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Differentiable Circle Maps with a Flat Interval

J. Graczyk[1,★], L.B. Jonker[2], G. Świątek[3], F.M. Tangerman[4], J.J.P. Veerman[5]

[1] Institute of Mathematics, Warsaw University, ul.Banacha 2,02-097 Warszawa, Poland
[2] Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
[3] Mathematics Department, Princeton University, Princeton, NJ 08544, USA
[4] Institute of Mathematical Sciences, SUNY at Stony Brook, Stony Brook, NY 11794, USA
[5] Depto de matematica, U.F.P.E., Cidade Universitaria, Rua Prof. Luis Freire, Recife, PE 50740-540, Brazil

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Abstract: We study weakly order preserving circle maps with a flat interval, which are differentiable even on the boundary of the flat interval. We obtain estimates on the Lebesgue measure and the Hausdorff dimension of the non-wandering set. Also, a sharp transition is found from degenerate geometry to bounded geometry, depending on the degree of the singularities at the boundary of the flat interval.

1. Introduction

1.1. Maps with a Flat Interval. We consider degree one weakly order-preserving circle endomorphisms which are constant on precisely one arc (called the flat interval). Maps of this kind appear naturally in the study of Cherry flows on the torus (see [1]) and non-invertible circle endomorphisms (see [14]). They have been less thoroughly researched than homeomorphisms.

Because the maps we consider are continuous and weakly order preserving, they have a rotation number. If f is a map with a flat interval, and F a lifting of f to the real line, the rotation number $\rho(f)$ is the limit

$$\lim_{n\to\infty} \frac{F^n(x)}{n} \pmod 1 .$$

This limit exists for every x and its value is independent of x. In the discussion that follows in this paper, it will often be convenient to identify f and F and subsets of $\mathbf{S}^1$ with corresponding subsets of $\mathbf{R}$. We will do this without warning, as it will simplify the presentation. The dynamics of f is most interesting when the rotation number is irrational.

We study first the topology of the non-wandering set, then its geometry. Where the geometry is concerned, we discover a dichotomy. Some of our maps show a "degenerate universality" akin to what was found in a similar case considered by [2] and [19], while others seem to be subject to the "bounded geometry" regime

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characteristic of critical homeomorphisms (maps which instead of the flat interval have a single critical point).

Before we can explain our results more precisely, it is necessary to define our class and fix some notation.

Almost Smooth Maps with a Flat Interval. We consider the class of continuous circle endomorphisms f of degree one for which an arc U exists so that the following properties hold:

1. The image of U is one point.
2. The restriction of f to $\mathbf{S}^1 \setminus \overline{U}$ is a C^3-diffeomorphism onto its image.
3. Let (a, b) be a preimage of U under the projection of the real line to $\mathbf{S}^1$. On some right-sided neighborhood of b, f can be represented as

$$h_r((x-b)^{l_r})$$

for $l_r \geqq 1$, where h_r is a C^3-diffeomorphism on a two-sided neighbourhood of b.

Analogously, on a left-sided neighborhood of a, f equals

$$h_l((a-x)^{l_l}) .$$

The ordered pair (l_l, l_r) will be called the *critical exponent of the map.* If $l_l = l_r$ the map will be referred to as *symmetric*.

In the future, we will deal exclusively with symmetric maps, and we will assume that $h_r(x) = h_l(x) = x$. Moreover, from now on we restrict our attention to maps with an irrational rotation number.

It is not difficult to show that it is possible to effect $\mathbf{C}^3$ coordinate changes near a and b that will allow us to replace both h_r and h_l by the identity function.

Additional Assumption. We will additionally assume in the proof of the first part of Theorem 2 that the map f has negative Schwarzian derivative,

$$S f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0 .$$

Basic Notations. The critical orbit is of paramount importance in studying any one-dimensional system, so we will introduce a simplified notation for backward and forward images of U. Instead of $f^i(U)$ we will simply write $\underline{i}$. For example, $\underline{0} = U$. This convention will also apply to more complex expressions. For example, $f^{-3q_n-20}(\underline{0})$ will be abbreviated to $\underline{-3q_n - 20}$. Thus, underlined positive integers represent points, and underlined non-positive integers represent intervals.

Distance between Points-Conventions of Notation. Denote by $(a, b) = (b, a)$ the open shortest interval between a and b regardless of the order of these two points. The length of that interval in the natural metric on the circle will be denoted by $|a - b|$. Let us adopt now the following conventions of notation:

- $|\underline{-i}|$ stands for the length of the interval $\underline{-i}$.
- Consider a point x and an interval $\underline{-i}$ not containing it. Then the distance from x to the closer endpoint of $\underline{-i}$ will be denoted by $|(x, \underline{-i})|$, and the distance to the more distant endpoint by $|(x, \underline{-i}]|$.

• We define the distance between the endpoints of two intervals $\underline{-i}$ and $\underline{-j}$ analogously. For example, $|(\underline{-i}, \underline{-j})|$ denotes the distance between the closest endpoints of these two intervals while $|[\underline{-i}, \underline{-j})|$ stands for $|\underline{-i}| + |(\underline{-i}, \underline{-j})|$.

Combinatorics. Let q_n denote the closest returns associated with the rotation number $\rho(F)$ (see Chapter I, Sect. 1 in [12]). The reader is reminded that $f^{q_n}(U)$ is the first forward iterate of U to fall in the interval between $f^{q_{n-1}}(U)$ and $f^{-q_{n-1}}(U)$, and that $f^{-q_n}(U)$ is the first backwards iterate to do so. The integers q_n are the denominators of the nearest rational approximants of the rotation number, and they are connected by the recursive relation

$$q_{n+1} = a_n q_n + q_{n-1} ,$$

where the integers $a_n \geqq 1$ are the coefficients in the continued fraction expansion of the rotation number.

Scalings near Critical Point. We define a sequence of scalings

$$\tau_n := \frac{|\underline{0}, \underline{q_n}|}{|\underline{0}, \underline{q_{n-2}}|} .$$

These quantities measure "the geometry" in the proximity of the critical point and their asymptotic properties will constitute the main subject of our investigations. When $\tau_n \to \infty$ we say that the geometry of the mapping is "degenerate." When τ_n is bounded away from zero we say that the geometry is "bounded." "Universal geometry" is said to occur when the sequence τ_n converges.

Uniform Constants. The letter K with subscripts will be reserved for "uniform constants." If we claim a statement which involves such constants we mean precisely that for each occurrence of such a constant a positive value can be inserted which will make the statement true. The choice is uniform in the sense that once f has been fixed, there is a choice of values which makes the statement true in all cases covered. The use of the symbol K will be local, in that the same symbol may signify different uniform constants in different parts of the paper.

A Summary of Previous Related Results. Maps with the critical exponent $(1, 1)$ were studied first. The most complete account can be found in [18], and was inspired by [8]. They turn out to be expanding apart from the flat interval. Therefore, the geometry can be studied relatively easily. One of the results is that the scalings τ_n tend to zero fast.

Next, critical exponents $(1, l)$ or $(l, 1)$ were investigated for $l > 1$ independently in [2] and [19]. The main result was that τ_n still tends to zero. This was shown to lead to "degenerate universality" of the first return map on $(\underline{q_{n-1}}, \underline{q_n})$. Namely, as n grows, the branches of this map become at least C^1 close to either affine strongly expanding maps, or a composition of $x \to x^l$ with such maps.

Finally, we need to be aware of the results for critical maps where U is a point and the singularity is symmetric. The scalings can still be defined by the same formula, but they certainly do not tend to zero (see [5, 17 and 3]). Moreover, if the rotation number is the golden mean, then they are believed to tend to a universal limit (see [13]). This is an example of bounded geometry, and conjectured "non-degenerate" universality.

In this context, we are ready to present our results.

1.2. Statement of Results. We investigate symmetric almost smooth maps with a flat interval with the critical exponent (l, l), $l > 1$. Also, we permanently assume that the rotation number is irrational. First, we get results about the non-wandering set which are true for any l. As in [2] it is easy to see that the non-wandering set of f is $S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i} U$.

Theorem 1. *For any f with the critical exponent (l, l), $l > 1$, the set $\mathbf{S}^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U)$ has zero Lebesgue measure. Moreover, if the rotation number is of bounded type (i.e. if q_n/q_{n-1} are uniformly bounded), the Hausdorff dimension of the non-wandering set is strictly less than* 1.

