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Integrable Structure of Conformal Field Theory, Quantum KdV Theory and Thermodynamic Bethe Ansatz

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Abstract: We construct the quantum versions of the monodromy matrices of KdV theory. The traces of these quantum monodromy matrices, which will be called as "**T**-operators," act in highest weight Virasoro modules. The **T**-operators depend on the spectral parameter λ and their expansion around $\lambda = \infty$ generates an infinite set of commuting Hamiltonians of the quantum KdV system. The **T**-operators can be viewed as the continuous field theory versions of the commuting transfer-matrices of integrable lattice theory. In particular, we show that for the values $c = 1 - 3\frac{(2n+1)^2}{2n+3}$, $n = 1, 2, 3\ldots$ of the Virasoro central charge the eigenvalues of the **T**-operators satisfy a closed system of functional equations sufficient for determining the spectrum. For the ground-state eigenvalue these functional equations are equivalent to those of the massless Thermodynamic Bethe Ansatz for the minimal conformal field theory $\mathcal{M}_{2,2n+3}$; in general they provide a way to generalize the technique of the Thermodynamic Bethe Ansatz to the excited states. We discuss a generalization of our approach to the cases of massive field theories obtained by perturbing these Conformal Field Theories with the operator $\Phi_{1,3}$. The relation of these **T**-operators to the boundary states is also briefly described.

The studies of the last decade revealed a deep relation between the structures of Conformal Field Theory (CFT) [1], Integrable Field Theory [2, 3] and Solvable Lattice Models [4]. The conformal symmetry of CFT is generated by its energy-momentum tensor $T(u)$, whose mode expansion

$$T(u) = -\frac{c}{24} + \sum_{-\infty}^{+\infty} L_{-n}\, e^{inu} \tag{1}$$

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is expressed in terms of the operators L_n satisfying the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{2}$$

of Virasoro algebra Vir [1]. Here c is the central charge which is the most important characteristic of CFT; in writing (1) we have chosen the periodic boundary conditions $T(u+2\pi) = T(u)$. The "chiral" space of states $\mathscr{H}_{\text{chiral}}$ is built up from a collection of suitably chosen irreducible highest weight modules[1],

$$\mathscr{H}_{\text{chiral}} = \bigoplus_a \mathscr{V}_a , \tag{3}$$

where $\mathscr{V}_a \equiv \mathscr{V}_{\Delta_a}$, the parameters Δ are the highest weights, and associated highest weight vectors $|\Delta\rangle \in \mathscr{V}_a$ satisfy the equations

$$L_n|\Delta\rangle = 0 \quad \text{for } n > 0; \qquad L_0|\Delta\rangle = \Delta|\Delta\rangle . \tag{4}$$

Along with this "conventional" characterization of CFT a quite different description of CFT in terms of "massless S-matrix" has been recently proposed [5, 6, 7]. In this approach the chiral states of CFT are described as the scattering states of a collection of "massless particles" and the CFT is characterized[2] by their factorizable S-matrix, rather than in terms of the Virasoro algebra and its highest weight representations. So far the relation between these two descriptions is understood to a very limited extent. The "massless S-matrix" allows one to compute, through the Thermodynamic Bethe Ansatz technique [9, 10], the asymptotic density of states and thus gives the central charge c (and sometimes few of the highest weights Δ [11]). In the general case even the correspondence between the "massless particle states" and vectors in (3) is not known. At the same time gaining more understanding about this relation is very important as the "massless S-matrix" description seems to capture much of the "integrable structure" of CFT, i.e. it is close to its description in terms of action-angle variables.

From the algebraic point of view these "massless particle states" are nothing else but the eigenstates of an infinite set of commuting Integrals of Motion (IM). It has been known for some while that if we consider the algebra *UVir* generated by the energy-momentum tensor (1) along with various "composite fields" built as a powers of $T(u)$ and its derivatives, this algebra contains an infinite-dimensional abelian subalgebra [12, 13, 45] spanned by the "local IM" $I_{2k-1} \in UVir$, $k = 1, 2, \ldots$ which have the form

$$I_{2k-1} = \int_0^{2\pi} \frac{du}{2\pi} T_{2k}(u) , \tag{5}$$

where the densities $T_{2k}(u)$ are appropriately regularized polynomials in $T(u)$ and its derivatives. The first few densities $T_{2k}(u)$ can be written as

$$T_2(u) = T(u), \quad T_4(u) =: T^2(u) : , \quad T_6(u) =: T^3(u) : + \frac{c+2}{12} : (T'(u))^2 : , \ \ldots . \tag{6}$$

[1] Of course the physical space of states $\mathscr{H}_{\text{phys}}$ is embedded (by diagonal or some other suitable embedding) into a tensor product $\mathscr{H}_{\text{chiral}} \otimes \bar{\mathscr{H}}_{\text{chiral}}$, see e.g. [1].

[2] Unlike the case of the massive field theory the "factorizable S-matrix" description of CFT in general is not unique; there can exist several different choices [8] of the "massless particle" basis in $\mathscr{H}_{\text{chiral}}$, with different particle contents and scattering amplitudes, which correspond to massless limits of different integrable perturbations of this CFT.

Here the prime stands for the derivative and : : denotes appropriately regularized products of the fields, for example

$$: T^2(u) := \oint_{\mathscr{C}} \frac{dw}{2\pi i} \frac{1}{w-u} \mathscr{T}(T(w)T(u)) \,, \tag{7}$$

where the symbol $\mathscr{T}$ denotes the "chronological ordering," i.e.

$$\mathscr{T}(A(w)B(u)) = \begin{cases} A(w)B(u), & \text{if } \Im m\, u > \Im m\, w \,; \\ B(u)A(w), & \text{if } \Im m\, w > \Im m\, u \,. \end{cases} \tag{8}$$

In writing (6) we disregarded all the terms which are total derivatives and do not contribute to (5). Although a general expression for all densities $T_{2k}(u)$ is not known they are uniquely determined (up to a normalization which we will fix later) by the requirement of the commutativity

$$[I_{2k-1}, I_{2l-1}] = 0 \tag{9}$$

and the "spin assignment"[3]

$$\oint_{\mathscr{C}} \frac{dw}{2\pi i} (w-u) \mathscr{T}(T(w)T_{2k}(u)) = 2k\, T_{2k}(u) \,. \tag{10}$$

