Commun. Math. Phys. 171, 547 – 588 (1995)

Communications in
Mathematical
Physics

A Lax Representation for the Vertex Operator and the Central Extension

M. Adler[1], T. Shiota[2], P. van Moerbeke[3]

[1] Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA.
The support of a National Science Foundation grant # DMS-9203246 is gratefully acknowledged.
[2] Department of Mathematics, Kyoto University, Kyoto 606–01, Japan.
The hospitality of the University of Louvain and Brandeis University is gratefully acknowledged.
[3] Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Department of Mathematics, Brandeis University, Waltham, Mass 02254, USA.
The support of a National Science Foundation grant # DMS-9203246, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged.

Received: 21 December 1993/in revised form: 6 June 1994

Abstract: Integrable hierarchies, viewed as isospectral deformations of an operator L may admit symmetries; they are time-dependent vector fields, transversal to and commuting with the hierarchy and forming an algebra. In this work, the commutation relations for the symmetries are shown to be based on a non-commutative Lie algebra splitting theorem. The symmetries, viewed as vector fields on L, are expressed in terms of a Lax pair.

This study introduces a "generating symmetry", a generating function for symmetries, both of the KP equation (continuous), and the two-dimensional Toda lattice (discrete), in terms of L and an operator M, introduced by Orlov and Schulman, such that $[L,M] = 1$. This "generating symmetry", acting on the wave function (or wave vector) lifts to a vertex operator *à la* Date-Jimbo-Kashiwara-Miwa, acting on the τ-function (or τ-vector). Lifting the algebra of symmetries, acting on wave functions, to an algebra of symmetries, acting on τ-functions, amounts to passing from an algebra to its central extension.

This provides a handy technology to find the constraints satisfied by various matrix integrals, arising in the context of $2d$-quantum gravity and moduli space topology. The point is to first prove the vanishing of symmetries at the Lax pair level, which usually turns out to be elementary and conceptual, and then use the lifting above to get the subalgebra of vanishing symmetries for the τ-function (or τ-vectors).

0. Introduction and Main Results

Most integrable equations are part of a hierarchy of equations, and can be viewed as isospectral deformations of a system of linear equations, with one or more sequences of scalar deformation parameters $t = (t_1, t_2, \ldots)$ as independent variables. They may come equipped with so-called *symmetries*, which are vector fields acting on the space of solutions of the hierarchy, which may explicitly depend on time, and which commute with the hierarchy, but not necessarily among themselves. Symmetries have played important roles in the study of integrable equations since early days of soliton theory, when the Bäcklund transformation of the K-dV equation led to the discovery of infinite sequence of conservation laws, and hence the K-dV hierarchy.

The purpose of this paper is to prove that the symmetries acting on the linear problem, as introduced by Fuchssteiner, Oevel and coworkers, and extensively studied by Orlov and coworkers, lifts to its central extension, the symmetries acting on τ-functions as introduced by Date, Jimbo, Kashiwara and Miwa, and which played the central role in the Kyoto school's theory of soliton equations in terms of an infinite dimensional Grassmann manifold. Indeed we get a generating function of the symmetries, a so-called vertex operator for the linear problem, which lifts to the vertex operator acting on τ-functions. The result is robust: it holds for continuous integrable systems (KP) as well as for discrete ones (two-dimensional Toda lattice), leading to a Lax vertex operator when operating on pseudo-differential operators or matrices. The proof is given at the same time for the discrete and the continuous cases, using the same algebraic formalism. In fact, non-Hamiltonian noncommuting generalization of the Adler-Kostant-Symes splitting theorem, which covers the situation of symmetries, is given in Sect. 2. Indeed, they correspond to two different Lie algebras, both denoted by $\mathscr{D}$, with splittings.

The KP equations

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_+, L\right], \quad n = 1,2,\ldots, \tag{0.0}$$

are deformations of a monic pseudodifferential operator L in x of order 1, $L = D + a_{-1}(x,t)D^{-1} + \cdots$, where $D = \partial/\partial x$. It is expressible in terms of the customary splitting of the algebra $\mathscr{D}$ of formal pseudodifferential operators into differential and negative order operators:

$$\begin{aligned} \mathscr{D} &= \mathscr{D}_+ + \mathscr{D}_-, \\ \sum a_i D^i &= \sum_{i \geqq 0} a_i D^i + \sum_{i<0} a_i D^i . \end{aligned} \tag{0.1}$$

A customary eigenfunction $\Psi = \Psi(x,t,z)$ of L with eigenvalue $z \sim \infty$, called the wave or Baker–Akhiezer function, plays an important role in this theory. In particular, it enabled Orlov and Shulman to introduce, besides L, another pseudodifferential operator M, which acting on Ψ amounts to differentiation by z, so that L, M and Ψ are related as follows:

$$\begin{gathered} L\Psi = z\Psi, \quad M\Psi = \frac{\partial \Psi}{\partial z}, \quad [L,M] = 1, \\ \frac{\partial M}{\partial t_n} = \left[(L^n)_+, M\right], \quad \frac{\partial \Psi}{\partial t_n} = \left[(L^n)_+, \Psi\right], \quad n = 1,2,\ldots . \end{gathered} \tag{0.2}$$

The two-dimensional Toda lattice equations

$$\frac{\partial L}{\partial t_n} = \left[(L_1^n, 0)_+, L\right] \quad \text{and} \quad \frac{\partial L}{\partial s_n} = \left[(0, L_2^n)_+, L\right], \quad n = 1,2,\ldots \tag{0.3}$$

are deformations of a pair of infinite matrices

$$L = (L_1, L_2) = \Big(\sum_{-\infty < i \leqq 1} a_i^{(1)} \Lambda^i, \sum_{-1 \leqq i < \infty} a_i^{(2)} \Lambda^i \Big) \in \mathscr{D} , \tag{0.4}$$

where $\Lambda = (\delta_{j-i,1})_{i,j\in\mathbb{Z}}$, and $a_i^{(1)}$ and $a_i^{(2)}$ are diagonal matrices depending on $t = (t_1, t_2, \dots)$ and $s = (s_1, s_2, \dots)$, such that

$$a_1^{(1)} = I \quad \text{and} \quad \left(a_{-1}^{(2)}\right)_{nn} \neq 0 \quad \forall n .$$

It is expressed in terms of the (less customary) splitting of the algebra $\mathscr{D}$ of pairs (P_1, P_2) of infinite ($\mathbb{Z} \times \mathbb{Z}$) matrices such that $(P_1)_{ij} = 0$ for $j - i \gg 0$ and $(P_2)_{ij} = 0$ for $i - j \gg 0$, used by us in [A-vM0], and also [vM-M]; to wit:

$$\begin{aligned}
\mathscr{D} &= \mathscr{D}_+ + \mathscr{D}_-, \\
\mathscr{D}_+ &= \{(P,P) \mid P_{ij} = 0 \text{ if } |i-j| \gg 0\} = \{(P_1,P_2) \in \mathscr{D} \mid P_1 = P_2\}, \\
\mathscr{D}_- &= \{(P_1,P_2) \mid (P_1)_{ij} = 0 \text{ if } j \geqq i,\ (P_2)_{ij} = 0 \text{ if } i > j\},
\end{aligned}$$

so that $(P_1,P_2) = (P_1,P_2)_+ + (P_1,P_2)_-$ is given by

$$\begin{aligned}
(P_1,P_2)_+ &= (P_{1u} + P_{2l}, P_{1u} + P_{2l}), \\
(P_1,P_2)_- &= (P_{1l} - P_{2l}, P_{2u} - P_{1u}),
\end{aligned} \tag{0.5}$$

where for a matrix P, P_u and P_l denote the upper (including diagonal) and strictly lower triangular parts of P, respectively. Note that $\mathscr{D}_\pm$ are actually associative rings as in the KP case.

In this context, one also introduces a pair of wave vectors $\Psi = (\Psi_1, \Psi_2)$ instead of a single wave function, and an operator $M = (M_1, M_2)$, all tied together as follows:

$$\begin{aligned}
&L\Psi = (z, z^{-1})\Psi, \qquad M\Psi = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial(z^{-1})}\right)\Psi, \quad [L,M] = (1,1), \\
&\frac{\partial M}{\partial t_n} = \left[(L_1^n, 0)_+, M\right], \quad \frac{\partial M}{\partial s_n} = \left[(0, L_2^n)_+, M\right], \\
&\hspace{20em} n = 1,2,\dots . \\
&\frac{\partial \Psi}{\partial t_n} = (L_1^n, 0)_+ \Psi, \qquad \frac{\partial \Psi}{\partial s_n} = (0, L_2^n)_+ \Psi ,
\end{aligned} \tag{0.6}$$

Sato's theory tells us that the KP or 2-Toda deformations of Ψ, and hence L, can ultimately all be expressed in terms of τ-functions

$$\tau(x + t_1, t_2, \dots) =: \tau_x(t), \quad x\text{: scalar} \quad \text{(KP case)},$$

$$\tau_n(t_1, t_2, \dots; s_1, s_2, \dots), \quad n \in \mathbb{Z} \quad \text{(2-Toda case)}:$$

<u>KP</u>

$$\Psi(t,z) = \frac{e^{-\eta}\tau_x(t)}{\tau_x(t)} e^{\sum t_i z^i} e^{xz}$$

<u>2-Toda</u>

$$\begin{aligned}
\Psi_1(t,z) &= \left(\frac{e^{-\eta}\tau_n(t,s)}{\tau_n(t,s)} e^{\sum t_i z^i} z^n\right)_{n\in\mathbb{Z}} \\
\Psi_2(t,z) &= \left(\frac{e^{-\tilde{\eta}}\tau_{n+1}(t,s)}{\tau_n(t,s)} e^{\sum s_i z^{-i}} z^n\right)_{n\in\mathbb{Z}}
\end{aligned} \tag{0.7}$$

where

$$\eta = \sum_1^\infty \frac{z^{-i}}{i} \frac{\partial}{\partial t_i} \quad \text{and} \quad \tilde{\eta} = \sum_1^\infty \frac{z^i}{i} \frac{\partial}{\partial s_i}, \tag{0.8}$$

so that

$$e^{a\eta+b\tilde{\eta}}f(t,s)=f(t+a[z^{-1}],s+b[z]) \tag{0.9}$$

with $[\alpha]=(\alpha,\alpha^2/2,\alpha^3/3,\dots)$.

The symmetries of the KP or 2-Toda hierarchy are conveniently expressed in terms of the operators L and M. In view of the relations

$$\underline{\text{KP :}} \qquad z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \Psi = M^\beta L^\alpha \Psi\ , \tag{0.10}$$

$$\underline{\text{2-Toda:}} \qquad z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \Psi_1 = M_1^\beta L_1^\alpha \Psi_1, \quad u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta\Bigg|_{u=z^{-1}} \Psi_2 = M_2^\beta L_2^\alpha \Psi_2\,,$$

the Lie algebra

$$w_\infty := \ \text{span}\ \left\{ z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \middle|\, \alpha,\beta\in\mathbb{Z},\beta\geqq 0\right\} \tag{0.11}$$

comes naturally into play. To be precise, denoting by ϕ the algebra antihomomorphism

$$\underline{\text{KP :}} \qquad w_\infty \longrightarrow \mathscr{D} : z^\alpha(\partial/\partial z)^\beta \longmapsto M^\beta L^\alpha,$$

$$\underline{\text{2-Toda:}} \qquad w_\infty\times w_\infty \longrightarrow \mathscr{D} : \begin{cases} \left(z^\alpha(\partial/\partial z)^\beta,0\right) \mapsto \left(M_1^\beta L_1^\alpha,0\right) \\ \left(0,u^\alpha(\partial/\partial u)^\beta\right) \mapsto \left(0,\ M_2^\beta L_2^\alpha\right) \end{cases}, \tag{0.12}$$

and by w either w_∞ (KP case) or $w_\infty\times w_\infty$ (2-Toda case), we have a Lie algebra antihomomorphism[1]

$$\begin{aligned} \mathbb{Y} : w &\longrightarrow \left\{ \text{ symmetries on } \Psi,\ L \text{ or } M\right\}\ , \\ p &\longmapsto \begin{cases} \mathbb{Y}_p\Psi = -\phi(p)_-\Psi, \\ \mathbb{Y}_pL\ = \left[-\phi(p)_-,L\right], \mathbb{Y}_pM = \left[-\phi(p)_-,M\right]\ . \end{cases} \end{aligned} \tag{0.13}$$

We shall also denote $\mathbb{Y}_p$ by $\mathbb{Y}_{\phi(p)}$, if there is no fear of confusion.

A generating function of the symmetries on the Ψ-manifold is given by

$$\mathbb{Y}_N\Psi = -N_-\Psi\ , \tag{0.14}$$

and on the L-manifold it takes the Lax form

$$\mathbb{Y}_NL = [-N_-,L]\ ,$$

with

$$\begin{aligned} \underline{\text{KP:}} \qquad N &= (\mu-\lambda)\phi(\delta(\lambda,z)e^{(\mu-\lambda)\partial/\partial z}) = (\mu-\lambda)e^{(\mu-\lambda)M}\delta(\lambda,L) \\ &= \sum_{k=1}^{\infty}\frac{(\mu-\lambda)^k}{k!}\sum_{l=-\infty}^{\infty}\lambda^{-l-k}kM^{k-1}L^{k-1+l}\ , \end{aligned} \tag{0.15}$$

[1] Thus $-\mathbb{Y}$ is a Lie algebra homomorphism.

$$\underline{\text{2-Toda:}} \qquad \begin{aligned} N &= (N_1, 0) \text{ or } (0, N_2)\ , \\ N_i &= (\mu - \lambda) e^{(\mu-\lambda)M_i} \delta(\lambda, L_i)\ , \end{aligned} \tag{0.16}$$

where $\delta(\lambda, z) := \sum_{-\infty}^{\infty} \lambda^{-n} z^{n-1}$.

We now turn to the τ-manifold symmetries; Date, Jimbo, Kashiwara and Miwa observed in their fundamental work that the symmetries of the τ-manifold are realized by the vertex operator

$$X(t, \lambda, \mu) := \exp\left(\sum_1^{\infty} t_i(\mu^i - \lambda^i)\right) \exp\left(\sum_1^{\infty} (\lambda^{-i} - \mu^{-i}) \frac{1}{i} \frac{\partial}{\partial t_i}\right). \tag{0.17}$$

Introduce now a small variation of X and generators[2] of a W_∞-algebra:

$$\begin{array}{ll} \underline{\text{KP}} & \underline{\text{2-Toda}} \\ \mathbb{X}(t, \lambda, \mu) := e^{(\mu-\lambda)x} X(t, \lambda, \mu), & \mathbb{X}(t, \lambda, \mu) := \left(\left(\frac{\mu}{\lambda}\right)^n X(t, \lambda, \mu)\right)_{n \in \mathbb{Z}}, \\ & \tilde{\mathbb{X}}(s, \lambda, \mu) := \left(\left(\frac{\lambda}{\mu}\right)^n X(s, \lambda, \mu)\right)_{n \in \mathbb{Z}}, \end{array} \tag{0.18}$$

$$\left(\frac{\mu}{\lambda}\right)^{\alpha} X(t, \lambda, \mu) = \sum_{k=0}^{\infty} \frac{(\mu - \lambda)^k}{k!} \sum_{l=-\infty}^{\infty} \lambda^{-l-k} W_{\alpha,l}^{(k)}, \quad \alpha \in \mathbb{Z}, \tag{0.19}$$

and

$$W_l^{(k)} := W_{0,l}^{(k)}, \quad \tilde{W}_{\alpha,l}^{(k)} := W_{-\alpha,l}^{(k)}\Big|_{t \to s}\ .$$

We now state the main theorem:

Theorem 0.1 *The vector fields of type $\mathbb{Y}_N$ on the Ψ-manifold and the vertex operators of type $\mathbb{X}(t, \lambda, \mu)$ on the τ-manifold are related as follows*:
(i) *continuous* (*KP*) *case*:

$$\frac{\mathbb{Y}_N \Psi}{\Psi} = (e^{-\eta} - 1) \frac{\mathbb{X}\tau}{\tau}\ ,$$

(ii) *discrete* (*2-Toda*) *case*:

$$\frac{\mathbb{Y}_{N_1} \Psi}{\Psi} = \left((e^{-\eta} - 1) \frac{\mathbb{X}(t, \lambda, \mu)\tau}{\tau}, (\Lambda e^{-\tilde{\eta}} - 1) \frac{\mathbb{X}(t, \lambda, \mu)\tau}{\tau}\right)\ ,$$

$$\frac{\mathbb{Y}_{N_2} \Psi}{\Psi} = \frac{\mu}{\lambda} \left((e^{-\eta} - 1) \frac{\tilde{\mathbb{X}}(s, \lambda, \mu)\tau}{\tau}, (\Lambda e^{-\tilde{\eta}} - 1) \frac{\tilde{\mathbb{X}}(s, \lambda, \mu)\tau}{\tau}\right)\ .$$