Corollary to Theorem 1 (Compare Theorem C in [9]). *There are no wandering intervals, and any two maps from our class with the same irrational rotation number are topologically conjugate.*

Theorem 2 consists of three parts.

Theorem 2.

1. *If $l \leqq 2$ and under additional assumption that $Sf < 0$ we have that the scalings τ_n tend to zero at least exponentially fast.*

2. *For rotation numbers of bounded type and the critical exponent $l > 2$ the sequence τ_n is bounded away from zero.*

3. *Under the same assumptions as above, for rotation numbers of bounded type, we have that*

if $l = 2$ then

the sequence τ tends to zero exponentially. That is, there are two sequences of the form λ^n, where $0 < \lambda < 1$, which bound τ_n asymptotically from above and below,

if $l < 2$ then

the sequence τ_n converges to zero exponentially. This means that there are two sequences of the form $(\lambda(l))^{\mu(l)^n}$, where $0 < \lambda(l) < 1$ and $\mu(l) > 1$ which eventually bound τ_n from below and above.

Theorem 2 shows that a transition occurs from the "degenerate geometry" case to the "bounded geometry" case as the exponent passes through 2. This is the first discovery of bounded geometry behavior in maps with a flat interval. Recently, the same phenomenon was observed in [6] for one important class of S-unimodal maps of the interval, so called Fibonacci maps (see also [7]).

Numerical Findings. A natural question appears whether bounded geometry, when it occurs, is accompanied by non-trivial universal geometry. More precisely, we have two conjectures:

Conjecture 1. *For a map f from our class with the golden mean rotation number, the scalings τ_n tend to a limit.*

We found this conjecture supported numerically, albeit only for one map considered. Moreover, the rate of convergence appears to be exponential. The reader is referred to the Appendix for a detailed description of our experiment.

There is a much bolder conjecture:

Conjecture 2. *Consider two maps from our class with the same critical exponent larger than 2 and the same irrational (bounded type?) rotation number. Then, the conjugacy between them is differentiable at* $\underline{1}$ *(the critical value according to our convention).*

This conjecture is motivated by the analogy with the critical case. The same analogy (see [15]) makes us expect that Conjecture 2 would be implied by Conjecture 1 if the convergence in Conjecture 1 is exponential and the limit is independent of f.

1.3. Technical Tools

Cross-ratios. In this paper we will use two different cross-ratios, **Cr** and **Poin**. If $a < b < c < d$, then define their *cross-ratio* **Cr** by

$$\mathbf{Cr}(a,b,c,d) := \frac{|b-a||d-c|}{|c-a||d-b|},$$

and their *cross-ratio* **Poin** by

$$\mathbf{Poin}(a,b,c,d) := \frac{|d-a||b-c|}{|c-a||d-b|}.$$

Negative Schwarzian and Expanding Cross-Ratios. Diffeomorphisms with negative Schwarzian expand cross-ratios **Poin**:

$$\mathbf{Poin}(f(a), f(b), f(c), f(d)) > \mathbf{Poin}(a,b,c,d).$$

Distortion of the Cross-Ratio **Cr.** Here, we formulate the result which enables us to control a growth of the iterates of cross-ratios **Cr** even if maps are no longer homeomorphisms with negative Schwarzian nor are invertible.

The Cross-Ratio Inequality. Consider a chain of quadruples

$$\{a_i, b_i, c_i, d_i\} \quad i = 0, \ldots, n$$

such that each is mapped onto the next by the map f. If the following conditions hold:

1. Each point of the circle belongs to at most k of the intervals (a_i, d_i).
2. Intervals (b_i, c_i) do not intersect $\underline{0}$.

Then

$$\log \frac{\mathbf{Cr}(a_n, b_n, c_n, d_n)}{\mathbf{Cr}(a_0, b_0, c_0, d_0)} \leqq K_{[k]},$$

where the constant $K_{[k]}$ does not depend on the set of quadruples.

This cross-ratio inequality for critical circle homeomorphisms was introduced and proved by several authors (cf. [10, 11, 17, 20]). The above form is an easy modification of that of [17] and was stated and proved in [2].

How to Control Distortion of Ratios. We want to control the distortion of iterates of the inclusive ratios

$$\mathbf{R}(a,b,c) = \frac{|a-b|}{|a-c|}$$

defined on ordered triples of points $a < b < c$. To achieve this we will frequently use the procedure which consists in replacing an ordered triple by an ordered quadruple by adding the outermost endpoint of an appropriately chosen preimage of the flat interval, usually so that the resulting cross-ratio includes the initial-ratio as one of the factors. Then, we will apply the cross-ratio inequality for **Cr** or the expanding property for **Poin**.

More Notation. In order to avoid both ambiguities in notation and long and unreadable formulas while discussing cross-ratios, we will introduce other symbols to denote cross-ratios. Namely, both cross-ratio **Cr** and **Poin** are uniquely defined if we point out the extreme intervals (a,b) and (c,d). Thus, from now on we adopt the following notation to describe quadruples defining cross-ratios:

$$\{(a,b),(c,d)\} := \{a,b,c,d\}\ .$$

For example, if $\underline{-i} = (a,b)$ and $\underline{-j} = (c,d)$ we will write $\mathbf{Cr}(\underline{-i},\underline{-j})$ in place of $\mathbf{Cr}(a,b,c,d)$.

1.4. Geometric Bounds. The orbit of $\underline{0}$ for $0 \leqq i \leqq q_{n+1} + q_n - 1$ together with open arcs lying between successive points of the orbit constitute a partition of the circle referred to as the n_r^{th} *dynamic partition* $\mathscr{P}_n$. The properties of f are studied by analysing the geometry of this partition and its counterpart using the backward orbit of $\underline{0}$. The intervals of $\mathscr{P}_n$ consist of two types:

• The set of "long" intervals consists of the interval between $\underline{0}$ and $\underline{q_n}$ along with its forward images:

$$\mathscr{A}_n := \{(\underline{i}, \underline{q_n + i}) : 1 \leqq i \leqq q_{n+1} - 1\}\ ;$$

• the set of "short" intervals consists of the interval between $\underline{0}$ and $\underline{q_{n+1}}$ together with its forward images

$$\mathscr{B}_n := \{(\underline{q_{n+1} + i}, \underline{i}) : 1 \leqq i \leqq q_n - 1\}\ .$$

The dynamic partition produced by the first $q_n + q_{n+1}$ preimages of U is denoted $\mathscr{P}_{-n}$. It consists of

$$\mathscr{I}_n = \{\underline{-i} : 0 \leqq i \leqq q_{n+1} + q_n - 1\}\ ,$$

together with the gaps between these sets. As in the case of $\mathscr{P}_n$ there are two kinds of gaps, "long" and "short":

• The "long" gaps are the interval I_0^n between $\underline{q_n}$ and $\underline{0}$ and its preimages,

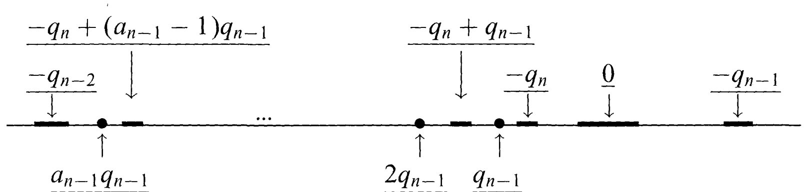
$$I_i^n := f^{-i}(I_0^n), \quad i = 0, 1, \ldots, q_{n+1} - 1\ .$$

• The "short" gaps are the interval I_0^{n+1} between $\underline{0}$ and $\underline{q_{n+1}}$ and its preimages,

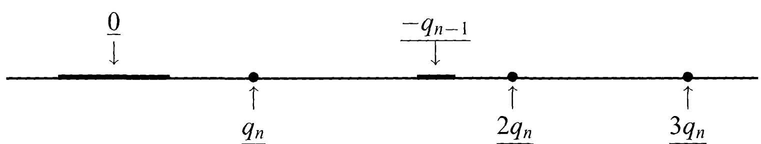
$$I_i^{n+1} := f^{-i}(I_0^{n+1}), \quad i = 0, 1, \ldots, q_n - 1\ .$$

When the order of the dynamic partition is increased, the short gaps in the old partition become long gaps in the new partition.