More representatives (beyond (6)) of this infinite set of densities $T_{2k}(u)$ can be found in [12]. The integrals (5) define operators $I_{2k-1} : \mathscr{V}_\Delta \to \mathscr{V}_\Delta$ which can be expressed in terms of the Virasoro generators L_n, for example

$$\begin{aligned} I_1 &= L_0 - \frac{c}{24} \,, \\ I_3 &= 2 \sum_{n=1}^{+\infty} L_{-n} L_n + L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} \,, \\ I_5 &= \sum_{n_1+n_2+n_3=0} : L_{n_1} L_{n_2} L_{n_3} : + \sum_{n=1}^{+\infty} \left(\frac{c+11}{6} n^2 - 1 - \frac{c}{4} \right) L_{-n} L_n \\ &\quad + \frac{3}{2} \sum_{r=1}^{+\infty} L_{1-2r} L_{2r-1} - \frac{c+4}{8} L_0^2 + \frac{(c+2)(3c+20)}{576} L_0 \\ &\quad - \frac{c(3c+14)(7c+68)}{290304} , \ldots \,. \end{aligned} \tag{11}$$

The "normal ordering" : : in these formulas means that the operators L_n with the bigger n are placed to the right. Note that although these operators are not polynomial in L_n their actions in $\mathscr{V}_\Delta$ are well defined.

In this work we study the problem of simultaneous diagonalization of the operators I_{2k-1} in $\mathscr{V}_\Delta$ by using the approach which can be regarded as a version of the Quantum Inverse Scattering Method [2, 3]. It was remarked many times [12, 13, 14, 45] that this problem can be thought of as the quantum version of the

[3] Another way to formulate this requirement is to assign the grade 2 to the field $T(u)$ and the grade 1 to the derivative; then the density $T_{2k}(u)$ is a homogeneous polynomial in $T(u)$ and its derivatives of the total grade $2k$.

KdV problem as it reduces to the classical KdV problem (with periodic boundary conditions) in its "classical limit" $c \to -\infty$. Indeed, in this limit the substitution

$$T(u) \to -\frac{c}{6} U(u), \qquad [\,,\,] \to \frac{6\pi}{ic}\{\,,\,\}$$

($U(u+2\pi) = U(u)$) reduces the algebra (1), (2) to the Poisson bracket algebra

$$\{U(u), U(v)\} = 2(U(u) + U(v))\delta'(u-v) + \delta'''(u-v)\,, \tag{12}$$

which is well known to describe the second Hamiltonian structure of the KdV equation provided we take one of the infinite set of classical IM $I^{(cl)}_{2k-1}$,

$$I^{(cl)}_1 = \int_0^{2\pi} \frac{du}{2\pi} U(u)\,, \qquad I^{(cl)}_3 = \int_0^{2\pi} \frac{du}{2\pi} U^2(u)\,,$$

$$I^{(cl)}_5 = \int_0^{2\pi} \frac{du}{2\pi} \left[U^3(u) - \frac{(U'(u))^2}{2} \right] \;\ldots\,, \tag{13}$$

as the Hamiltonian. The classical IM (13), which form a commutative Poisson bracket algebra $\{I^{(cl)}_{2k-1}, I^{(cl)}_{2l-1}\} = 0$, evidently are the classical versions of the operators (5), (11).

It is also well known [15] that the KdV equations

$$\partial_{t_{2k-1}} U(t_1, t_3, \ldots) = \{I_{2k-1}, U(t_1, t_3, \ldots)\}, \quad t_1 \equiv u \tag{14}$$

describe isospectral deformations of the second order differential operator

$$L = \partial_u^2 + U(u) - \lambda^2\,. \tag{15}$$

In particular, if we define the 2×2 Monodromy Matrix $\mathbf{M}(\lambda)$, which belongs to the group $SL(2)$, as

$$(\psi_1(u+2\pi), \psi_2(u+2\pi)) = (\psi_1(u), \psi_2(u))\,\mathbf{M}(\lambda)\,, \tag{16}$$

where $\psi_1(u), \psi_2(u)$ are two linearly independent solutions to the equation $L\psi = 0$, then the eigenvalues of $\mathbf{M}(\lambda)$ are involutive (with respect to the Poisson structure (12)) Integrals of Motion of the KdV flows, and the trace

$$\mathbf{T}(\lambda) = \mathrm{tr}\,\mathbf{M}(\lambda) \tag{17}$$

can be thought of as the generating function for the local IM (13) as it expands in the asymptotic series

$$\frac{1}{2\pi} \log[\mathbf{T}(\lambda)] \simeq \lambda - \sum_{n=1}^{\infty} c_n I^{(cl)}_{2n-1} \lambda^{1-2n}\,, \tag{18}$$

here $c_1 = \frac{1}{2}$, $c_n = \frac{(2n-3)!!}{2^n n!}$, $n > 1$.

It is more convenient for our purposes to start with the first order differential operator

$$\mathscr{L} = \partial_u - \phi'(u) H - \lambda(E + F)\,, \tag{19}$$

where $\phi(u)$ are the canonical variables with the Poisson brackets

$$\{\phi(u), \phi(v)\} = \frac{1}{2}\varepsilon(u-v)\,;$$

$$\varepsilon(u) = 2n+1 \quad \text{for } 2\pi n < u < 2\pi(n+1)\,; \quad n \in \mathbf{Z}\,, \tag{20}$$

which are related to $U(u)$ by the Miura transform [16]

$$U(u) = -\phi'(u)^2 - \phi''(u)\,, \tag{21}$$

and E, F, H are the generators of the Lie algebra $sl(2)$,

$$[H,E] = 2E, \qquad [H,F] = -2F, \qquad [E,F] = 2H\,. \tag{22}$$

In general, the classical field $\phi(u)$ has to be taken quasiperiodic,

$$\phi(u+2\pi) = \phi(u) + 2\pi i p\,, \tag{23}$$

to guarantee the periodicity of $U(u)$. In order to define the monodromy matrix for the operator (19) one has to pick some matrix representation for the $sl(2)$ algebra (22). Let $\pi_j[E], \pi_j[F], \pi_j[H]$ be $(2j+1)\times(2j+1)$ matrices, $j = 0, 1/2, 1, 3/2, \ldots$, representing (22), such that $\pi_j[H] = \mathrm{diag}(2j, 2j-1, \ldots, -2j)$. Then, for given j, the solution to the equation $\mathscr{L}\Psi(u) = 0$ is

$$\Psi(u) = \pi_j\left[e^{\phi(u)H}\mathscr{P}\exp\left(\lambda\int_0^u dv\,(e^{-2\phi(v)}E + e^{2\phi(v)}F)\right)\right]\Psi_0\,, \tag{24}$$

where the symbol $\mathscr{P}$ denotes the "path ordered" exponential and Ψ_0 is an arbitrary vector in $\mathbf{C}^{2j+1}$. The associated monodromy matrices have the form

$$\mathbf{M}_j(\lambda) = \pi_j\left[e^{2\pi i p H}\mathscr{P}\exp\left(\lambda\int_0^{2\pi} dv\,(e^{-2\phi(v)}E + e^{2\phi(v)}F)\right)\right]\,. \tag{25}$$