Note that in general the action of a vector field on τ induces its action on Ψ via (0.7). By logarithmic derivative we have

$$\frac{\Psi'}{\Psi} = (e^{-\eta} - 1) \frac{\tau'}{\tau}, \quad \frac{\Psi_2'}{\Psi_2} = (\Lambda e^{-\tilde{\eta}} - 1) \frac{\tau'}{\tau}\ ,$$

respectively in the KP and Ψ_1 case, and in the Ψ_2 case. The theorem means in this sense that $\mathbb{X}$ on τ induces $\mathbb{Y}_N$ on Ψ in the KP case, and $\mathbb{X}$ and $(\mu/\lambda)\tilde{\mathbb{X}}$ induce

[2] Using $(\mu/\lambda)^{\alpha} = \sum_{k \geqq 0} \binom{\alpha}{k} \left(\frac{\mu-\lambda}{\lambda}\right)^k$

$\mathbb{Y}_{N_1}$ and $\mathbb{Y}_{N_2}$, respectively, in the 2-Toda case. Expanding $\mathbb{X}, \tilde{\mathbb{X}}$ and $\mathbb{Y}$ in terms of the W-generators and $M_i^\alpha L_i^\beta$, we have

Corollary 0.1.1. *For* $n, l \in \mathbb{Z}, n \geqq 0$, *we have in terms of the W-generators* (0.19)
(i) *continuous case*:

$$-\frac{(M^n L^{n+l})_- \Psi}{\Psi} = (e^{-\eta} - 1)\frac{\frac{1}{n+1} W_l^{(n+1)}(\tau)}{\tau}, \tag{0.20}$$

(ii) *discrete case*:

$$-\frac{((M_1^n L_1^{n+l})_{(l)} \Psi_1)_m}{(\Psi_1)_m} = \frac{1}{n+1}(e^{-\eta} - 1)\frac{W_{m,l}^{(n+1)}(\tau_m)}{\tau_m},$$

$$\frac{((M_1^n L_1^{n+l})_{(u)} \Psi_2)_m}{(\Psi_2)_m} = \frac{1}{n+1}\left(e^{-\tilde{\eta}}\frac{W_{m+1,l}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{W_{m,l}^{(n+1)}(\tau_m)}{\tau_m}\right),$$

$$-\frac{((M_2^n L_2^{n+l})_{(l)} \Psi_1)_m}{(\Psi_1)_m} = \frac{1}{n+1}(e^{-\eta} - 1)\frac{\tilde{W}_{m-1,l}^{(n+1)}(\tau_m)}{\tau_m},$$

$$\frac{((M_2^n L_2^{n+l})_{(u)} \Psi_2)_m}{(\Psi_2)_m} = \frac{1}{n+1}\left(e^{-\tilde{\eta}}\frac{\tilde{W}_{m,l}^{(n+1)}(\tau_{m+1})}{\tau_{m+1}} - \frac{\tilde{W}_{m-1,l}^{(n+1)}(\tau_m)}{\tau_m}\right).$$

The proof comprises both the continuous and the discrete cases. A similar result can also be established for the one-dimensional symmetric Toda lattice; this will be reported elsewhere (see [A-vM2]).

Besides its intrinsic, conceptual relevance, this study has been motivated by questions arising in string theory. A mathematical formulation of string theory includes the following: Given a differential operator L, evolving according to the KP-equations, find another differential operator Q satisfying $[L, Q] = 1$ or $= f(L)$, for some reasonable function f. It has been noticed that the exceptional L's or corresponding Ψ's for which this is possible, form a locus on the manifold of L's or Ψ's, which are fixed points for a certain symmetry vector field; in other terms, this symmetry vanishes all along that locus. For KP, a simple argument, based on the fact that differential operators form an associative algebra for multiplication, besides being a Lie algebra, shows that a whole algebra of symmetries vanishes along that same locus. The vanishing of these Ψ-manifold symmetries implies the vanishing of the corresponding τ-manifold symmetries (often called constraints); since it is also known that, in certain circumstances the corresponding τ-functions are certain matrix integrals, this provides a set of constraints for the latter. These ideas can now be applied to the 1-matrix models (see [A-vM2] and [vM]), the Kontsevich integrals and its generalizations ([A-vM1]) and to the 2-matrix models ([A-vM3]). These applications are sketched in Sect. 6.

These results have been lectured on by PvM at various stages of their evolution, at a very early stage at Sophia-Antipolis (June 1991), then later at Como (October 1992), Utrecht (November 1992), Brandeis and MIT (spring 1992); see [vM]. The present proof, lectured on at Cortona (Sept. 93), simplifies the argument by an effective use of the bilinear identity.

In the pioneering work [O, O-Sc], Orlov and Schulman have conjectured the relation (i) for the KP-equation, but formulated in a different language, and proved partial results in this direction including: (a) the action of $\mathbb{Y}_N$ on the first nontrivial coefficient (the coefficient of z^{-1}) in $\exp(-xz - \sum t_i z^i)\,\Psi = 1 + O(z^{-1})$, and (b) the action of $\mathbb{Y}_N$ on the trivial solution $\Psi = \exp(xz + \sum t_i z^i)$ (or $L = D$). Aoyama and Kodama [AK] have also conjectured the relation (i). After completion of this work, the authors found out that Dickey [D3] provided a slightly different proof of the relation (i).

1. Preliminaries

Consider for an additive group $\mathbb{F}$ the function (character)

$$\begin{aligned} \chi : \mathbb{F} \times \mathbb{C}^* &\to \mathbb{C}^* \quad (\text{ or } \mathbb{F} \times \mathbb{C} \to \mathbb{C}^*) \\ (x,z) &\mapsto \chi_x(z) \end{aligned} \tag{1.1}$$

satisfying

$$\chi_{x+y}(z) = \chi_x(z)\chi_y(z) \quad \text{and} \quad \chi_{-x}(z) = (\chi_x(z))^{-1}\ . \tag{1.2}$$

Regarded as a function of z parametrized by x, this induces a group homomorphism of $\mathbb{F}$ into the multiplicative group of non-vanishing functions

$$\chi : x \mapsto \chi_x\ .$$

Now we switch the point of view and regard χ as a family of functions of x parametrized by z. Assume χ is holomorphic in z, and the linear map

$$\rho : \mathbb{C}[z, (\partial/\partial z)] \to \{ \text{ functions in } x, z \} : P(z, \partial/\partial z) \mapsto P(z, \partial/\partial z)\chi_x(z)$$

is injective.

Consider a pair of linear operators ∂ and ε acting on the space of functions in x spanned by $\{(\partial/\partial z)^n \chi_x(z) \mid z \in \mathbb{C}^*,\ n = 0, 1, 2, \dots\}$, such that

$$\partial\chi = z\chi, \quad \varepsilon\chi = \frac{\partial}{\partial z}\chi\,. \tag{1.3}$$

One computes that

$$\begin{aligned} [\partial, \varepsilon]\chi = \partial\varepsilon\chi - \varepsilon\partial\chi &= \partial\frac{\partial}{\partial z}\chi - \varepsilon z\chi \\ &= \frac{\partial}{\partial z}\partial\chi - z\varepsilon\chi \\ &= \frac{\partial}{\partial z}z\chi - z\frac{\partial}{\partial z}\chi \\ &= [\frac{\partial}{\partial z}, z]\chi = \chi\,, \end{aligned}$$

which, by the injectivity of ρ, leads to

$$[\partial, \varepsilon] = 1\ , \tag{1.4}$$

and to the fact that $P(\partial, \varepsilon) \mapsto \rho^{-1}(P(\partial, \varepsilon)\chi)$ gives an antiisomorphism of rings $\mathbb{C}[\partial, \varepsilon] \to \mathbb{C}[z, \partial/\partial z]$.

Since $[\cdot, \varepsilon]$ acts as a derivation in "$\cdot$", and since by (1.4) this derivation is continuous in the ∂-adic and ∂^{-1}-adic topologies, the following holds for any formal series $f(z)$ in z or z^{-1}:

$$[f(\partial),\varepsilon] = f'(\partial), \quad \text{and hence} \quad e^{f(\partial)}\varepsilon e^{-f(\partial)} = \varepsilon + f'(\partial) \,. \tag{1.5}$$

Here in the latter formula we assume $e^{f(z)} = \sum(1/n!)f(z)^n$ makes sense, e.g., f consists of either strictly negative or strictly positive powers in z, i.e, $f(z) \in z^{\pm 1}\mathbb{C}[[z^{\pm 1}]]$.

We also have by (1.3) that

$$f(\partial)\chi = f(z)\chi \,, \tag{1.6}$$

and by Taylor's theorem, as formal power series in a,

$$e^{a\varepsilon}\chi(z) = \chi(z+a) \,. \tag{1.7}$$

The two major examples of this situation are

(i) $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or Spec $\mathbb{C}[[x]]$ <u>(continuous case)</u>: $\chi := e^{xz}$,

$$\partial := D := \partial/\partial x, \quad \varepsilon := x \,. \tag{1.8}$$

(ii) $\mathbb{F} = \mathbb{Z}$ <u>(discrete case)</u>: $\chi := (z^i)_{i\in\mathbb{Z}} = (\ldots, z^{-1}, 1, z, \ldots)^\top$,

$$\partial := \Lambda := (\delta_{i,j-1})_{i,j\in\mathbb{Z}} = \begin{pmatrix} \ddots & & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & \\ & & & & & & \ddots \end{pmatrix} , \tag{1.9}$$

and

$$\varepsilon := \ \mathrm{diag}\ (i)_{i\in\mathbb{Z}} \cdot \Lambda^{-1} = \begin{pmatrix} \ddots & & & & & & \\ & 0 & & & & & \\ & -1 & 0 & & & & \\ & & 0 & 0 & & & \\ & & & 1 & 0 & & \\ & & & & 2 & 0 & \\ & & & & & & \ddots \end{pmatrix} . \tag{1.10}$$

Introducing the operator $j := (\delta_{i+j,0})_{i,j\in\mathbb{Z}} : (x_i) \to (x_{-i})$ we also define

$$\chi^*(z) := j\chi(z) = \chi(z^{-1}), \quad \partial^* := j\partial j^{-1} = \partial^{-1} \quad \text{and } \varepsilon^* := j\varepsilon j^{-1} \,. \tag{1.11}$$

They satisfy

$$\begin{aligned} \partial^*\chi^* &= z\chi^*, \quad \varepsilon^*\chi^* = (\partial/\partial z)\chi^* \,, \\ \partial^*\chi &= z^{-1}\chi, \quad \varepsilon^*\chi = (\partial/\partial z^{-1})\chi \,, \\ [\partial^*,\varepsilon^*] &= j[\partial,\varepsilon]j^{-1} = 1 \,. \end{aligned} \tag{1.12}$$

Pseudodifferential Operators. Let $C(\mathbb{F})$ be a ring of "smooth" functions on $\mathbb{F}$; e.g., $\mathbb{C}[[x]]$ if $\mathbb{F} = $ Spec $\mathbb{C}[[x]]$, and $\mathbb{C}^{\mathbb{Z}}$ (vectors, and also regarded as the ring of diagonal matrices) if $\mathbb{F} = \mathbb{Z}$. The ring of pseudodifferential operators and its discrete analogue[3], both denoted by $\mathscr{D}$:

$$\mathscr{D} = \begin{cases} C(\mathbb{F})((\partial^{-1})) = C(\mathbb{F})\left(\left(\left(\frac{\partial}{\partial x}\right)^{-1}\right)\right) & \text{(continuous case)} \\ C(\mathbb{F})((\partial^{-1})) \oplus C(\mathbb{F})((\partial)) = C(\mathbb{F})((\Lambda^{-1})) \oplus C\mathbb{F}((\Lambda)) & \text{(discrete case),} \end{cases}$$

are associative algebras over $C(\mathbb{F})$. The action of $\mathscr{D}$ on χ, simply defined by the formula[4]

$$\sum a_i(x)\partial^i \chi = \sum a_i(x) z^i \chi \ ,$$

is a formal Laurent series in z^{-1} times χ in the continuous case, and a pair of formal Laurent series in $z^{\mp 1}$ times χ in the discrete case; i.e., geometrically the action is well-defined on the infinitesimal punctured disc(s) around $z = \infty$ (continuous case), or around $z = 0, \infty$ (discrete case). Clearly, $\mathscr{D}\chi$ is a left $\mathscr{D}$-module isomorphic to $\mathscr{D}$ itself.

The splitting of $\mathscr{D}$ into two subalgebras, $\mathscr{D}_+ \oplus \mathscr{D}_-$, as explained in the preceding section, is characterized as follows: every element of $\mathscr{D}_+\chi$ extends holomorphically to $z \in \mathbb{C}$ (continuous case) or $z \in \mathbb{C}^*$ (discrete case), while every element of $\mathscr{D}_-\chi$, still a formal series, has no poles at $z = \infty$ (continuous case) or $z = \infty, 0$ (discrete case), and vanishes at $z = \infty$. The last property (vanishing at ∞) is not a canonical choice, and it is only to exclude constants from $\mathscr{D}_-$. A different choice was made in [A-vM0, vM-M].

In the discrete case the following simple observation will be useful.

Remark 1.1. For $P = (P_1, P_2) \in \mathscr{D}$, we have $P_- = 0$ if and only if $P_1 = P_2$. Moreover, if we define an additive group isomorphism $Y : \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}} \to \mathscr{D}_-$ by $Y(A) = (A_{(l)}, -A_{(u)})$, then $P_- = Y(P_1 - P_2)$.

KP Equations (Continuous Case). A solution

$$L = \partial + \sum_{j=-1}^{-\infty} a_j(x,t)\partial^j \in \mathscr{D}, \quad \text{with} \quad x \in \mathbb{F},\ t = (t_1, t_2, \dots\ ), \tag{1.13}$$

to the KP equations (0.0) has, in terms of the wave operator $S \in 1 + \mathscr{D}_-$, the associated

$$W := S e^{\sum_{i=1}^{\infty} t_i \partial^i} \ , \tag{1.14}$$

or the wave (Baker-Akhiezer) function

$$\Psi := W\chi = S e^{\sum_{i=1}^{\infty} t_i z^i} \chi(z) \ , \tag{1.15}$$

the following representation

$$L = S\partial S^{-1} = W\partial W^{-1} \ \mathit{or}\ L\Psi = z\Psi \ , \tag{1.16}$$

---

[3] Note that abstractly the three rings $C_{\text{cont}}(\mathbb{F})((D^{-1}))$, $C_{\text{discrete}}(\mathbb{F})((\Lambda^{\pm 1}))$ are just different completions of $\mathbb{C}[\partial, \varepsilon]$.

[4] In the continuous case, $a_i = a_i(x)$ and in the discrete case, $a_i = \text{diag}\ (a_i(n))_{n\in\mathbb{Z}}$

(note that the last condition determines L uniquely) subjected to the following equivalent conditions:

$$\frac{\partial S}{\partial t_n} = -(L^n)_- S\,, \tag{1.17}$$

$$\frac{\partial W}{\partial t_n} = (L^n)_+ W\,, \tag{1.18}$$

$$\frac{\partial \Psi}{\partial t_n} = (L^n)_+ \Psi\,. \tag{1.19}$$

Conversely, given (equivalently) S, W *or* Ψ satisfying these conditions, L given by (1.16) satisfies the KP equations (0.0). Indeed, the equivalence of (1.17) and (1.18) follows from (1.14) and (1.16):

$$\frac{\partial W}{\partial t_n} W^{-1} - \frac{\partial S}{\partial t_n} S^{-1} = S \frac{\partial e^{\sum t_i \partial^i}}{\partial t_n} e^{-\sum t_i \partial^i} S^{-1} = S \partial^n S^{-1} = L^n\,,$$

and the equivalence of (1.18) and (1.19) follows from the definition (1.15) of Ψ.

