The Integrable Hierarchy Constructed from a Pair of KdV-Type Hierarchies and its Associated W Algebra

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Abstract: For any two arbitrary positive integers "n" and "m", using the mth KdV hierarchy and the $(n+m)^{\text{th}}$ KdV hierarchy as building blocks, we are able to construct another integrable hierarchy (referred to as the $(n,m)^{\text{th}}$ KdV hierarchy). The W-algebra associated to the second Hamiltonian structure of the $(n,m)^{\text{th}}$ KdV hierarchy (called $W(n,m)$ algebra) is isomorphic via a Miura map to the direct sum of a W_m-algebra, a W_{n+m}-algebra and an additional $U(1)$ current algebra. In turn, from the latter, we can always construct a representation of a W_∞-algebra.

1. Introduction

Our purpose in this paper is to show how to construct new integrable hierarchies starting from a couple of KdV-type hierarchies plus a $U(1)$ current. Also in order to give the coordinates of our paper with respect to the current literature, let us recall a few fundamental things about KdV hierarchies.

There are two different descriptions of the nth KdV hierarchy. One is based on the so-called pseudodifferential operator analysis (see [1]), in which we start from a differential operator L, called scalar Lax operator,

$$L = \partial^n + \sum_{i=1}^{n-1} u_i \partial^{n-i-1}, \qquad \partial = \frac{\partial}{\partial x}, \tag{1.1}$$

where the u_i's are functions of the "space" coordinate x. Throughout the paper the symbol L will mean (1.1). After introducing the inverse ∂^{-1} of the derivative ∂ (i.e. the formal integration operator),

$$\partial\partial^{-1} = \partial^{-1}\partial = 1,$$

$$\partial^{-1}f(x) = \sum_{l=0}^{\infty} (-1)^l f^{(l)} \partial^{-l-1},$$

we can calculate the fractional powers of L.

In general for a pseudodifferential operator, $A = \sum_{i \leq n} a_i \partial^i$, we define

$$A_+ = \sum_{i \geq 0} a_i \partial^i, \qquad A_- = A - A_+, \quad \text{res}(A) = a_{-1} .$$

Since the operator $[(L^{\frac{r}{n}})_+, L] = [L, (L^{\frac{r}{n}})_-]$ is a purely differential operator of order $(n-2)$ for any positive integer r, it naturally defines a series of infinitely many differential equations,

$$\frac{\partial}{\partial t_r} L = [(L^{\frac{r}{n}})_+, L], \qquad r \geq 1 , \tag{1.2}$$

where $t_1 \equiv x$, while $t_2, t_3, \ldots$, are real time parameters. This set of equations is usually referred to as the nth KdV hierarchy or, simply, the n-KdV hierarchy.

Another presentation of the n-KdV hierarchy is by means of the Drinfeld–Sokolov construction [2]. In such an approach, we begin with a first order matrix differential operator

$$\mathscr{L} = \partial + q - \Lambda , \tag{1.3}$$

where q and Λ are $n \times n$ matrices and

$$\Lambda = \lambda E_{n1} + I, \qquad I = \sum_{i=1}^{n-1} E_{i,i+1}, \qquad (E_{ij})_{kl} = \delta_{ik}\delta_{jl} , \tag{1.4}$$

where λ is the spectral parameter, while q is a lower triangular matrix. One can find a formal series

$$T = 1 + \sum_{i=1}^{\infty} T_i \Lambda^{-i} ,$$

such that

$$\mathscr{L}_{(0)} = T \mathscr{L} T^{-1} = \partial + \Lambda + \sum_{i=0}^{\infty} f_i \Lambda^{-i} ,$$

with all f_i being functions. The centralizer of $\mathscr{L}_{(0)}$ contains nothing but the constant elements of the Heisenberg subalgebra, thus we can easily get the centralizer of the operator $\mathscr{L}$, and we can define a series of flow equations

$$\frac{\partial}{\partial t_r} \mathscr{L} = [\mathscr{M}_r^+, \mathscr{L}], \qquad \mathscr{M}_r = T^{-1} \Lambda^r T , \tag{1.5}$$

where the superscript "+" means that we keep only non-negative powers of Λ. These equations are only defined for classes of gauge equivalence, i.e. up to transformations which leave (1.5) form invariant. If we suitably fix the gauge, Eq. (1.5) reduces to the nth KdV hierarchy (1.2).

One can generalize the above construction in different directions. In fact this construction is based on the Lie algebra sl_n, and Λ can be understood as an element of the associated affine algebra which enjoys particular properties. The generalization in which $\Lambda = \sum_{i=0}^{n-1} e_i$, where e_i are the standard Chevalley generators of an affine Kac–Moody algebra (in this case, Λ is a grade one element of the principal Heisenberg subalgebra), and q is an element belonging to the relevant non-positive graded Borel subalgebra, has been studied in [2].

Recently there have been several attempts to generalize the KdV-type integrable hierarchies in other directions. One possibility is to replace Λ in (1.3) by any constant regular element of any Heisenberg subalgebra of the Kac–Moody algebra.

The hierarchy constructed in such a way is called type I; if the element Λ is not regular the hierarchy is called type II [3]. It has been shown that, in the gl_n case, the graded regular elements exist only in some very special cases; furthermore, after gauge fixing, this extended Drinfeld–Sokolov hierarchy reduces to the Gelfand–Dickii matrix hierarchy, which is a simple extension of (1.2) obtained by replacing the scalar Lax operator (1.1) by a matrix valued one [4]. On the other hand, there has been so far no detailed discussion about type II integrable hierarchies, due to their complexity (see however [5]).

Moving from a completely different starting point, in a recent paper we have considered another type of extension: we have modified the Lax operator (1.1) by adding some suitable pseudodifferential terms; in this way we have obtained a new integrable hierarchy, which we have called the $(n,m)^{\text{th}}$ KdV hierarchy, [7]. Actually these hierarchies are not an artifact of ours. They naturally appear in two- (and multi-) matrix models describing 2D gravity coupled to conformal matter. Two-matrix models are in fact characterized by Toda lattice hierarchies [6] – each different hierarchy being specified by a particular choice of the two relevant Jacobi matrices. There is a duality between Toda lattice integrable hierarchies and differential integrable hierarchies, [8,9], which enables us to extract KP-type differential hierarchies from the lattice hierarchies and vice versa. Moreover the differential hierarchies obtained in this way can be reduced to new hierarchies while preserving integrability. The full set of KP-type integrable hierarchies obtained from the Toda lattice hierarchy together with their integrable reductions turn out to fill up exactly the set of the $(n,m)^{\text{th}}$ KdV hierarchies.

In this paper we present the $(n,m)^{\text{th}}$ KdV hierarchy from the point of view of extending the KdV-type hierarchies: given any two KdV type hierarchies, say an mth KdV hierarchy and an $(n+m)^{\text{th}}$ KdV hierarchy, plus a $U(1)$ *gluon* current J, we show how to construct another integrable hierarchy and that the latter is exactly the $(n,m)^{\text{th}}$ KdV hierarchy. Moreover we will show that there exists a Miura map which establishes an isomorphism between the W-algebra associated to the second Hamiltonian structure of the $(n,m)^{\text{th}}$ KdV hierarchy to the direct sum of the W_{n+m}-algebra and W_m-algebra, as well as an additional $U(1)$ current algebra (when $m = 1$ the isomorphism is simply with the direct sum of a W_{n+1}-algebra and the $U(1)$ current algebra.)

The paper is organized as follows. In Sect. 2, we review some well-known facts about W_n algebras. Our main results are presented in Sect. 3, where we first construct the $(n,m)^{\text{th}}$ KdV hierarchy from the $(n+m)^{\text{th}}$ KdV hierarchy and m^{th} KdV hierarchy, then we show that the $W(n,m)$-algebra is related to $W_{n+m} \oplus W_m \oplus U(1)$ by a Miura map. In Sect. 4, we will analyze the conformal spectrum, and construct the Drinfeld–Sokolov representation of the $(n,m)^{\text{th}}$ KdV hierarchy. We show that the $(n,m)^{\text{th}}$ KdV hierarchies correspond in part to a type I and in part to type II generalized Drinfeld–Sokolov hierarchy, [3]. Several examples and some remarks are presented in Sect. 5.

2. The nth KdV Hierarchy and W_n Algebra

In this section we will review some well-known results on the nth KdV hierarchy and the W_n-algebra. More precisely, we will show how to derive the W_n-algebra from: 1) the nth KdV hierarchy; 2) a suitable infinitesimal deformation of the corresponding

differential operator; 3) field-dependent gauge transformations (or residual gauge symmetry).

2.1. The nth KdV Hierarchy. There is a natural inner product on the pseudodifferential operator algebra, defined by

$$\langle A \rangle \equiv \int dx \ \text{res}(A) \,, \tag{2.1}$$

which enables us to define two compatible Poisson structures

$$\{f_X, f_Y\}_1(L) = \langle L[Y,X]\rangle, \quad f_X(L) = \langle LX\rangle, \quad X = \sum_{i=1}^{n-1} \partial^{-i}\chi_i \,, \tag{2.2}$$

$$\{f_X, f_Y\}_2(L) = \langle (XL)_+ YL\rangle - \langle (LX)_+ LY\rangle + \frac{1}{n} \int \partial^{-1}([L,X]_{(-1)})[L,Y]_{(-1)} \,. \tag{2.3}$$

These two Poisson structures give rise to two Poisson bracket algebras of the basic independent fields u_i,

$$\{u_i(x), u_j(y)\}_1 = (\hat{\mathcal{H}}_1)_{ij}[u(x)]\delta(x - y) \,, \tag{2.4}$$

$$\{u_i(x), u_j(y)\}_2 = (\hat{\mathcal{H}}_2)_{ij}[u(x)]\delta(x - y) \,. \tag{2.5}$$

We call $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ Hamiltonian operators. They are $(n - 1) \times (n - 1)$ matrix operators and only contain the derivative ∂ and the basic fields. In particular, Eq. (2.5) is referred to as the W_n algebra.