Let us introduce auxiliary matrices

$$\mathbf{L}_j(\lambda) = \pi_j[e^{-\pi i p H}]\,\mathbf{M}_j(\lambda)\,. \tag{26}$$

They satisfy the "r-matrix" Poisson bracket algebra [17]

$$\{\mathbf{L}_j(\lambda) \underset{,}{\otimes} \mathbf{L}_{j'}(\mu)\} = [\mathbf{r}_{jj'}(\lambda\mu^{-1}), \mathbf{L}_j(\lambda) \otimes \mathbf{L}_{j'}(\mu)]\,, \tag{27}$$

where $\mathbf{r}_{jj'}(\lambda) = \pi_j \otimes \pi_{j'}[\mathbf{r}]$ is the "classical r-matrix"

$$\mathbf{r}(\lambda) = \frac{\lambda+\lambda^{-1}}{\lambda-\lambda^{-1}}\frac{H\otimes H}{2} + \frac{2}{\lambda-\lambda^{-1}}(E\otimes F + F\otimes E)\,. \tag{28}$$

It follows immediately that the quantities

$$\mathbf{T}_j(\lambda) = \mathrm{tr}\,\mathbf{M}_j(\lambda) \tag{29}$$

are in involution with respect to the Poisson bracket (12), (20):

$$\{\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\mu)\} = 0\,. \tag{30}$$

In particular, $\mathbf{T}_{\frac{1}{2}}(\lambda)$ coincides with (17).

After this brief review of known classical results we turn to the quantum case. The quantum version of the Miura transform (21) is the Feigin–Fuchs "free field representation" of Vir [18, 19]

$$-\beta^2\, T(u) =: \varphi'(u)^2 : +(1-\beta^2)\varphi''(u) + \frac{\beta^2}{24}\,, \tag{31}$$

in terms of the free field operator

$$\varphi(u) = iQ + iPu + \sum_{n\neq 0} \frac{a_{-n}}{n} e^{inu}\,, \tag{32}$$

where the mode operators Q, P and a_n satisfy the Heisenberg algebra

$$[Q,P] = \frac{i}{2}\beta^2; \qquad [a_n, a_m] = \frac{n}{2}\beta^2\, \delta_{n+m,0} \tag{33}$$

(P and Q commute with a_n) and the parameter β is related to the central charge c as

$$\beta = \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}}\,. \tag{34}$$

Let $\mathscr{F}_p$ ("Fock space") be the highest weight module over the Heisenberg algebra (33) with the highest weight vector $|p\rangle$ ("vacuum") defined by the equations

$$P|p\rangle = p|p\rangle; \qquad a_n|p\rangle = 0 \quad \text{for } n > 0\,. \tag{35}$$

Equation (31) defines the action of Vir in $\mathscr{F}_p$. For generic c and p the space $\mathscr{F}_p$ is isomorphic to the highest weight Virasoro module $\mathscr{V}_\Delta$ with [18]

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24}\,. \tag{36}$$

The space $\mathscr{F}_p$ naturally splits into the sum of finite-dimensional "level subspaces"

$$\mathscr{F}_p = \bigoplus_{l=0}^{\infty} \mathscr{F}_p^{(l)}; \qquad L_0\, \mathscr{F}_p^{(l)} = (\Delta + l)\, \mathscr{F}_p^{(l)}\,. \tag{37}$$

The "normal ordering" suitable for this representation is implied in (31), $: \; :$ means that the operators a_n with the bigger n are placed to the right. Of course the IM (5), (11) can be expressed in terms of the Feigin–Fuchs free field $\varphi'(u)$. We adopt the following normalization of the operators I_{2k-1}:

$$I_{2k-1} = (-1)^k \beta^{-2k} \int_0^{2\pi} \frac{du}{2\pi} \left(: (\varphi'(u))^{2k} : + \cdots \right), \tag{38}$$

where the omitted terms contain higher derivatives of $\varphi(u)$. These operators act invariantly in the level subspaces $\mathscr{F}_p^{(l)}$. Therefore diagonalization of I_{2k-1} in a given level subspace reduces to a finite algebraic problem which however rapidly becomes very complex for higher levels. Here we list only the vacuum eigenvalues for the first few I_{2k-1},

$$I_{2k-1}|p\rangle = I_{2k-1}^{(\mathrm{vac})}(\Delta)|p\rangle\,, \tag{39}$$

where

$$
\begin{aligned}
I_1^{\mathrm{vac}}(\Delta) &= \Delta - \frac{c}{24}\,,\\
I_3^{\mathrm{vac}}(\Delta) &= \Delta^2 - \frac{c+2}{12}\Delta + \frac{c(5c+22)}{2880}\,,\\
I_5^{\mathrm{vac}}(\Delta) &= \Delta^3 - \frac{c+4}{8}\Delta^2 + \frac{(c+2)(3c+20)}{576}\Delta - \frac{c(3c+14)(7c+68)}{290304}\,,\\
I_7^{\mathrm{vac}}(\Delta) &= \Delta^4 - \frac{c+6}{6}\Delta^3 + \frac{15c^2+194c+568}{1440}\Delta^2 - \frac{(c+2)(c+10)(3c+28)}{10368}\Delta\\
&\quad + \frac{c(3c+46)(25c^2+426c+1400)}{24883200}\,. \qquad (40)
\end{aligned}
$$