Let

$$[\alpha] := \left(\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \dots\right), \quad \bar{t} := (x + t_1, t_2, t_3, \dots)\,,$$
$$\tilde{\partial} := (\partial/\partial t_1, (1/2)\partial/\partial t_2, (1/3)\partial/\partial t_3, \dots)\,,$$

and

$$e^{\sum_1^\infty t_n z^n} = \sum_0^\infty p_n(t) z^n\,.$$

According to [DJKM], Ψ has the following representation in terms of the τ-function τ:

$$\begin{aligned}\Psi(t,z) &= \frac{\tau(\bar{t} - [z^{-1}])}{\tau(\bar{t})} e^{\sum_1^\infty t_i z^i} \chi(z) \\ &= \sum_{n=0}^\infty \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} z^{-n} e^{\sum_1^\infty t_i z^i} \chi(z) \\ &= \sum_{n=0}^\infty \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} (\partial^{-n}\chi(z)) e^{\sum_1^\infty t_i z^i}\,,\end{aligned} \tag{1.20}$$

implying in view of (1.15)

$$S = \frac{\tau(\bar{t} - [\partial^{-1}])}{\tau(\bar{t})} := \sum_{n=0}^\infty \frac{p_n(-\tilde{\partial})\tau(\bar{t})}{\tau(\bar{t})} \partial^{-n}\,. \tag{1.22}$$

Note that the last equation in (1.16):

$$\begin{aligned}z\Psi &= zW\chi = Wz\chi = W\partial\chi = W\partial W^{-1}\Psi \\ &= L\Psi\end{aligned} \tag{1.25}$$

is a half of the dressed version of the relation (1.3). The other half:

$$\frac{\partial}{\partial z}\Psi = W\frac{\partial}{\partial z}\chi = W\varepsilon\chi = W\varepsilon W^{-1}\Psi \\ = M\Psi \tag{1.26}$$

naturally leads to the operator (use (1.5))

$$M := W\varepsilon W^{-1} = Se^{\sum t_k\partial^k}\varepsilon e^{-\sum t_k\partial^k}S^{-1} = S(\varepsilon + \sum kt_k\partial^{k-1})S^{-1} \tag{1.27}$$

satisfying

$$[L,M] = W[\partial,\varepsilon]W^{-1} = 1. \tag{1.28}$$

Thus for any formal series f in ε and ∂ we have

$$f(M,L) = Wf(\varepsilon,\partial)W^{-1}. \tag{1.29}$$

Two-Dimensional Infinite Toda Lattice (Discrete Case). A solution $L = (L_1, L_2)$ of the two-dimensional Toda lattice equations (0.3) has a representation in terms of the pair of wave operators $S = (S_1, S_2) \in \exp\mathscr{D}_-$, i.e.,

$$\begin{cases} S_1 = \sum\limits_{i\leqq 0} c_i\Lambda^i, \quad S_2 = \sum\limits_{i\geqq 0} c_i'\Lambda^i, \\ c_i, c_i'\text{: diagonal matrices, } c_0 = I,\ (c_0')_{ii} \neq 0\ \forall i, \end{cases} \tag{1.32}$$

the associated $W = (W_1, W_2)$:

$$W_1(t,s) = S_1(t,s)e^{\sum_1^\infty t_k\Lambda^k}, \quad W_2(t,s) = S_2(t,s)e^{\sum_1^\infty s_k\Lambda^{-k}}, \tag{1.33}$$

or the pair of vector wave functions $\Psi = (\Psi_1, \Psi_2)$:

$$\Psi_i(t,s;z) := W_i(t,s)\chi(z),$$

as follows (here recall $\partial = \Lambda$ and $\partial^* = \Lambda^{-1}$)

$$L := W(\partial,\partial^*)W^{-1} = S(\partial,\partial^*)S^{-1}, \quad \text{or } L\Psi = (z,z^{-1})\Psi$$

with the following equivalent conditions satisfied:

$$\frac{\partial S_i}{\partial t_n} = -(L_1^n,0)_-S_i, \qquad \frac{\partial S_i}{\partial s_n} = -(0,L_2^n)_-S_i \tag{1.34}$$

$$\frac{\partial W_i}{\partial t_n} = (L_1^n,0)_+W_i, \qquad \frac{\partial W_i}{\partial s_n} = (0,L_2^n)_+W_i, \tag{1.35}$$

$$\frac{\partial \Psi_i}{\partial t_n} = (L_1^n,0)_+\Psi_i, \qquad \frac{\partial \Psi_i}{\partial s_n} = (0,L_2^n)_+\Psi_i. \tag{1.36}$$

In accordance with (1.26), we also define

$$M := (M_1, M_2) = W(\varepsilon,\varepsilon^*)W^{-1}, \tag{1.37}$$

satisfying $[L_i, M_i] = 1$ and

$$M\Psi = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z^{-1}}\right)\Psi. \tag{1.38}$$

Throughout this paper, for a vector $\tau = (\tau_n)_{n\in\mathbb{Z}}$ we shall denote

$$\tau_\Lambda := \Lambda\tau, \text{ i.e., } (\tau_\Lambda)_n = \tau_{n+1}. \tag{1.39}$$

According to [UT], Ψ has the following representation in terms of the vector of τ-functions $\tau = (\tau_n)_{n\in\mathbb{Z}}$:

$$\begin{aligned}\Psi_1(t,s;z) &= S_1 e^{\sum t_i \Lambda^i}\chi(z) \\ &= \left(\frac{\tau_n(t-[z^{-1}],s)}{\tau_n(t,s)} e^{\sum t_i z^i} z^n\right)_{n\in\mathbb{Z}} \\ &= e^{\sum t_i z^i}\frac{\tau(t-[\Lambda^{-1}],s)}{\tau}\chi(z),\end{aligned} \tag{1.40}$$

$$\begin{aligned}\Psi_2(t,s;z) &= S_2 e^{\sum s_i \Lambda^{-i}}\chi(z) \\ &= \left(\frac{\tau_{n+1}(t,s-[z])}{\tau_n(t,s)} e^{\sum s_i z^{-i}} z^n\right)_{n\in\mathbb{Z}} \\ &= e^{\sum s_i z^{-i}}\frac{\tau_\Lambda(t,s-[\Lambda])}{\tau}\chi.\end{aligned} \tag{1.41}$$

So we have

$$S_1 = \frac{\tau(t-[\partial^{-1}],s)}{\tau}, \quad S_2 = \frac{\tau_\Lambda(t,s-[\partial])}{\tau}. \tag{1.42}$$

Adjoint Wave Function. In either KP (continuous) or 2-Toda (discrete) case, given a wave function $\Psi = W\chi$, the adjoint wave function is defined by

$$\Psi^* = (W^\top)^{-1}\chi^*. \tag{1.43}$$

It satisfies similar equations as Ψ, represented in terms of τ (see [DJKM, UT]), and, together with Ψ, plays an important role in the bilinear identities (see Sects. 3 and 4).

δ-function. According to [DJKM], the following formal series:

$$\delta(\lambda,z) = \frac{1}{z}\sum_{n=-\infty}^{\infty}\left(\frac{z}{\lambda}\right)^n = \frac{1}{z}\frac{1}{1-\lambda/z} + \frac{1}{\lambda}\frac{1}{1-z/\lambda} \tag{1.44}$$

is a δ-function in the following sense: given a function $f(z) = \sum_{i=-\infty}^{\infty} a_i z^i$,

$$f(\lambda)\delta(\lambda,z) = f(z)\delta(\lambda,z), \tag{1.45}$$

as is seen from $z^i\sum_n (z/\lambda)^n = \lambda^i\sum_n (z/\lambda)^{n+i}$. Thus

$$(\mu-\lambda)\delta(\lambda,z) = (\mu-z)\delta(\lambda,z) = -\frac{1-\mu/z}{1-\lambda/z} + \frac{\mu}{\lambda}\frac{1-z/\mu}{1-z/\lambda}, \tag{1.46}$$

and

$$\delta(\lambda,z)\chi(z+\mu-\lambda) = \delta(\lambda,z)\chi(\mu)\chi^*(\lambda)\chi(z). \tag{1.47}$$

Later we shall use

$$\delta(\lambda, \partial) = \sum_{n=-\infty}^{\infty} \lambda^{-n} \partial^{n-1}$$

for which formally

$$f(\partial)\delta(\lambda, \partial) = f(\lambda)\delta(\lambda, \partial). \tag{1.48}$$

2. Symmetries and a Non-Commutative Splitting Theorem

We shall need the following Lie algebra splitting lemma, dealing with operators and their eigenfunctions, which generalizes the Adler–Konstant–Symes Theorem.

Lemma 2.1 *Let $\mathscr{D}$ be a Lie algebra with a vector space decomposition $\mathscr{D} = \mathscr{D}_+ \oplus \mathscr{D}_-$ into two Lie subalgebras $\mathscr{D}_+$ and $\mathscr{D}_-$; let V be a representation space of $\mathscr{D}$, and let $\mathfrak{M} \subset V$ be a submanifold preserved under the vector fields defined by the action of $\mathscr{D}_-$, i.e.,*

$$\mathscr{D}_- x \subset T_x\mathfrak{M} \quad \forall x \in \mathfrak{M}.$$

For any function $p : \mathfrak{M} \to \mathscr{D}$, let $\mathbb{Y}_p$ be the vector field on $\mathfrak{M}$ defined by

$$\mathbb{Y}_p(x) := -p(x)_- x, \quad x \in \mathfrak{M}.$$

(a) Consider a set $\mathscr{A}$ of functions $p : \mathfrak{M} \to \mathscr{D}$ such that

$$\mathbb{Y}_q p = [-q_-, p], \quad \forall p, q \in \mathscr{A},$$

holds. Then $\mathbb{Y} : p \mapsto \mathbb{Y}_p$ gives a Lie algebra homomorphism of the Lie algebra generated by $\mathscr{A}$ to the Lie algebra $\mathscr{X}(\mathfrak{M})$ of vector fields on $\mathfrak{M}$:

$$\left[\mathbb{Y}_{p_1}, \mathbb{Y}_{p_2}\right] = \mathbb{Y}_{[p_1, p_2]}, \quad \forall p_1, p_2 \in \mathscr{A},$$

and hence we can assume without loss of generality that $\mathscr{A}$ itself is a Lie algebra.
(b) Suppose for a subset $\mathscr{B} \subset \mathscr{A}$ of functions

$$\mathbb{Z}_q(x) := q(x)_+ x \in T_x\mathfrak{M}, \quad \forall x \in \mathfrak{M}, q \in \mathscr{B},$$

and hence defines another vector field $\mathbb{Z}_q \in \mathscr{X}(\mathfrak{M})$ when $q \in \mathscr{B}$, and such that

$$\mathbb{Z}_q p = [q_+, p], \quad \forall p \in \mathscr{A}, q \in \mathscr{B},$$

holds. Then

$$[\mathbb{Y}_p, \mathbb{Z}_q] = 0, \quad \forall p \in \mathscr{A}, q \in \mathscr{B}.$$

Remark. A special case of this which applies to many integrable systems is: $V = \mathscr{D}'$, a Lie algebra containing $\mathscr{D}$, and $\mathscr{D}$ acts on $\mathscr{D}'$ by Lie bracket, i.e., $\mathbb{Y}_p(x) = [-p(x)_-, x]$, etc.

Proof. Let $p_1, p_2 \in \mathscr{A}$, and first consider the infinitesimal flow

$$\exp(\varepsilon_1 \mathbb{Y}_{p_1}) \exp(\varepsilon_2 \mathbb{Y}_{p_2}) \bmod (\varepsilon_1^2, \varepsilon_2^2).$$

Since $\big(\exp(\varepsilon\mathbb{Y}_p(x))\big)x = (1+\varepsilon\mathbb{Y}_p(x))\cdot x \bmod \varepsilon^2$, at $x\in\mathfrak{M}$ this becomes

$$\begin{aligned}
&(1+\varepsilon_1\mathbb{Y}_{p_1}(x+\varepsilon_2\mathbb{Y}_{p_2}x))\cdot(x+\varepsilon_2\mathbb{Y}_{p_2}(x))\\
&\quad=(1-\varepsilon_1 p_1(x+\varepsilon_2\mathbb{Y}_{p_2}x)_-)\cdot(x-\varepsilon_2 p_2(x)_-x)\\
&\quad=(1-\varepsilon_1 p_1(x)_- - \varepsilon_1\varepsilon_2\mathbb{Y}_{p_2}(x)(p_1(x))_-)\cdot(x-\varepsilon_2 p_2(x)_-x)\\
&\quad=(1-\varepsilon_1 p_1(x)_- - \varepsilon_1 p_2(x)_- + \varepsilon_1\varepsilon_2(-\mathbb{Y}_{p_2}(x)(p_1(x))_- + p_1(x)_- p_2(x)_-))x.
\end{aligned}$$

Hence

$$\begin{aligned}
[\mathbb{Y}_{p_1},\mathbb{Y}_{p_2}](x) &= \text{coefficient of } \varepsilon_1\varepsilon_2 \text{ in}\\
&\quad \exp(\varepsilon_1\mathbb{Y}_{p_1})\exp(\varepsilon_2\mathbb{Y}_{p_2})x - \exp(\varepsilon_2\mathbb{Y}_{p_2})\exp(\varepsilon_1\mathbb{Y}_{p_1})x \bmod (\varepsilon_1^2,\varepsilon_2^2)\\
&\quad = Z_1 x,
\end{aligned}$$

where

$$\begin{aligned}
Z_1 &:= (\mathbb{Y}_{p_1}(p_2))_- - (\mathbb{Y}_{p_2}(p_1))_- + [p_{1-},p_{2-}]\\
&= [-p_{1-},p_2]_- - [-p_{2-},p_1]_- + [p_{1-},p_{2-}]\\
&= -[p_{1-},p_2]_- - [p_1,p_{2-}]_- + [p_{1-},p_{2-}]_-\,,\\
&\qquad\qquad\qquad\qquad \text{using } \mathscr{D}_- \text{ is a Lie subalgebra,}\\
&= (-[p_{1-},p_2] - [p_1,p_{2-}] + [p_{1-},p_{2-}] - [p_{1+},p_{2+}])_-\,,\\
&\qquad\qquad\qquad\qquad \text{using } \mathscr{D}_+ \text{ is a Lie subalgebra,}\\
&= -([p_{1-},p_{2+}] + [p_1,p_{2-}] + [p_{1+},p_{2+}])_-\,,\\
&\qquad\qquad\qquad\qquad \text{combining the 1st and 3rd terms}\\
&= -([p_1,p_{2+}] + [p_1,p_{2-}])_-\,, \text{ combining the 1st and 3rd terms}\\
&= -[p_1,p_2]_-.
\end{aligned}$$

Similarly,

$$[\mathbb{Z}_q,\mathbb{Y}_p](x) = Z_2 x\,,$$

where

$$\begin{aligned}
Z_2 &:= (\mathbb{Z}_q(p))_- + (\mathbb{Y}_p(q))_+ - [q_+,p_-]\\
&= [q_+,p]_- + [-p_-,q]_+ - [q_+,p_-]\\
&= [q_+,p_-]_- + [-p_-,q_+]_+ - [q_+,p_-]\,,\\
&\qquad\qquad\qquad\qquad \text{since } \mathscr{D}_\pm \text{ are Lie subalgebras}\\
&= [q_+,p_-] - [q_+,p_-] = 0\,,
\end{aligned}$$

ending the proof of the lemma.

Remark 2.1.0. In the setup of the lemma, if we are given a Lie algebra (anti)homomorphism $\phi : \mathrm{g}\to\mathscr{A}$, we denote $\mathbb{Y}_{\phi(x)}$ by $\mathbb{Y}_x$ and $\mathbb{Z}_{\phi(x)}$ by $\mathbb{Z}_x$ if there is no fear of confusion.

Consider the Lie algebra w_∞ (introduced in Sect. 1) of ordinary differential operators with Laurent polynomial coefficients spanned by

$$\left\{ z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \,\middle|\, \alpha,\beta\in\mathbb{Z}, \beta\geqq 0\right\}.$$

The Lie algebra $\text{span}\{z^\alpha \partial/\partial z \mid \alpha \in \mathbb{Z}\}$ of vector fields on the circle S^1 is a subalgebra of w_∞.