The conserved quantities (or Hamiltonians) have very simple form

$$H_r = \frac{n}{r}\langle L^{\frac{r}{n}}\rangle, \quad \forall r \geqq 1 \,. \tag{2.6}$$

They generate the Hamiltonian flows (1.2) through the Poisson brackets.

2.2. Infinitesimal Deformations of the Lax Operator. The nth KdV hierarchy (1.2) can be viewed as the consistency conditions of the following spectral evolution problem:

$$L\psi = \lambda\psi \,, \tag{2.7a}$$

$$\frac{\partial}{\partial t_r}\psi = (L^{\frac{r}{n}})_+\psi \,. \tag{2.7b}$$

The function $\psi(\lambda, t)$ is usually referred to as the Baker–Akhiezer function.

As observed in [10], calculating a W_n-algebra is equivalent to finding two infinitesimal differential operators P and Q, such that

$$\delta L = QL - LP \tag{2.8}$$

and $(L + \delta L)$ still has the same form as (1.1). This is equivalent to saying that Eq. (2.7a) with vanishing spectral parameter $\lambda = 0$ is invariant under the infinitesimal deformations

$$\psi \longrightarrow \psi + \delta\psi, \qquad L \longrightarrow L + \delta L$$

with δL specified by Eq. (2.8), and

$$\delta\psi = P\psi \,. \tag{2.9}$$

In other words, after such infinitesimal deformation, we still have

$$(L + \delta L)(\psi + \delta \psi) = 0 \,. \tag{2.10}$$

Proposition 2.1. *If we choose*

$$P = (YL)_+ - \frac{1}{n}Z, \qquad Q = (LY)_+ - \frac{1}{n}Z \,, \tag{2.11}$$

where $Y = \sum_{i=1}^{n-1} \partial^{i-n} \varepsilon_i$ is an arbitrary infinitesimal pseudodifferential operator, and

$$Z = \int^x ([L, Y]_{(-1)}) \,,$$

then the deformation (2.8) *coincides with the one derived from the second Poisson structure* (2.3), *i.e.*

$$\delta u_i = \sum_{j=1}^{n-1} (\hat{\mathscr{H}}_2)_{ij}[u] \cdot \varepsilon_j \,. \tag{2.12}$$

Proof. Let X and Y be two arbitrary pseudodifferential operators (with Y infinitesimal), then the variation of the functional $f_X(L)$ under the transformation generated by $f_Y(L)$ with respect to the second Poisson structure is as follows:

$$\delta f_X(L) = \{f_X, f_Y\}_2(L)$$

$$= \langle X((LY)_+ L - L(YL)_+) \rangle - \frac{1}{n} \langle Z(LX - XL) \rangle \,.$$

Let X be independent of the basic fields u_i, we obtain

$$\delta L = \left((LY)_+ - \frac{1}{n}Z \right) L - L \left((YL)_+ - \frac{1}{n}Z \right) \,,$$

which is just the formula (2.8) with the identification (2.11). On the other hand, Eq. (2.12) is a direct consequence of (2.3), so it must be the solution of Eq. (2.8) – remember that

$$f_Y(L) = \sum_{i=1}^{n-1} \int u_i \varepsilon_i, \qquad \delta f_X(L) = \sum_{i=1}^{n-1} \int \delta u_i \chi_i \,. \tag{2.13}$$

This ends the proof.

Let us see a simple example. Choose $Y = \partial^{1-n} \varepsilon$. A straightforward calculation shows that

$$Z = -\frac{n(n-1)}{2} \varepsilon', \qquad P = \varepsilon\partial - \frac{n-1}{2} \varepsilon', \qquad Q = \varepsilon\partial + \frac{n+1}{2} \varepsilon' \,. \tag{2.14}$$

Plugging this into Eq. (2.8), and using Eq. (2.12), we get

$$\delta L = \sum_{i=1}^{n-1} ((\hat{\mathscr{H}}_2)_{i1}[u] \cdot \varepsilon)\partial^{n-i-1} = \left(\varepsilon\partial + \frac{n+1}{2} \varepsilon' \right) L - L \left(\varepsilon\partial - \frac{n-1}{2} \varepsilon' \right) \,. \tag{2.15}$$

In particular we find that u_1 satisfies the Virasoro algebra

$$\{u_1(x), u_1(y)\}_2 = (c_n\partial^3 + u_1(x)\partial + \partial u_1(x))\partial(x-y), \qquad c_n = \frac{1}{2} \binom{n+1}{3} \,. \tag{2.16}$$

Thus the deformation we considered is nothing but an infinitesimal conformal transformation. Furthermore

$$\delta\psi = \left(\varepsilon\partial - \frac{n-1}{2}\varepsilon'\right)\psi, \qquad \delta(L\psi) = \left(\varepsilon\partial + \frac{n+1}{2}\varepsilon'\right)(L\psi)$$

indicates that ψ and $L\psi$ are primary fields with conformal weights $\frac{1-n}{2}$ and $\frac{n+1}{2}$, respectively.

We may write down the global form of this diffeomorphism

$$x \longrightarrow f(x), \qquad L \longrightarrow (f'(x))^{-\frac{n+1}{2}}\widetilde{L}(f'(x))^{-\frac{n-1}{2}}, \qquad (2.17)$$

where $\widetilde{L} = \partial^n + \widetilde{u}_1\partial^{n-2} + \cdots$. From this transformation law, one can obtain the conformal properties of all fields [11].

2.3. Drinfeld–Sokolov Representation. The basic idea to construct a matrix version of the nth KdV hierarchy is to find a vector space representation of the differential (or pseudodifferential) operator algebra. For such a purpose, we may linearize the spectral equation (2.7a) by introducing $(n-1)$ supplementary fields as follows:

$$\psi_1 \equiv \psi, \qquad \psi_{i+1} = \partial\psi_i = \partial^i\psi, \quad 1 \leqq i \leqq n-1. \qquad (2.18)$$

Define

$$U = \sum_{i=1}^{n-1} u_i E_{n,n-i}, \qquad \Lambda = \lambda E_{n1} + I, \quad I = \sum_{i=1}^{n-1} E_{i,i+1}.$$

Further denote by Ψ and $\mathscr{L}$ the column vector $(\psi, \psi_2, \ldots, \psi_n)^t$ and the $(n \times n)$ matrix operator $(\partial + U - \Lambda)$, respectively. Then the linear system (2.7a, 2.7b) can be rewritten in matrix form as follows:

$$\mathscr{L}\Psi = 0 ; \qquad (2.19a)$$

$$\frac{\partial}{\partial t_{ni+r}}\Psi = \mathscr{M}_{ni+r}\Psi, \quad i \geqq 0, \quad n-1 \geqq r \geqq 1, \qquad (2.19b)$$

where $\mathscr{M}_{ni+r}$ is a uniquely determined $n \times n$ matrix field, which is a differential polynomial in the u_i's and only contains non-negative powers of λ. In particular, the first $n-1$ elements of its last column (these are the important ones) take the form,

$$(\mathscr{M}_{ni+r})_{n-j,n-1} = \sum_{0 \leqq l \leqq i} \lambda^{i-l}\frac{\delta H_{r+nl}}{\delta u_j}, \quad 1 \leqq j \leqq n-1. \qquad (2.20)$$

It is perhaps worth giving a proof of Eq. (2.19b) and Eq. (2.20). To this end we first prove the following lemma:

Lemma 2.2. *Let (i,j,r) be non-negative integers, and let $1 \leqq j$, $r \leqq n-1$. Then*

$$\partial^j(L^{\frac{r}{n}+i})_+ = \sum_{l=0}^{i+1}((\partial^j L^{\frac{r}{n}+l-1})_- L)_+ L^{i-l} - (\partial^j(L^{\frac{r}{n}+i})_-)_+ + (\partial^j L^{\frac{r}{n}-1})_+ L^{i+1}. \qquad (2.21)$$

Proof. Since the operator $\partial^j(L^{\frac{r}{n}+i})_+$ is a purely differential operator, we have

$$\partial^j(L^{\frac{r}{n}+i})_+ = \partial^j L^{\frac{r}{n}} L^i - \partial^j(L^{\frac{r}{n}+i})_- = ((\partial^j L^{\frac{r}{n}} L^i)_+ - (\partial^j(L^{\frac{r}{n}+i})_-)_+.$$

The second term on the extreme right-hand side has order less than n, while the first term can be rewritten as

$$((\partial^j L^{\frac{r}{n}} L^i)_+ = (\partial^j L^{\frac{r}{n}-1})_+ L^{i+1} + ((\partial^j L^{\frac{r}{n}-1})_- L^{i+1})_+ \,. \qquad (2.22)$$

Now suppose that l is also a positive integer and $l \leqq i$. Due to the fact that L is a purely differential operator, we obviously have

$$((\partial^j L^{\frac{r}{n}+l})_+ L)_- = 0$$

and

$$((\partial^j L^{\frac{r}{n}+l})_- L)_- = (\partial^j L^{\frac{r}{n}+l+1})_- \,,$$

which immediately lead to the following recursion relation:

$$((\partial^j L^{\frac{r}{n}+l})_- L^{i-l})_+ = ((\partial^j L^{\frac{r}{n}+l})_- L)_+ L^{i-l-1} + ((\partial^j L^{\frac{r}{n}+l+1})_- L^{i-l-1})_+ \,. \qquad (2.23)$$

Using this identity and Eq. (2.22), it is straightforward to show that Eq. (2.21) is true. This ends the proof of Lemma 2.1.