After these preparations we can define a quantum version of the Monodromy Matrices (25) and the operators (26), (29); we are going to use the same symbols $\mathbf{L}_j, \mathbf{T}_j$ for the quantum counterparts of (26), (29). Consider the following operator valued matrices [20];

$$
\mathbf{L}_j(\lambda) = \pi_j\left[e^{i\pi PH}\mathscr{P}\exp\left(\lambda\int_0^{2\pi} du(:e^{-2\varphi(u)}:q^{\frac{H}{2}}E+ :e^{2\varphi(u)}:q^{-\frac{H}{2}}F)\right)\right]\,, \qquad (41)
$$

where the vertex operators

$$
:e^{\pm 2\varphi(u)}: \quad :\mathscr{F}_p \to \mathscr{F}_{p\pm\beta^2} \qquad (42)
$$

are defined as

$$
:e^{\pm 2\varphi(u)}:\equiv \exp\left(\pm 2\sum_{n=1}^{\infty}\frac{a_{-n}}{n}e^{inu}\right)\exp(\pm 2i(Q+Pu))\exp\left(\mp 2\sum_{n=1}^{\infty}\frac{a_n}{n}e^{-inu}\right), \qquad (43)
$$

and E, F and H are the generating elements of the quantum universal enveloping algebra $U_q(sl(2))$ [21]

$$
[H,E]=2E, \qquad [H,F]=-2F, \qquad [E,F]=\frac{q^H-q^{-H}}{q-q^{-1}} \qquad (44)
$$

with

$$
q = e^{i\pi\beta^2}\,. \qquad (45)
$$

The symbol π_j in (41) stands again for the $(2j+1)$ dimensional representation of $U_q(sl(2))$ so that (41) is in fact a $(2j+1)\times(2j+1)$ matrix whose elements are the operators in

$$
\hat{\mathscr{F}}_p = \bigoplus_{n=-\infty}^{+\infty}\mathscr{F}_{p+n\beta^2}\,. \qquad (46)
$$

These operators are understood as the power series in λ,

$$
\mathbf{L}_j(\lambda) = \pi_j\left[e^{i\pi PH}\sum_{k=0}^{\infty}\lambda^k\int_{2\pi\geqq u_1\geqq u_2\geqq\cdots\geqq u_k\geqq 0}K(u_1)K(u_2)\cdots K(u_k)\,du_1du_2\cdots du_k\right];
$$

$$
K(u) =: e^{-2\varphi(u)}:q^{\frac{H}{2}}E+ :e^{2\varphi(u)}:q^{-\frac{H}{2}}F\,. \qquad (47)
$$

The integrals in (47) are convergent for

$$-\infty < c < -2\,. \tag{48}$$

In fact, the operators (41) can be defined for a wider range of c by appropriate regularization of divergent integrals in (47). In this paper we restrict our attention to the domain (48).

The operator matrices (41) are designed in such a way that they satisfy the Quantum Yang–Baxter Equation

$$\mathbf{R}_{jj'}(\lambda\mu^{-1})(\mathbf{L}_j(\lambda)\otimes 1)(1\otimes \mathbf{L}_{j'}(\mu)) = (1\otimes \mathbf{L}_{j'}(\mu))(\mathbf{L}_j(\lambda)\otimes 1)\mathbf{R}_{jj'}(\lambda\mu^{-1})\,, \tag{49}$$

where the matrix $\mathbf{R}_{jj'}(\lambda)$ is a trigonometric solution of the Yang–Baxter equation which acts in the space $\pi_j\otimes\pi_{j'}$. In particular

$$\mathbf{R}_{\frac{1}{2}\frac{1}{2}}(\lambda) = \begin{pmatrix} q^{-1}\lambda - q\lambda^{-1} & & & \\ & \lambda-\lambda^{-1} & q^{-1}-q & \\ & q^{-1}-q & \lambda-\lambda^{-1} & \\ & & & q^{-1}\lambda - q\lambda^{-1} \end{pmatrix}\,. \tag{50}$$

The validity of (49) can be checked explicitly for the first few orders of the expansion of (49) in λ and μ with the use of (47). One can prove (49) to all orders by taking the discrete approximations to the $\mathscr{P}$-ordered integral in (41) [20].

Let us define now the operators

$$\mathbf{T}_j(\lambda) = \mathrm{tr}_{\pi_j}(e^{i\pi PH}\mathbf{L}_j(\lambda))\,, \tag{51}$$

which satisfy

$$[\mathbf{T}_j(\lambda), \mathbf{T}_{j'}(\mu)] = 0 \tag{52}$$

as a simple consequence of (49). It is easy to see that the operators (51) commute with the operator P and hence they act invariantly in $\mathscr{F}_p$. Moreover, one can check by direct calculations that

$$[\mathbf{T}_j(\lambda), I_{2k-1}] = 0\,. \tag{53}$$

It follows, in particular, that the level subspaces $\mathscr{F}_p^{(l)}$ are the eigenspaces of $\mathbf{T}_j(\lambda)$.

Let us concentrate first on the simplest nontrivial $\mathbf{T}$-operator $\mathbf{T}(\lambda) = \mathbf{T}_{\frac{1}{2}}(\lambda)$ which corresponds to the two-dimensional representation of $U_q(sl(2))$. In this case

$$\pi_{\frac{1}{2}}(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \pi_{\frac{1}{2}}(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \pi_{\frac{1}{2}}(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{54}$$

Substituting (47) into (51) and computing the trace one finds

$$\mathbf{T}(\lambda) = 2\cos(2\pi P) + \sum_{n=1}^{\infty} \lambda^{2n} Q_n\,, \tag{55}$$

where

$$Q_n = q^n \int_{2\pi\geqq u_1\geqq u_2\geqq\cdots\geqq u_{2n}\geqq 0} \big(e^{2i\pi P} :e^{-2\varphi(u_1)}::e^{2\varphi(u_2)}::e^{-2\varphi(u_3)}:\ldots:e^{2\varphi(u_{2n})}: \\ + e^{-2i\pi P} :e^{2\varphi(u_1)}::e^{-2\varphi(u_2)}::e^{2\varphi(u_3)}:\ldots:e^{-2\varphi(u_{2n})}:\big)\,du_1\cdots du_{2n}\,. \tag{56}$$

So, the operator $\mathbf{T}(\lambda)$ can be considered as the generating function for the "nonlocal IM" Q_n which commute among themselves,

$$[Q_n, Q_m] = 0\,, \tag{57}$$

and also commute with all the "local IM" I_{2k-1},

$$[I_{2k-1}, Q_n] = 0\,. \tag{58}$$

The operators Q_n invariantly act on each of the level subspaces $\mathscr{F}_p^{(l)}$; in particular, the vacuum state $|p\rangle$ is the eigenstate of all Q_n,

$$Q_n|p\rangle = Q_n^{(\mathrm{vac})}(p)\,|p\rangle\,, \tag{59}$$

where the eigenvalues $Q_n^{(\mathrm{vac})}(p)$ are given by the integrals

$$\begin{aligned} Q_n^{(\mathrm{vac})}(p) = \int_0^{2\pi} du_1 \int_0^{u_1} dv_1 \int_0^{v_1} du_2 \int_0^{u_2} dv_2 \cdots \int_0^{v_{n-1}} du_n \int_0^{u_n} dv_n \\ \times \prod_{j>i}^{n} \left[\left(2\sin\left(\frac{u_i-u_j}{2}\right)\right)^{2\beta^2} \left(2\sin\left(\frac{v_i-v_j}{2}\right)\right)^{2\beta^2} \right] \\ \times \prod_{j\geqq i}^{n} \left(2\sin\left(\frac{u_i-v_j}{2}\right)\right)^{-2\beta^2} \prod_{j>i}^{n} \left(2\sin\left(\frac{v_i-u_j}{2}\right)\right)^{-2\beta^2} \\ \times 2\cos\left(2p\left(\pi + \sum_{i=1}^{n}(v_i-u_j)\right)\right)\,. \end{aligned} \tag{60}$$

