname{Log}_b \sum_{U\in A_n^s} \mu^s(U)^2 + \frac{1}{n}\operatorname{Log}_b \sum_{V\in A_n^u} \mu^u(V)^2 . \tag{5.7}$$

We introduce therefore the sequences $L^s(b^{-n})$ and $L^u(b^{-n})$ corresponding to μ^s and μ^u. It is clear that

$$L^s(b^{-n}) \geqq L^s(c^{-n}) \quad \text{and} \quad L^s(b^{-n}) \geqq L^s(d^{-n}) .$$

From (2.4) we have

$$L^s(b^{-n}) \geqq \frac{1}{n}\operatorname{Log}_c \sum_{U\in A_n^s} \mu^s(U)^2 + \frac{1}{n}\operatorname{Log}_d \sum_{V\in A_n^u} \mu^u(V)^2 . \tag{5.8}$$

We introduce therefore the sequences $L^s(c^{-n})$ and $L^u(d^{-n})$ corresponding to μ^s and μ^u. It is clear that

$$L^s(c^{-n}) \geqq \frac{1}{n}\operatorname{Log}_c \sum_{U\in A_n^s} \mu^s(U)^2 = a_n^s$$

and

$$L^s(d^{-n}) \geqq \frac{1}{n}\operatorname{Log}_d \sum_{V\in A_n^u} \mu^u(V)^2 = a_n^u .$$

We have then from the above formulas and (2.7)

$$\underline{\lim}_{n\to+\infty} L^s(c^{-n}) \geqq \lim_{n\to+\infty} a_n^s = \frac{h_\rho - 2h - 2P_\varphi}{\int -J^s\, d\rho}$$

and

$$\underline{\lim}_{n\to+\infty} L^u(d^{-n}) \geqq \lim_{n\to+\infty} a_n^u = \frac{h_\rho - 2h - 2P_\varphi}{\int -J^u\, d\rho} .$$

Since the measure ρ is arbitrary and with (2.3) we get

$$\underline{\lim}_{n\to+\infty} L^s(c^{-n}) \geqq \sup_{\substack{\rho\in M_g(\Lambda)\\ \rho \text{ ergodic}}} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^s\, d\rho}\right] = \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^s\, d\rho}\right]$$

and

$$\underline{\lim}_{n\to+\infty} L^u(d^{-n}) \geqq \sup_{\substack{\rho\in M_g(\Lambda)\\ \rho \text{ ergodic}}} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^u\, d\rho}\right] = \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^u\, d\rho}\right] .$$

Using (5.7) it becomes

$$\underline{\lim}_{r\to 0} L(r) \geqq \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^s\, d\rho}\right] + \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^u\, d\rho}\right] . \tag{5.9}$$

We prove a sort of converse of (5.9) in the same way as Theorem 3.1, i.e.

$$\overline{\lim}_{r\to 0} L^s(r) \leqq \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^s\, d\rho}\right]$$

and

$$\overline{\lim_{r\to 0}} L^u(r) \leqq \sup_{\rho\in M_g(\Lambda)} \left[\frac{h_\rho - 2h - 2P_\varphi}{\int -J^u\, d\rho}\right] .$$

We have thus obtained

$$\lim_{r\to 0} L(r) = \lim_{r\to 0} L^s(r) + \lim_{r\to 0} L^u(r) ,$$

or equivalently

$$-\lim_{r\to 0} L(r) = M(1) = M^s(1) + M^u(1) .$$

This proves the theorem for $\beta = 1$. The proof is analogous for any real β. □

This function M verifies the following properties:

Proposition 5.2. *The function M is real analytic on $\mathbb{R}$.*

Proof of Proposition 5.2. Consider the functions defined on $\mathbb{R}^2$ by

$$G^s(x, y) = P[(x + 1)h + yJ^s]$$

and

$$G^u(x, y) = P[(x + 1)h + yJ^u] .$$

From Proposition 4.2, we have for any real β,

$$G^s(\beta, -M^s(\beta)) = G^u(\beta, -M^u(\beta)) = 0 . \tag{5.10}$$

Consider the Hölder continuous functions

$$\varphi^s_\beta = (\beta + 1)h - M^s(\beta) \quad \text{and} \quad \varphi^u_\beta = (\beta + 1)h - M^u(\beta) ,$$

and their Gibbs measures μ^s_β and μ^u_β. We have then, differentiating (5.10),

$$\left(\frac{\partial G^s}{\partial y}\right)(\beta, -M^s(\beta)) = \int J^s\, d\mu^s_\beta < 0 \tag{5.11}$$

and

$$\left(\frac{\partial G^u}{\partial y}\right)(\beta, -M^u(\beta)) = \int J^u\, d\mu^u_\beta < 0 , \tag{5.12}$$

and the analycity of the functions M^s and M^u by the implicit function theorem [R]. The functions M^s and M^u are therefore real analytic on $\mathbb{R}$, and $M = M^s + M^u$ also. □

Proposition 5.3. *The function M is strictly increasing and*

- *either M is linear: this is the case when J^s and J^u are homologous to a constant, i.e. $= c + K \circ g - K$ (where $c \in \mathbb{R}$ and $K \in C^\gamma(\Lambda)$),*
- *or the function M is strictly concave.*