In order to prove Eq. (2.20), we write

$$((\partial^j L^{\frac{r}{n}+l-1})_- L)_+ = \sum_{k=0}^{n-1} \alpha_{l,k} \partial^k \,.$$

Then

$$\alpha_{l,k} = \mathrm{res}\,[((\partial^j L^{\frac{r}{n}+l-1})_- L)_+ \partial^{-k-1}] \,.$$

In particular for $k = n - 1$, we have

$$\alpha_{l,n-1} = \mathrm{res}\,[((\partial^j L^{\frac{r}{n}+l-1})_- L)_+ \partial^{-n}]$$

$$= \mathrm{res}\,[(\partial^j L^{\frac{r}{n}+l-1})_- L \partial^{-n}] = (\partial^j L^{\frac{r}{n}+l-1})_{(-1)} \,.$$

On the other hand, from

$$\delta H_{r+nl} = \int dx\,\mathrm{res}\,((\delta L) L^{\frac{r}{n}+l-1}) = \int dx \sum_{j=1}^{n-1} (\delta u_{n-j})(\partial^{j-1} L^{\frac{r}{n}+l-1})_{(-1)} \,,$$

we get

$$\frac{\delta H_{r+nl}}{\delta u_{n-j}} = (\partial^{j-1} L^{\frac{r}{n}+l-1})_{(-1)} = \alpha_{l,n-1} \,. \qquad (2.24)$$

Therefore

$$\frac{\partial}{\partial t_{ni+r}} \psi_j = \frac{\partial}{\partial t_{ni+r}} \partial^{j-1} \psi = \partial^{j-1} (L^{\frac{r}{n}+i})_+ \psi$$

$$= \sum_{l=0}^{i} \lambda^{i-l} ((\partial^{j-1} L^{\frac{r}{n}+l-1})_- L)_+ \psi - (\partial^{j-1} (L^{\frac{r}{n}+i})_-)_+ \psi + \lambda^{i+1} (\partial^{j-1} L^{\frac{r}{n}-1})_+$$

$$= \sum_{l=0}^{i} \lambda^{i-l+1} \frac{\delta H_{r+nl}}{\delta u_{n-j}} \psi_n + \cdots \,. \qquad (2.25)$$

The dots contain auxiliary fields ψ_k with $1 \leqq k \leqq n - 1$. This ends the proof of Eq. (2.20).

The consistency conditions of Eqs. (2.19a, 2.19b) give rise to the Drinfeld–Sokolov representation of the nth KdV hierarchy,

$$\frac{\partial}{\partial t_{r+ni}} \mathscr{L} = [\mathscr{M}_{r+ni}, \mathscr{L}] \,. \tag{2.26}$$

To be precise, the term Drinfeld–Sokolov integrable hierarchy is utilized for the flow equations (1.5) with reference to the operator $\mathscr{L}$ defined in (1.3). As noticed above, (1.5) are gauge dependent; if we choose a suitable gauge we recover Eq. (2.26).

A specification is necessary at this point. The linearization of Eq. (2.7a) is not unique. For example, we may introduce supplementary fields in the following way:

$$\widetilde{\psi}_1 = \psi, \qquad \widetilde{\psi}_{i+1} = (\partial + h_i)\widetilde{\psi}_i, \quad 1 \leq i \leq n-1 \,;$$

$$(\partial + h_n)\widetilde{\psi}_n = \lambda\psi, \qquad -h_n = h_1 + h_2 + \cdots + h_{n-1} \,.$$

Then we have another linearized spectral equation

$$\widetilde{\mathscr{L}}\widetilde{\Psi} = 0, \quad \widetilde{\mathscr{L}} = \partial + \widetilde{U} - \Lambda, \quad \widetilde{U} = \sum_{i=1}^{n} h_i E_{ii} \,.$$

The difference between these two linearizations is just a Miura map. In general we call a Miura transformation a (non-invertible) gauge transformation which maps a minimal set of independent coordinates onto another minimal set. Therefore, modulo Miura transformations, the linearization is unique. Hereafter we only focus our attention on the linearization (2.19a).

Example. For the second flow, we have

$$\mathscr{M}_2 = (U - \Lambda)^2 - U' + \frac{2}{n} \sum_{i \geq J} \binom{i-1}{j-1} u_1^{(i-j)} E_{ij} \,.$$

The LHS of Eq. (2.26) is independent of λ, therefore we can set $\lambda = 0$ on both sides of Eq. (2.26). In the remaining part of this section, we will show that in the case $\lambda = 0$, the spectral equation completely determines the W_n-algebra.

Proposition 2.3. *The spectral equation*

$$\mathscr{L}_0\Psi = 0, \quad \mathscr{L}_0 = \mathscr{L}(\lambda = 0) = \partial + U - I \tag{2.27}$$

is invariant under the following infinitesimal transformations:

$$\Psi \longrightarrow G\Psi, \quad \mathscr{L}_0 \longrightarrow G\mathscr{L}_0 G^{-1}, \quad G = 1 + R \,, \tag{2.28}$$

where the infinitesimal matrix field R satisfies

$$\delta U = [R, \partial + U - I] \,, \tag{2.29}$$

and $(U + \delta U)$ has the same form as U. The elements of R are polynomials of the basic fields u_i. In particular we can choose

$$R_{n-j,n} = \frac{\delta f_Y(L)}{\delta u_j} \equiv \varepsilon_j, \quad 1 \leq j \leq n-1 \,, \tag{2.30}$$

where f_Y is defined in Eq. (2.13).

Eq. (2.29) *and the choice* (2.30) *completely determines the structure of the* W_n-*algebra.*

Proof. The first statement follows directly from the invariance (2.10), i.e. we have

$$(\mathscr{L}_0 + \delta\mathscr{L}_0)(\Psi + \delta\Psi) = 0 .\tag{2.31}$$

This equation requires that R satisfy Eq. (2.29), which in turn determines R up to $(n-1)$ arbitrary elements. In particular setting

$$R_{n-j,n} = \sum_{i,r} \frac{\delta H_{r+ni}}{\delta u_j} \delta t_{ni+r}$$

and comparing with (2.20), we get the $(r+ni)^{\text{th}}$ Hamiltonian flow.

We remark that the choice of the independent elements of R is not unique as long as the only requirement is to recover a W_n-algebraic structure, no matter what the coordinates are. For example, we can choose the first row to be independent, then the variation of the Baker–Akhiezer function can be derived from Eq. (2.28),

$$\delta\psi = \sum_{j=1}^{n} R_{1j}\psi_{j-1} = P\psi, \quad P = \sum_{j=1}^{n} R_{1j}\partial^{j-1} .\tag{2.32}$$

However choosing the first $n-1$ elements of the last column as independent, as in (2.30), is of particular importance, because the relation (2.30) leads to coincidence with the second Poisson bracket (2.3).

To see this point let $\delta\psi = P\psi$ be given by Eq. (2.11). We notice that

$$\partial^j P = \partial^j \left((YL)_+ - \frac{1}{n}Z \right) = \partial^j YL - \partial^j (YL)_- - \frac{1}{n}\partial^j Z$$

$$= \left(\partial^j YL - \partial^j (YL)_- - \frac{1}{n}\partial^j Z \right)_+$$

$$= (\partial^j Y)_+ L + ((\partial^j Y)_- L)_+ - (\partial^j (YL)_-)_+ - \frac{1}{n}\partial^j Z$$

$$= \left(\sum_{i=n-j}^{n-1} \partial^{i+j-n}\varepsilon_i \right) L + \sum_{i=1}^{n-j-1} (\partial^{i+j-n}\varepsilon_i L)_+$$

$$\qquad - (\partial^j (YL)_-)_+ - \frac{1}{n}\partial^j Z, \quad 0 \leqq j \leqq n-2 ,$$

which shows that

$$\delta\psi_j = \partial^{j-1}(\delta\psi) = \partial^{j-1}P\psi = \left(\sum_{\iota=n-j+1}^{n-1} \partial^{i+j-n-1}\varepsilon_i \right) \lambda\psi + \varepsilon_{n-j}\psi_n + \cdots .$$

The first term disappears in the case $\lambda = 0$, the last term only contains the auxiliary fields ψ_k with $1 \leqq k \leqq n-1$. Thus we obtain

$$R_{j,n} = \varepsilon_{n-j} = \frac{\delta f_Y(L)}{\delta u_{n-j}} ,$$

which shows that the choice (2.30) coincides with the choices made in Proposition 2.1, i.e. with the second Poisson structure (2.5), alias the W_n algebra. This ends the proof.

The definition of the W_n-algebra contained in the above proposition exhibits the intimate relation between the W-algebra and Kac–Moody current algebra [12]. Let $\mathscr{A}$ be a gauge field (WZNW current) valued on some Lie algebra $\mathscr{G}$, and G be an element of the corresponding Lie group. Then the general gauge transformation reads

$$\mathscr{A} \to G(\partial + \mathscr{A})G^{-1} . \tag{2.33}$$

This gauge symmetry implies that the components of the $\mathscr{A}$ field obey a Kac–Moody current algebra. After (partially) fixing the gauge[1], the gauge symmetry (2.33) will be reduced to the form (2.29). From this point of view we may call the symmetry considered in Proposition 2.2 the *residual gauge symmetry*. On the other hand, since the symmetry (2.28) (characterized by the matrix field R) explicitly depends on the basic fields u_i, we may also call it *field dependent gauge transformation*. In practice, deriving the W-algebra by solving Eq. (2.29) seems to be easier than calculating the Poisson bracket (2.3) (for example, see [13]).

Example. The functional $F = \int dx\, u_1(x)\varepsilon(x)$ generates the following field dependent gauge transformation:

$$R_{\mathrm{conf}} = \varepsilon I - \varepsilon' I_0 + \text{lower triangular part} , \tag{2.34}$$

where I_0 is a diagonal matrix with elements $(I_0)_{ii} = \frac{n-2i+1}{2}$. One can easily check R_{conf} indeed leads to the diffeomorphism (2.14).

3. The $(n,m)^{\mathrm{th}}$ KdV Hierarchy and $W(n,m)$-Algebra

In this section we will construct the $(n,m)^{\mathrm{th}}$ KdV hierarchy from a pair of ordinary higher KdV hierarchies plus a $U(1)$ current J. We also discuss the $W(n,m)$-algebra, which is the algebra associated to the second Hamiltonian structure of the $(n,m)^{\mathrm{th}}$ KdV hierarchy.