In particular

$$Q_1^{(\mathrm{vac})}(p) = \frac{4\pi^2\,\Gamma(1-2\beta^2)}{\Gamma(1-\beta^2-2p)\Gamma(1-\beta^2+2p)}\,. \tag{61}$$

Using the power series expansion (55) one can show that the operator $\mathbf{T}(\lambda)$ is an entire function of λ^2 (just as it was in the classical case) in the sense that all its matrix elements and its eigenvalues $t(\lambda)$ are entire functions of this variable. In fact, the eigenvalues $t(\lambda)$ exhibit essential singularity at the infinity as the result of accumulation of zeroes of these functions along the real axis in λ^2-plane as $\lambda^2 \to -\infty$[4]. The asymptotic form of the operator $\mathbf{T}(\lambda)$ at $\lambda^2 \to \infty$ is of primary interest because, in view of (18), it is in this limit one anticipates to get in touch with the "local IM" (5), (38). A rough estimate of the matrix elements of the operators (56) along the lines proposed in [22] as well as physical arguments based on the thermodynamic treatment of the "Coulomb gas partition function" (60) gives the leading asymptotic

$$\log[\mathbf{T}(\lambda)] \sim m\,\lambda^{1+\xi}\,, \tag{62}$$

where

$$\xi = \frac{\beta^2}{1-\beta^2}\,. \tag{63}$$

[4] Validity of this picture is restricted to the domain (48).

The constant m is expected to depend on c but not on the particular matrix element of this operator; this asymptotic form holds in the domain

$$-\pi+\varepsilon < \arg(\lambda^2) < \pi-\varepsilon, \quad \lambda\to\infty \tag{64}$$

with arbitrary small positive ε. So far we could not find any direct way to compute the constant m. Therefore we used the Algebraic Bethe Ansatz [2] to study the discrete approximations to the integral in (41); in taking the continuous limit the integral form of the Bethe Ansatz equations proposed in [23] is particularly useful. This way we obtain

$$\mathbf{T}(\lambda) = \Lambda(q^{1/2}\lambda) + \Lambda^{-1}(q^{-1/2}\lambda)\,, \tag{65}$$

where $\Lambda(\lambda)$ expands in the domain (64) into the asymptotic series

$$\log\Lambda(q^{1/2}\lambda) \simeq m\,\lambda^{1+\xi} - \sum_{n=1}^{\infty} C_n\, I_{2n-1}\, \lambda^{(1-2n)(1+\xi)}\,, \tag{66}$$

where

$$m = \frac{2\sqrt{\pi}\,\Gamma(\frac{1}{2}-\frac{\xi}{2})}{\Gamma(1-\frac{\xi}{2})}\left(\Gamma\left(\frac{1}{1+\xi}\right)\right)^{1+\xi},$$

$$C_n = \frac{1+\xi}{n!}\left(\frac{\pi\xi}{1+\xi}\right)^n \left(\frac{2\,\Gamma(\frac{1}{2}-\frac{\xi}{2})}{m\,\Gamma(1-\frac{\xi}{2})}\right)^{2n-1} \frac{\Gamma((n-\frac{1}{2})(1+\xi))}{\Gamma(1+(n-\frac{1}{2})\xi)}\,, \tag{67}$$

and I_{2k-1} are exactly the "local IM" (5), (38). The coefficients C_n here are closely related to the canonical on-shell normalization of the local IM of the quantum Sine–Gordon theory [24]. Note that the coefficients c_n in (18) can be recovered in the classical limit $\xi\to 0$ ($c\to-\infty$). Appearance of the fractional powers of λ in (66) should not be very surprising – the exponentials in (41) have anomalous dimensions and so the spectral parameter λ is to be thought of as carrying the dimension $[length]^{-\frac{1}{1+\xi}}$. The expansions (55) and (66) give a highly nontrivial analytic relation between the "nonlocal IM" (56) and "local IM" (5), (38).

The higher spin operators $\mathbf{T}_j(\lambda)$ also admit the power series expansion similar to (55), the coefficients being of course algebraically dependent from the operators Q_n in (55). For the first few coefficients we find

$$\begin{aligned}\mathbf{T}_j(\lambda) = {} & \frac{\sin(2\pi P(2j+1))}{\sin(2\pi P)} + \lambda^2 A_j(2\pi P, \pi\beta^2)\, Q_1 \\ & + \lambda^4[A_j(2\pi P, 2\pi\beta^2)\, Q_2 + B_j(2\pi P, \pi\beta^2)\, Q_1^2] + O(\lambda^6)\,,\end{aligned} \tag{68}$$

where numerical coefficients read explicitly

$$A_j(x,a) = \frac{1}{4\sin x\sin a}\left(\frac{\sin(2j+1)(x-a)}{\sin(x-a)} - \frac{\sin(2j+1)(x+a)}{\sin(x+a)}\right),$$

$$\begin{aligned}B_j(x,a) = {} & \frac{1}{16\sin x\sin a\sin 2a}\left(\frac{\sin(2j+1)(x-2a)}{\sin(x-a)\sin(x-2a)} + \frac{\sin(2j+1)(x+2a)}{\sin(x+a)\sin(x+2a)}\right. \\ & \left. - \frac{2\sin(2j+1)x\cos a}{\sin(x-a)\sin(x+a)}\right).\end{aligned}$$

The operators Q_1 and Q_2 in are the same as in (56). This algebraic dependence can be summarized by the functional relations

$$\mathbf{T}_j(q^{\frac{1}{2}}\lambda)\mathbf{T}_j(q^{-\frac{1}{2}}\lambda) = 1 + \mathbf{T}_{j-\frac{1}{2}}(\lambda)\mathbf{T}_{j+\frac{1}{2}}(\lambda)\,. \tag{69}$$

Note that these relations are identical to the functional relation obeyed by the commuting transfer-matrices of the integrable *XXZ* model [25, 26, 27]. This is well expected as our $\mathbf{T}$-operators appear to be the continuous field theory versions of the lattice transfer-matrices. The relations (69) can be derived from (49) essentially the same way they are obtained in the lattice theory, by using the R-matrix fusion procedure [21]. The simple form of (69) is due to the fact that the quantum determinant [3] of the operator $\mathbf{L}_{\frac{1}{2}}(\lambda)$ (41) is equal to 1. Using the relation (69) one can show that all the operators $\mathbf{T}_j(\lambda)$ with $j \geqq 1$ are also entire functions of λ^2 as well as $\mathbf{T}(\lambda)$.