Several of the proofs in the following will depend strongly on the relative positions of the points and intervals of $\mathscr{P}_n$ and $\mathscr{P}_{-n}$. In reading the proofs the reader is advised to keep the following pictures in mind, which show some of these objects near the flat interval $\underline{0}$.



In the next picture we have enlarged the right-hand part of this picture to show the location of the points $\underline{q_n}, \underline{2q_n}$ and $\underline{3q_n}$ for the case $a_n = 1$.



We start by stating a standard fact which will be used frequently in the paper.

Fact 1. *Let three points with y between x and z be arranged so that, of the three, the point x is the closest to the flat interval. If f is a diffeomorphism on (x,z), the following inequality holds*:

$$\frac{|f(z)-f(y)|}{|f(z)-f(x)|} \leqq K\frac{|y-z|}{|z-x|},$$

where K is a uniform constant.

Lemma 1.1. *The ratio*

$$\frac{|(\underline{q_n}, \underline{3q_n})|}{|(\underline{0}, \underline{q_n})|}$$

is uniformly bounded away from zero.

Proof. In the proof of this lemma we will use the well-known method of "the shortest interval" (see [16 and 17]). Let J be the shortest arc belonging to the set $\mathscr{A}_n$. If J coincides with the interval between $\underline{q_n}$ and $\underline{0}$ then clearly the ratio is larger than 1/2, so we assume that one of the other elements of $\mathscr{A}_n$ is shorter.

By Fact 1 we have that

$$\frac{|(\underline{q_n+1}, \underline{3q_n+1})|}{|(\underline{1}, \underline{3q_n+1})|} \leqq K_1\frac{|(\underline{q_n}, \underline{3q_n})|}{|(\underline{0}, \underline{3q_n})|}. \tag{1}$$

Let J be the i^{th} iterate of $(\underline{q_n+1}, \underline{1})$. Observe that the i^{th} image of each of the intervals $(\underline{q_n+1}, \underline{3q_n+1})$ and $(\underline{q_{n+1}-q_n+1}, \underline{1})$ covers an interval belonging to $\mathscr{A}_n$ and adjacent (but not necessarily contiguous) to J. These images lie on different sides of J. Therefore, by the choice of J, we conclude that the cross-ratio **Cr** of the i^{th} image of the quadruple marked out by the endpoints of these two intervals is larger than 1/4. Since all intermediate images of $(\underline{q_{n+1} - q_n + 1}, \underline{3q_n + 1})$ cover the circle at most three times, we conclude, by the cross-ratio inequality, that the cross-ratio **Cr** of the initial quadruple is greater than a uniform constant. Thus, the same holds for the ratio on the right-hand side of (1). The lemma follows. □

Proposition 1. *The sequence* $|(\underline{0}, \underline{q_n})|$ *tends to zero at least exponentially fast.*

Proof. Lemma 1.1 implies that there is a constant $K < 1$ so that

$$|(\underline{0}, \underline{q_n})| \leqq K|(\underline{0}, \underline{3q_n})| \,.$$

But $\underline{3q_n}$ lies between $\underline{0}$ and $\underline{q_{n-4}}$. Thus

$$|(\underline{0}, \underline{q_n})| \leqq K|(\underline{0}, \underline{q_{n-4}})| \,. \qquad \square$$

Remark. Proposition 1 implies in particular that for large values of n the restriction of f to the interval between $\underline{-q_n}$ and $\underline{0}$ is of the form x^l. We will use this fact repeatedly in the rest of the paper.

Proposition 2. *If A is a preimage of U belonging to $\mathscr{P}_{-n}$, and if B is one of the gaps adjacent to A, then $|A|/|B|$ is bounded away from zero by a constant that does not depend on n, A, or B.*

Proof. Let $\underline{-i}$ and $\underline{-j}$ be successive members of $\mathscr{I}_n$. Form the initial quadruple by taking the endpoints of these two intervals. Iterate the quadruple by f until one of the intervals $\underline{-i}$ and $\underline{-j}$ is mapped to $\underline{0}$. By the cross-ratio inequality, the cross-ratio $\mathbf{Cr}(\underline{-i}, \underline{-j})$ is larger than

$$K_1 \frac{|\underline{-\varepsilon}|}{|[\underline{-\varepsilon}, \underline{0})|} \,, \tag{2}$$

where ε is equal to either q_n or q_{n+1} and K_1 is a uniform constant. By Fact 1, (2) is larger than

$$K_2 \frac{|\underline{-\varepsilon + 1}|}{|[\underline{-\varepsilon + 1}, \underline{1})|} \,. \tag{3}$$

We now take the quadruple consisting of the endpoints of $\underline{-\varepsilon + 1}$ along with $\underline{1}$ and $\underline{2\varepsilon + 1}$. We iterate this quadruple $\varepsilon - 1$ times, and apply the cross-ratio inequality for **Cr**. If we then drop one of the factors in the initial cross-ratio **Cr** we obtain that (3) is larger than

$$K_3 \frac{|(\underline{\varepsilon}, \underline{3\varepsilon})|}{|(\underline{0}, \underline{3\varepsilon})|} \,,$$

which is, by Lemma 1.1, greater than a uniform constant. Combining all the above inequalities we finally get that the cross-ratio $\mathbf{Cr}(\underline{-i}, \underline{-j})$ is greater than a uniform constant which means in particular that the same holds for each of the two factors

$$\frac{|\underline{-i}|}{|[\underline{-i}, \underline{-j})|} \,, \quad \frac{|\underline{-j}|}{|(\underline{-i}, \underline{-j}]|} \,.$$

This establishes the proposition. □

The First Corollary to Proposition 2. *The sequence $\{\tau_n\}_{n=1}^{\infty}$ is bounded away from 1.*

Proof. To derive the corollary, we note that $\underline{0}, \underline{-q_{n-1}}$ and $\underline{-q_{n-1}+q_{n-2}}$ are adjacent elements of $\mathscr{I}_{n-2}$, and that $\underline{q_n}$ and $\underline{q_{n-2}}$ each lie in one of the gaps between them. This completes the proof. □

The Second Corollary to Proposition 2. *The lengths of the gaps of the dynamic partition $\mathscr{P}_{-n}$ tend to zero at least exponentially fast with n.*

Now, we are in position to refine the claim of Lemma 1.1.

Lemma 1.2. *The ratio*

$$\frac{|(\underline{q_n}, \underline{2q_n})|}{|(\underline{0}, \underline{2q_n})|}$$

is bounded away from zero by a uniform constant.

Proof. We may assume that

$$|(\underline{q_n}, \underline{2q_n})| \leqq \frac{1}{2}|(\underline{0}, \underline{2q_n})| .$$

Observe that if $q_{n+1} = q_n + q_{n-1}$, then $\underline{-q_{n-1}}$ lies between $\underline{q_n}$ and $\underline{2q_n}$ and so the ratio is bounded away from 0 by Proposition 2. In the opposite case, first apply f and then note that the resulting ratio is greater than the cross-ratio

$$\mathbf{Cr}(\underline{-q_n+1}, (\underline{q_n+1}, \underline{2q_n+1})) .$$

By the cross-ratio inequality, f^{q_n-1} can increase this cross-ratio by at most a uniform multiplicative constant. Therefore, by Fact 1,

$$\frac{|(\underline{q_n}, \underline{2q_n})|}{|(\underline{0}, \underline{2q_n})|} \geqq K_1 \frac{|(\underline{2q_n}, \underline{3q_n})|}{|(\underline{q_n}, \underline{3q_n})|} .$$

Now, the reasoning falls into two parts. First, if $|(\underline{2q_n}, \underline{3q_n})|$ is larger than $|(\underline{q_n}, \underline{2q_n})|$ then we are done by the above inequality. If not, then by Lemma 1.1,

$$\frac{|(\underline{q_n}, \underline{2q_n})|}{|(\underline{0}, \underline{2q_n})|} \geqq 2\frac{|(\underline{q_n}, \underline{3q_n})|}{|(\underline{0}, \underline{2q_n})|} \geqq K\frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{2q_n})|} \geqq \frac{K}{2} .$$ □

Analysing the position of the points q_n and $2q_n$ with respect to intervals of the dynamical partition $\mathscr{P}_{-n}$ we easily derive from Lemma 1.2 the following fact.