3.1. Constructing the $(n,m)^{\mathrm{th}}$ KdV Hierarchy. Our construction of the $(n,m)^{\mathrm{th}}$ KdV hierarchy is based on the following theorem:

Theorem 3.1. *Let A and B be two purely differential operators*

$$A = \partial^{n+m} + \sum_{i=1}^{n+m-1} u_i \partial^{n+m-i-1}, \qquad B = \partial^m - \sum_{i=1}^{m-1} v_i \partial^{m-i-1} , \tag{3.1}$$

and J be a function of x. Define

$$\frac{\partial}{\partial t_2} A = [(A^{\frac{2}{n+m}})_+, A] + m\left(\left(J\partial + \frac{n+m+1}{2}J' \right) A - A\left(J\partial - \frac{n+m-1}{2}J' \right) \right) ; \tag{3.2a}$$

---

[1] For $gl(n)$, the standard gauge fixing conditions consist of restricting $\mathscr{A}$ to be of the form $(U - I)$ as exhibited in Eq. (2.27). This is equivalent to imposing a set of first and second class constraints; after reduction à la Dirac we obtain a W_n-algebra.

$$\frac{\partial}{\partial t_2}B = [(B^{\frac{2}{m}})_+, B] + (n+m)\left(\left(J\partial + \frac{m+1}{2}J'\right)B - B\left(J\partial - \frac{m-1}{2}J'\right)\right) ; \quad (3.2b)$$

$$\frac{\partial}{\partial t_2}J = \left(\frac{4}{n(n+m)}u_1 + \frac{4}{nm}v_1 + \frac{n+2m}{2}J^2\right)' . \quad (3.2c)$$

This set of differential equations gives rise to an integrable system.

Proof. We will show that Eqs. (3.2a–3.2c) admit a Lax pair representation. First we observe that Eqs. (3.2a) can be rewritten as

$$\frac{\partial}{\partial t_2}A = \left[\left(\partial + \frac{m}{2}J\right)^2 + \frac{2}{n+m}u_1 - \frac{m^2}{4}J^2 + \frac{m(n+m)}{2}J'\right]A$$

$$- A\left[\left(\partial + \frac{m}{2}J\right)^2 + \frac{2}{n+m}u_1 - \frac{m^2}{4}J^2 - \frac{m(n+m)}{2}J'\right] . \quad (3.3)$$

Similarly we re-express Eqs. (3.2b) as

$$\frac{\partial}{\partial t_2}B = \left[\left(\partial + \frac{n+m}{2}J\right)^2 - \frac{2}{m}v_1 - \frac{(n+m)^2}{4}J^2 + \frac{(n+m)m}{2}J'\right]B$$

$$- B\left[\left(\partial + \frac{n+m}{2}J\right)^2 - \frac{2}{m}v_1 - \frac{(n+m)^2}{4}J^2 - \frac{m(n+m)}{2}J'\right] . \quad (3.4)$$

Next we introduce a pseudo-differential operator as follows:

$$L_{[n,m]} = \phi^{-\frac{m}{2}}A\phi^{-\frac{n}{2}}B^{-1}\phi^{\frac{n+m}{2}}, \quad (\ln\phi)' = -J , \quad (3.5)$$

and expand $L_{[n,m]}$ in powers of ∂

$$L_{[n,m]} = \partial^n + \sum_{i=0}^{\infty} w_i\partial^{n-i-1} . \quad (3.6)$$

Due to the identity

$$\phi^{-1}\partial\phi = \partial - J ,$$

all the coefficients w_i turn out to be differential polynomials of the fields u_i, v_j and J. For example

$$w_0 = 0, \quad w_1 = u_1 + v_1 + \frac{1}{4}nm(n+m)\left(\frac{1}{2}J^2 + J'\right) ,$$

$$w_2 = u_2 + v_2 + mJu_1 + (n+m)Jv_1 + nv_1'$$

$$+ \frac{1}{4}nm(n+m)\left(\frac{n+2m}{6}J^3 + (n+m-1)JJ' + \frac{2n+m-3}{3}J''\right) \quad (3.7)$$

and so on. The RHS of Eq. (3.2c) is a derivative. Therefore we can extract the equation of motion of the field ϕ,

$$\frac{\partial}{\partial t_2}(\ln\phi) = -\left(\frac{4}{n(n+m)}u_1 + \frac{4}{nm}v_1 + \frac{n+2m}{2}J^2\right) . \quad (3.8)$$

In this passage we have ignored possible integration constants: this is part of the definition of ϕ.

Now using Eq. (3.3) and Eq. (3.4), as well as Eq. (3.8), we can derive the equation of motion of the operator $L_{[n,m]}$,

$$\frac{\partial}{\partial t_2} L_{[n,m]} = \left[\partial^2 + \frac{2}{n} w_1, L_{[n,m]} \right] .$$

Noting that

$$\partial^2 + \frac{2}{n} w_1 = (L_{[n,m]}^{\frac{2}{n}})_+ ,$$

we finally get the following Lax pair representation:

$$\frac{\partial}{\partial t_2} L_{[n,m]} = \left[(L_{[n,m]}^{\frac{2}{n}})_+, L_{[n,m]} \right] . \tag{3.9}$$

Once we know the Lax operator any higher flow is completely determined. This ends the proof of integrability.

Two remarks are in order.

1) If we define a map (Miura map) as follows:

$$L_1 = \phi^{-\frac{m}{2}} A \phi^{\frac{m}{2}} , \tag{3.10a}$$

$$L_2 = \phi^{-\frac{n+m}{2}} B \phi^{\frac{n+m}{2}} = (\partial - S_1)(\partial - S_2) \cdots (\partial - S_m) , \tag{3.10b}$$

$$S_1 + S_2 + \cdots + S_m = \frac{(n+m)m}{2} J , \tag{3.10c}$$

then

$$L_{[n,m]} = L_1 L_2^{-1} = \partial^n + \sum_{i=1}^{n-1} a_i \partial^{n-i-1} + \sum_{l=1}^{m} a_{n+l-1} \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_2} \frac{1}{\partial - S_1} . \tag{3.11}$$

This is exactly the Lax operator considered in [7]. The a_i fields in the first sum coincide with w_i. The full integrable hierarchy is

$$\frac{\partial}{\partial t_r} L_{[n,m]} = [(L_{[n,m]}^{\frac{r}{n}})_+, L_{[n,m]}] . \tag{3.12}$$

Following [7], we will call it the $(n, m)^{\text{th}}$ KdV hierarchy. The W-algebra associated to its second Hamiltonian structure is referred to as a $W(n, m)$-algebra.

2) In our construction of the $(n, m)^{\text{th}}$ KdV hierarchy, the field J plays an essential role. It allows us to merge two KdV-type hierarchies into a new one. Pictorially, we can say that the field J behaves like a *gluon*, which mediates the interaction between the mth KdV hierarchy and the $(n + m)^{\text{th}}$ KdV hierarchy.

3.2. W(n, m)-Algebra. It has been shown that there exists a bi-Hamiltonian structure connected with the pseudodifferential operator (3.6) [14]. From it we can extract two compatible Poisson structures for the $(n, m)^{\text{th}}$ KdV hierarchy, as was shown in [7],

$$\{f_X, f_Y\}_1(L_{[n,m]}) = \langle L_{[n,m]}[X, Y]_{\mathscr{R}} \rangle , \tag{3.13}$$

and

$$\{f_X, f_Y\}_2(L_{[n,m]}) = \langle (XL_{[n,m]})_+ YL_{[n,m]} \rangle - \langle (L_{[n,m]}X)_+ L_{[n,m]}Y \rangle$$

$$+ \frac{1}{n} \int [L_{[n,m]}, Y]_{-1} (\partial^{-1}[L_{[n,m]}, X]_{-1}) , \qquad (3.14)$$

where

$$[X, Y]_{\mathscr{R}} \equiv \frac{1}{2}([\mathscr{R}X, Y] + [X, \mathscr{R}Y]) = [X_+, Y_+] - [X_-, Y_-], \qquad \mathscr{R}A \equiv A_+ - A_- .$$

These two Poisson structures lead to two infinite dimensional Poisson algebras among the fields w_i, which are

1) a $W_{1+\infty}$-algebra

$$\{w_i(x), w_j(y)\}_1 = (\hat{\mathscr{W}}_1)_{ij}(x)\delta(x - y) , \qquad (3.15)$$

and

2) a W_∞-algebra

$$\{w_i(x), w_j(y)\}_2 = (\hat{\mathscr{W}}_2)_{ij}(x)\delta(x - y) . \qquad (3.16)$$

Since in the $(n, m)^{\text{th}}$ KdV hierarchy we only have $(n + 2m - 1)$ fundamental independent fields, these two ∞-field Poisson algebras are realized via two finite-field algebras on the dynamical variables $(S_i; a_i)$, [7]. We denote the corresponding Poisson brackets by

$$\{q_i(x), q_j(y)\}_1 = (\hat{\mathscr{H}}_1)_{ij}(x)\delta(x - y) , \qquad (3.17)$$

and

$$\{q_i(x), q_j(y)\}_2 = (\hat{\mathscr{H}}_2)_{ij}(x)\delta(x - y) . \qquad (3.18)$$

q_i are the components of a $(n + 2m - 1)$-dimensional vector $q = (a_1, \ldots, a_{n+m-1}; S_1, \ldots, S_m)$. $\hat{\mathscr{H}}_1$ and $\hat{\mathscr{H}}_2$ are the appropriate Hamiltonian operators: they are $(n + 2m - 1) \times (n + 2m - 1)$ matrix operators. We are particularly interested in the second Poisson structure. Let us rewrite the Hamiltonian operator $\hat{\mathscr{H}}_2$ in block form

$$\begin{pmatrix} \hat{\mathscr{P}}_1 & \hat{\mathscr{K}}_1 \\ \hat{\mathscr{K}}_2 & \hat{\mathscr{P}}_2 \end{pmatrix} ,$$

where $\hat{\mathscr{P}}_1$ and $\hat{\mathscr{P}}_2$ are $(n + m - 1) \times (n + m - 1)$ and $(m \times m)$ matrix operators. The antisymmetry of the Poisson bracket implies

$$\hat{\mathscr{K}}_1^\dagger = -\hat{\mathscr{K}}_2 ,$$

where the superscript "$\dagger$" means the following conjugation operation

$$\partial^\dagger = -\partial, \qquad f^\dagger = f, \qquad (AB)^\dagger = B^\dagger A^\dagger, \qquad (M^\dagger)_{ij} = M_{ji}^\dagger$$

for any ordinary function f, differential operators A, B, and matrix operator M.