At generic values of c the relations (69) just allow one to express the higher $\mathbf{T}$ operators in terms of the lower ones. However it is well known in the lattice theory that at particular values of the parameters, when the *XXZ* system can be reduced to the *RSOS* model [28], the functional relations truncate to become a finite system of functional equations [29, 26]. Of course, a similar phenomenon happens in our continuous theory. Restricting our attention to the domain (48) we find that the most simple truncation occurs at

$$c = 1 - 3\frac{(2n+1)^2}{2n+3}, \qquad \xi = \frac{2}{2n+1}; \quad n = 1, 2, 3, \ldots\,. \tag{70}$$

For given n in (70) consider the finite collection of the Fock spaces,

$$\mathscr{F}_{p_k}\ ;\ p_k = \frac{2k-2n-3}{2(2n+3)}, \quad k = 1, \ldots, n+1\,. \tag{71}$$

At these values of p the Fock spaces (71) are known to be reducible with respect to the action of the Virasoro algebra (they correspond to the $(1, k)$ degenerate representations in the Kac classification [30, 18]). Let us denote $\mathscr{V}_{p_k}$ the associated irreducible Virasoro module obtained from (71) by factoring out all the submodules. Then the space

$$\mathscr{H}_{\text{chiral}}(\mathscr{M}_{2,2n+3}) = \bigoplus_{k=1}^{n+1} \mathscr{V}_{p_k} \tag{72}$$

coincides with the space of chiral states of the "minimal CFT" $\mathscr{M}_{2,2n+3}$. It is possible to show that the operators $\mathbf{T}_j(\lambda)$ with $j = 0, \frac{1}{2}, 1, \ldots, n+\frac{1}{2}$ invariantly act in the space (72), and being restricted to this space, these operators satisfy the symmetry relation

$$\mathbf{T}_{n+\frac{1}{2}-j}(\lambda) = \mathbf{T}_j(\lambda); \quad j = 0, \frac{1}{2}, \ldots, n+\frac{1}{2}\,, \tag{73}$$

in particular $\mathbf{T}_{n+\frac{1}{2}}(\lambda) = \mathbf{I}$. Under these circumstances the relations (69) become a finite system of functional equations

$$\begin{aligned} &t_j(q^{\frac{1}{2}}\lambda)\,t_j(q^{-\frac{1}{2}}\lambda) = 1 + t_{j+\frac{1}{2}}(\lambda)\,t_{j-\frac{1}{2}}(\lambda)\,; \\ &t_0(\lambda) = t_{n+\frac{1}{2}}(\lambda) = 1; \qquad t_{n+\frac{1}{2}-j}(\lambda) = t_j(\lambda)\,, \end{aligned} \tag{74}$$

which is satisfied by all eigenvalues $t_j(\lambda)$ of the operators $\mathbf{T}_j(\lambda)$ in the space (72). We conjecture that any solution $t_j(\lambda)$ to the functional equations (74) which is an entire function of λ^2 and has the asymptotic behavior (62) in the domain (64) corresponds to some eigenstate of $\mathbf{T}_j(\lambda)$ in the space (72).

It was recognized before [27] that the substitution

$$Y_j(\theta) = t_{j+\frac{1}{2}}(\lambda)\, t_{j-\frac{1}{2}}(\lambda); \quad \lambda = \exp\left(\frac{\theta}{1+\xi}\right) \tag{75}$$

brings the system (74) to the form

$$\begin{aligned} &Y_j\left(\theta + \frac{i\pi\xi}{2}\right) Y_j\left(\theta - \frac{i\pi\xi}{2}\right) = (1 + Y_{j+\frac{1}{2}}(\theta))(1 + Y_{j-\frac{1}{2}}(\theta))\,; \\ &Y_0(\theta) = Y_{n+\frac{1}{2}}(\theta) = 0; \qquad Y_{n+\frac{1}{2}-j}(\theta) = Y_j(\theta)\,, \end{aligned} \tag{76}$$

which coincides with the functional form of the Thermodynamic Bethe Ansatz (TBA) equations [31] for the "massless S-matrix" theory associated with the minimal CFT $\mathcal{M}_{2,2k+1}$ [32]. In this "massless TBA" approach one assumes that the states of the CFT $\mathcal{M}_{2,2k+1}$ *in infinite volume* can be interpreted as the scattering states of a collection of n mass-less right-moving particles A_j; $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{n}{2}$, with the energy-momentum spectrum $E = P$. Parameterizing the energy-momentum for the particle A_j in terms of rapidity θ as

$$E_j(\theta) = P_j(\theta) = \frac{m_j}{2} e^{\theta}, \qquad m_j = \frac{2m}{\pi} \cot\left(\frac{\pi\xi}{2}\right) \sin(\pi\xi j)\,, \tag{77}$$

one conjectures the purely elastic and factorizable S-matrix for these particles with the two-particle elements $S_{jj'}(\theta - \theta')$ (describing the scattering processes $A_j(\theta) + A_{j'}(\theta') \to A_j(\theta) + A_{j'}(\theta')$, in obvious notations) given by [33, 34],

$$S_{jj'}(\theta) = F_{j+j'}(\theta) F_{|j-j'|}(\theta) \prod_{k=1}^{2\min(j,j')-1} F^2_{|j-j'|+k}(\theta)\,, \tag{78}$$

where the notation

$$F_x(\theta) = \frac{\sinh\theta + i\sin(\pi\xi x)}{\sinh\theta - i\sin(\pi\xi x)}$$

is used. The S-matrix (78) allows one to determine the spectral density of states in this infinite-volume system and hence to compute the free energy of this system at finite temperature R^{-1}. As the thermal ensemble corresponds to the circular compactification of "imaginary time" one can interchange the roles of the space and the "imaginary time" and reinterpret this free energy as the ground-state energy $e_0(R)$ of the finite-volume system with periodic boundary condition defined on the circle with the circumference R[5]. This way one obtains

$$e_0(R) = -\sum_j \frac{m_j}{2} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{\theta} \log(1 + e^{-\varepsilon_j(\theta)})\,, \tag{79}$$

[5] Of course R can be arbitrarily changed by the scale transformation which is the symmetry of CFT. Note that in our previous discussion of the $\mathbf{T}$-operators this parameter was set equal to 2π.