We now apply the above setup to

$$\mathscr{D} := \text{the algebra of} \begin{cases} \text{pseudodifferential operators} & \text{(KP case)}, \\ \text{pairs of infinite matrices as in preceding sections} & \text{(2-Toda case)}, \end{cases}$$

$$\mathscr{D}' := \mathscr{D} \times \mathscr{D}, \quad \text{(see remark at the end of Lemma 2.1)}$$

on which $\mathscr{D}$ acts via diagonal embedding $\mathscr{D} \hookrightarrow \mathscr{D}' : \; p \mapsto (p, p)$,

$V := \mathscr{D}, \mathscr{D}\chi$ or $\mathscr{D}'$,

$\mathfrak{M} :=$ respectively, the space of wave operators W, of wave functions Ψ or of pairs (L, M) such that $[L, M]{=}1$,

$\mathscr{A} :=$ the space of polynomials in

$$\begin{cases} L \text{ and } M & \text{(KP case)}, \\ (L_1, 0), (0, L_2), (M_1, 0) \text{ and } (0, M_2) & \text{(2-Toda case)}, \end{cases}$$

$\mathscr{B} :=$ the space of polynomials in

$$\begin{cases} L & \text{(KP case)}, \\ (L_1, 0), (0, L_2) & \text{(2-Toda case)}, \end{cases}$$

and

$$\mathfrak{g} := w = \begin{cases} w_\infty & \text{(KP case)} \\ w_\infty \times w_\infty & \text{(2-Toda case)}, \end{cases}$$

with the antihomomorphism $\phi : \mathfrak{g} \to \mathscr{A}$ given by

$$\phi(z^\alpha \partial_z^\beta) := M^\beta L^\alpha \quad \text{(KP case), or}$$

$$\left.\begin{aligned} \phi(z^\alpha \partial_z^\beta, 0) &:= (M_1^\beta L_1^\alpha, 0) \\ \phi(0, u^\alpha \partial_u^\beta) &:= (0, M_2^\beta L_2^\alpha) \end{aligned}\right\} \quad \text{(2-Toda case)},$$

where $\partial_z = \partial/\partial z$, $u = z^{-1}$, $\partial_u = \partial/\partial(z^{-1})$. Noting

$$\mathbb{Z}_{L^n} \Psi = (L^n)_+ \Psi = \frac{\partial}{\partial t_n} \Psi, \quad \text{(KP case)}$$

$$\left.\begin{aligned} \mathbb{Z}_{(L_1^n, 0)} \Psi &= (L_1^n, 0)_+ \Psi = \frac{\partial}{\partial t_n} \Psi, \\ \mathbb{Z}_{(0, L_2^n)} \Psi &= (0, L_2^n)_+ \Psi = \frac{\partial}{\partial s_n} \Psi, \end{aligned}\right\} \quad \text{(2-Toda case)}$$

etc., yields

Theorem 2.1 (i) *KP case. There is an antihomomorphism of Lie algebras*

$$w_\infty \to \left\{ \begin{array}{l} \textit{Lie algebra of vector fields on} \\ \textit{the } \Psi\textit{-manifold commuting} \\ \textit{with the KP-flows} \end{array} \right\}$$

$$\simeq \left\{ \begin{array}{l} \textit{Lie algebra of vector fields on} \\ \textit{the } (L, M)\textit{-manifold commuting} \\ \textit{with the KP-flows} \end{array} \right\}$$

$$z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta \mapsto \mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta},\quad \mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}\Psi := -(M^\beta L^\alpha)_-\Psi,$$
$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}L := [-(M^\beta L^\alpha)_-, L],$$
$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}M := [-(M^\beta L^\alpha)_-, M], \tag{2.1}$$

i.e., it satisfies

$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \mathbb{Y}_{z^{\alpha'}\left(\frac{\partial}{\partial z}\right)^{\beta'}}\right] = \mathbb{Y}_{\left[z^{\alpha'}\left(\frac{\partial}{\partial z}\right)^{\beta'}, z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta\right]},$$
$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \frac{\partial}{\partial t_n}\right] = 0.$$

(ii) *Two-dimensional Toda lattice. There is an antihomomorphism of Lie algebras*

$$w_\infty \times w_\infty \to \left\{\begin{array}{l} \textit{Lie algebra of vector fields on the} \\ \Psi = (\Psi_1, \Psi_2) \textit{ manifold commuting} \\ \textit{with the } \partial/\partial t_i \textit{ and } \partial/\partial s_i \textit{ flows} \end{array}\right\}$$
$$\simeq \left\{\begin{array}{l} \textit{Lie algebra of vector fields on the} \\ (L, M) = \big((L_1, L_2), (M_1, M_2)\big) \textit{ manifold} \\ \textit{commuting with the } \partial/\partial t_i \textit{ and } \partial/\partial s_i \textit{ flows} \end{array}\right\}$$

$$\left(z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta, 0\right) \mapsto \mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta} := \mathbb{Y}_{\left(z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta, 0\right)},\quad \mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}\Psi := -(M_1^\beta L_1^\alpha, 0)_-\Psi,$$
$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}L := [-(M_1^\beta L_1^\alpha, 0)_-, L],$$
$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}M := [-(M_1^\beta L_1^\alpha, 0)_-, M],$$

$$\left(0, u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta\right) \mapsto \mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta} := \mathbb{Y}_{\left(0, u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta\right)},\quad \mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}\Psi := -(0, M_2^\beta L_2^\alpha)_-\Psi,$$
$$\mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}L := [-(0, M_2^\beta L_2^\alpha)_-, L],$$
$$\mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}M := [-(0, M_2^\beta L_2^\alpha)_-, M],$$

i.e., it satisfies

$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \mathbb{Y}_{z^{\alpha'}\left(\frac{\partial}{\partial z}\right)^{\beta'}}\right] = \mathbb{Y}_{\left[z^{\alpha'}\left(\frac{\partial}{\partial z}\right)^{\beta'}, z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta\right]},$$
$$\left[\mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}, \mathbb{Y}_{u^{\alpha'}\left(\frac{\partial}{\partial u}\right)^{\beta'}}\right] = \mathbb{Y}_{\left[u^{\alpha'}\left(\frac{\partial}{\partial u}\right)^{\beta'}, u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta\right]},$$
$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \mathbb{Y}_{u^{\alpha'}\left(\frac{\partial}{\partial u}\right)^{\beta'}}\right] = 0,$$
$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \frac{\partial}{\partial t_n}\right] = \left[\mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}, \frac{\partial}{\partial t_n}\right] = 0 \quad \textit{and}$$
$$\left[\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}, \frac{\partial}{\partial s_n}\right] = \left[\mathbb{Y}_{u^\alpha\left(\frac{\partial}{\partial u}\right)^\beta}, \frac{\partial}{\partial s_n}\right] = 0. \tag{2.2}$$

Remark 2.1.1. These vector fields induce vector fields on S and $M^kL^j, j,k \in \mathbb{Z}$, $k \geqq 0$, as follows:

$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}(S) = -(M^\beta L^\alpha)_- S$$

and

$$\mathbb{Y}_{z^\alpha\left(\frac{\partial}{\partial z}\right)^\beta}(M^kL^j) = [-(M^\beta L^\alpha)_-, M^kL^j] \ . \tag{2.3}$$

Remark 2.1.2. It will be necessary to understand the relationship between the flows $\mathbb{Y}_n := \mathbb{Y}_{z^n}$ and the KP flows $\mathbb{Z}_n := \mathbb{Z}_{z^n} = \partial/\partial t_n$, in the notation of Lemma 2.1 and Remark 2.1.0. To do so, take the partial derivative with regard to the initial condition s in $\tau_s := \tau(s+t)$, where $s = (s_1, s_2, \ldots)$ is another sequence of scalar variables, bearing no relation to the s-variables in the 2-Toda hierarchy, and

$$\Psi_s(t,z) := e^{\sum t_i z^i} \frac{\tau_s(t - [z^{-1}])}{\tau_s(t)} \ ,$$

yielding

$$\frac{\partial \Psi}{\partial s_n} = \frac{\partial \Psi}{\partial t_n} - z^n \Psi = (L^n)_+ \Psi - L^n \Psi = -(L^n)_- \Psi \ .$$

This relation implies

$$\frac{\partial S}{\partial s_n} e^{\sum t_i z^i} = \frac{\partial \Psi}{\partial s_n} = -(L^n)_- S\, e^{\sum t_i z^i} \ ,$$

and so

$$\frac{\partial S}{\partial s_n} = -(L^n)_- S = \frac{\partial S}{\partial t_n} \ .$$

So, at the level of Ψ, the vector fields $\mathbb{Y}_n = \partial/\partial s_n$ and $\mathbb{Z}_n = \partial/\partial t_n$ are different, but they coincide at the level of S and L.

Special Subalgebra: There are other realizations of w_∞, namely $w_\infty(u(z))$, where $u(z)$ is a formal series in z: (see [FKN])

$$\begin{aligned} w_\infty \to w_\infty(u(z)) : y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta &\mapsto \sqrt{u'} \left(y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta \Big|_{y=u} \right) \frac{1}{\sqrt{u'}} \\ &= u^\alpha \left(\frac{1}{u'} \frac{\partial}{\partial z} - \frac{1}{2} \frac{u''}{u'^2} \right)^\beta . \end{aligned}$$

In particular for $u(z) = z^p$

$$w_\infty \to w_\infty(z^p) : y^\alpha \left(\frac{\partial}{\partial y}\right)^\beta \mapsto z^{\alpha p} \left(\frac{1}{p} \left(z^{-p+1} \frac{\partial}{\partial z} - \frac{p-1}{2} z^{-p} \right) \right)^\beta .$$

Fix $p \geqq 1$. Since

$$\left\{ z^{(\alpha-1)p+1} \frac{\partial}{\partial z} - \frac{p-1}{2} z^{(\alpha-1)p} \middle| \alpha - 1 \geqq -1 \right\} \subset w_\infty(z^p)$$

is a subalgebra, it is natural to consider the subalgebra of (Virasoro) vector fields on Ψ generated by

$$\mathbb{L}_n(\Psi) = -\frac{1}{p}\left(ML^{np+1} - \frac{p-1}{2}L^{np}\right)_-\Psi, \quad n \geqq -1,$$

inducing the following vector fields on L:

$$\mathbb{L}_n(L) = \left[-\frac{1}{p}\left(ML^{np+1} - \frac{p-1}{2}L^{np}\right)_-, L\right], \quad n \geqq -1.$$

Corollary 2.2. *If L^p is a differential operator, then L^p remains differential under the vector fields $\mathbb{L}_n(L)$, $n \geqq -1$; the latter form a Virasoro algebra (with zero central charge) and they interact with the symmetry $\mathbb{Y}_k = \partial/\partial_{s_k}$ induced by the KP flows (using the notation of Remark 2.1.2) as follows:*

$$[\mathbb{L}_n, \mathbb{L}_m] = (n-m)\mathbb{L}_{n+m} \quad \text{and } [\mathbb{L}_n, \mathbb{Y}_k] = -\frac{k}{p}\mathbb{Y}_{np+k}. \tag{2.9}$$

Proof. The vector fields ∂/∂_{s_k} and $\mathbb{L}_n$ correspond respectively to the elements z^k and

$$v_n = \frac{1}{p}\left(z^{np+1}\frac{\partial}{\partial z} - \frac{p-1}{2}z^{np}\right) = \sqrt{z^{p-1}}\left(y^{n+1}\frac{\partial}{\partial y}\bigg|_{y=z^p}\right)\frac{1}{\sqrt{z^{p-1}}}, \quad n \geqq -1, \tag{2.10}$$

of w_∞, which satisfy

$$[v_n, v_m] = (m-n)v_{n+m} \quad \text{and} \quad [v_n, z^k] = \frac{k}{p}z^{np+k},$$

and we apply (2.2) to get (2.9).

The vector field $\mathbb{L}_n$ acting on L^p reads:

$$\begin{aligned}\mathbb{L}_n(L^p) &= \left[-\frac{1}{p}(ML^{np+1})_-, L^p\right], \text{ which has order at most } p-1,\\ &= \left[-\frac{1}{p}ML^{np+1}, L^p\right] + \frac{1}{p}[(ML^{np+1})_+, L^p]\\ &= (L^p)^{n+1} + \frac{1}{p}[(ML^{np+1})_+, L^p],\end{aligned} \tag{2.11}$$

using $[z^{np+1}\frac{\partial}{\partial z}, z^p] = pz^{(n+1)p}$; this expression is a differential operator as long as $n+1 \geqq 0$. Therefore $\mathbb{L}_n(L^p)$ is a differential operator of order at most $p-1$, implying the statement.

Example. The Korteweg–de Vries equation ($p=2$) and its symmetries. The symmetries for KdV were also used, although in a different context, by Duistermaat–Grünbaum and Magri–Zubelli; we now show that their vector fields coincide with ours, i.e. (2.11) for $p=2$ and evaluated at $t=0$. Since

$$-\frac{1}{2}(ML^{2n+1})\bigg|_{t=0} = -\frac{1}{2}S\left(x + \sum_1^\infty kt_kD^{k-1}\right)D^{2n+1}S^{-1}\bigg|_{t=0} = -\frac{1}{2}SxD^{2n+1}S^{-1},$$

the vector fields $\mathbb{L}_n^{(0)} = \mathbb{L}_n\big|_{t=0}$ for $n \geqq -1$ act on the differential operator $L^2 = D^2 + q = D^2 + 2(\log \tau)''$ as in (2.11):

$$\mathbb{L}_n^{(0)}(L^2) = \left[\frac{1}{2}(SxD^{2n+1}S^{-1})_+, L^2\right] + (L^2)^{n+1} . \tag{2.12}$$

We now compute a few examples; note that

$$\begin{aligned}
&\frac{1}{2}(SxD^{-1}S^{-1})_+ = 0 , \\
&\frac{1}{2}(SxDS^{-1})_+ = \frac{x}{2}D , \\
&\frac{1}{2}(SxD^3S^{-1})_+ = \frac{x}{2}D^3 - aD - b ,
\end{aligned} \tag{2.13}$$

for some functions $a(t)$ and $b(t)$ to be determined. Expanding SxD^nS^{-1} up to order D^{n-2},

$$\begin{aligned}
SxD^nS^{-1} &= \left(1 - \frac{\tau'}{\tau}D^{-1} + \cdots\right)xD^n\left(1 + \frac{\tau'}{\tau}D^{-1} + \cdots\right) \\
&= xD^n + (nx(\log \tau)'' + (\log \tau)')D^{n-2} + \cdots \\
&= xD^n + \frac{1}{2}\left(nxq + \int_{-\infty}^{x} q\right)D^{n-2} + \cdots ,
\end{aligned}$$

yields

$$a = -\frac{1}{4}\left(3xq + \int_{-\infty}^{x} q\right) . \tag{2.14}$$

The function b can be determined, either by expanding further or by the following (more efficient) argument. The bracket appearing in (2.12) is a symmetric operator, since L^2 and $\mathbb{L}_n^{(0)}(L^2)$ are symmetric. Since the bracket of a symmetric and a skew-symmetric operator is symmetric, one expects $\frac{1}{2}(SxD^{2n+1}S^{-1})_+$ to become skew-symmetric, after adding an appropriate differential operator commuting with L^2, to wit an operator of the form $\sum_{k \geqq 0} c_k L^{2k}$. For instance, if we assume

$$\begin{aligned}
\frac{1}{2}(SxDS^{-1})_+ + c_0 &= \frac{1}{4}(xD + Dx) \\
\frac{1}{2}(SxD^3S^{-1})_+ + c_1L^2 &= \frac{1}{4}(xD^3 + D^3x) + c'(fD + Df) ,
\end{aligned} \tag{2.15}$$

then the constants c_i and c' and the function f are readily obtained by setting the expression (2.13) for $\frac{1}{2}(SxD^3S^{-1})_+$ into (2.15) and comparing both sides, yielding $c_0 = \frac{1}{4}, c_1 = \frac{3}{4}, c' = -\frac{1}{2}, f = a$ and

$$b = \frac{1}{2}a' + \frac{3}{4}q = \frac{3}{8}xq' + \frac{1}{4}q, \text{ setting (2.14) for } a . \tag{2.16}$$

Since Corollary 2.2 implies that the D-part of (2.12) is absent, it suffices to compute the independent term in:

$$\begin{aligned}\mathbb{L}_1^{(0)}(L^2) = \dot{q} &= \left[\frac{1}{2}(SxD^3S^{-1})_+, L^2\right] + (L^2)^2 \\ &= \left[\frac{x}{2}D^3 - aD - b, D^2 + q\right] + (D^2+q)^2 \\ &= \left(\frac{x}{2}q''' - aq' + b'' + q'' + q^2\right) . \end{aligned} \tag{2.17}$$

Using (2.13) and substituting (2.14) and (2.16) for a and b into (2.17), lead to

$$\begin{aligned}\mathbb{L}_{-1}^{(0)}(q) &= 1 , \\ \mathbb{L}_0^{(0)}(q) &= \frac{x}{2}q' + q , \\ \mathbb{L}_1^{(0)}(q) &= \frac{1}{8}\left(xq''' + 6xqq' + 4q'' + 8q^2 + 2q'\int_{-\infty}^{x} q\right) .\end{aligned}$$

3. Bilinear Identities (Continuous)

As a foreshadowing of the future, we provide a proof of the bilinear identities, essentially the same as in [DJKM], depending on the following two lemmas:

Lemma 3.1. *The wave operator* $W(t)$ *(see* (1.14)) *satisfies*

$$\left(W(t)W^{-1}(t')\right)_- = 0,$$

where $t = (t_1, t_2, \ldots)$ *and* $t' = (t_1', t_2', \ldots)$ *are two independent sequences of variables.*

Proof. The proof proceeds by induction: assume for some

$$\nabla := \prod_i \left(\frac{\partial}{\partial t_i}\right)^{\nu_i}, \quad \nu_i \geqq 0,$$

we have

$$\left((\nabla W(t)W^{-1}(t))\right)_- = 0. \tag{3.1}$$

Then

$$\begin{aligned}\mathscr{D}_+ \ni &\frac{\partial}{\partial t_k}((\nabla W(t))W^{-1}(t)) \\ &= \left(\frac{\partial}{\partial t_k}\nabla W(t)\right)W^{-1}(t) - (\nabla W(t))W^{-1}(t)\frac{\partial W}{\partial t_k}W^{-1}(t) .\end{aligned}$$

Using (3.1), that

$$\frac{\partial W}{\partial t_k}W^{-1}(t) = (L^k)_+ \in \mathscr{D}_+ ,$$

and that $\mathscr{D}_+$ is a ring, this implies

$$\left(\frac{\partial}{\partial t_k}\nabla W(t)\right)W^{-1}(t) \in \mathscr{D}_+ .$$

Since $\left(W(t)W^{-1}(t)\right)_- = 1_- = 0$, the induction establishes (3.1) for all partials ∇. It follows that

$$\left(W(t')W^{-1}(t)\right)_-$$

has a zero Taylor series about $t' = t$, and so is identically zero, proving the lemma.