Fact 2. *The ratio*

$$\frac{|(\underline{0}, \underline{-q_{n+1}+q_n})|}{|(\underline{0}, \underline{-q_{n+1}+q_n}]|}$$

is uniformly less than 1.

Let us introduce a quantity, which will play an important role in the proof of Theorem 2,

$$\sigma_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{q_{n-1}})|} .$$

Lemma 1.3. *The sequence σ_n is bounded.*

Proof. We will show that the sequence $1/\sigma_n$ is uniformly bounded away from zero. Clearly,

$$\frac{1}{\sigma_n} \geqq \frac{|\underline{(q_{n-1},0)}|}{|\underline{(-q_{n-1},0)}|} \, .$$

By applying f to both intervals in this ratio we raise it to the l^{th} power (assuming that n is large). The resulting ratio is certainly larger than the cross-ratio $\mathbf{Cr}(\underline{(q_{n-1}+1,1),-q_{n-1}+1})$. We iterate it $q_{n-1}-1$ times. By the cross-ratio inequality and Fact 1 we have that

$$\left(\frac{1}{\sigma_n}\right)^l \geqq K_1 \frac{\underline{(2q_{n-1},q_{n-1})}|}{|\underline{(2q_{n-1},0)}|} \, ,$$

which is uniformly bounded away from zero by Lemma 1.2. □

2. Proof of Theorem 1

Proof of the First Claim. By Proposition 2 the complement of all preimages of I does not contain any density point with respect to the Lebesque measure. Hence, by the Lebesque Density Lemma the set of non-wandering points is of zero Lebesque measure.

Proof of the Second Claim. The claim concerning the Hausdorff dimension requires a longer argument. Suppose that the rotation number is of bounded type. Take the n^{th} partition $\mathscr{P}_{-n}$. The elements of $\mathscr{P}_{-(n+1)}$ subdivide the gaps $\mathscr{P}_{-n}$ in the following way:

$$I_i^n \subset \bigcup_{j=0}^{a_{n+1}-1} I_{i+q_n+jq_{n+1}}^{n+1} \cup I_i^{n+2} \, .$$

We pick α so that $0 < \alpha < 1$ and estimate

$$\sum (|I_i^n|^\alpha + |I_i^{n+1}|^\alpha) \, , \tag{4}$$

where $\sum$ denotes the sum over all gaps of the n^{th} partition $\mathscr{P}_{-n}$. By Proposition 2 it follows that there is a constant $\beta < 1$ so that

$$\sum_{j=0}^{a_{n+1}-1} |I_{i+q_n+jq_{n+1}}^{n+1}| \leqq \beta |I_i^n|$$

holds for all "long" gaps $I_{i+q_n+jq_{n+1}}^n$ of the n^{th} partition. In particular it means that the gaps of $\mathscr{P}_{-n}$ decrease uniformly and exponentially fast to zero while n tends to infinity. We use concavity of the function x^α to obtain that

$$\sum_{j=0}^{a_{n+1}-1} |I_{i+q_n+jq_{n+1}}^{n+1}|^\alpha \leqq |a_{n+1}|^{1-\alpha} \beta^\alpha |I_i^n|^\alpha \leqq |I_i^n|^\alpha$$

if α is close to 1. Hence (4) is a decreasing function of n. Consequently, the sum is bounded above. The only remaining point is to prove that for a given ε the gaps of $\mathscr{P}_{-n}$ constitute an ε-cover of Ω if n is large enough. But this is so by the first corollary to Proposition 2. This completes the proof of Theorem 1.

3. Proof of the First Part of Theorem 2

In this section we will introduce a set of inductive parameters which will describe the geometry of the dynamic partitions. Successive partitions are not independent and their geometries are related to each other by a recurrence formula. We will observe that the scalings in this recurrence will diminish to zero provided some initial estimates are satisfied. This finally will force at least exponentially fast decay of the geometry of dynamic partitions. The reader easily notes that the plan of the proof is very similar to that already used in [6 and 4]. The results are also in the same spirit even though the objects studied in these two other papers are very different from ours. Namely, the degree of the critical point at which a sudden change between "degenerate geometry" and "bounded geometry" takes place is equal to $l = 2$. Compare this result with the one of [3] stated for critical circle homeomorphisms. In that case the passage from the geometry of diffeomorphisms to "bounded geometry" occurs for $l = 1$.

The following set of ratios will be frequently used throughout this section. Along with τ_n these ratios serve as scalings relating the geometries of successive dynamic partitions:

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|} .$$

It is easy to see that $\alpha_n > \tau_n$. Thus, it is enough to prove the first claim of Theorem 2 for α_n in place of τ_n.

3.1. A Recursive Inequality. In this section we prove an inequality of the form

$$(\alpha_n)^l \leqq M_n(l)\, \alpha_{n-2}^2 ,$$

analogous to inequalities studied in [4 and 6]. The coefficients $M_n(l)$ depend on the sequence α_n.

Distortion of Ratios. In this section we will need only upper estimates of the distortion of ratios. Hence, when a ratio is replaced by a cross-ratio, it will always be the cross-ratio **Poin**, which is then expanded by iterates of f.

A Priori Estimates. Degenerate geometry will be forced if we can show that the product $\prod_{k=1}^{n} M_k(l)$ is less than 1 for large n. To do this we will need some initial estimates on α_n. As it turns out, it is sufficient to find an upper bound solely for $l = 2$. We have the following proposition:

Proposition 3. *The following upper bounds work asymptotically*:

- *for all* α_n

$$\alpha_n(l) < (0.55)^{2/l} ,$$

- *for at least every other* α_n

$$\alpha_n < (0.3)^{2/l} ,$$

- *If* $\alpha_n(l) > (0.3)^{2/l}$ *then either* $\alpha_n(l) < (0.44)^{2/l}$ *or* $\alpha_n(l) < (0.16)^{2/l}$.

The analysis of the a priori bounds does not use the main ideas of the paper and thus will be postponed until the next section.

Auxiliary Constants. We use the following auxiliary constants:

$$\sigma_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{q_{n-1}})|},$$

$$s_n = \frac{|[\underline{-q_{n-2}}, \underline{0}]|}{|\underline{0}|}.$$

By Proposition 1, the sequence s_n tends exponentially fast to one.

Proposition 4. *For n sufficiently large the following inequality holds*:

$$(\alpha_n)^l \leqq M_n(l)\,\alpha_{n-2}^2,$$

where

$$M_n(l) = s_{n-1}^2 \cdot \frac{2}{l} \cdot \left(\frac{1}{1+\sqrt{1-\frac{2(l-1)}{l}s_{n-1}\alpha_{n-1}}}\right) \cdot \frac{1}{1-\alpha_{n-2}} \cdot \frac{\sigma_n}{\sigma_{n-2}}.$$

3.1.1. Proof of the Proposition. By applying f to the intervals defining the ratio α_n we see that for large n,

$$(\alpha_n)^l = \frac{|(\underline{-q_n+1}, \underline{1})|}{|[\underline{-q_n+1}, \underline{1})|},$$

which is certainly less than the cross-ratio

$$\mathbf{Poin}(\underline{-q_n+1}, (\underline{1}, \underline{-q_{n-1}+1}]).$$

The cross ratio **Poin** is expanded by $f^{q_{n-1}-1}$. Consequently, We obtain that

$$(\alpha_n)^l \leqq \delta_n(1) \cdot s_n(1), \tag{5}$$

where

$$\delta_n(k) = \frac{|(\underline{-q_n+kq_{n-1}}, \underline{kq_{n-1}})|}{|[\underline{-q_n+kq_{n-1}}, \underline{kq_{n-1}})|}$$

and

$$s_n(k) = \frac{|[\underline{-q_n+kq_{n-1}}, \underline{0}]|}{|(\underline{-q_n+kq_{n-1}}, \underline{0}]|}.$$