where the functions $\varepsilon_j(\theta)$ solve the integral equations of TBA:

$$\frac{Rm_j}{2}e^{\theta} = \varepsilon_j(\theta) + \sum_{j'} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi_{jj'}(\theta - \theta') \log(1 + e^{-\varepsilon_{j'}(\theta')}) \tag{80}$$

with the kernel

$$\varphi_{jj'}(\theta) = -i\,\partial_\theta \ln S_{jj'}(\theta)\,.$$

By inverting the integral operator in the right-hand side of (80) one can show, following [31], that the functions

$$Y_j^{\text{gr.st.}}(\theta) = e^{\varepsilon_j(\theta)} \tag{81}$$

satisfy the functional equations (76) and agree with the asymptotic condition (62) provided the parameter m in (77) is the same as in (62). The ground state (the state of the lowest eigenvalue of L_0) of the CFT $\mathcal{M}_{2,2n+3}$ is the vacuum state $|p_{n+1}\rangle$ from (72). We have checked (for the case of $\mathcal{M}_{2,5}$) that for this value of p the numerical solution of the integral equations (80) (which is named the "kink solution" in [10]) matches perfectly both the expansions (66) and (68) [35]. We see that the functions $\varepsilon_j(\theta)$ of TBA admit an interpretation in terms of the eigenvalues of the $\mathbf{T}$-operators in the "cross-channel."

Let us stress that while the solution to the integral equation (80) is unique, the functional equations (76) admit infinitely many solutions even in the class of entire functions with the asymptotic behavior (62). From the analytic point of view these solutions (which are in one-to-one correspondence with the eigenstates of the $\mathbf{T}$ operators in the space (72), by our conjecture) can be characterized by the patterns of zeroes of the functions $t_j(\lambda)$ in the complex λ^2 plane. All the solutions have infinitely many zeroes at the negative segment of the real axis in this plane (with $-\infty$ as their accumulation point) and some finite number of zeroes away from this locus. It is a characteristic feature of the ground state solution (81) that *all* zeroes of the functions $t_j^{\text{gr.st.}}(\lambda)$ are located at $\Im m\lambda^2 = 0$; $\Re e\lambda^2 < 0$; under this condition (80) follow from (76). The eigenvalue functions $t_j(\lambda)$ associated with the other eigenstates in (72) have more complex patterns of zeroes. For example, for the vacuum states $|p_k\rangle$; $k = 1, 2, \ldots, n$ the eigenvalue functions $t_j(\lambda)$ have a finite number of zeroes at $\Im m\lambda^2 = 0$; $\Re e\lambda^2 > 0$ and no complex zeroes, and for the excited states in (72) these functions exhibit also complex zeroes which can be interpreted as the rapidities of the massless particles in these states[6]. In all cases one can use the technique developed in [27] to convert the functional equations (76) into integral equations similar to (80) but with additional terms in the left-hand sides. We have analyzed numerically these "excited state TBA equations" for a few simplest states in the CFT $\mathcal{M}_{2,5}$ and found again a perfect agreement with the expansions (66) and (68) [35].

Strictly speaking, the above discussion concerned the case of CFT and the associated massless TBA. It is known however that the integrable structure of CFT remains essentially intact in the more general massive Quantum Field Theory obtained by perturbing this CFT with the relevant primary field $\Phi_{1,3}$ [37]. Namely, this perturbed field theory exhibits two infinite sets of commuting local Integrals of Motion, I_{2k-1} and $\bar{I}_{2k-1}$, $k = 1, 2, \ldots$, where I_{2k-1} are obtained by appropriate

[6] It would be extremely interesting to find out how the classification of states by the patterns of zeroes relates to the "fermionic sum representations" of the Virasoro characters [36].

deformations of the "right" conformal IM (5) and $\bar{I}_{2k-1}$ are the deformations of the same IM from the "left" sector of the CFT (see [37]). There are good reasons to believe that the above operators $\mathbf{T}_j(\lambda)$, with suitable deformations[7], can be extended to the case of the perturbed theory in such a way that they satisfy Eqs. (52) and (69). They also enjoy the same asymptotic expansion (66) in terms of (deformed) "right" IM I_{2k-1}. The major difference is that the deformed operators $\mathbf{T}_j(\lambda)$ are no longer entire functions of λ; instead they are expected to have an essential singularity at $\lambda = 0$ (being regular everywhere else in the finite part of the λ plane) which is controlled by the asymptotic expansion in terms of the "left" IM $\bar{I}_{2k-1}$, similar to (66), with λ replaced by μ/λ, where μ is related in a simple way to the coupling constant of the perturbed theory. In the case of the perturbed minimal CFT $\mathcal{M}_{2,2n+3}$ the functional relations (69) become again the closed set of functional equations (74), but now one has to look for the solutions which have an essential singularity at $\lambda = 0$ and enjoy the asymptotic behavior

$$\log \mathbf{T}_j(\lambda) \sim \begin{cases} m\,\lambda^{1+\xi}, & \text{if } \lambda \to \infty\,; \\ m\,(\frac{\mu}{\lambda})^{1+\xi}, & \text{if } \lambda \to 0\,. \end{cases} \tag{82}$$

Like in the case of CFT, it is very plausible that there is one to one correspondence between the solutions of (74) with these analytic characteristics and the stationary states of the perturbed CFT $\mathcal{M}_{2,2n+3}$ on a finite circle. We studied numerically [35] a few simplest solutions in the case of perturbed $\mathcal{M}_{2,5}$, again with the excellent agreement with the data available through the Truncated Conformal Space method [39].

We can not resist the temptation to mention here a remarkable relation between the operators (51) and CFT with non-conformal interactions at the boundary. According to the analysis in [40], in a minimal CFT (the one where the sum (3) contains a finite number of terms) with a boundary only finitely many *conformally invariant* boundary conditions (CBC) is possible. These CBC are in one to one correspondence with the terms in (3); we denote B_a the CBC associated with $\mathcal{V}_a$. The boundary state $|B_a\rangle \in \mathcal{H}_{\text{phys}}$ describing CBC B_a has the form

$$|B_a\rangle = \sum_b \frac{\mathbf{S}_{ab}}{\sqrt{\mathbf{S}_{0b}}} \sum_\mu |a_\mu\rangle \otimes |\bar{a}^\mu\rangle\,, \tag{83}$$

where $\mathbf{S}$ is the matrix of a modular transformation of characters, $\{|a_\mu\rangle\}$ is an arbitrary basis in $\mathcal{V}_a$ and $\{|\bar{a}^\mu\rangle\}$ is the dual basis in $\bar{\mathcal{V}}_a$. Furthermore, with a given conformal boundary condition B_a, the space of local boundary operators[8] forms the subspace $\mathcal{H}_{B_a}$ in $\mathcal{H}_{\text{chiral}}$ isomorphic to the direct sum $\bigoplus_b \mathcal{V}_b$, where only those b which have nonvanishing fusion constants N_{aa}^b are admitted. One can obtain more general non-conformal boundary conditions by perturbing the CBC B_a with relevant operators $\psi_b \in \mathcal{H}_{B_a}$. Let us denote $B_a(g)$ the CBC B_a perturbed with the operator