Lemma 3.2. *If $U(x,D)$ and $V(x,D)$ are pseudodifferential operators, then*[5]

$$\begin{aligned}&(U(x,D_x)V(x,D_x))_-\delta(x-y)\\ &= \oint_{z=\infty}\left(U(x,D_x)\chi_x(z)\right)\left(V^\top(y,D_y)\chi_y^*(z)\right)\frac{dz}{2\pi i}Y(x-y)\ ,\end{aligned}$$

where the integrals are taken over a small circle around $z=\infty$, $\delta(x)$ is the usual Dirac delta function supported at the origin, and $Y(x) = D^{-1}\delta(x)$ is the Heaviside function.

Proof. Using the following representations

$$U(x,D) = \sum_i a_i(x)D_x^i \quad \text{and} \quad V(x,D) = \sum_i D_x^i b_i(x),$$

we have

$$\begin{aligned}&(U(x,D_x)V(x,D_x))_-\delta(x-y)\\ &= \sum_{i+j\leqq -1} a_i(x)b_j(y)D_x^{i+j}\delta(x-y)\\ &= \sum_{i+j\leqq -1} a_i(x)b_j(y)\frac{(x-y)^{-i-j-1}}{(-i-j-1)!}Y(x-y),\end{aligned}$$

using

$$D^{-k}\delta(x) = \frac{x^{k-1}}{(k-1)!}Y(x) \quad (k>0)\ .$$

Also, using $\chi_x(z) = e^{xz} = \sum x^\ell z^\ell/\ell!$ we have

$$\begin{aligned}&\oint_{z=\infty}\left(U(x,D_x)\chi_x(z)\right)\left(V^\top(y,D_y)\chi_y^*(z)\right)\frac{dz}{2\pi i}\\ &= \frac{1}{2\pi i}\sum_{\substack{i,j\\ \ell\geqq 0}} a_i(x)b_j(y)\frac{(x-y)^\ell}{\ell!}\oint_{z=\infty} z^{i+j+\ell}dz\\ &= \sum_{\substack{i,j\\ i+j\leqq -1}} a_i(x)b_j(y)\frac{(x-y)^{-i-j-1}}{(-i-j-1)!},\end{aligned}$$

proving the lemma.

---

[5] Remember $\chi_x(z) = e^{xz}$ and $\chi_x^*(z) = e^{-xz}$

Proposition 3.1. *If $P(t,x,D)$ is an arbitrary pseudodifferential operator depending on the parameters t, then*

(i) $$\oint_{z=\infty} P\Psi(x,t,z)\Psi^*(x',t',z)\frac{dz}{2\pi i}Y(x-x')$$
$$= \left(P(t,x,D)W(x,t)W(x,t')^{-1}\right)_-\delta(x-x'), \quad \textit{for all } x,x',t,t'.$$

(ii) *We have $P \in \mathscr{D}_+$ if and only if*

$$\oint_{z=\infty} P\Psi(x,t,z)\Psi^*(x',t',z)\frac{dz}{2\pi i} = 0, \quad \textit{for all } x,x',t,t'. \tag{3.2}$$

Proof. By setting $U(x,D) = P(t,x,D)W(x,t)$ and $V(x,D) = W^{-1}(x,t')$ in Lemma 3.2, and using (1.15) and (1.43), we obtain (i). If $P \in \mathscr{D}_+$, then since by Lemma 3.1 $W(x,t)W(x,t')^{-1} \in \mathscr{D}_+$, (i) implies (3.2), for $\mathscr{D}_+$ is a ring. Conversely, (3.2) implies $P \in \mathscr{D}_+$, because the right-hand side of (i) reduces to $P_-\delta(x-x')$ when $t = t'$. Thus we have (ii), ending the proof.

Proposition 3.2. *For a wave function and its adjoint wave function, we have*

$$\oint_{z=\infty} \Psi(x,t,z)\Psi^*(x',t',z)dz = 0\ , \tag{3.3}$$

and also a modified version (useful in Sect. 5)

$$\oint_{z=\infty} \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau(\bar{t})}{e^{-\eta}\tau(\bar{t})} \cdot \frac{1-z/\mu}{1-z/\lambda}\Psi(x,t,z)\Psi^*(x',t',z)dz = 0\ . \tag{3.4}$$

Proof. Identity (3.3) follows at once from Proposition 3.1 (ii) by setting $P = I$ and noting $P_- = I_- = 0$. Version (3.4) is obtained by shifting t in (3.3):

$$\int \Psi\left(x,t+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right],z\right)\Psi^*(x',t',z)dz = 0,$$

multiplying both sides by $e^{\sum(\mu^i-\lambda^i)\bar{t}_i}\tau\left(\bar{t}+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)/\tau(\bar{t})$, and noting

$$e^{\sum_1^\infty(\mu^i-\lambda^i)\bar{t}_i}\frac{\tau\left(\bar{t}+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)}{\tau(\bar{t})}\Psi\left(x,t+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right],z\right)$$
$$= e^{\sum_1^\infty(\mu^i-\lambda^i)\bar{t}_i}\frac{\tau\left(\bar{t}+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)}{\tau(\bar{t})}e^{\sum_1^\infty(\bar{t}_i+\frac{1}{i\lambda^i}-\frac{1}{i\mu^i})z^i}\frac{\tau\left(\bar{t}-\left[z^{-1}\right]+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)}{\tau\left(\bar{t}+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)}$$
$$= e^{\sum_1^\infty(\mu^i-\lambda^i)\bar{t}_i}\frac{\tau\left(\bar{t}-\left[z^{-1}\right]+\left[\lambda^{-1}\right]-\left[\mu^{-1}\right]\right)}{\tau\left(\bar{t}-\left[z^{-1}\right]\right)}e^{\sum_1^\infty(\bar{t}_i+\frac{1}{i\lambda^i}-\frac{1}{i\mu^i})z^i}\frac{\tau\left(\bar{t}-\left[z^{-1}\right]\right)}{\tau(\bar{t})}$$
$$= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau(\bar{t})}{e^{-\eta}\tau(\bar{t})} \cdot \frac{1-z/\mu}{1-z/\lambda}\Psi(x,t,z), \tag{3.5}$$

where we used (1.20) and $e^{-\sum_1^\infty a^i/i} = 1-a$. This ends the proof of Proposition 3.2.

4. Bilinear Identities (Discrete)

In this section we present the bilinear identities for the discrete case as initiated by Ueno & Takesaki; properly reformulated, they will become strikingly similar to the continuous case, and will play a crucial role in the proof of the main theorem.

Lemma 4.1. *The pair of matrices* $W = (W_1, W_2)$ (*see* (1.33)) *satisfies the bilinear relation*

$$\left(W(t,s)W(t',s')^{-1}\right)_- = 0 , \tag{4.1}$$

or equivalently (*see Remark* 1.1)

$$W_1(t,s)W_1(t',s')^{-1} = W_2(t,s)W_2(t',s')^{-1} .$$

Proof. Noting that

$$\frac{\partial W}{\partial t_k} W^{-1} = (L_1^k, 0)_+ \in \mathscr{D}_+ \quad \text{and} \quad \frac{\partial W}{\partial s_k} W^{-1} = (0, L_2^k)_+ \in \mathscr{D}_+ ,$$

and that $\mathscr{D}_+$ is a ring, we can apply the proof of Lemma 3.1 to the present case, after replacing ∇ by

$$\prod_i \left(\frac{\partial}{\partial t_i}\right)^{p_i} \left(\frac{\partial}{\partial s_i}\right)^{q_i}$$

and $\partial/\partial t_k$ by either $\partial/\partial t_k$ or $\partial/\partial s_k$.

We now state the discrete analogue of Lemma 3.2:

Lemma 4.2. *Given two operators* $U := (U_1, U_2)$, $V := (V_1, V_2) \in \mathscr{D}$ *whose coefficients depend on* t *and* s,

(i) *the following matrix identities hold*[6]

$$U_1 V_1 = \oint_{z=\infty} \left(U_1 \chi(z)\right) \otimes \left(V_1^{\mathrm{T}} \chi^*(z)\right) \frac{dz}{2\pi i z} ,$$

$$U_2 V_2 = \oint_{z=0} \left(U_2 \chi(z)\right) \otimes \left(V_2^{\mathrm{T}} \chi^*(z)\right) \frac{dz}{2\pi i z} ;$$

(ii) *We have* $(UV)_- = 0$ *if and only if*

$$\oint_{z=\infty} \left(U_1 \chi(z)\right) \otimes \left(V_1^{\mathrm{T}} \chi^*(z)\right) \frac{dz}{2\pi i z} = \oint_{z=0} \left(U_2 \chi(z)\right) \otimes \left(V_2^{\mathrm{T}} \chi^*(z)\right) \frac{dz}{2\pi i z} .$$

Remark. Actually this is a precise analogue of Lemma 3.2. The analogue of $\delta(x - y)$ is not missing, but it is hidden. Indeed for a pseudodifferential operator (or any linear operator) $P(x, D)$, the two variable "function" $k(x, y) = P(x, D)\delta(x - y)$ is nothing but its distribution kernel:

$$\int k(x, y) f(y) dy = P(x, D) f(x) ;$$

---

[6] $(A \otimes B)_{ij} = A_i B_j$ and remember $\chi^*(z) = \chi(z^{-1})$.

in the discrete case it merely associates to a linear operator the corresponding matrix as we implicitly do. Having this in mind, and using Remark 1.1, we can restate part (ii) of the lemma as

the matrix pair representing $(UV)_-$

$$= Y\left(\oint_{z=\infty}\left(U_1\chi(z)\right)\otimes\left(V_1^{\top}\chi^*(z)\right)\frac{dz}{2\pi iz}-\oint_{z=0}\left(U_2\chi(z)\right)\otimes\left(V_2^{\top}\chi^*(z)\right)\frac{dz}{2\pi iz}\right).$$

Proof of Lemma 4.2. Set

$$U_i=\sum_{\alpha}u_{i,\alpha}\Lambda^{\alpha}\quad\text{and}\quad V_i=\sum_{\beta}\Lambda^{\beta}v_{i,\beta},\quad i=1,2,$$

where $u_{i,\alpha}$ and $v_{i,\alpha}$ are diagonal matrices. It suffices to compare the (n,n')-entries on each side. On the left side of the first formula we have

$$\begin{aligned}(U_1V_1)_{n,n'}&=\left(\sum_{\alpha,\beta}u_{1,\alpha}\Lambda^{\alpha+\beta}v_{1,\beta}\right)_{n,n'}\\&=\sum_{\alpha,\beta}u_{1,\alpha}(n)(\Lambda^{\alpha+\beta})_{n,n'}v_{1,\beta}(n')\\&=\sum_{\substack{\alpha,\beta\\ \alpha+\beta=n'-n}}u_{1,\alpha}(n)v_{1,\beta}(n'),\end{aligned}$$

and on the right side

$$\begin{aligned}&\oint_{z=\infty}\left(U_1\chi(z)\right)_n\left(V_1^{\top}\chi(z^{-1})\right)_{n'}\frac{dz}{2\pi iz}\\&=\oint_{z=\infty}\left(\sum_{\alpha}u_{1,\alpha}z^{\alpha}\chi(z)\right)_n\left(\sum_{\beta}v_{1,\beta}z^{\beta}\chi(z^{-1})\right)_{n'}\frac{dz}{2\pi iz}\\&=\oint_{z=\infty}\sum_{\alpha,\beta}u_{1,\alpha}(n)v_{1,\beta}(n')z^{\alpha+\beta-(n'-n)}\frac{dz}{2\pi iz}\\&=\sum_{\substack{\alpha,\beta\\ \alpha+\beta=n'-n}}u_{1,\alpha}(n)v_{1,\beta}(n'),\end{aligned}$$

establishing the first formula in (i); the second formula is proved in exactly the same way. Then (i) implies (ii) in view of Remark 1.1.

In the rest of this section, for simplicity we shall denote $\Psi_i(t,s;z)\otimes\Psi_i^*(t',s';z)$ by $\Psi_i\otimes\Psi_i^*$.

Proposition 4.1. *Given $P=(P_1,P_2)\in\mathscr{D}$, we have*

$$\begin{aligned}P_1W_1(t,s)W_1(t',s')^{-1}&=\oint_{z=\infty}P_1\Psi_1\otimes\Psi_1^*\frac{dz}{2\pi iz},\\P_2W_2(t,s)W_2(t',s')^{-1}&=\oint_{z=0}P_2\Psi_2\otimes\Psi_2^*\frac{dz}{2\pi iz}.\end{aligned}\tag{4.2}$$

Thus we have $P \in \mathscr{D}_+$, i.e., $P_1 = P_2$, if and only if

$$\oint_{z=\infty} P_1\Psi_1 \otimes \Psi_1^* \frac{dz}{2\pi i z} = \oint_{z=0} P_2\Psi_2 \otimes \Psi_2^* \frac{dz}{2\pi i z}\,. \tag{4.3}$$

Proof. Setting

$$U_i(t,s) = P_i W_i(t,s) \quad \text{and} \quad V_i(t',s') = W_i(t',s')^{-1}, \quad i = 1,2,$$

in Lemma 4.2 and using (1.43) immediately imply (4.2), while (4.3) is proven in the same way as (3.2) in Proposition 3.1.

Proposition 4.2. *For a wave function and its adjoint wave function, we have*

$$\oint_{z=\infty} \Psi_1 \otimes \Psi_1^* \frac{dz}{2\pi i z} = \oint_{z=0} \Psi_2 \otimes \Psi_2^* \frac{dz}{2\pi i z}\,. \tag{4.4}$$

In analogy with (3.4), *we also have modified versions of* (4.4):

$$\oint_{z=\infty} \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\frac{1-z/\mu}{1-z/\lambda}\Psi_1 \otimes \Psi_1^* \frac{dz}{2\pi i z} = \oint_{z=0} \frac{\mathbb{X}(t,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\Psi_2 \otimes \Psi_2^* \frac{dz}{2\pi i z}, \tag{4.5}$$

and

$$\oint_{z=\infty} \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\Psi_1 \otimes \Psi_1^* \frac{dz}{2\pi i z} = \oint_{z=0} \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\frac{1-1/\mu z}{1-1/\lambda z}\Psi_2 \otimes \Psi_2^* \frac{dz}{2\pi i z}. \tag{4.6}$$

Proof. For $P_i = I$, the relation (4.3) becomes (4.4). Relation (4.5) follows from the identity

$$\begin{aligned}
&\oint_{z=\infty} \Psi_1\big(t+[\lambda^{-1}]-[\mu^{-1}],s;z\big) \otimes \Psi_1^*(t',s';z)\frac{dz}{2\pi i z} \\
&= \oint_{z=0} \Psi_2\big(t+[\lambda^{-1}]-[\mu^{-1}],s;z\big) \otimes \Psi_2^*(t',s';z)\frac{dz}{2\pi i z}
\end{aligned}$$

by means of the same calculation as (3.5), with Ψ replaced by $\Psi_1, e^{\sum(\mu^i-\lambda^i)\tilde{t}_i}$ by $\chi(\mu)\chi^*(\lambda)e^{\sum(\mu^i-\lambda^i)\tilde{t}_i}$ and τ by the vector τ, together with

$$\begin{aligned}
&\chi(\mu)\chi^*(\lambda)e^{\sum_1^\infty(\mu^i-\lambda^i)t_i}\frac{\tau\big(t+[\lambda^{-1}]-[\mu^{-1}],s\big)}{\tau(t,s)}\Psi_2\big(t+[\lambda^{-1}]-[\mu^{-1}],s;z\big) \\
&= \chi(\mu)\chi^*(\lambda)e^{\sum_1^\infty(\mu^i-\lambda^i)t_i}\frac{\tau_\Lambda\big(t+[\lambda^{-1}]-[\mu^{-1}],s-[z]\big)}{\tau_\Lambda\big(t,s-[z]\big)}e^{\sum_1^\infty s_i z^{-i}}\frac{\tau_\Lambda\big(t,s-[z]\big)}{\tau(t,s)}\chi^*(z) \\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\Psi_2(t,s;z).
\end{aligned} \tag{4.7}$$