Next, we estimate $\delta_n(k)$: By the Mean Value Theorem, f transforms the intervals defining the ratio $\delta_n(k)$ into a pair whose ratio is

$$\left(\frac{u_k}{v_k}\right)\delta_n(k),$$

and u_k is the derivative of x^l at a point in the interval U_k between $\underline{-q_n+kq_{n-1}}$ and $\underline{kq_{n-1}}$, while v_k is the derivative at a point in

$$V_k = U_k \cup \underline{-q_n+kq_{n-1}}.$$

Note that for n sufficiently large,

$$u_1 < v_1 < u_2 < v_2 < \cdots < u_{a_{n-1}} < v_{a_{n-1}} \, .$$

The image of $\delta_n(k)$ by f is evidently smaller than the following cross-ratio.

$$\frac{|(\underline{-q_n + kq_{n-1} + 1}, \underline{kq_{n-1} + 1})| \, |[\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1}]|}{|[\underline{-q_n + kq_{n-1} + 1}, \underline{kq_{n-1} + 1})| \, |(\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1}]|} \, ,$$

which is then mapped by $f^{q_{n-1}-1}$. By the expanding cross-ratios property we get that

$$\frac{u_k}{v_k} \delta_n(k) \leqq s_n(k+1) \cdot \delta_n(k+1) \, . \tag{6}$$

Multiplying (6) for $k = 1, 2, \ldots, a_{n-1} - 1$, and substituting the resulting estimate of δ_1 into (5) we obtain that

$$(\alpha_n)^l \leqq \delta_n(a_{n-1}) \cdot \frac{v_{a_{n-1}-1}}{u_1} \cdot s_n(1) \cdots s_n(a_{n-1}) \, .$$

It is easy to see that $s_n(1) \cdots s_n(a_{n-1}) \leqq s_n$, and that

$$\frac{v_{a_{n-1}-1}}{u_1} \leqq \left(\frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|} \right)^{l-1} \leqq \frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|} \, .$$

Thus

$$(\alpha_n)^l \leqq s_n \cdot \frac{|(\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|} \cdot \delta_n(a_{n-1}) \, ,$$

which can be rewritten in the form

$$(\alpha_n)^l \leqq s_n \, v_{n-2} \, \mu_{n-2} \, \alpha_{n-2} \, , \tag{7}$$

where

$$v_{n-2} = \frac{|[\underline{-q_{n-2}}, \underline{0})|}{|(\underline{q_{n-1}}, \underline{0})|} \cdot \frac{|[\underline{-q_{n-2}}, \underline{0})|}{|[\underline{-q_{n-2}}, \underline{a_{n-1}q_{n-1}})|} \, ,$$

and

$$\mu_{n-2} = \frac{|(\underline{-q_{n-2}}, \underline{a_{n-1}q_{n-1}})|}{|(\underline{-q_{n-2}}, \underline{0})|} \, .$$

We finish by estimating v_{n-2} and μ_{n-2}. For v_{n-2}, observe that

$$|[\underline{-q_{n-2}}, \underline{0})| \leqq |(\underline{-q_{n-3}}, \underline{0})| \, ,$$

so that

$$v_{n-2} \leqq \frac{1}{\sigma_{n-1}\sigma_{n-2}} \cdot \frac{1}{1 - \alpha_{n-2}} \, . \tag{8}$$

To estimate μ_{n-2}, we will need that following lemma.

Lemma 3.1. *For any pair of numbers $x > y$ the inequality*

$$\frac{x^l - y^l}{x^l} \geqq \left(\frac{x-y}{x} \right) \left(l - \frac{l(l-1)}{2} \left(\frac{x-y}{x} \right) \right)$$

holds.

Proof. Let $u = 1 - \frac{y}{x}$, and let

$$k(u) = 1 - (1-u)^l = \frac{x^l - y^l}{x^l} \ .$$

Clearly $k(0) = 0$, $k'(0) = l$, and for $u > 0$,

$$k''(u) = -l(l-1)(1-u)^{l-2} < 0 \ .$$

In particular, for $u \in (0, 1)$,

$$k''(u) \geqq -l(l-1) \ .$$

Hence

$$1 - (1-u)^l \geqq lu - \frac{l(l-1)}{2} u^2 \ . \qquad \square$$

Now, apply f to the intervals defining the ratio μ_{n-2}. By Lemma 3.1 the resulting ratio is larger than

$$\mu_{n-2}\left(l - \frac{l(l-1)}{2}\mu_{n-2}\right) \ .$$

The cross-ratio

$$\frac{|(\underline{-q_{n-2}+1}, \underline{a_{n-1}q_{n-1}+1})|}{|[\underline{-q_{n-2}+1}, \underline{a_{n-1}q_{n-1}+1})|} \cdot \frac{|[\underline{-q_{n-2}+1}, \underline{1})|}{|(\underline{-q_{n-2}+1}, \underline{1})|}$$

is larger again. To this we apply $f^{q_{n-2}-1}$, giving us

$$\mu_{n-2}\left(l - \frac{l(l-1)}{2}\mu_{n-2}\right) \leqq s_{n-1} \cdot \sigma_n \sigma_{n-1} \ . \tag{9}$$

Solving this quadratic inequality we have that

$$\mu_{n-2} \leqq \frac{2}{l} \cdot \frac{1}{1 + \sqrt{1 - \frac{2(l-1)}{l} s_{n-1}\sigma_{n-1}\sigma_n}} s_{n-1}\sigma_n\sigma_{n-1} \ . \tag{10}$$

Clearly, $\alpha_{n-1} > \sigma_{n-1}\sigma_n$. Combining (7), (8), (9), and (10), we obtain the claim of Proposition 4.

3.2. Convergence of the Sequence α_n

Technical Reformulation of Proposition 4. Define a sequence W_n by the condition:

$$M_n(l) = W_n(l)\frac{\sigma_n}{\sigma_{n-2}} \ .$$

Since, by Lemma 1.3, the sequence σ_n is bounded, $\prod_{i=0}^{n} M_i$ tends to zero if only $\prod_{i=0}^{n} W_n$ does so.

We will rewrite now the recursive formula of Proposition 4 as follows:

$$(\alpha_n)^l \leqq M_n'(l)\,\alpha_{n-2}^l \ , \tag{11}$$

where $M_n'(l) = M_n(l)\alpha_{n-2}^{2-l}$. Let $W_n'(l) = W_n(l)\alpha_{n-2}^{2-l}$.

To determine the size of $W_n'(l)$ we will study the function:

$$W'(x,y,l) = \frac{1}{\frac{l}{2}+\frac{l}{2}\sqrt{1-\frac{2(l-1)}{l}x^{\frac{2}{l}}}} \cdot \frac{y^{\frac{4}{l}-2}}{1-y^{\frac{2}{l}}} .$$

The next lemma shows the advantage of the technical reformulation.

Lemma 3.2. *For any* $0 < y < \frac{1}{\sqrt{e}}$, $x \in (0,1)$, *and* $l \in (1,2]$ *the function* $W'(x,y,l)$ *is increasing with respect to* l.

Proof. Set $s = 1/l$. It suffices to show that $(1/W'(x,y,1/s))$ is an increasing function of s. By algebra, we find that

$$1/W'(x,y,1/s) = \frac{(1-y^{2s})(1+\sqrt{1-2(1-s)x^{2s}})}{2sy^{4s-2}} .$$

Clearly, the numerator is positive and increasing. We compute the derivative of the denominator:

$$\frac{d}{ds}2sy^{4s-2} = 2y^{4s-2} + 8s(\ln y)y^{4s-2} ,$$

which is less than zero for $y < \frac{1}{\sqrt{e}}$. This completes the proof. □

A Priori Bounds Infer Convergence. Since the assumptions of Lemma 3.2 are satisfied, the only remaining point is the verification of the convergence of $\prod_{i=1}^n W_i'(2)$. To complete this we analyse the asymptotic size of $W_n'(2)$. If $\alpha_{n-2} < 0.3$ then $W_n'(2) < W'(0.55,0.3,2) < 0.9$. If not, then by Proposition 3 we have that $W_n'(2)$ is less than $W'(0.3,0.44,2) < 0.98$, or else that the product $W_{n+1}'(2)W_n'(2)$ is less than $W'(0.55,0.16,2)W'(0.16,0.55,2) < 1.16 \cdot 0.73 < 0.85$. By Proposition 2 the sequence $W_n'(2)$ is bounded above which finally concludes the proof of the first part of Theorem 2.