Theorem 3.2. *The Miura map*

$$\hat{\mathscr{M}} : q = (a_1, \ldots, a_{n+m-1}; S_1, \ldots, S_m) \rightarrow \tilde{q} = (u_1, \ldots, u_{n+m-1}; v_1, \ldots, v_{m-1}; J) \quad (3.19)$$

transforms the second Hamiltonian structure into the following block diagonal form:

$$\hat{\mathscr{M}} \hat{\mathscr{H}}_2 (\hat{\mathscr{M}})^\dagger = \begin{pmatrix} \hat{\hat{\mathscr{P}}}_1 & 0 & 0 \\ 0 & \hat{\hat{\mathscr{P}}}_2 & 0 \\ 0 & 0 & \hat{\hat{\mathscr{P}}}_3 \end{pmatrix}, \tag{3.20}$$

where $\hat{\hat{\mathscr{P}}}_1, \hat{\hat{\mathscr{P}}}_2$ and $\hat{\hat{\mathscr{P}}}_3$ are the Hamiltonian operators of a W_{n+m}, W_m-algebra and a $U(1)$ current algebra, respectively. In other words, the fields u_i form a W_{n+m} algebra, the v_i form a W_m algebra and J a $U(1)$ algebra, respectively. The remaining Poisson brackets vanish.

In order to prove this theorem, we proceed first to prove the following:

Proposition 3.3. *Suppose that a W-algebra be defined by $(n + 2m - 1)$ basic fields $(u_1, \ldots, u_{n+m-1}; v_1, \ldots, v_{m-1}; J)$, which satisfy the following properties:*

(i) *The fields $u_i (1 \leq i \leq n + m - 1)$ form a W_{n+m}-algebra*

$$\{u_i, u_j\}_2 = \hat{\Omega}_{ij}[u]\delta(x - y). \tag{3.21}$$

(ii) *The fields $(-v_j)(1 \leq j \leq m - 1)$ form a W_m-algebra*

$$\{v_i, v_j\}_2 = \hat{\sigma}_{ij}[-v]\delta(x - y). \tag{3.22}$$

(iii) *The field J forms a $U(1)$ current algebra*

$$\{J, J\}_2 = \frac{4}{nm(n + m)}\delta'(x - y), \tag{3.23}$$

while the three groups of fields u_i, v_i and J mutually commute

$$\{u_i, v_j\}_2 = 0, \qquad \{u_i, J\}_2 = 0, \qquad \{v_j, J\}_2 = 0, \quad \forall i, j.$$

Then Eqs. (3.2a–3.2c) are Hamiltonian equations ensuing from the above Poisson brackets and the Hamiltonian $H = \int w_2(x)\, dx$, where $w_2(x)$ is given by Eqs. (3.7).

Proof. With respect to the W-algebra specified in the above proposition, the Hamiltonian H generates the following equations:

$$\frac{\partial}{\partial t_2} u_i = \hat{\Omega}_{i2}[u] \cdot 1 + m \hat{\Omega}_{i1}[u] \cdot J, \tag{3.24a}$$

$$\frac{\partial}{\partial t_2} v_i = -\hat{\sigma}_{i2}[-v] \cdot 1 - (n + m)\hat{\sigma}_{i1}[-v] \cdot J, \tag{3.24b}$$

$$\frac{\partial}{\partial t_2} J = \left(\frac{4}{n(n + m)} u_1 + \frac{4}{nm} v_1 + \frac{n + 2m}{2} J^2 \right)'. \tag{3.24c}$$

Now we are going to show that Eqs. (3.2a–3.2c) can be re-expressed in this form.

We recall that the Lax pair form of the second flow equations of the mth KdV hierarchy is

$$\frac{\partial}{\partial t_{b2}} B = [(B^{\frac{2}{m}})_+, B], \tag{3.25}$$

with Lax operator B given in Eqs. (3.1). This flow is generated by the second Hamiltonian $H_{b2} = -\int v_2(x)\,dx$ through the Poisson structure (3.22), i.e.

$$\frac{\partial}{\partial t_{b2}}v_i = \{v_i, H_{b2}\}_2 = -\hat{\sigma}_{i2}[-v] \cdot 1 \,,$$

which leads to

$$\frac{\partial}{\partial t_{2b}}B = -\sum_{i=1}^{m-1}\left(\frac{\partial}{\partial t_{b2}}v_i\right)\partial^{m-i-1} = \sum_{i=1}^{m-1}(\hat{\sigma}_{i2}[-v]\cdot 1)\partial^{m-i-1}\,. \tag{3.26}$$

Comparing Eqs. (3.25) and (3.26), we get the following identity:

$$\sum_{i=1}^{m-1}(\hat{\sigma}_{i2}[-v]\cdot 1)\partial^{m-i-1} = [(B^{\frac{2}{m}})_+, B]\,. \tag{3.27}$$

On the other hand, in Eq. (2.15), if we choose $\varepsilon = J(x)\widetilde{\varepsilon}$ with $\widetilde{\varepsilon}$ being an x-independent infinitesimal parameter, then we immediately get another identity,

$$\sum_{i=1}^{m-1}(\hat{\sigma}_{i1}[-v]\cdot J)\partial^{m-i-1} = \left(J\partial + \frac{m+1}{2}J'\right)B - B\left(J\partial - \frac{m-1}{2}J'\right)\,. \tag{3.28}$$

These two identities, (3.27) and (3.28), tell us that Eq. (3.2a) is exactly the same as Eq. (3.24a).

Similarly for the $(n+m)^{\text{th}}$ KdV hierarchy, we have

$$\sum_{i=1}^{n+m-1}(\hat{\Omega}_{i2}[u]\cdot 1)\partial^{n+m-i-1} = [(A^{\frac{2}{n+m}})_+, A]\,, \tag{3.29}$$

and

$$\sum_{i=1}^{n+m-1}(\hat{\Omega}_{i1}[u]\cdot J)\partial^{n+m-i-1} = \left(J\partial + \frac{n+m+1}{2}J'\right)A - A\left(J\partial - \frac{n+m-1}{2}J'\right)\,, \tag{3.30}$$

which guarantees the coincidence between Eq. (3.2b) and Eq. (3.24b). This completes the proof of Proposition 3.3.

Proposition 3.3 means that $W_{n+m} \oplus W_m \oplus U(1)$ is, modulo a Miura transformation, the second Hamiltonian structure of the $(n,m)^{\text{th}}$ KdV hierarchy. Due to the uniqueness of the second Poisson structure, it is just the $W(n,m)$-algebra (3.14), up to a Miura transformation. This completes the proof of Theorem 3.2.

As a direct consequence of Theorem 3.2, we have

Corollary 3.4. *Suppose that we have a W-algebra specified in Proposition 3.3, introduce an infinite set of fields w_i's by Eq. (3.6), then the w_i's satisfy the infinite dimensional Poisson algebras (3.15) and (3.16).*

Let us make one final remark. The relation between ϕ and J is the typical vertex operator relation that allows us to express interacting fields in terms of free fields in chirally split 2D conformal field theories. In this case ϕ plays the role of the vertex operator and J is the derivative of a free field. Since it is well-known that W_n algebras are representable by means of free fields, this implies that $W(n,m)$ algebras can also be represented by means of free fields.

4. Drinfeld–Sokolov Representation

In this section we will derive the Drinfeld–Sokolov representation of the $(n, m)^{\text{th}}$ KdV hierarchy, and analyze other ways to determine the $W(n, m)$-algebra.

4.1. Infinitesimal Deformation of the Lax Operator. As we did in Sect. 2, we first construct the associated linear spectral system of Eqs. (3.12),

$$L_{[n,m]}\psi = \lambda\psi \,, \tag{4.1}$$

$$\frac{\partial}{\partial t_r}\psi = (L_{[n,m]}^{\frac{r}{n}})_+\psi \,. \tag{4.2}$$

Once again ψ is a Baker–Akhiezer function. We are going to show that calculating the $W(n, m)$-algebra is equivalent to finding two infinitesimal differential operators P and Q, such that

$$\delta L_{[n,m]} = QL_{[n,m]} - L_{[n,m]}P \,, \tag{4.3}$$

and $(L_{[n,m]} + \delta L_{[n,m]})$ has the same form as $L_{[n,m]}$. This property reflects the symmetry of the spectral equation (4.1) at $\lambda = 0$, i.e.

$$\delta\psi = P\psi, \qquad (L_{[n,m]} + \delta L_{[n,m]})(\psi + \delta\psi) = 0 \,. \tag{4.4}$$

Proposition 4.1. *Equation* (4.3) *coincides with the second Poisson structure* (3.14), *if we choose*

$$P = (YL_{[n,m]})_+ - \frac{1}{n}Z, \qquad Q = (L_{[n,m]}Y)_+ - \frac{1}{n}Z \,, \tag{4.5}$$

where $Y = \sum_{i=1}^{\infty}\partial^{i-n}\varepsilon_i$ *is an arbitrary infinitesimal pseudodifferential operator, and*

$$Z = \int^{x}([L_{[n,m]}, Y]_{(-1)}) \,. \tag{4.6}$$

The proof is the same as the proof of Proposition 2.1. Although the formulas (4.3) and (4.5) have the same form as Eq. (2.8) and Eq. (2.11), one should keep in mind that the differential part of Y now plays an important role.