Finally (4.6) follows from the identity

$$\begin{aligned}
&\oint_{z=\infty} \Psi_1\big(t,s+[\lambda^{-1}]-[\mu^{-1}];z\big)\otimes \Psi_1^*(t',s';z)\frac{dz}{2\pi iz}\\
&\quad=\oint_{z=0} \Psi_2\big(t,s+[\lambda^{-1}]-[\mu^{-1}];z\big)\otimes \Psi_2^*(t',s',z)\frac{dz}{2\pi iz}\,,
\end{aligned}$$

upon noticing a modified version of (3.5):

$$\begin{aligned}
&\chi(\lambda)\chi^*(\mu)e^{\sum_1^\infty(\mu^i-\lambda^i)s_i}\frac{\tau\big(t,s+[\lambda^{-1}]-[\mu^{-1}]\big)}{\tau(t,s)}\Psi_2\big(t,s+[\lambda^{-1}]-[\mu^{-1}];z\big)\\
&=\chi(\lambda)\chi^*(\mu)e^{\sum_1^\infty(\mu^i-\lambda^i)s_i}\frac{\tau_A\big(t,s+[\lambda^{-1}]-[\mu^{-1}]-[z]\big)}{\tau(t,s)}\\
&\quad\times e^{\sum_1^\infty\left(s_i+\frac{\lambda^{-i}}{i}-\frac{\mu^{-i}}{i}\right)z^{-i}}\chi^*(z)\\
&=\chi(\lambda)\chi^*(\mu)e^{\sum_1^\infty(\mu^i-\lambda^i)s_i}\frac{\tau_A\big(t,s+[\lambda^{-1}]-[\mu^{-1}]-[z]\big)}{\tau_A\big(t,s-[z]\big)}\frac{1-1/\mu z}{1-1/\lambda z}\\
&\quad\times e^{\sum_1^\infty s_iz^{-i}}\frac{\tau_A\big(t,s-[z]\big)}{\tau(t,s)}\chi^*(z)\\
&=\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde\eta}\tau_A(t,s)}{e^{-\tilde\eta}\tau_A(t,s)}\cdot\frac{1-1/\mu z}{1-1/\lambda z}\Psi_2(t,s;z)\,,
\end{aligned}$$

and a modified version of (4.7):

$$\begin{aligned}
&\chi(\lambda)\chi^*(\mu)e^{\sum_1^\infty(\mu^i-\lambda^i)s_i}\frac{\tau\big(t,s+[\lambda^{-1}]-[\mu^{-1}]\big)}{\tau\big(t,s\big)}\Psi_1\big(t,s+[\lambda^{-1}]-[\mu^{-1}];z\big)\\
&=\chi(\lambda)\chi^*(\mu)e^{\sum_1^\infty(\mu^i-\lambda^i)s_i}\frac{\tau\big(t-[z^{-1}],s+[\lambda^{-1}]-[\mu^{-1}]\big)}{\tau\big(t-[z^{-1}],s\big)}\\
&\quad\times e^{\sum_1^\infty t_iz^i}\frac{\tau\big(t-[z^{-1}],s\big)}{\tau(t,s)}\chi(z)\\
&=\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\Psi_1(t,s;z)\,,
\end{aligned}$$

thus ending the proof of Proposition 4.2.

5. Proof of the Main Theorem

In this section we prove, at the same time, the continuous and discrete statements, contained in the main theorem. The continuous case uses the same algebra as the discrete one, with M_1, L_1, N_1 and Ψ_1 replaced by M, L, N and Ψ respectively.

Step 1. In view of the definition of N_t in (0.16), we first prove the following relations:

$$\begin{aligned}
\frac{1}{\mu-\lambda}N_1\Psi_1 &= W_1\delta(\lambda,z)\chi(z+\mu-\lambda)\\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\delta(\lambda,z)\Psi_1,\\
\frac{1}{\mu-\lambda}N_2\Psi_2 &= W_2\delta(\lambda,z^{-1})\chi^*(z^{-1}+\mu-\lambda)\\
&= \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde{\eta}}\tau_A}{e^{-\tilde{\eta}}\tau_A}\delta(\lambda,z^{-1})\Psi_2\ ,
\end{aligned}\tag{5.1}$$

where the first relation also serves the continuous case. Indeed, remembering that ∂ and ε act on the x-component of the character $\chi_x(z)$, we have

$$\begin{aligned}
\frac{1}{\mu-\lambda}N_1\Psi_1 &= W_1e^{(\mu-\lambda)\varepsilon}\delta(\lambda,\partial)\chi(z) && \text{using (0.16), (1.29) and (1.15)}\\
&= W_1\delta(\lambda,z)e^{(\mu-\lambda)\partial/\partial z}\chi(z)\\
&= W_1\delta(\lambda,z)\chi(z+\mu-\lambda) && \text{using (1.6) and (1.7)}\\
&= \frac{\tau(t-[\partial^{-1}],s)}{\tau(t,s)}e^{\sum_1^\infty t_k\partial^k}\delta(\lambda,z)\chi(z+\mu-\lambda)\\
& && \text{using (1.33) and (1.42)}\\
&= \frac{\tau(t-[(z+\mu-\lambda)^{-1}],s)}{\tau(t,s)}e^{\sum_1^\infty t_k(z+\mu-\lambda)^k}\delta(\lambda,z)\chi(z+\mu-\lambda)\\
& && \text{using (1.6)}\ |_{z\to z+\mu-\lambda}\\
&= \frac{\tau(t-[z^{-1}]-[\mu^{-1}]+[\lambda^{-1}],s)}{\tau(t,s)}\\
&\quad\cdot e^{\sum t_k(\mu^k-\lambda^k)}e^{\sum t_kz^k}\delta(\lambda,z)\chi(\mu)\chi^*(\lambda)\chi(z)\\
& && \text{using (1.45) (and (1.47))}\\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\delta(\lambda,z)\frac{e^{-\eta}\tau}{\tau}e^{\sum t_k\partial^k}\chi(z) && \text{using (0.18)}\\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\delta(\lambda,z)\Psi_1(t,s;z), && \text{using (1.40)}
\end{aligned}$$

proving the first relation. To prove the second relation, just simultaneously substitute in the former argument $\cdot_1\to\ \cdot_2\,(\cdot=N,\Psi,W)$, $\varepsilon\to\varepsilon^*$, $\partial\to\partial^*=\partial^{-1}$, $\chi\to\chi^*$, $z\to z^{-1}$, $t\leftrightarrow s$, and in the last two lines $e^{-\eta}\tau\to e^{-\tilde{\eta}}\tau_A$, keeping in mind the differing formulas (1.40), (1.41) and (1.42) for Ψ_1 and Ψ_2 in terms of τ, and the relationships (1.12) between ε^*,χ^*,z and d/dz^{-1}, and also $\chi^*(z^{-1})=\chi(z)$; this confirms (5.1).

Step 2. Next we prove the following:

$$
\begin{aligned}
(N_1 - f_1(L_1))\Psi_1 &= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\frac{\mu}{\lambda}\frac{1-z/\mu}{1-z/\lambda}\Psi_1\,,\\
(N_2 - f_2(L_2^{-1}))\Psi_2 &= \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde\eta}\tau_A}{e^{-\tilde\eta}\tau_A}\frac{\mu}{\lambda}\frac{1-1/\mu z}{1-1/\lambda z}\Psi_2\,,\\
g_1(L_1)\Psi_1 &= -\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\Psi_1\,,\\
g_2(L_2^{-1})\Psi_2 &= -\frac{\mathbb{X}(t,\lambda,\mu)e^{-\tilde\eta}\tau_A}{e^{-\tilde\eta}\tau_A}\Psi_2\,,
\end{aligned}
\tag{5.2}
$$

where[7]

$$
\begin{aligned}
f_1(z) &:= -e^{-\eta}\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau} = -\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau} + (\text{negative powers in } z)\,,\\
f_2(z) &:= -e^{-\tilde\eta}\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau_A}{\tau_A} = -\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau_A}{\tau_A} + (\text{positive powers in } z)\\
g_1(z) &:= -e^{-\eta}\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau} \quad \text{and} \quad g_2(z) := -e^{-\tilde\eta}\frac{\mathbb{X}(t,\lambda,\mu)\tau_A}{\tau_A}\,,
\end{aligned}
\tag{5.3}
$$

and so, for some a_i, b_i, c_i and d_i,

$$
\begin{aligned}
f_1(L_1) &= \sum_0^\infty a_i L_1^{-i} = f_1(L_1)_l + f_1(\infty),\\
f_2(L_2^{-1}) &= \sum_0^\infty b_i L_2^{-i} = f_2(L_2^{-1})_u,\\
g_1(L_1) &= \sum_0^\infty c_i L_1^{-i} = g_1(L_1)_l + g_1(\infty),\\
g_2(L_2^{-1}) &= \sum_0^\infty d_i L_2^{-i} = g_2(L_2^{-1})_u.
\end{aligned}
\tag{5.4}
$$

The $f_i(z)$ and $g_i(z)$ are functions in the continuous case and diagonal matrices in the discrete case; and it is interpreted that in the same development, the variable z appears on the right when one substitutes L_1 or L_2^{-1} for z.

Proof of (5.2). We shall need the following identities for the operators $e^{\pm\eta}$ and $X(t,\lambda,\mu)$:

$$
\begin{aligned}
e^{\pm\eta}X(t,\lambda,\mu) &= e^{\pm\eta}e^{\sum_1^\infty(\mu^i-\lambda^i)t_i}e^{\sum_1^\infty i^{-1}(\lambda^{-i}-\mu^{-i})\partial/\partial t_i}\\
&= e^{\sum_1^\infty(\mu^i-\lambda^i)(t_i\pm 1/(iz^i))}e^{\pm\eta}e^{\sum_1^\infty i^{-1}(\lambda^{-i}-\mu^{-i})\partial/\partial t_i}\\
&= e^{\pm\sum_1^\infty(\mu^i-\lambda^i)/(iz^i)}X(t,\lambda,\mu)e^{\pm\eta}\\
&= \left(\frac{1-\lambda/z}{1-\mu/z}\right)^{\pm 1}X(t,\lambda,\mu)e^{\pm\eta}\,,
\end{aligned}
\tag{5.5}
$$

[7] The variable z appears in η and $\tilde\eta$

and similarly

$$e^{\pm\tilde{\eta}}X(s,\lambda,\mu) = \left(\frac{1-\lambda z}{1-\mu z}\right)^{\pm 1} X(s,\lambda,\mu)e^{\pm\tilde{\eta}} \ . \tag{5.6}$$

Note also the trivial commutation relations

$$[e^{\pm\eta}, X(s,\lambda,\mu)] = 0 = [e^{\pm\tilde{\eta}}, X(t,\lambda,\mu)] \ , \tag{5.7}$$

$$\mathbb{X}(t,\lambda,\mu)\Lambda = \frac{\lambda}{\mu}\Lambda\mathbb{X}(t,\lambda,\mu) \ \text{ and } \ \tilde{\mathbb{X}}(s,\lambda,\mu)\Lambda = \frac{\mu}{\lambda}\Lambda\tilde{\mathbb{X}}(s,\lambda,\mu)\,. \tag{5.8}$$

We have

$$\begin{aligned}
&\big(N_1 - f_1(L_1)\big)\Psi_1 \\
&= (N_1 - f_1(z))\Psi_1 && \text{since } L_1\Psi_1 = z\Psi_1 \\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\left(\frac{\mu}{\lambda}\frac{1-z/\mu}{1-z/\lambda} - \frac{1-\mu/z}{1-\lambda/z}\right)\Psi_1 + \frac{e^{-\eta}\mathbb{X}(t,\lambda,\mu)\tau}{e^{-\eta}\tau}\Psi_1 \\
& && \text{by (5.1), (5.3) and the expression (1.46) for } (\mu-\lambda)\delta(\lambda,z) \\
&= \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau}\frac{\mu}{\lambda}\frac{1-z/\mu}{1-z/\lambda}\Psi_1 \ , && \text{by (5.5)}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\big(N_2 - f_2(L_2^{-1})\big)\,\Psi_2 \\
&= (N_2 - f_2(z))\Psi_2 && \text{since } L_2^{-1}\Psi_2 = z\Psi_2 \\
&= \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\left(\frac{\mu}{\lambda}\frac{1-1/\mu z}{1-1/\lambda z} - \frac{1-\mu z}{1-\lambda z}\right)\Psi_2 + \frac{e^{-\tilde{\eta}}\tilde{\mathbb{X}}(s,\lambda,\mu)\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\Psi_2 \\
& && \text{by (5.1), (5.3) and (1.46)} \\
&= \frac{\tilde{\mathbb{X}}(s,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda}\frac{\mu}{\lambda}\frac{1-1/\mu z}{1-1/\lambda z}\Psi_2 && \text{by (5.6)}\,.
\end{aligned}$$

The last two relations in (5.2) follow from the last line in (5.3) and the commutation relations (5.7), confirming (5.2).

Step 3. Next we establish the following relations:
in the continuous case

$$(N - f(L))_- = 0 \ ; \tag{5.9}$$

in the discrete case

$$\begin{aligned}
&(N_1 - f_1(L_1), -(\mu/\lambda)g_2(L_2^{-1}))_- = 0 \ , \\
&(-(\mu/\lambda)g_1(L_1), N_2 - f_2(L_2^{-1}))_- = 0 \ ,
\end{aligned} \tag{5.10}$$

i.e.,

$$\begin{aligned}
N_1 &= f_1(L_1) - (\mu/\lambda)g_2(L_2^{-1})\,, \\
N_2 &= f_2(L_2^{-1}) - (\mu/\lambda)g_1(L_1)
\end{aligned}$$

(recall that $(P_1,P_2)_- = (P_{1l} - P_{2l}, P_{2u} - P_{1u})$, with diagonals included in the upper part).

Indeed, in the continuous case, integrating (5.2) against $\Psi^*(x',t',z)$, and using the modified version (3.4) of the bilinear identity, we have

$$\oint_{z=\infty} (N - f(L))\Psi(x,t,z)\Psi^*(x',t',z)dz$$

$$= \int \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau} \frac{\mu}{\lambda} \frac{1 - z/\mu}{1 - z/\lambda} \Psi(x,t,z)\Psi^*(x',t',z)dz = 0 .$$

Applying Proposition 3.1 (ii) to this identity yields (5.9).

In the discrete case, the first relation of (5.10) follows from the computation:

$$\oint_{z=\infty} (N_1 - f_1(L_1))\Psi_1(t,s;z) \otimes \Psi_1^*(t',s';z)\frac{dz}{2\pi iz}$$

$$= \oint_{z=\infty} \frac{\mu}{\lambda} \frac{\mathbb{X}(t,\lambda,\mu)e^{-\eta}\tau}{e^{-\eta}\tau} \frac{1 - z/\mu}{1 - z/\lambda} \Psi_1(t,s;z) \otimes \Psi_1^*(t',s';z)\frac{dz}{2\pi iz} \quad \text{by (5.2)}$$

$$= \oint_{z=0} \frac{\mu}{\lambda} \frac{\mathbb{X}(t,\lambda,\mu)e^{-\tilde{\eta}}\tau_\Lambda}{e^{-\tilde{\eta}}\tau_\Lambda} \Psi_2(t,s;z) \otimes \Psi_2^*(t',s';z)\frac{dz}{2\pi iz} \quad \text{by (4.5)}$$

$$= \oint_{z=0} -\frac{\mu}{\lambda} g_2(L_2^{-1})\Psi_2(t,s;z) \otimes \Psi_2^*(t',s';z)\frac{dz}{2\pi iz} , \quad \text{by (5.2)}$$

and then applying (4.3) to this identity. The second relation of (5.10) follows similarly.