3.3. A Priori Estimates when $l \leqq 2$. Our goal is to prove Proposition 3. It will be achieved by introducing new inductive parameters and a new recursive inequality.

Let

$$\beta_n(k) = \frac{|(\underline{-q_n + kq_{n-1}, 0})|}{|[\underline{-q_n + kq_{n-1}, 0})|} ,$$

and

$$\gamma_n(k) = \frac{|(\underline{-q_n + kq_{n-1}, 0})|}{|(\underline{-q_{n-1}, 0})|} .$$

We start with the following lemma.

Lemma 3.3. *The inequality*

$$\frac{(\beta_n(k)^l + (\alpha_{n-1})^l\gamma_n(k)^l)(1+\gamma_n(k)^l)}{(1+(\alpha_{n-1})^l\gamma_n(k)^l)(\beta_n(k)^l+\gamma_n(k)^l)} \leqq s_n\beta_n(k+1)$$

holds for n sufficiently large.

Proof. The left-hand side of the inequality is equal to the cross-ratio

$$\frac{|[\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1}]|\ |(\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1})|}{|[\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1})|\ |(\underline{-q_n + kq_{n-1} + 1}, \underline{-q_{n-1} + 1}]|} .$$

We apply $f^{q_{n-1}-1}$ to obtain the inequality. □

The left-hand side of the inequality of Lemma 3.3 is a function of the three variables $\beta_n(k), \alpha_{n-1}, \gamma_n(k)$. It is easy to see that the function increases monotonically with each of the first two variables. However, relative to the third variable, the function reaches a minimum. To see this take the logarithm of the function and check that the first derivative is equal to zero only when

$$\gamma_n(k)^2 = \frac{\beta_n(k)}{\alpha_{n-1}} .$$

By substituting this value for $\gamma_n(k)$ we get that

$$\left(\frac{\beta_n(k)^{1/2} + (\alpha_{n-1})^{1/2}}{1 + \beta_n(k)^{1/2}(\alpha_{n-1})^{1/2}} \right)^2 \leqq s_n \beta_n(k+1) . \tag{12}$$

Set two additional parameters $y_n(k)$ and x_n to be equal to

$$y_n(k) =: \beta_n(k)^{1/2} \quad \text{and} \quad x_n(k) =: \min((\alpha_{n-1})^{1/2}, \beta_n(k)^{1/2}) .$$

Observe that the right-hand side of (12) is not greater than $s_n y_n(k+1)$. Substituting the above variables into (12) gives rise to a quadratic inequality in $x_n(k)$ whose only root in the interval $(0, 1)$ is given by

$$\frac{\sqrt{s_n y_n(k+1)}}{1 + \sqrt{1 - s_n y_n(k+1)}} .$$

Lemma 3.4. *The function*

$$h_n(z) = \frac{\sqrt{s_n z}}{1 + \sqrt{1 - s_n z}}$$

moves points to the left, $h(z) < z$, if $z \geqq 0.3$ and n is large enough.

Proof. The function $h_n(z)$ has a unique attractive fixed point in the interval $(0, 1)$ whose domain of attraction is the whole interval. Set temporarily $s_n = 1$. Then check directly that $h_n(0.3) < 0.3$. This proves the lemma. □

Lemma 3.5. *There is a subsequence of $\{\alpha_n\}$ including at least every other α_n, such that* $\limsup(\alpha_n)^{1/2} \leqq 0.3$.

Proof. We select the subsequence. Suppose α_{n-2} has been selected. Note that $y_n(a_{n-1}) = \alpha_{n-2}^{1/2}$. If for any $0 \leqq k < a_{n-1}$ we have $x_n(k) = (\alpha_{n-1})^{1/2}$, we select α_{n-1} as the next element. Otherwise, by the definition of $x_n(k)$ we have

$$(\alpha_n)^{1/2} = y_n(0) \leqq h_n^{a_{n-1}}(\alpha_{n-2}^{1/2}) .$$

In this case we choose α_n as the next element of the subsequence. It is clear that the resulting subsequence $\{\alpha_n\}$ satisfies the properties claimed. □

Corollary to Lemma 3.5. *For the whole sequence* $\{\alpha_n\}$ *we have* $\limsup(\alpha_n)^{l/2} < 0.3^{1/2}$. *Moreover, if* α_{n-1} *does not belong to the subsequence defined in Lemma* 3.5 *then either* $(\alpha_{n-1})^{l/2} < 0.44$ *or* $(\alpha_n)^{l/2} < 0.16$.

Proof. Suppose that α_{n-1} does not belong to the subsequence chosen in the proof of Lemma 3.5. Then for large n,

$$\frac{(\alpha_n)^{l/2} + (\alpha_{n-1})^{l/2}}{1 + (\alpha_n)^{l/2}(\alpha_{n-1})^{l/2}} \leqq \sqrt{s_n y_n(1)} < \sqrt{0.3}\ , \tag{13}$$

as a consequence of the choice of the subsequence in Lemma 3.5. Setting $\alpha_n = 0$ we get the first estimate of the corollary. In order to derive the pair of alternatives, let us assume that $\alpha_n^{l/2} \geqq 0.16$. Substituting 0.16 into (13), we obtain the desired estimate. □

This proves Proposition 3 and ends the analysis of initial bounds.

4. Proof of the Second Claim of Theorem 2

In this section we prove that if the degree l of the singularity is greater than 2, and if the rotation number is of bounded type, then f exhibits bounded geometry behaviour, in that both α_n and τ_n are bounded away from zero. We start with Proposition 5 which shows that the scalings τ_n and α_n are equivalent for bounded type rotation numbers.

The proof presented here depends strongly on the assumption that the continued fraction coefficients a_n are bounded. However, we no longer need the assumption about the negative Schwarzian.

4.1. Relation between Two Scalings. We will introduce a new parameter which measures the relative size of α_n and τ_n. Let

$$\kappa_n = \frac{|(\underline{0}, \underline{q_n})|}{|(\underline{0}, \underline{-q_{n-1}})|}\ .$$

The point q_{n-2} lies in the gap between $\underline{-q_{n-1}}$ and $\underline{-q_{n-1} + q_{n-2}}$ of the dynamical partition $\mathscr{P}_{n-2}$. Thus, by Proposition 2, τ_n/α_{n-1} is comparable with κ_n.

Proposition 5. *For any rotation number of bounded type, there is a uniform constant K so that*

$$\kappa_n > K(\alpha_{n-1})^{\frac{1-l^{-a_n+1}}{l-1}}\ .$$

4.1.1. Proof of the Proposition. We start the proof with the following lemma.

Lemma 4.1. *The ratio*

$$1 - \beta_n(i) = \frac{|\underline{-q_n + iq_{n-1}}|}{|[\underline{-q_n + iq_{n-1}}, \underline{0})|}$$

is bounded away from zero by a uniform constant for all $i = 0, \ldots, a_{n-1}$.

Proof. Apply f to the intervals defining this ratio. By Fact 1, the resulting ratio is larger than

$$\mathbf{Cr}(\underline{-q_n + iq_{n-1} + 1}, \underline{-q_{n-1} + 1}) .$$

Applying $f^{q_{n-1}-1}$ to this, and using the cross-ratio inequality, we find that the initial ratio multiplied by a uniform constant is larger than the next in the sequence of Lemma 4.1. Since the last is equal to $1 - \alpha_{n-2}$, the lemma follows. □

Lemma 4.2.

$$\beta_n(i)^l \geqq K\beta_n(i+1)$$

for a uniform constant K and for $i = 1, 2, \ldots, a_{n-1} - 1$.