In the case $Y = \partial^{1-n}\varepsilon$, we have

$$\delta L_{[n,m]} = \left(\varepsilon\partial + \frac{n+1}{2}\varepsilon'\right)L_{[n,m]} - L_{[n,m]}\left(\varepsilon\partial - \frac{n-1}{2}\varepsilon'\right) \,.$$

As a consequence, a_1 satisfies the Virasoro algebra

$$\{a_1, a_1\} = (c_n\partial^3 + a_1\partial + \partial a_1)\delta(x - y) \,, \tag{4.7}$$

moreover ψ and $(L_{[n,m]}\psi)$ transform like primary fields with conformal weights $(\frac{1-n}{2})$ and $(\frac{n+1}{2})$, respectively. In other words, the scalar Lax operator (3.11) transforms covariantly under diffeomorphisms

$$\begin{cases} x \to f(x) \,, \\ L_{[n,m]} \to (f'(x))^{-\frac{n+1}{2}}\widetilde{L}_{[n,m]}(f'(x))^{-\frac{n-1}{2}} \,, \end{cases} \tag{4.8}$$

where

$$\widetilde{L}_{[n,m]} = \partial^n + \widetilde{a}_1(x)\partial^{n-2} + \cdots \,.$$

Obviously, under this transformation, the differential part and pseudo-differential part do not intertwine. In other words, they transform separately in a covariant way. Therefore we have following results:

(i) The a_l fields can be separated into two subsets; the fields $a_l(1 \leqq l \leqq n-1)$ (the first subset) have the conformal properties as the fields of the ordinary W_n-algebra.

(ii) Each factor of the pseudodifferential part must be a conformal covariant operator, which implies that $a_{n+l}(0 \leqq l \leqq m-1)$ are primary fields, and

$$\frac{1}{\partial - S_j} \to (f'(x))^{\frac{n+2j-1}{2}} \frac{1}{\partial - \widetilde{S_j}} (f'(x))^{-\frac{n+2j-3}{2}} \ ,$$

which results in the following Poisson brackets:

$$\{a_1, S_j\}_2 = \left(\frac{n+2j-1}{2} \partial^2 + S_j \partial \right) \delta(x-y) \ . \tag{4.9}$$

Summing over j, and recalling the definition of the field J, we get

$$\{a_1, J\}_2 = (\partial^2 + J\partial)\delta(x-y), \qquad \{a_1, \phi\}_2 = \phi\delta'(x-y) \ . \tag{4.10}$$

In other words, the field ϕ is a primary field with conformal weight one. If we define

$$w = \frac{1}{4}nm(n+m)\left(\frac{1}{2}J^2 + J'\right) \ ,$$

then it is easy to see that

$$\{w, a_1\} = (c_\phi \partial^3 + w\partial + \partial w)\delta(x-y), \qquad c_\phi = -\frac{1}{4}nm(n+m) \ . \tag{4.11}$$

(iii) The conformal properties of the operator $L_{[n,m]}$ and the field ϕ imply that

$$\begin{cases} A \to (f'(x))^{-\frac{n+m+1}{2}} \widetilde{A}(f'(x))^{-\frac{n+m-1}{2}} \ , \\ B \to (f'(x))^{-\frac{m+1}{2}} \widetilde{B}(f'(x))^{-\frac{m-1}{2}} \ . \end{cases} \tag{4.12}$$

We immediately recognize that the differential operators A, B transform in the same way as the Lax operator of the ordinary KdV hierarchy under diffeomorphisms. As a result, we have

$$\{u_1, a_1\} = (c_{n+m}\partial^3 + u_1\partial + \partial u_1)\delta(x-y), \qquad c_{n+m} = \frac{1}{2}\binom{n+m+1}{3} \ , \tag{4.13}$$

$$\{v_1, a_1\} = (-c_m\partial^3 + v_1\partial + \partial v_1)\delta(x-y), \qquad c_m = \frac{1}{2}\binom{m+1}{3} \ . \tag{4.14}$$

Combining (4.11), (4.13), and (4.14), we get (4.7), which guarantees consistency. For $a_1 = u_1 + v_1 + w$ and the set of fields $(u_i), (v_j)$ and J are mutually commutative. Equations (4.7), (4.11) and (4.13, 4.14) are four copies of Virasoro algebras; the total central charge

$$c = c_{n+m} - c_m + c_\phi = c_n \tag{4.15}$$

depends only on the order of Lax operator $L_{[n,m]}$, no matter what m is. The field ϕ just tunes the conformal weight so that $L_{[n,m]}$ is a conformal operator with weight n.

4.2. Drinfeld–Sokolov Representation. In order to construct the Drinfeld–Sokolov representation of the $(n,m)^{\text{th}}$ KdV hierarchy, we introduce a set of auxiliary fields

$$\psi_1 = \psi, \qquad \psi_{i+1} = \partial \psi_i = \partial^i \psi, \quad 1 \leq i \leq n-1;$$

$$(\partial - S_{i+1})\psi_{-i} = \psi_{1-i}, \quad 0 \leq i \leq m-1. \tag{4.16}$$

In this way, we obtain the linearized version of the spectral equation (4.1)

$$\mathscr{L}\Psi = 0, \quad \mathscr{L} = \partial + U - \Lambda; \tag{4.17}$$

with

$$U = -\sum_{i=0}^{m-1} S_{m-i}E_{-i,-i} + \sum_{i=1}^{n+m-1} a_i E_{n,n-i}, \tag{4.18}$$

and

$$\Lambda = \lambda E_{n,1} + \sum_{i=-m+1}^{n-1} E_{i,i+1}. \tag{4.19}$$

In order to derive the flow equation of Ψ, we need the following lemmas.

Lemma 4.2. *Let P be a differential operator $P = \sum_{i=0}^{k} p_i \partial^i$ with order k smaller than m, then it is always possible to find a set of $\widetilde{p}_i$ which are differential polynomials of p_i and S_i, such that*

$$P = \widetilde{p}_0 + \sum_{i=1}^{k} \widetilde{p}_i (\partial - S_{m-i+1}) \cdots (\partial - S_{m-1})(\partial - S_m).$$

The proof is straightforward.

Lemma 4.3. *Let (i,j,r) be non-negative integers and $1 \leq j,r \leq n-1$. Then*

$$\partial^j (L_{[n,m]}^{\frac{r}{n}+i})_+ = (\partial^j L_{[n,m]}^{\frac{r}{n}-1})_+ L_{[n,m]}^{i+1} - (\partial^j (L_{[n,m]}^{\frac{r}{n}+i})_-)_+$$

$$+ \sum_{l=0}^{i} [((\partial^j L_{[n,m]}^{\frac{r}{n}+l-1})_- L_{[n,m]})_+ + ((\partial^j L_{[n,m]}^{\frac{r}{n}+l-1})_+ L_{[n,m]})_-] L_{[n,m]}^{i-l}. \tag{4.20}$$

Furthermore,

$$\left(\partial^{j-1} L_{[n,m]}^{\frac{r}{n}+l-1}\right)_{(-1)} = \frac{\delta H_{r+nl}}{\delta a_{n-j}}. \tag{4.21}$$

The proof is similar to the proof of Lemma 2.2. The appearance of the second term in the square bracket reflects the fact that $L_{[n,m]}$ is not a purely differential operator. Since

$$\frac{\partial}{\partial t_{r+ni}} \psi_j = \partial^{j-1} \frac{\partial}{\partial t_{r+ni}} \psi = \partial^{j-1} (L_{[n,m]}^{r+ni})_+ \psi$$

the above two lemmas imply that the RHS can be represented linearly in $\psi_k(-m+1 \leq k \leq n)$, and the coefficient in front of ψ_n is $\sum_{l=0}^{i} \lambda^{i-l} \frac{\delta H_{r+nl}}{\delta a_{n-j}}$.

Now let us turn our attention to the derivation of the equations of motion for the elements ψ_{-j} $(0 \leqq j \leqq m - 1)$. From the definition (4.16), we have

$$\frac{\partial}{\partial t_{r+ni}} \psi_{-j} = \left(\sum_{l=1}^{j+1} \frac{1}{\partial - S_j} \cdots \frac{1}{\partial - S_l} \left(\frac{\partial}{\partial t_{r+ni}} S_l \right) \frac{1}{\partial - S_l} \cdots \frac{1}{\partial - S_1} \right.$$

$$\left. + \frac{1}{\partial - S_{j+1}} \cdots \frac{1}{\partial - S_1} (L_{[n,m]}^{\frac{r}{n}+l})_+ \right) \psi .$$

Define $O_1 = (L_{[n,m]}^{\frac{r}{n}})_+$, then Eq. (3.12) guarantees the following recursion relations:

$$(\partial - S_i) O_{i+1} - O_i (\partial - S_i) = \frac{\partial}{\partial t_{r+ni}} S_i, \quad 1 \leqq i \leqq m$$

or equivalently

$$O_{i+1} \frac{1}{\partial - S_i} = \frac{1}{\partial - S_i} O_i + \frac{1}{\partial - S_i} \left(\frac{\partial}{\partial t_{r+ni}} S_i \right) \frac{1}{\partial - S_i} .$$

These equations completely determine the operators O_{i+1} and the equations of motion $\frac{\partial}{\partial t_r} S_i$. Applying this procedure, we are able to get

$$\frac{\partial}{\partial t_{r+ni}} \psi_{-j} = O_{j+1} \frac{1}{\partial - S_j} \cdots \frac{1}{\partial - S_1} \psi .$$

In the spirit of the above lemmas, we can rewrite the RHS of the above expression into the form $V_1 \cdot \Psi$, where the elements of row vector V_1 are Taylor series in λ and differential polynomials in q. All these results together lead to the following proposition:

Proposition 4.4. (i) *Equation (4.2) can be uniquely rewritten in matrix form,*

$$\frac{\partial}{\partial t_{r+ni}} \Psi = \mathcal{M}_{r+ni} \Psi , \tag{4.22}$$

where the elements of M_{r+ni} are Taylor series in λ, while they are differential polynomials of the fields (a_i, S_j), in particular

$$(\mathcal{M}_{r+ni})_{n-j,n} = \sum_{0 \leqq l \leqq i} \lambda^{i-l} \frac{\delta H_{r+nl}}{\delta a_j} . \tag{4.23}$$

(ii) *The consistency conditions of Eqs. (4.17, 4.22) form the generalized Drinfeld–Sokolov hierarchy*

$$\frac{\partial}{\partial t_{r+ni}} \mathcal{L} = [\mathcal{M}_{r+ni}, \mathcal{L}] . \tag{4.24}$$

Example. The second flow corresponds to the following matrix field:

$$\mathcal{M}_2 = (U - \Lambda)^2 + \frac{2}{n} \sum_{m+1 \leqq j \leqq l \leqq n+m} \binom{i - m - 1}{j - m - 1} a_1^{(l-j)} E_{ij} - 2 \sum_{l=1}^{m-i+1} S_l' E_{il} .$$

Since Λ in general is irregular, so the integrable hierarchy (4.24) is a type II generalized Drinfeld–Sokolov integrable hierarchy in the terminology of reference 3.