Step 4. Finally we prove
in the continuous case:

$$-\frac{N_-\Psi}{\Psi} = (e^{-\eta} - 1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau} ,$$

in the discrete case:

$$-\frac{(N_1,0)_-\Psi}{\Psi} = \left((e^{-\eta} - 1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}, (\Lambda e^{-\tilde{\eta}} - 1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}\right) ,$$

$$-\frac{(0,N_2)_-\Psi}{\Psi} = \left(\frac{\mu}{\lambda}(e^{-\eta} - 1)\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}, \frac{\mu}{\lambda}(\Lambda e^{-\tilde{\eta}} - 1)\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}\right) .$$

Proof. In the continuous case (5.9) implies, since $f(L) = \sum_0^\infty a_i L^{-i}$,

$$(N)_- = f(L)_- = f(L) - f(\infty) ,$$

which, applied to Ψ, yields

$$\begin{aligned}(N)_-\Psi &= f(z)\Psi - f(\infty)\Psi && \text{using } L\Psi = z\Psi\\ &= -\Psi(e^{-\eta}-1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau} && \text{using (5.3).}\end{aligned}$$

In the discrete case (5.10) implies

$$\begin{aligned}(N_1,0)_- &= \left(f_1(L_1), \frac{\mu}{\lambda}g_2(L_2^{-1})\right)_-\\ &= \left(f_1(L_1)_l - \frac{\mu}{\lambda}g_2(L_2^{-1})_l, \frac{\mu}{\lambda}g_2(L_2^{-1})_u - f_1(L_1)_u\right)\\ &= \left(f_1(L_1) - f_1(\infty), \frac{\mu}{\lambda}g_2(L_2^{-1}) - f_1(\infty)\right) && \text{using (5.4)},\\ (0,N_2)_- &= \left(\frac{\mu}{\lambda}g_1(L_1), f_2(L_2^{-1})\right)_-\\ &= \left(\frac{\mu}{\lambda}g_1(L_1)_l - f_2(L_2^{-1})_l, f_2(L_2^{-1})_u - \frac{\mu}{\lambda}g_1(L_1)_u\right)\\ &= \left(\frac{\mu}{\lambda}g_1(L_1) - \frac{\mu}{\lambda}g_1(\infty), f_2(L_2^{-1}) - \frac{\mu}{\lambda}g_1(\infty)\right) && \text{using (5.4)},\end{aligned}$$

which, applied to $\Psi = (\Psi_1, \Psi_2)$, yield

$$\begin{aligned}(N_1,0)_-\Psi &= \left(f_1(z) - f_1(\infty), \frac{\mu}{\lambda}g_2(z) - f_1(\infty)\right)\Psi\\ &= -\left((e^{-\eta}-1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}, (Ae^{-\tilde{\eta}}-1)\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}\right)\Psi,\\ (0,N_2)_-\Psi &= \left(\frac{\mu}{\lambda}g_1(z) - \frac{\mu}{\lambda}g_1(\infty), f_2(z) - \frac{\mu}{\lambda}g_1(\infty)\right)\Psi\\ &= -\left(\frac{\mu}{\lambda}(e^{-\eta}-1)\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}, \frac{\mu}{\lambda}(Ae^{-\tilde{\eta}}-1)\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}\right)\Psi,\end{aligned}$$

using $L_1\Psi_1 = z\Psi_1$, $L_2^{-1}\Psi_2 = z\Psi_2$, (5.3) and (5.8), ending the proof of the main theorem.

Proof of Corollary 0.1.1. Comparing the expansions for the vertex operators $\mathbb{Y}_N$ as in (0.15) and for $\mathbb{X}$ as in (0.19) establishes the corollary.

Remark. By (5.10) and (5.3), we immediately have the decomposition

$$
\begin{aligned}
N_1 &:= (\mu-\lambda)e^{(\mu-\lambda)M_1}\delta(\lambda, L_1) = f_1(L_1) - \frac{\mu}{\lambda}g_2(L_2^{-1}) \\
&= -\frac{\mathbb{X}(t-[L_1^{-1}],\lambda,\mu)\tau(t-[L_1^{-1}],s)}{\tau(t-[L_1^{-1}],s)} + \frac{\mu}{\lambda}\frac{\mathbb{X}(t,\lambda,\mu)\tau_A(t,s-[L_2^{-1}])}{\tau_A(t,s-[L_2^{-1}])} \\
&:= -\sum_{k=0}^{\infty} p_k(-\tilde{\partial}_t)\left(\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}\right)L_1^{-k} + \sum_{k=0}^{\infty} p_k(-\tilde{\partial}_s)\left(A\frac{\mathbb{X}(t,\lambda,\mu)\tau}{\tau}\right)L_2^{-k}, \\
N_2 &:= (\mu-\lambda)e^{(\mu-\lambda)M_2}\delta(\lambda, L_2) = f_2(L_2^{-1}) - \frac{\mu}{\lambda}g_1(L_1) \\
&= -\frac{\tilde{\mathbb{X}}(s-[L_2^{-1}],\lambda,\mu)\tau_A(t,s-[L_2^{-1}]]}{\tau_A(t,s-[L_2^{-1}])} + \frac{\mu}{\lambda}\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau(t-[L_1^{-1}],s)}{\tau(t-[L_1^{-1}],s)} \\
&:= -\frac{\mu}{\lambda}\left(\sum_{k=0}^{\infty} p_k(-\tilde{\partial}_s)\left(A\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}\right)L_2^{-k} - \sum_{k=0}^{\infty} p_k(-\tilde{\partial}_t)\left(\frac{\tilde{\mathbb{X}}(s,\lambda,\mu)\tau}{\tau}\right)L_1^{-k}\right),
\end{aligned}
$$

where

$$
e^{\sum_1^\infty t_i z^i} = \sum_0^\infty p_k(t)z^k, \quad \tilde{\partial}_t = \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right)
$$

as in Sect. 1. The appearance of the same vector, $\mathbb{X}(t,\lambda,\mu)\tau/\tau$ or $\tilde{\mathbb{X}}(s,\lambda,\mu)\tau/\tau$, on the right-hand side of each formula is the key to the consistency, i.e., that the actions of $\mathbb{Y}_{N_i}$ can be lifted to actions on the vector of τ-functions.

6. Remarks and Applications

Let Gr be the Grassmann manifold of subspaces V_0 in $\mathbb{C}((z^{-1}))$ of relative dimension 0 with respect to $\mathbb{C}[z]$, i.e., $\sigma : V_0 \to \mathbb{C}((z^{-1}))/\mathbb{C}[[z^{-1}]]z^{-1}(\simeq \mathbb{C}[z])$ satisfies $\dim \operatorname{Ker}\sigma = \dim \operatorname{Coker}\sigma < \infty$. Let $V_0 \in$ Gr, and let $\{v_0, v_1, \dots\}$ be a basis of V_0. Since a perturbation δV_0 of V_0 is given by assigning to each v_n a vector δv_n mod $V_0 \in \mathbb{C}((z^{-1}))/V_0$, we have

$$
T_{V_0}\,\mathrm{Gr} = \mathrm{Hom}_{\mathbb{C}}(V_0, \mathbb{C}((z^{-1}))/V_0)\,.
$$

Thus given

$$
A \in w_\infty = \mathbb{C}[z,z^{-1}][\partial/\partial z]\ ,
$$

we have a vector field $\tilde{\mathbb{Y}}_A$ on Gr induced by a family of linear maps parametrized by $V_0 \in$ Gr:

$$
\tilde{\mathbb{Y}}_A(V_0) = \left(\pi_{V_0} \circ A : V_0 \to \mathbb{C}((z^{-1}))/V_0\right) \in T_{V_0}\,\mathrm{Gr}\,,
$$

where $\pi_{V_0} : \mathbb{C}((z^{-1})) \to \mathbb{C}((z^{-1}))/V_0$ is the quotient map. The fixed points of $\tilde{\mathbb{Y}}_A$, $\tilde{\mathbb{Y}}_A(V_0) = 0$, are given by the condition

$$
AV_0 \subset V_0\ .
$$

We show that $\tilde{\mathbb{Y}}_A$ can be identified with the vector field $-\mathbb{Y}_A$ on the Ψ-manifold studied in preceding sections.

Consider the subspace $V_0 \subset \mathbb{C}((z^{-1}))$ generated by the wave function Ψ:

$$\begin{aligned} V_0 &= \text{“span}\,\{\Psi(t,z) \mid t \in \mathbb{C}^{\infty}\}\text{”} \\ &= \text{span}\left\{ \prod_n \left(\frac{\partial}{\partial t_n}\right)^{k_n} \Psi(0,z) \middle| k_n = 0,1,\dots \right\} \\ &= \text{span}\left\{ \left(\frac{\partial}{\partial x}\right)^{k} \Psi(0,z) \middle| k = 0,1,\dots \right\}, \end{aligned} \tag{6.1}$$

where the last equality, which follows from Eq. (1.19), implies that $V_0 \in \mathrm{Gr}$, and gives a specific choice of basis[8] $\{v_0, v_1, \dots\}$ of V_0

$$v_n = \left(\frac{\partial}{\partial x}\right)^n \Psi(0,z), \quad \text{or} \quad \Psi(x,0,z) = \sum \frac{x^n}{n!} v_n \; . \tag{6.2}$$

Conversely, each $V_0 \in \mathrm{Gr}$ is obtained by a unique choice of KP wave function Ψ in this way. Hence Gr can be identified with the Ψ-manifold, and a vector field on Gr with one on the Ψ-manifold. The latter looks more specific because the condition

$$\delta\Psi(x,0,z) = \sum \frac{x^n}{n!} \delta v_n$$

determines δv_n itself rather than δv_n mod V_0, and defines a linear map

$$\phi_{V_0} : V_0 \longrightarrow \mathbb{C}((z^{-1})) : v_n \longmapsto \delta v_n \; .$$

In view of the fact that Ψ is determined uniquely by V_0, and that $\pi_{V_0} \circ \phi_{V_0} (= \phi_{V_0}$ mod $V_0)$ gives the corresponding δV_0, we observe that

$$\phi_{V_0} = 0 \quad \text{if and only if} \quad \phi_{V_0} \equiv 0 \bmod V_0 \; . \tag{6.3}$$

Now for $A \in w_\infty$, let $P_A = P_A(V_0) \in \mathscr{D}$ be the unique pseudodifferential operator such that

$$A\Psi(x,0,z) = P_A \Psi(x,0,z) \; . \tag{6.4}$$

More explicitly, if $A = \sum a_{ij} z^i (\partial/\partial z)^j$, then $P_A = \sum a_{ij} M^j L^i |_{t=0}$, and the map $A \mapsto P_A$ gives an antihomomorphism of rings, $P_{AB} = P_B P_A$. Clearly from (6.1) and (6.4), A is given in terms of the basis $\{v_n\}$ by

$$A v_n = \left(\frac{\partial}{\partial x}\right)^n P_A \Psi(0,z) \; ,$$

so that $\tilde{\mathbb{Y}}_A(V_0) = \pi_{V_0} \circ A$ is given by the generating function $P_A \Psi(x,0,z)$ mod V_0. But since $V_0 = \text{span}\{v_n\} = \mathscr{D}_+ \Psi(0,z)$, the Taylor coefficients in x of $(P_A)_+ \Psi(x,0,z)$ are in V_0, so that

$$\begin{aligned} P_A \Psi(x,0,z) &\equiv (P_A)_- \Psi(x,0,z) \bmod V_0 \\ &= -\mathbb{Y}_A \Psi(x,0,z) \; . \end{aligned}$$

Hence in view of (6.3) above, $\tilde{\mathbb{Y}}_A$ precisely corresponds to $-\mathbb{Y}_A$.

[8] These formulas are for V_0 in the big stratum Gr_0 of Gr. When $V_0 \in \mathrm{Gr} \backslash \mathrm{Gr}_0$ we have similar explicit formulas depending on the Schubert stratum to which V_0 belongs (see [Sa1]).

Noting that $(P_A)_-\Psi = (P_A)_- W\chi = 0 \Leftrightarrow (P_A)_- W = 0 \Leftrightarrow (P_A)_- = 0$ since W is invertible, we observe that the following conditions are equivalent:

(i) $\tilde{\mathbb{Y}}_A(V_0) = 0$,
(ii) $AV_0 \subset V_0$,
(iii) $P_A(V_0) \in \mathscr{D}_+$,
(iv) $\mathbb{Y}_A \Psi(0,z) = 0$.

We can define $P_A(t) = P_A(V_0, t)$ by

$$A\Psi(t,z) = P_A(t)\Psi(t,z)\,,$$

so that $P_A(0)$ is the old P_A, and noting that

$$P_A(t) \in \mathscr{D}_+ \Leftrightarrow P_A(0) \in \mathscr{D}_+\,,$$

we can introduce t in (iii) and (iv) of the above equivalence statement. This follows from the above argument itself (since, up to the t-adic completion $\{(\partial/\partial x)^n \Psi(t,z)\}_{n=0}^{\infty}$ generates the same space V_0 as $\{(\partial/\partial x)^n \Psi(0,z)\}_{n=0}^{\infty}$ by (6.1)), or from the differential equations $P_A(t)$ satsify:

$$\frac{\partial}{\partial t_n} P_A(t) = [(L^n)_+, P_A(t)]\,.$$

Given V_0, such A's leading to fixed points (vanishing of $\mathbb{Y}_A(V_0)$) form an associative subalgebra of w_∞, the stabilizer algebra Z_{V_0} (or w-constraint).

Kontsevich integral. Along with Kac and Schwarz [K-S, Sc], let us consider the case where Z_{V_0} contains two elements of the form

$$\begin{aligned} A_1 &= z^p\,,\\ A_2 &= \exp\left(-\frac{p}{p+q} z^{p+q}\right) z^{(p-1)/2} \frac{\partial}{\partial(z^p)} z^{(1-p)/2} \exp\left(\frac{p}{p+q} z^{p+q}\right)\\ &= \frac{1}{p}\left(z^{1-p}\frac{\partial}{\partial z} + \frac{1-p}{2} z^{-p} + p z^q\right)\\ &= z^q + \text{lower degree in } z\,, \end{aligned}$$

i.e., V_0 is fixed by the corresponding vector fields:

$$A_1 V_0 \subset V_0, \quad A_2 V_0 \subset V_0\,,$$

called the (p,q) case. Observe

$$A_i V_0 \subset V_0 \quad \text{yields} \quad P_{A_i} \in \mathscr{D}_+,$$

and

$$[A_2, A_1] = 1 \quad \text{yields} \quad [P_{A_1}, P_{A_2}] = 1,$$

the so-called string relation. As seen from (6.5), the differential operators P_{A_i} take the form

$$\begin{aligned}
P := P_{A_1} &= L^p = SD^pS^{-1} \\
&= D^p + \text{lower order terms}, \\
Q := P_{A_2} &= \frac{1}{p}\left(ML^{1-p} + \frac{1-p}{2}L^{-p} + pL^q\right) \\
&= \frac{1}{p}\left(MP^{(1-p)/p} + \frac{1-p}{2}P^{-1} + pP^{q/p}\right) \\
&= ((p+q)/p)D^q + \text{lower order terms (when } t=0).
\end{aligned}$$

Specializing to the case where $q = 1$ leads to matrix integrals of the Kontsevich type [K, A-vM1]. Indeed the precise form of A_2 implies, by the method of stationary phase, the following asymptotics:

$$\begin{aligned}
A_2^k \Psi(0,z) &= z^{\frac{p-1}{2}} e^{-\frac{p}{p+1}z^{p+1}} \left(\frac{\partial}{\partial y}\right)^k \Bigg|_{y=z^p} z^{-\frac{p-1}{2}} e^{\frac{p}{p+1}z^{p+1}} \Psi(0,z) \\
&= z^k \Psi(0,z) + O(z^{-p+k-1}), \quad k = 0,1,\dots ;
\end{aligned}$$

we have $V_0 = \operatorname{span}\{A_2^k \Psi(0,z) \mid k = 0,1,\dots\} \in \mathrm{Gr}_0$ (the big stratum), which in terms of the Vandermonde $\Delta(z) = \det\left(z_i^{j-1}\right)_{1 \leqq i,j \leqq N'}$ has two crucial implications:

(1) $A_1 \Psi = \sum\limits_0^p \alpha_i A_2^i \Psi, \ \alpha_i \in \mathbb{C}, \alpha_p \neq 0$,

(2) $\tau(t) = \dfrac{\det\left(A_2^{j-1}\Psi(0,z_i)\right)_{1\leqq i,j \leqq N}}{\det\left(z_i^{j-1}\right)_{1 \leqq i,j \leqq N}}$, where $t_n = -\sum\limits_{i=1}^N z_i^{-n}/n$ (Miwa time).

The asymptotics above for $A_2^k \Psi(0,z)$ implies that $\alpha_i = 0$ for $0 \leqq i \leqq p-1$, and $\alpha_p = 1$; setting $\rho(z) := z^{\frac{p-1}{2}} e^{-\frac{p}{p+1}z^{p+1}}$, it follows that the function

$$\varphi(y) = \rho(z)^{-1} \Psi(0,z)\big|_{z=y^{1/p}} ,$$

satisfies the higher Airy differential equation:

$$\left(\frac{d}{dy}\right)^p \varphi(y) = y\varphi(y) .$$

The only solution satisfying the asymptotics above is given by the higher Airy function, where the integral is taken appropriately:

$$\varphi(y) = \int e^{-\frac{x^{p+1}}{p+1} + xy} dx .$$

Since $A_2^k\Psi(0,z)=\rho(z)\left(\frac{\partial}{\partial y}\right)^k\varphi(y)$ for $y=z^p$, we have, modulo constant prefactors,

$$\begin{aligned}
\tau(t) &= \frac{\prod_1^N\rho(z_i)}{\Delta(z)}\det\left(\left(\frac{\partial}{\partial y}\right)^{j-1}\varphi(y_i)\right)_{1\leqq i,j\leqq N'} \quad y_i=z_i^p \\
&= \frac{\prod_1^N\rho(z_i)}{\Delta(z)}\det\left(\int x^{j-1}e^{-\frac{x^{p+1}}{p+1}+xz_i^p}\,dx\right)_{1\leqq i,j\leqq N} \\
&= \frac{\prod_1^N\rho(z_i)}{\Delta(z)}\int_{\operatorname{diag}X} e^{\operatorname{Tr}\left(-\frac{X^{p+1}}{p+1}+XZ^p\right)}\Delta(X)dX \\
&= \frac{\Delta(Z^p)\prod_1^N\rho(z_i)}{\Delta(Z)}\int_{\mathscr{H}_N} e^{\operatorname{Tr}\left(-\frac{X^{p+1}}{p+1}+XZ^p\right)}dX \\
&= \frac{\int_{\mathscr{H}_N}dYe^{-\operatorname{Tr}\left(\frac{(Y+Z)^p}{p+1}\right)_{\geqq 2}}}{\int_{\mathscr{H}_N}dYe^{-\operatorname{Tr}\left(\frac{(Y+Z)^p}{p+1}\right)_2}}, \quad \text{upon setting } X=Y+Z\,,
\end{aligned}$$

where $\mathscr{H}_N$ denotes the space $N\times N$ Hermitean matrices and where the notation $(\ )_k$ refers to taking the terms of degree k in Y; in the computations above, we used Mehta's formula and we performed an explicit Gaussian integration. For more details the reader is referred to [K, A-vM1].