Proof. We apply f to the intervals defining $\beta_n(i)$ producing a ratio equal to $\beta_n(i)^l$. We then replace this ratio by the smaller cross-ratio

$$\mathbf{Cr}([\underline{-q_{n-2} + 1}, \underline{-q_n + iq_{n-1} + 1}), (\underline{-q_n + iq_{n-1} + 1}, \underline{1})) .$$

To this we apply $f^{q_{n-2}-1}$, after which we discard the intervals that contain $\underline{0}$. We apply f next, then replace the resulting ratio by the cross-ratio

$$\mathbf{Cr}([\underline{-q_{n-2} + 1}, \underline{-q_n + iq_{n-1} + q_{n-2} + 1}),$$
$$(\underline{-q_n + iq_{n-1} + q_{n-2} + 1}, \underline{q_{n-2} + 1})) .$$

We now repeat this sequence of steps $a_{n-2} - 1$ times: Apply $f^{q_{n-2}-1}$, discard the intervals containing $\underline{0}$, apply f, and replace the result by a cross-ratio spanning the interval $\underline{-q_{n-2} + 1}$. At the end this will produce the coss ratio

$$\mathbf{Cr}([\underline{-q_{n-2} + 1}, \underline{-q_n + iq_{n-1} + a_{n-2}q_{n-2} + 1}),$$
$$(\underline{-q_n + iq_{n-1} + a_{n-2}q_{n-2} + 1}, \underline{a_{n-2}q_{n-2} + 1})) .$$

As a final step we apply $f^{q_{n-3}-1}$ and then discard the intervals containing $\underline{-q_{n-2} + q_{n-3}}$. This last operation is justified by Lemma 4.1. The resulting ratio is

$$\frac{|(\underline{-q_n + (i+1)q_{n-1}}, \underline{q_{n-1}})|}{|[\underline{-q_n + (i+1)q_{n-1}}, \underline{q_{n-1}})|} .$$

Since $\underline{-q_n + q_{n-1}}$ lies between $\underline{-q_n + (i+1)q_{n-1}}$ and $\underline{q_{n-1}}$, it follows from Lemma 4.1 that this ratio is comparable to $\beta_n(i+1)$. □

To conclude Proposition 5, note that by Proposition 2

$$|(\underline{-q_n + (i+1)q_{n-1}}, \underline{0})|$$

is comparable to

$$|[\underline{-q_n + iq_{n-1}}, \underline{0})| .$$

From this it follows immediately that κ_{n-1} is comparable to the product $\beta_n(1)$ $\beta_n(2) \cdots \beta_n(a_{n-1} - 1)$. The proof of Proposition 5 follows immediately if we combine this with Lemma 4.2.

4.2. The Recursive Relation. The following proposition is the main step in the proof of the other parts of Theorem 2. It represents the second of our two main recursive relations. Similar inequalities were used in [2, 19, 6 and 4].

Proposition 6. *If $\rho(f)$ is of bounded type, then*

$$\alpha_n \geqq K(\alpha_{n-1})^{\frac{1-l^{-a_n}}{l-1}}(\alpha_{n-2})^{l^{-a_{n-1}}},$$

where K is a uniform constant.

4.2.1. Proof of the Proposition. If n is large $(\alpha_n)^l$ is equal to the ratio

$$\frac{|(\underline{-q_n+1},\underline{1})|}{|[\underline{-q_n+1},\underline{1})|}$$

which in turn is larger than the product of two ratios

$$\xi_1 = \frac{|(\underline{-q_n+1},\underline{1})|}{|(\underline{-q_n+1},\underline{-q_{n-1}+1})|} \quad \text{and} \quad \xi_2 = \frac{|(\underline{-q_n+1},\underline{-q_{n-1}+1})|}{|[\underline{-q_n+1},\underline{-q_{n-1}+1})|}.$$

The estimates fall into two parts. First we bound ξ_1.

Lemma 4.3. *There is a constant K so that for all n large enough*

$$\xi_1 \geqq K\tau_n .$$

Proof. The proof is similar to the proof of Lemma 4.2: Replace ξ_1 by the cross-ratio

$$\mathbf{Cr}((\underline{-q_n+1},\underline{1}),\underline{-q_{n-1}+1}),$$

apply $f^{q_{n-1}-1}$, and discard the intervals containing $\underline{0}$. Repeat this $a_{n-1}-1$ times more: Apply f, replace the result by a cross-ratio of the form

$$\mathbf{Cr}((\underline{-q_n+iq_{n-1}+1},\underline{iq_{n-1}+1}),\underline{-q_{n-1}+1}),$$

apply $f^{q_{n-1}-1}$, and again discard the intervals containing $\underline{0}$. For the last step we apply f, replace the resulting ratio by

$$\mathbf{Cr}((\underline{-q_{n-2}+1},\underline{-q_{n-2}+q_n+1}),(\underline{1},\underline{-q_{n-1}+1}]),$$

and apply $f^{q_{n-2}-1}$. By Fact 2, the resulting cross-ratio is larger than $K\tau_n$. □

Lemma 4.4. *There is a constant K so that for all n large enough*

$$\xi_2 \geqq K(\alpha_{n-2})^{l^{-a_{n-1}+1}}.$$

Proof. First suppose $a_{n-1} > 1$. The ratio ξ_2 is larger than the cross-ratio

$$\mathbf{Cr}([\underline{-q_n+q_{n-1}+1},\underline{-q_n+1}),(\underline{-q_n+1},\underline{-q_{n-1}+1})).$$

We apply $f^{q_{n-1}-1}$ to this and then discard the intervals containing $\underline{2q_{n-1}-q_n}$. Using the cross-ratio inequality and Lemma 4.1 we obtain that

$$\xi_2 \geqq K_1\beta_n(1).$$

The lemma follows from this by using Lemma 4.2 and the fact that

$$\beta_n(a_{n-1}) = \alpha_{n-2} .$$

We now consider the case $a_{n-1} = 1$. We replace ξ_2 by the cross-ratio

$$\mathbf{Cr}([\underline{q_{n-2}+1}, \underline{-q_n+1}), (\underline{-q_n+1}, \underline{-q_{n-1}+1})) ,$$

apply $f^{q_{n-2}-1}$, discard the intervals containing $\underline{0}$, and apply f. We repeat this sequence of steps a total of a_{n-2} times, at the end of which we have the ratio

$$\frac{|(\underline{-q_{n-3}-q_{n-2}+1}, \underline{q_{n-3}+1})|}{|[\underline{-q_{n-3}-q_{n-2}+1}, \underline{q_{n-3}+1}]|} .$$

Again we replace the result by a cross-ratio that spans the interval $\underline{-q_{n-2}+1}$. However, this time we apply $f^{q_{n-3}-1}$. By a combination of Fact 1 and the cross-ratio inequality we conclude that the result is less than a universal constant times the initial ratio ξ_2. By Lemma 4.1 we may discard the two intervals spanning $\underline{-q_{n-2}+q_{n-3}}$. This results in α_{n-2} and completes the proof. □

Combining the estimates of Lemma 4.3 and Lemma 4.4 we obtain that

$$(\alpha_n)^l \geqq K_3(\alpha_{n-2})^{l^{-a_{n-1}+1}} \tau_n .$$

By Proposition 5,

$$\tau_n \geqq K_4(\alpha_{n-1})^{\frac{l-l^{-a_n+1}}{l-1}} .$$

The above two inequalities yield the claim of Proposition 6.

4.3. Analysis of the Recurrence. Define the quantity

$$v_n = -\ln \alpha_n .$$

Proposition 6 implies that there is a uniform constant K so that

$$v_n - \frac{1-l^{-a_n}}{l-1} v_{n-1} - l^{-a_{n-1}} v_{n-2} \leqq K_1 . \tag{14}$$

We will prove that the sequence v_n is bounded. To this end consider the sequence of vectors $\{\upsilon_n\}$:

$$\upsilon_n = \begin{pmatrix} v_n \\ v_{n-1} \end{pmatrix}$$

and the sequence of matrices $\{A_l(n)\}$:

$$A_l(n) = \begin{pmatrix} \frac{1-l^{-a_n}}{l-1} & l^{-a_{n-1}} \\ 1 & 0 \end{pmatrix} ,$$

and the vector

$$\kappa = \begin{pmatrix} K_1 \\ 0 \end{pmatrix} .$$

Now, we can rewrite (14) in the form

$$v_n \leqq A_l(n)A_l(n-1)\cdots A_l(2)v_1 + \left(\sum_{i=2}^{n-1} A_l(n-1)A_l(n-2)\cdots A_l(i)\right)\kappa ,$$

where the inequality is understood component-wise.

By this relation it follows that the proof of the second part of Theorem 2 will be completed as soon as we can show that a long composition of $A_l(n)$ contracts the Euclidean metric exponentially.

Long Compositions of $A_l(n)$. Fix $l > 1$. Consider a more general sequence of matrices defined by the formula

$$B_l(n) = \begin{pmatrix} \frac{1-b_n}{l-1} & b_{n-1} \\ 1 & 0 \end{pmatrix} ,$$

where $\{b_n\}$ is a sequence of positive numbers bounded above by $1/l$. We have the following lemma.