Proof of Proposition 5.3. Using (5.10), (5.11), (5.12) and (4.3) we get for any real β, that $(M^s)'(\beta)$ which is given by:

$$(M^s)'(\beta) = \frac{h}{\int -J^s\, d\mu^s_\beta} \tag{5.13}$$

is positive, and $(M^u)'(\beta)$ which is given by:

$$(M^u)'(\beta) = \frac{h}{\int -J^u \, d\mu^u_\beta} \tag{5.14}$$

is also positive. Therefore for any real β we have $M'(\beta)$ positive since

$$M'(\beta) = (M^s)'(\beta) + (M^s)'(\beta) \, .$$

In the case when J^s and J^u are homologous to a constant, then the measures μ^s_β and μ^u_β are constant when β varies in $\mathbb{R}$, and by (5.13) and (5.14) the functions $(M^s)'$ and $(M^u)'$ are constants as well. Therefore M^s, M^u and thus M are linear on $\mathbb{R}$. If we are in the case where the transformation g is Anosov, then the measure μ is absolutely continuous with respect to the Lebesgue measure [B], and g is differentiably conjugate to an automorphism of the torus [De].

In the other case, we have for any real β either

$$(M^s)''(\beta) = \frac{h^2}{(\int J^s \, d\mu^s_\beta)^3} \left(\frac{\partial^2 G^s}{\partial y^2} \right) (\beta, -M^s(\beta)) < 0 \tag{5.15}$$

or

$$(M^u)''(\beta) = \frac{h^2}{(\int J^u \, d\mu^u_\beta)^3} \left(\frac{\partial^2 G^u}{\partial y^2} \right) (\beta, -M^u(\beta)) < 0 \, , \tag{5.16}$$

from [M; Ma; R; SI]. We have then

$$M''(\beta) = (M^s)''(\beta) + (M^u)''(\beta) < 0 \, . \qquad \square$$

We shall study the correlation dimension M associated to a Gibbs measure μ. Like in [SII], we define the Hölder continuous functions ξ^s (resp. ξ^u) satisfying $P(\xi^s) = 0$ $(= P(\xi^u))$ in such a way that the associated family of Gibbs measures μ^s (resp. μ^u) on the stable (resp. unstable) manifold verify: there exist constants c and C such that

$$c \leqq \frac{d\mu}{d(\mu^s \times \mu^u)} \leqq C \, . \tag{5.17}$$

We decompose as in (5.1) along the stable and unstable manifolds, and following the same steps we prove

Theorem 5.4. *We have for any real* β,

$$\begin{aligned} M(\beta) &= \lim_{r \to 0} \frac{\operatorname{Log} \int \mu(B(x,r))^\beta \mu(dx)}{\operatorname{Log} r} \\ &= M^s(\beta) + M^u(\beta) \\ &= \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + (\beta+1) \int (\xi^s - P_\varphi) \, d\rho}{\int J^s \, d\rho} \right] \\ &\quad + \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho + (\beta+1) \int (\xi^u - P_\varphi) \, d\rho}{\int J^u \, d\rho} \right] . \qquad \square \end{aligned}$$

We have also

Proposition 5.5. *The function M is real analytic on $\mathbb{R}$ and strictly increasing moreover*

- *either M is linear: it is the case when J^s is homologous to $c\xi^s$, i.e $J^s = c\xi^s + K \circ g - K$, where $K \in C^\gamma(\Lambda)$, and J^u is homologous to $c'\xi^u$.*
- *or the function M is strictly concave.*

Proof of Proposition 5.2. Consider the Hölder continuous functions ($\in C^\gamma(\Lambda)$)

$$\varphi^s_\beta = (\beta+1)\xi^s - M^s(\beta)J^s \quad \text{and} \quad \varphi^u_\beta = (\beta+1)\xi^u - M^u(\beta)J^u$$

and the associated Gibbs measures μ^s_β and μ^u_β. We have then for any real β,

$$G^s(\beta, -M^s(\beta)) = G^u(\beta, -M^u(\beta)) = 0\,,$$

where the functions G^s and G^u are defined on $\mathbb{R}^2$ by

$$G^s(x,y) = P[(x+1)\xi^s + yJ^s] \tag{5.18}$$

and

$$G^u(x,y) = P[(x+1)\xi^u + yJ^u]\,. \tag{5.19}$$

We have then for any real β,

$$(M^s)'(\beta) = \frac{\int (\xi^s - P_\varphi)\, d\mu^s_\beta}{\int J^s\, d\mu^s_\beta} > 0 \tag{5.20}$$

and symmetrically

$$(M^u)'(\beta) = \frac{\int (\xi^u - P_\varphi)\, d\mu^u_\beta}{\int J^u\, d\mu^u_\beta} > 0\,, \tag{5.21}$$

which implies

$$\forall \beta \in \mathbb{R}, \quad M'(\beta) = (M^u)'(\beta) + (M^s)'(\beta) > 0\,.$$

When J^s is homologous to $c\xi^s$, the measures μ^s_β are constant when β varies, and then the function $(M^s)'$ is constant (the same property for J^u and $c'\xi^u$). Otherwise following (4.4) and [SII] we prove that M^s or M^u is strictly concave, and à fortiori $M(= M^s + M^u)$. □

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