[7] In fact, it is not very difficult to figure out what these **T**-operators in the perturbed theory should look like, if one recalls the known relation between the integrable structures of KdV and Sine–Gordon theories [17, 12, 13, 14] and the relation between the quantum Sine–Gordon theory and the perturbed minimal CFT explained in [38, 34]. As we did not yet elaborate all the details we do not present these deformed **T**-operators here.

[8] We call here *local* the boundary operators which exist as the insertions at the points where two components of the boundary with the *same* boundary conditions meet, in contrast with the operators associated with the juxtapositions of *different* boundary conditions, see [40].

$\psi_{1,3}$, the primary field associated with the highest weight vector in $\mathscr{V}_{1,3}$; here g stands for the coupling parameter. The corresponding boundary state is

$$|B_a(g)\rangle = \mathscr{P}\exp\left(-g\int_0^{2\pi} du\, \psi_{1,3}(u)\right)|B_a\rangle\,, \tag{84}$$

where we have assumed the geometry of a half-infinite cylinder with u being the coordinate along the boundary, $\psi_{1,3}(u+2\pi)=\psi_{1,3}(u)$. It is known [41] that a minimal CFT with the perturbed boundary condition $B_a(g)$ is integrable. Now we recall [40] the standard correspondence between the vectors in $\mathscr{H}_{\mathrm{chiral}}\otimes\bar{\mathscr{H}}_{\mathrm{chiral}}$ and endomorphisms of $\mathscr{H}_{\mathrm{chiral}}$ and denote $\mathbf{B}_j(g)$ the operator in $\mathscr{H}_{\mathrm{chiral}}$ associated with the boundary state (84) with $a=(1,2j+1)$. Using (83) with the explicit form of the matrix $\mathbf{S}$ for the minimal model $\mathscr{M}_{n,n'}$ ($\beta^2=\frac{n}{n'}$, $n'>n=2,3,\ldots$) we obtain

$$\mathbf{B}_j(0)=\sum_{k,l}\left(\frac{nn'}{8}\right)^{-\frac{1}{4}}\left[\frac{\sin(\frac{\pi}{2\beta^2}\,p_{k,l})}{\sin(2\pi\, p_{k,l})}\right]^{\frac{1}{2}}\sin(2\pi\, p_{k,l}(2j+1))\;\mathbf{P}_{k,l}\,, \tag{85}$$

where $\mathbf{P}_{k,l}$ are the projectors on the subspaces $\mathscr{V}_{k,l}$ and

$$2p_{k,l}=\beta^2 l-k$$

are the eigenvalues of the operator P on these subspaces. Comparing this expression with (68) we see that

$$\mathbf{B}_j(0)=\left(\frac{nn'}{8}\right)^{-\frac{1}{4}}(\sin(2\pi P)\,\sin(2\pi\beta^{-2}P))^{\frac{1}{2}}\,\mathbf{T}_j(0)\,. \tag{86}$$

Of course this relation is not very surprising as the operators $\mathbf{T}_j(0)$ are essentially the Verlinde operators ϕ_a [42] for $a=(1,2j+1)$ and the eigenvalues of the latter are known to be $\mathbf{S}_{ab}/\mathbf{S}_{0b}$. What is less trivial is that the $\mathscr{P}$-exponential in (47) gives exactly the Feigin–Fuchs realization for the $\mathscr{P}$-exponential in (84), and so this relation extends to the case of a full boundary state $\mathbf{B}_j(g)$, i.e.

$$\mathbf{B}_j(g)=\left(\frac{nn'}{8}\right)^{-\frac{1}{4}}(\sin(2\pi P)\,\sin(2\pi\beta^{-2}P))^{\frac{1}{2}}\,\mathbf{T}_j(\lambda)\,, \tag{87}$$

where the relation between λ and g depends on the normalization of the boundary field $\psi_{1,3}(u)$ in (84). Using the operator product expansion

$$\psi_{1,3}(u)\psi_{1,3}(u')=(u-u')^{-2\Delta_{1,3}}+\text{less singular terms}$$

to fix this normalization we find

$$g^2=\lambda^4\,\frac{\sin(2\pi(j+1)\beta^2)\sin(2\pi j\beta^2)}{\pi\sin(2\pi\beta^2)}\;\frac{\Gamma^3(1-\beta^2)\Gamma(3\beta^2-1)}{1-2\beta^2}\,. \tag{88}$$

The relation (87) allows one to interpret many properties of the $\mathbf{T}$-operators discussed in this paper in terms of Renormalization Group flows between different CBC. We believe it also throws new light on the functional equations for the boundary partition functions obtained in [22] by TBA approach. More details about this relation will be given in [35].

The above discussion was restricted to the domain (48) of c as for $c > -2$ the integrals in (47) become divergent. However this limitation is not very significant. It is possible to change the integration contours in (47) in such a way that the operator (41) will make sense for any c (this can be thought of as an analytic continuation in c). Moreover, with this redefinition most of the above properties of the $\mathbf{T}_j(\lambda)$ operators will still hold in the larger domain $-\infty < c < 1$, the most significant change being in the nature of the essential singularity of $\mathbf{T}_j(\lambda)$ at $\lambda = \infty$ for $-2 < c < 1$ ((66) remains valid but in the smaller domain $-2\pi(1-\beta^2) < \arg(\lambda^2) < 2\pi(1-\beta^2)$ while completely different asymptotic behavior emerges in the complimentary sector). Although we did not yet complete the analysis for this domain $-2 < c < 1$, for the cases of unitary minimal models $\mathcal{M}_{n,n+1}$ [43] we expect to observe the "truncation" of the functional relations (69) similar to the one for $\mathcal{M}_{2,2n+3}$ discussed above. We hope to return to this point elsewhere.

And, of course, the above approach can be generalized in a straightforward way to describe the quantum theory of generalized KdV [44] associated with the higher-rank simply laced Lie algebras, or equivalently, to describe the integrable structure of the CFT with the extended W-symmetry.

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