Lemma 4.5. *Assume that $l \geqq 2$. Then the sequence $\{B_l^{\circ n}\}$*

$$B_l^{\circ n} := B_l(n) \circ \cdots \circ B_l(1)$$

is relatively compact.

Proof. Both $B_l(n)$ and $B_2(n) - B_l(n)$ are non-negative. Therefore, it suffices to consider the case when $l = 2$. $B_l^{\circ n}$ can be represented in the form

$$B_2^{\circ n} = \begin{pmatrix} d_n & e_n \\ d_{n-1} & e_{n-1} \end{pmatrix} .$$

By induction, we check easily that d_n and e_n are given by

$$d_n = 1 - b_n + b_n b_{n-1} + \cdots + (-1)^n b_n b_{n-1} \cdots b_2 ,$$

$$e_n = b_1(1 - b_n + b_n b_{n-1} + \cdots + (-1)^n b_n b_{n-1} \cdots b_2) .$$

By the definition, $b_n \leqq 1/2$ and consequently, $d_n \leqq 2$ and $e_n \leqq 1$. □

Proposition 7. *When $l > 2$, then $B_l^{\circ n}$ contracts the Euclidean metric provided n is large enough. The scale of contraction is bounded away from 1 independently of l and the particular sequence $\{b_n\}$, whereas the moment when the contraction starts depends on the upper bound of b_n.*

4.3.1. Proof of the Proposition. Each $B_l^{\circ n}$ can be expressed as

$$B_l(n) = \begin{pmatrix} d_n(z) & e_n(z) \\ d_{n-1}(z) & e_{n-1}(z) \end{pmatrix} ,$$

where z stands for $1/(l-1)$. Then $d_n(z)$ and $e_n(z)$ are polynomials of degree n in the variable z. Denote coefficients of these two polynomials by $d_{n,i}$ and $e_{n,i}$ respectively. All these coefficients are nonnegative. By Lemma 4.5, the sums

$$\sum_{i=0}^{n} d_{n,i} \quad \text{and} \quad \sum_{i=0}^{n} e_{n,i}$$

are uniformaly bounded. Therefore

$$\sum_{i=k}^{n} d_{n,i} \frac{1}{(l-1)^i} \quad \text{and} \quad \sum_{i=k}^{n} e_{n,i} \frac{1}{(l-1)^i}$$

are arbitrary close to zero provided k is large enough. We are left with the task of determining the behavior of finitely many first coefficients of polynomials $d_n(z)$ and $e_n(z)$ if n goes to infinity.

Lemma 4.6. *for any fixed k the coefficients*

$$\{d_{n,0}, \ldots, d_{n,k}, e_{n,0}, \ldots, e_{n,k}\}$$

tend to zero at least exponentially fast.

Proof. We will concentrate only on the polynomials $d_n(z)$ since the proof of the other ones is very much the same. By the definition of the polynomial $d_n(z)$, we have that

$$d_n(z) = (1 - b_n)z\, d_{n-1}(z) + b_{n-1}d_{n-2}(z)\,. \tag{15}$$

The absolute value of the coefficient $d_n(0)$ is zero if n is odd, and is equal to the product $b_1 b_3 \cdots b_{n-1}$ if n is even and therefore goes exponentially fast to zero. Assume that for given $i > 0$ and n all coefficients $d_{m,j}$ with $j < i$ and $m < n$ tend to zero. Then by (15), we get that

$$d_{n,k} = (1 - b_n)d_{n-1,k-1} + b_{n-1}d_{n-2,k}\,,$$

and the lemma follows by induction. □

5. Proof of the Third Claim of Theorem 2

The proof of the third claim is very short and consists in the observation that for $1 < l < 2$ the values of polynomials $d_n(z)$ and $e_n(z)$ at $1/(l-1)$ are bounded from above by $K_1/(l-1)^n$. This results in the inequality

$$\|v_n\| \leqq K_2/(l-1)^n\,.$$

For $l = 2$, by Lemma 4.5, we have that

$$\|v_n\| \leqq K_3 n\,,$$

which gives the desired lower estimates on α_n. By Proposition 5, the same sort of estimates can be obtained for the sequence τ_n.

Proposition 4 and the analysis of the initial bounds imply that there is a constant $\lambda(l) < 1$ so that at least asymptotically the following inequality holds

$$(\alpha_n)^l \leqq \lambda(l)\,\alpha_{n-2}^2\,.$$

This concludes the proof of Theorem 2.

Appendix

Description of the Procedure. A numerical experiment was performed in order to check Conjecture 1 of the introduction. To this end, a family of almost smooth maps with a flat spot was considered given by the formula

$$x \to \left(\frac{x-1}{b}\right)^3 \left(1 - 3\frac{x+b-1}{b} + 6\left(\frac{x+b-1}{b}\right)^2 - 10\left(\frac{x+b-1}{b}\right)^3 + (x-1)^3\right) + t\,(\mathrm{mod}\,1)\,.$$

These are symmetric maps with the critical exponent $(3, 3)$. The parameter b controls the length of the flat spot, while t must be adjusted to get the desired rotation number.

In our experiment, b was chosen to be 0.5, which corresponds to the flat spot of the same length. By binary search, a value t_{Au} was found which approximated the parameter value corresponding to the golden mean rotation number $\frac{\sqrt{5}-1}{2}$. Next, the forward orbit of the flat spot was studied and the results are given in the table below.

It should finally be noted that the experiment presents serious numerical difficulties as nearest returns to the critical value tend to 0 very quickly so that the double precision is insufficient when one wants to see more than 15 nearest returns. This problem was avoided, at a considerable expense of computing time, by the use of an experimental package which allows for floating-point calculations to be carried out with arbitrarily prescribed precision.

Results. Below the results are presented. The column y_i is defined by $y_i := \mathrm{dist}(\underline{0}, \underline{q_i})$. The μ_i is given by $\mu := \frac{\tau_{i+2}-\tau_{i+1}}{\tau_{i+1}-\tau_i}$.

n	y_n	τ_n	μ_n
10	$3.010 \cdot 10^{-3}$	0.2637	0.5869
11	$1.544 \cdot 10^{-3}$	0.2450	1.683
12	$0.7044 \cdot 10^{-3}$	0.2340	0.4527
13	$0.3328 \cdot 10^{-3}$	0.2156	1.775
14	$0.1460 \cdot 10^{-3}$	0.2072	0.5079
15	$64.04 \cdot 10^{-6}$	0.1924	1.285
16	$26.99 \cdot 10^{-6}$	0.1849	0.6396
17	$11.22 \cdot 10^{-6}$	0.1752	0.9773
18	$4.562 \cdot 10^{-6}$	0.1690	0.7485
19	$1.829 \cdot 10^{-6}$	0.1630	0.8634
20	$0.7229 \cdot 10^{-6}$	0.1585	0.8015
21	$0.2826 \cdot 10^{-6}$	0.1546	0.8307
22	$0.1095 \cdot 10^{-6}$	0.1514	0.8191
23	$42.07 \cdot 10^{-9}$	0.1488	0.8243
24	$16.06 \cdot 10^{-9}$	0.1467	0.8241
25	$6.097 \cdot 10^{-9}$	0.1449	0.8172
26	$2.305 \cdot 10^{-9}$	0.1435	0.9982
27	$1.070 \cdot 10^{-9}$	0.1423	−2.54
28	$0.8677 \cdot 10^{-9}$	0.1411	−25.9
29	$0.3252 \cdot 10^{-9}$	0.1441	−10.9

Interpretation. The most interesting is the third column which shows the scalings. They seem to decrease monotonically. The last column attempts to measure the exponential rate at which the differences between consecutive scalings change. Here, the last three numbers are obviously out of line which, however, is explained by the fact that t_{Au} is just an approximation of the parameter value which generates the golden mean dynamics. Other than that, the numbers from the last column seem to be firmly below 1, which indicates geometric convergence. If 0.82 is accepted as the limit rate, this projects to the scalings limit of about 0.137 which is consistent with rough theoretical estimates of [19].

Thus, we conclude that Conjecture 1 has a numerical confirmation.

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