Proposition 4.5. *The spectral equation with* $\lambda = 0$,

$$(\partial + U - I)\Psi = 0 \tag{4.25}$$

is invariant under the following infinitesimal transformations:

$$\Psi \longrightarrow G\Psi, \qquad \mathscr{L}_0 \longrightarrow G\mathscr{L}_0 G^{-1}, \quad G = 1 + R. \tag{4.26}$$

The infinitesimal matrix field R *satisfies*

$$\delta U = [R, \partial + U - I], \tag{4.27}$$

and $(U + \delta U)$ *has the same form as* U. *The elements of* R *are polynomials of the basic fields* u_i's, *in particular*

$$R_{n-j,n} = \frac{\delta F}{\delta a_j} \equiv \varepsilon_j \tag{4.28}$$

with F *being an arbitrary infinitesimal functional.*

  Equation (4.27) *completely determines the upper triangular part of the matrix field* R, *and its main diagonal line* (*except the trace*). *Further,*

$$\delta a_i = \sum_{j=1}^{m} (\widetilde{\mathscr{N}}_1)_{ij} \delta S_j + \sum_{j=1}^{n+m-1} (\widetilde{\mathscr{N}}_2)_{ij} \varepsilon_j, \quad 1 \leqq i \leqq n + m - 1, \tag{4.29}$$

where $\widetilde{\mathscr{N}}_1$ *and* $\widetilde{\mathscr{N}}_2$ *are certain operatorial matrices of dimensions* $(n + m - 1) \times m$, *and* $(n + m - 1) \times (n + m - 1)$ *respectively.*

Proof. The first and the second statements can be proved in the exactly the same way as Proposition 2.3. In order to prove the third statement, we denote $R = \sum_{1-n-m}^{n+m-1} R_l$ and $U = \sum_{1-n-m}^{n+m-1} U_l$, where $R_l(U_l)$ means the pseudo-diagonal line $R_{i,j}(U_{i,j})$ with $i - j = l$. Then Eq. (4.27) shows

$$[R_l, I] = -R'_{l+1} + \sum_{k=1-n-m} [R_{l-k+1}, U_k], \quad \forall l \geqq 0. $$

Therefore we can recursively solve this equation for all $R_l(l \geqq 0)$, which are differential polynomials in (ε_i, q_j). When $l < 0$, Eq. (4.27) means

$$[R_l, I] = -\delta U_{l+1} - R'_{l+1} + \sum_{k=1-n-m}^{0} [R_{l-k+1}, U_k], \quad \forall l < 0. \tag{4.30}$$

Solving these equations recursively, we obtain Eq. (4.29). This ends the proof.

  Now let us suppose F to be an arbitrary infinitesimal functional, define

$$\frac{\delta F}{\delta a_j} = \varepsilon_j, \quad (1 \leqq j \leqq n + m - 1), \qquad \frac{\delta F}{\delta S_j} = \theta_j, \quad 1 \leqq j \leqq m.$$

With respect to the second Poisson structure (3.14), F generates the infinitesimal transformation

$$\delta a_i = \sum_{j=1}^{n+m-1} (\hat{\mathscr{P}}_1)_{ij} \varepsilon_j + \sum_{j=1}^{m} (\hat{\mathscr{K}}_1)_{ij} \theta_j; \tag{4.31}$$

$$\delta S_i = \sum_{j=1}^{n+m-1} (\hat{\mathscr{K}}_2)_{ij} \varepsilon_j + \sum_{j=1}^{m} (\hat{\mathscr{P}}_2)_{ij} \theta_j. \tag{4.32}$$

Since the operatorial matrices $\hat{\mathcal{K}}_1, \hat{\mathcal{K}}_2$ are conjugate to each other, there are only three independent matrix differential operators, say, $\hat{\mathcal{P}}_1, \hat{\mathcal{P}}_2$ and $\hat{\mathcal{K}}_1$, which completely exhibit the structure of the $W(n,m)$-algebra. However, as shown in the previous proposition, the field-dependent gauge symmetry can only determine the matrix operators $\widetilde{\mathcal{N}}_1, \widetilde{\mathcal{N}}_2$ (in Eq. (4.29)). Comparing Eq. (4.29) and Eqs. (4.31, 4.32), we obtain

$$\hat{\mathcal{P}}_1 = \widetilde{\mathcal{N}}_1 \hat{\mathcal{K}}_2 + \widetilde{\mathcal{N}}_2, \quad \hat{\mathcal{K}}_1 = \widetilde{\mathcal{N}}_1 \hat{\mathcal{P}}_2, \tag{4.33}$$

while $\hat{\mathcal{P}}_2$ remains undetermined. Therefore the field-dependent gauge symmetry (4.27) is *necessary* but not sufficient to completely determine the structure of the $W(n,m)$-algebra (compare with Subsect. 2.2). This is quite a distinguished feature of the $(n,m)^{\text{th}}$ KdV hierarchy with $m \neq 0$. We would like to point out that this is not a negative aspect. In fact it is just this leftover arbitrariness which provides room for the W_∞-algebra.

4.3. On the Regularity Properties of Λ. Let us come now to the distinction mentioned in the introduction between type I and type II integrable hierarchies according to whether the constant element Λ in the DS system is regular or not. We have to study the regularity properties of the Λ's defined in Eq. (4.19).

We recall that an element Ξ, belonging to a finite dimensional Lie algebra $\mathcal{G}$ of rank r, is regular if

$$\dim \mathcal{G}(\Xi) = r, \tag{4.34}$$

where

$$\mathcal{G}(\Xi) = \{X \in \mathcal{G} : (\mathrm{ad}_\Xi)^k X = 0 \text{ for some } k = 1,2,\ldots\}.$$

It is known that, if Ξ is regular, then $\mathcal{G}(\Xi)$ is a Cartan subalgebra of $\mathcal{G}$.

Our elements Λ belong to the loop algebra (with loop parameter λ): the regularity property has to be understood with respect to their projection on the relevant finite dimensional Lie algebra.

We have checked this regularity property for Λ, defined by Eq. (4.19), by direct calculation in the simplest cases, up to $n + m = 5$ (Λ is imbedded in sl_{n+m} except when $n = 1$, in which case it is understood to be imbedded in gl_{n+m}). It turns out that Λ *is regular if $m = 0,1$, while it is not regular in the other cases.* We have further studied the diagonalizability properties of Λ in more general cases, which confirms the above statement. This leads us to the conjecture that the above statement is true for any n and m. We are therefore oriented to believe that the (n,m) hierarchies are type I if $m = 0,1$, while they are type II in the other cases.

Altogether we can conclude that the distinction between type I and II integrable hierarchies has a technical meaning in the Drinfeld–Sokolov context, but does not seem to have any relevance in the Gelfand–Dickii context.

5. Examples

In this section we present several examples to exhibit our construction of the $(n,m)^{\text{th}}$ KdV hierarchy, and to show the decomposition of the $W(n,m)$-algebra into the direct sum. For practical reasons we will proceed in reverse order with respect to the demonstrations given so far. We will consider the hierarchies and the $W(n,m)$-algebras explicitly given in [7,20], and, subsequently, work out the corresponding Miura maps. We show that the modified Poisson algebras are a direct

sum of W_m, W_{n+m} and $U(1)$, and the modified integrable equations coincide with the ones given by Eqs. (3.2a–3.2c).

$(1,1)^{\text{th}}$ *KDV hierarchy.* The simplest case of the $(n,m)^{\text{th}}$ KdV hierarchy is with $n = m = 1$. We choose it as our first example because it has attracted a lot of attention both from mathematicians [15] and physicists [16] in the past few years. The $(1,1)^{\text{th}}$ KdV hierarchy is derived from one random matrix model [8],

$$\frac{\partial}{\partial t_r} L_{[1,1]} = [(L_{[1,1]}^r)_+, L_{[1,1]}], \quad L_{[1,1]} = \partial + a_1 \frac{1}{\partial - S_1} . \tag{5.1}$$

The second flow equations are

$$\frac{\partial}{\partial t_2} a_1 = a_1'' + 2(a_1 S_1)', \qquad \frac{\partial}{\partial t_2} S_1 = (2a_1 + S_1^2 - S_1')' . \tag{5.2}$$

The $W(1,1)$-algebra is

$$\{a_1, a_1\} = (a_1 \partial + \partial a_1)\delta(x - y), \qquad \{a_1, S_1\} = (\partial^2 + S_1 \partial)\delta(x - y),$$

$$\{S_1, S_1\} = 2\delta'(x - y) . \tag{5.3}$$

Define a map

$$L_{[1,1]} = \phi^{-\frac{1}{2}}(\partial^2 + u)\phi^{-\frac{1}{2}}\partial^{-1}\phi, \qquad u = a_1 - \frac{1}{4}S_1^2 - \frac{1}{2}S_1', \qquad J = S_1 , \tag{5.4}$$

then the field u satisfies Virasoro algebra

$$\{u, u\} = \left(\frac{1}{2}\partial^3 + u\partial + \partial u\right)\delta(x - y), \quad \{J, J\} = 2\delta'(x - y) . \tag{5.5}$$

The flow equations become

$$\frac{\partial}{\partial t_2} u = \frac{1}{2}J''' + 2uJ' + u'J, \qquad \frac{\partial}{\partial t_2} J = 2u' + 3JJ' . \tag{5.6}$$

It is perhaps worth giving the third flow equations:

$$\frac{\partial}{\partial t_3} u = u''' + 6uu' + \frac{3}{4}(J^2)''' + 6uJJ' + \frac{3}{2}J^2u' ,$$

$$\frac{\partial}{\partial t_3} J = \left(6Ju + \frac{5}{2}J^3 + J''\right)' .$$

It is easy to see that this set of integrable equations extend the famous KdV equation by an additional boson field. The Gelfand–Dickii Poisson brackets give rise to two infinite dimensional Poisson algebras, which are referred to as two boson representations of the $W_{1+\infty}$-algebra and the W_∞-algebra, respectively. This hierarchy has also a modified Lax pair representation [17]

$$\frac{\partial}{\partial t_r} L_{\text{mod}} = [((L_{\text{mod}}^r)_{\geqq 1}^\dagger)^\dagger, L_{\text{mod}}] \tag{5.7}$$

with the modified Lax operator

$$L_{\text{mod}} = \phi L_{[1,1]}\phi^{-1} = \partial + J + h\partial^{-1}, \quad h = u + \frac{1}{4}J^2 + \frac{1}{2}J' . \tag{5.8}$$

As a consequence, the Gelfand–Dickii Poisson bracket should be modified simultaneously [18].

$(2,1)^{\text{th}}$ *KdV hierarchy.* This is the next simpler case. The hierarchy is [19, 20]

$$\frac{\partial}{\partial t_r}L_{[2,1]} = [(L_{[2,1]}^{\frac{r}{2}})_+, L_{[2,1]}], \quad L_{[2,1]} = \partial^2 + a_1 + a_2\frac{1}{\partial - S_1}. \tag{5.9}$$

The second flow equations are

$$\frac{\partial}{\partial t_2}a_1 = 2a_2', \qquad \frac{\partial}{\partial t_2}a_2 = (a_2' + 2a_2S_1)', \qquad \frac{\partial}{\partial t_2}S_1 = (a_1 + S_1^2 - S_1')'. \tag{5.10}$$

The Poisson algebra is

$$\{a_1, a_1\} = \left(2a_1\partial + a_1' + \frac{1}{2}\partial^3\right)\delta(x-y), \qquad \{a_1, a_2\} = (3a_2\partial + 2a_2')\delta(x-y),$$

$$\{a_1, S_1\} = \left(\frac{3}{2}\partial^2 + S_1\partial\right)\delta(x-y), \qquad \{S_1, S_1\} = \frac{3}{2}\delta'(x-y),$$

$$\{a_2, a_2\} = [(2a_2' + 4a_2S_1)\partial + a_2'' + 2(a_2S_1)']\delta(x-y),$$

$$\{a_2, S_1\} = (a_1\partial + (\partial + S_1)^2\partial)\delta(x-y). \tag{5.11}$$

The relevant map is

$$L_{[2,1]} = \phi^{-\frac{1}{2}}(\partial^3 + u_1\partial + u_2)\phi^{-1}\partial^{-1}\phi^{\frac{3}{2}}$$

with

$$J = \frac{2}{3}S_1, \qquad u_1 = a_1 - \frac{1}{3}S_1^2 - S_1',$$

$$u_2 = a_2 - \frac{2}{3}a_1S_1 - \frac{2}{27}S_1^3 - \frac{2}{3}S_1S_1' - \frac{2}{3}S_1''. \tag{5.12}$$

Then the fields (u_1, u_2) satisfy the W_3-algebra

$$\{u_1, u_1\} = (2\partial^3 + u_1\partial + \partial u_1)\delta(x-y),$$

$$\{u_1, u_2\} = (u_2\partial + 2\partial u_2 - \partial^2 u_1 - \partial^4)\delta(x-y),$$

$$\{u_2, u_2\} = [\partial^2 u_2 - u_2\partial^2 - \frac{2}{3}(u_1 + \partial^2)(\partial u_1 + \partial^3)]\delta(x-y). \tag{5.13}$$

Equation (5.10) becomes

$$\frac{\partial}{\partial t_2}u_1 = 2u_2' - u_1'' + 2J''' + 2u_1J' + u_1'J,$$

$$\frac{\partial}{\partial t_2}u_2 = u_2'' - \frac{2}{3}(u_1u_1' + u_1''') + J'''' + u_1J'' + 3u_2J' + u_2'J,$$

$$\frac{\partial}{\partial t_2}J = \frac{2}{3}u_1' + 4JJ'. \tag{5.14}$$

Once again this coincides with Eqs. (3.2a–3.2c). This equation is an extended version of the Boussinesq equation (in parametric form) by the addition of one further boson field. The various free field representations of the Poisson algebra have been given in [19].

$(1,2)^{th}$ *KdV hierarchy.* This hierarchy has been studied in [20]:

$$\frac{\partial}{\partial t_r}L_{[1,2]} = [(L^r_{[1,2]})_+, L_{[1,2]}], \quad L_{[1,2]} = \partial + a_1\frac{1}{\partial - S_1} + a_2\frac{1}{\partial - S_2}\frac{1}{\partial - S_1}. \quad (5.15)$$

The second flow equations are

$$\frac{\partial}{\partial t_2}a_1 = a_1'' + 2a_2' + 2(a_1S_1)',$$

$$\frac{\partial}{\partial t_2}a_2 = a_2'' + 2a_2'S_2 + 2a_2(S_1 + S_2)',$$

$$\frac{\partial}{\partial t_2}S_1 = 2a_1' + 2S_1S_1' - S_1'',$$

$$\frac{\partial}{\partial t_2}S_2 = 2a_1' + 2S_2S_2' - S_2'' - 2S_1''. \quad (5.16)$$

The $W(1,2)$-algebra is

$$\{a_1, a_1\}_2 = (2a_1\partial + a_1')\delta(x-y), \quad \{a_1, a_2\}_2 = (3a_2\partial + 2a_2')\delta(x-y),$$

$$\{a_1, S_1\}_2 = (\partial^2 + S_1\partial)\delta(x-y), \quad \{a_1, S_2\}_2 = (2\partial^2 + S_2\partial)\delta(x-y),$$

$$\{a_2, a_2\}_2 = [(2a_2' + 4a_2S_2 - 2a_2S_1)\partial + a_2'' + (2a_2S_2 - a_2S_1)']\delta(x-y),$$

$$\{a_2, S_2\}_2 = (a_1\partial + (\partial + S_2)(\partial + S_2 - S_1)\partial)\delta(x-y),$$

$$\{S_1, S_1\}_2 = 2\delta'(x-y), \quad \{a_2, S_1\}_2 = 0,$$

$$\{S_1, S_2\}_2 = \delta'(x-y), \quad \{S_2, S_2\}_2 = 2\delta'(x-y). \quad (5.17)$$

The Miura map we need is

$$L_{[1,2]} = \phi^{-1}(\partial^3 + u_1\partial + u_2)\phi^{-\frac{1}{2}}(\partial^2 - v_1)^{-1}\phi^{\frac{3}{2}}$$

with

$$J = \frac{1}{3}(S_1 + S_2), \quad v_1 = \frac{1}{4}(S_2 - S_1)^2 + \frac{1}{2}(S_2 - S_1)',$$

$$u_1 = a_1 - \frac{1}{3}(S_1^2 + S_2^2 - S_1S_2) - S_2',$$

$$u_2 = a_2 + \frac{1}{3}a_1(S_1 - 2S_2) + \frac{1}{27}(S_1 + S_2)(5S_1S_2 - 2S_1^2 - 2S_2^2)$$

$$\qquad - \frac{1}{3}(S_1 - 2S_2)(S_1 - S_2)' + \frac{1}{3}(S_1 - 2S_2)''. \quad (5.18)$$

We remark that this is the first non-trivial Miura map we encountered till now, since the maps (5.4) and (5.12) are invertible, so as to be just redefinitions of

the fields. The fields (u_1, u_2) satisfy the W_3-algebra (5.13), while v_1 satisfies the Virasoro algebra

$$\{v_1, v_1\} = \left(-\frac{1}{2}\partial^3 + v_1\partial + \partial v_1\right)\delta(x - y) \tag{5.19}$$

with negative central charge. The Poisson bracket of the field J is

$$\{J, J\} = \frac{2}{3}\delta'(x - y). \tag{5.20}$$

The modified second flow equations are

$$\frac{\partial}{\partial t_2}u_1 = 2u_2' - u_1'' + 4J''' + 4u_1J' + 2u_1'J,$$

$$\frac{\partial}{\partial t_2}u_2 = u_2'' - \frac{2}{3}(u_1u_1' + u_1''') + 2J''' + 2u_1J'' + 6u_2J' + 2u_2'J,$$

$$\frac{\partial}{\partial t_2}v_1 = -\frac{3}{2}J''' + 6v_1J' + 3v_1'J,$$

$$\frac{\partial}{\partial t_2}J = \frac{4}{3}u_1' + 2v_1' + 5JJ'. \tag{5.21}$$

This set of equations describe the coupling of the Boussinesq equation and KdV equation. It is worthwhile studying further.

In this paper, we have shown how to construct a new integrable hierarchy from two KdV hierarchies, in particular we shown that the corresponding $W(n, m)$-algebra can be decomposed into a direct sum of the ordinary W_n-algebras. The Lie algebra structure of the $W(n, m)$-algebra has been discussed in [22]. There are still many problems which should be understood better. First, it has been noticed that the $W(1, 1)$-algebra can be constructed from $sl(2)$ Kac–Moody algebra through coset construction [21]. It is interesting to check if this is true for all the other $W(n, m)$-algebras. Second, Dickey has observed that the $(n, 1)^{\text{th}}$ KdV can be viewed as a reduction of the KP hierarchy by fixing the additional symmetry [23]. It is perhaps also true for the $m > 1$ case. Finally the $(n, m)^{\text{th}}$ KdV hierarchy might play roles in the study of the low dimensional quantum gravity.

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