Moreover $A_1,A_2\in Z_{V_0}$ implies $A_1^jA_2^i\in Z_{V_0}$, or $Q^iP^j\in\mathscr{D}_+$ $(P=P_{A_1},\ Q=P_{A_2})$, so in particular

$$\begin{aligned}
0 &= (Q^iP^j)\Psi \\
&= \left[\frac{1}{p^i}\left(MP^{(1-p)/p}+\frac{1-p}{2}P^{-1}+pP^{q/p}\right)^iP^j\right]_-\Psi\,,
\end{aligned}$$

leading to w_∞-constraints on τ via Corollary 0.1.1. For instance, the Virasoro constraints, $i=1,\ j=k+1$ $(k=-1,0,1,\dots)$, become

$$\begin{aligned}
0 &= -\left(ML^{kp+1}+\frac{1-p}{2}L^{kp}+pL^{1+p(k+1)}\right)_-\Psi \\
&= \Psi(e^{-\eta}-1)\ \tau^{-1}\left(\frac{1}{2}W_{kp}^{(2)}+\frac{1-p}{2}\frac{\partial}{\partial t_{kp}}+p\frac{\partial}{\partial t_{(k+1)p+1}}\right)\tau\,,
\end{aligned}$$

yielding

$$\mathscr{L}_k^{(2)}\tau := \left(\frac{1}{2}W_{kp}^{(2)}+p\frac{\partial}{\partial t_{(k+1)p+1}}\right)\tau = c_k\tau$$

for $k=-1,0,1,\dots$. Since (by adding an appropriate constant to $\mathscr{L}_0^{(2)}$)

$$[\mathscr{L}_k^{(2)},\mathscr{L}_j^{(2)}]=(k-j)\mathscr{L}_{j+k}^{(2)},\quad k\geqq -1\,,$$

we conclude by substituting $\mathscr{L}_k^{(2)}\tau=c_k\tau$ into the above that $c_k=0$.

2-Matrix Integral. *Consider the matrix integral* [A-vM3, Me]

$$Z_N(t,s,c) := \int_{\{N\times N \text{ Hermitian}\}^2} \int dM_1 dM_2 \exp(\operatorname{tr}(V(M_1,M_2))) ,$$

$$V(x,y) := V_1(x) + V_2(y) + V_{12}(x,y) := \sum_1^\infty t_i x^i + \sum_1^\infty s_i y^i + cxy . \tag{6.6}$$

By the Mehta formula this becomes ($\bar{\lambda} = \operatorname{diag}(\lambda_1,\dots,\lambda_N)$)

$$Z_N(t,s,c) = \operatorname{vol}(U(N)) \cdot \int_{(\mathbb{R}^N)^2} \int d\eta d\lambda \Delta(\lambda)\Delta(\eta) e^{tr\, V(\bar{\lambda},\bar{\eta})} . \tag{6.7}$$

Consider the monic biorthogonal polynomials $\{p_n\}$, $\{p_n^*\}$:

$$\begin{aligned}\langle p_n(x), p_m^*(y)\rangle : &= \int\int p_n(x)p_m^*(y)e^{V(x,y)}dxdy \\ &= \delta_{n,m}h_n(t,s,c), \quad n,m \geqq 0 .\end{aligned}$$

Then

$$Z(t,s,c) = \operatorname{vol}(U(N)) \cdot h_0 \cdot h_1 \cdots h_{N-1} .$$

Defining also $L = (L_1,L_2)$, and (Q_1,Q_2^*) by

$$\begin{aligned} &zp = L_1 p,\ \tfrac{\partial}{\partial z} p = Q_1 p , \\ &zp^* = L_2^* p^*,\ \tfrac{\partial}{\partial z} p^* = Q_2^* p^* , \end{aligned} \qquad [L_i, Q_i] = I,$$

where $p = (p_0, p_1, \dots)^\top$, $h = \operatorname{diag}(h_0, h_1, \dots)$. Then the pair of matrices L as functions of (t,s) satisfies the Toda flow relations (0.3), and

$$Z_N(t,s,c) = \operatorname{vol}(U(N)) \cdot \tau_N(t,s,c), \quad 0 \leqq N < \infty ;$$

moreover, we may consider in the formula (6.7) (but not in (6.6)) the more general potential

$$V(x,y) = V_1 + V_2 + V_{12}(x,y) := \sum_i^\infty t_i x^i + \sum_i^\infty s_i y^i + \sum_{i,j>0} c_{ij} x^i y^j ,$$

which has the effect of introducing more time evolutions

$$\frac{\partial L}{\partial c_{ij}} = [-Y(L_1^i L_2^j), L] ,$$

where $Y(M) = (M_l - M_u)$ is as in Remark 1.1, and which commute with the Toda flows and amongst themselves.

For the 2-Toda lattice with generic initial conditions, the products $L_1^i L_2^j$ and thus the vector fields $\partial/\partial c_{ij}$ will not make sense. However in this context, $L_1^i L_2^j$ is well-defined, because L_1 (resp. L_2) has a finite number of non-zero entries in each row (resp. column).

Define (Ψ_1, Ψ_2^*) and (M_1, M_2^*) as in equation (0.6), with $[L_i, M_i] = 1$. In the context of bi-orthogonal polynomials, we find

$$M_1 = Q_1 + \frac{\partial V_1}{\partial x}(L_1) \quad \text{and} \quad M_2^* = Q_2^* + L_2^{*-1}\,, \quad M_2 = M_2^{*\top}.$$

The four matrices M_i and L_i are constrained by two relations ("string equations"):

$$M_1 + \frac{\partial V_{12}}{\partial x}(L_1, L_2) = 0, \qquad M_2 - L_2^{-1} - \frac{\partial V_{12}}{\partial y}(L_1, L_2) = 0\,. \tag{6.8}$$

From (6.8), we deduce immediately:

$$\begin{aligned} P_k &:= L_1^{k+1} M_1 + \sum i c_{ij} L_1^{i+k} L_2^j \\ &= M_1 L_1^{k+1} + (k+1) L_1^k + \sum i c_{ij} L_1^{i+k} L_2^j = 0\,, \\ Q_k &:= M_2 L_2^{k+1} - L_2^k - \sum j c_{ij} L_1^i L_2^{j+k} = 0\,, \end{aligned} \tag{6.9}$$

and one checks that the vector fields:

$$\begin{aligned} \mathbb{X}_k L &= [-Y(P_k), L]\,, \\ \mathbb{Y}_k L &= [-Y(Q_k), L] \end{aligned}$$

form two sets of decoupled Virasoro vector fields, i.e.,

$$\begin{aligned} &[\mathbb{X}_j, \mathbb{X}_k] = (j-k)\mathbb{X}_{j+k}, \qquad [\mathbb{Y}_j, \mathbb{Y}_k] = (k-j)\mathbb{Y}_{j+k}\,, \\ &[\mathbb{X}_j, \mathbb{Y}_k] = 0\,. \end{aligned}$$

Using Corollary 0.1.1 conclude that

$$\begin{aligned} 0 &= -\Big((P_k, 0)_- \Psi\Big)_m \\ &= \Bigg((e^{-\eta} - 1)\frac{(\frac{1}{2}W_{m,k}^{(2)} + (k+1)W_k^{(1)} + \sum i c_{ij}\partial/\partial c_{i+k,j})\tau_m}{\tau_m}\,, \\ &\qquad e^{-\tilde{\eta}}\frac{(\frac{1}{2}W_{m+1,k}^{(2)} + (k+1)W_k^{(1)} + \sum i c_{ij}\partial/\partial c_{i+k,j})\tau_{m+1}}{\tau_{m+1}} \\ &\qquad - \frac{(\frac{1}{2}W_{m,k}^{(2)} + (k+1)W_k^{(1)} + \sum i c_{ij}\partial/\partial c_{i+k,j})\tau_m}{\tau_m}\Bigg) \cdot (\Psi)_m\,, \end{aligned}$$

and similarly

$$\begin{aligned} 0 = \Bigg(&(e^{-\eta} - 1)\frac{(\frac{1}{2}\tilde{W}_{m,k}^{(2)} - \tilde{W}_k^{(1)} - \sum j c_{ij}\partial/\partial c_{i+k,j})\tau_m}{\tau_m}\,, \\ &e^{-\tilde{\eta}}\frac{(\frac{1}{2}\tilde{W}_{m,k}^{(2)} - \tilde{W}_k^{(1)} - \sum j c_{ij}\partial/\partial c_{i+k,j})\tau_{m+1}}{\tau_{m+1}}\,. \end{aligned} \tag{6.10}$$

The generators W and $\tilde{W}$ can be expressed in terms of the customary Virasoro generators,

$$J_n^{(1)} = \frac{\partial}{\partial t_n} + (-n)t_{-n} \quad \text{and} \quad J_n^{(2)} = \sum_{i+j=n} :J_i^{(1)}J_j^{(1)}: ,$$

the $\tilde{J}$'s being the same as above, but with t's replaced by s's. The relation (6.10) imply the following constraints for the two-matrix integrals τ_m, for $k \geqq -1$ and $m \geqq 0$:

$$\left(\frac{1}{2}J_k^{(2)} + \left(m + \frac{k+1}{2}\right)J_k^{(1)} + \sum_{i,j \geqq 1} ic_{ij} \sum_{\alpha=0}^{m-1} (L_1^{k+i}L_2^j)_{\alpha\alpha}\right)\tau_m = -\frac{m(m+1)}{2}\tau_m \delta_{k0}$$

$$\left(\frac{1}{2}\tilde{J}_k^{(2)} - \left(m + \frac{k+1}{2}\right)\tilde{J}_k^{(1)} - \sum_{i,j \geqq 1} jc_{ij} \sum_{\alpha=0}^{m-1} (L_1^{i}L_2^{j+k})_{\alpha\alpha}\right)\tau_m = -\frac{m(m+1)}{2}\tau_m \delta_{k0} \,, \quad (6.11)$$

The details on the formulae above can be found in [A-vM3]. It is interesting to note that equations (6.10) also provide the exact values of the constants on the right hand side of (6.11).

Appendix

In this appendix we give an indication of an alternative proof for Corollary 0.1.1 in the continuous case; in particular we provide a proof "by hand" of statement (0.20) for $n = 0$ and 1.

Theorem A.1.

$$\left.\begin{aligned} \frac{-(L^{-m})_-\Psi}{\Psi} &= (e^{-\eta} - 1)\frac{W_{-m}^{(1)}(\tau)}{\tau} \\ -\frac{(ML^{-m+1})_-\Psi}{\Psi} &= (e^{-\eta} - 1)\frac{\frac{1}{2}W_{-m}^{(2)}(\tau)}{\tau} \end{aligned}\right\} \text{all } m \in \mathbb{Z} \,. \quad (A.1)$$

The proof of this theorem requires several crucial lemmas and it will be postponed until later.

Lemma A.2. *For* $m \geqq 0$

$$(ML^{-m+1})_-\Psi = B_{-m}\Psi \,, \quad (A.2)$$

where

$$B_{-m} := -\sum_{n>m} nt_n \frac{\partial}{\partial t_{n-m}} - mt_m + z^{1-m}\frac{\partial}{\partial z} \quad (m \geqq 0) \,.$$

Proof. By straightforward computation, one finds

$$\begin{aligned}\mathbb{Y}_{-m+1,1}\Psi &= (ML^{-m+1})_-\Psi = \left(ML^{-m+1} - (ML^{-m+1})_+\right)\Psi \\ &= z^{1-m}\frac{\partial}{\partial z}\Psi - \left(S\sum_1^\infty kt_k D^{k-m}S^{-1}\right)_+\Psi \\ &= z^{1-m}\frac{\partial}{\partial z}\Psi - \left(S\sum_{k\geqq m} kt_k D^{k-m}S^{-1}\right)_+\Psi \\ &= z^{1-m}\frac{\partial}{\partial z}\Psi - \sum_{k\geqq m} kt_k(L^{k-m})_+\Psi \\ &= \left(z^{1-m}\frac{\partial}{\partial z} - \sum_{k\geqq m} kt_k\frac{\partial}{\partial t_{k-m}} - mt_m\right)\Psi = B_{-m}\Psi ,\end{aligned}$$

ending the proof of Lemma A.2.

Proof of Theorem A.1. Since the first line in (A.1) is trivial, we now proceed to the second line for $m \geqq 0$. In order to prove this, we need a few computational facts: since the operator B_{-m} is a derivation except for the term $-mt_m$, write

$$B_{-m} = B^0_{-m} - mt_m ,$$

and observe that

$$\begin{aligned}W^{(2)}_{-m}\tau &= \left(2\sum_{n>m} nt_n\frac{\partial}{\partial t_{n-m}} + \sum_{n=1}^{m-1} nt_n(m-n)t_{m-n} + (m-1)mt_m\right)\tau \\ &= -2B^0_{-m}\tau + \sum_{n=1}^{m-1} nt_n(m-n)t_{m-n}\tau + (m-1)mt_m\tau ;\end{aligned}$$

therefore

$$\begin{aligned}(e^{-\eta}-1)\frac{B^0_{-m}(\tau)}{\tau} &= -\frac{1}{2}(e^{-\eta}-1)\frac{W^{(2)}_{-m}\tau}{\tau} + \frac{1}{2}(e^{-\eta}-1)\sum_{n=1}^{m-1} nt_n(m-n)t_{m-n} \\ &\quad + \frac{1}{2}(e^{-\eta}-1)m(m-1)t_m \\ &= -\frac{1}{2}(e^{-\eta}-1)\frac{W^{(2)}_{-m}(\tau)}{\tau} \\ &\quad + \frac{1}{2}\sum_{n=1}^{m-1} n(m-n)\left((t_n - \frac{1}{n}z^{-n})(t_{m-n} - \frac{1}{m-n}z^{-m+n}) - t_nt_{m-n}\right) \\ &\quad + \frac{m(m-1)}{2}\left((t_m - \frac{1}{m}z^{-m}) - t_m\right) \\ &= -\frac{1}{2}(e^{-\eta}-1)\frac{W^{(2)}_{-m}(\tau)}{\tau} - \sum_{n=1}^{m-1} nt_nz^{-m+n} .\end{aligned} \tag{A.3}$$

Using these facts yields

$$\begin{aligned}(ML^{-m+1})_-\Psi &= B_{-m}\left(e^{\sum t_iz^i}\frac{e^{-\eta}\tau}{\tau}\right) \quad \text{using Lemma A.2} \\ &= \frac{e^{-\eta}\tau}{\tau}(B^0_{-m} - mt_m)e^{\sum t_iz^i} + e^{\sum t_iz^i}\frac{B^0_{-m}(e^{-\eta}\tau)}{\tau} - e^{\sum t_iz^i}\frac{(B^0_{-m}\tau)(e^{-\eta}\tau)}{\tau^2} \\ &\quad \text{since } B^0_{-m} \text{ is a derivation ,}\end{aligned}$$

$$= e^{\sum t_i z^i} \frac{e^{-\eta} \tau}{\tau} \sum_{n=1}^{m-1} n t_n z^{n-m} + e^{\sum t_i z^i} \frac{e^{-\eta} \tau}{\tau} (e^{-\eta} - 1) \frac{B^0_{-m}(\tau)}{\tau} ,$$

using the definition of B^0_{-m} and $[B^0_{-m}, \eta] = 0$

$$= \Psi \left(\sum_{n=1}^{m-1} n t_n z^{n-m} + (e^{-\eta} - 1) \frac{B^0_{-m}(\tau)}{\tau} \right)$$

$$= - \Psi(e^{-\eta} - 1) \frac{\frac{1}{2} W^{(2)}_{-m}(\tau)}{\tau}, \text{ using (A.3) ;}$$

this ends the proof of Theorem A.1 for $m \geqq 0$; the case $m < 0$ and the full corollary 0.1.1 can be established using the Lie algebra structures of w_∞ and W_∞; see [vM].

References

[A-vM0] Adler, M., van Moerbeke, P.: Completely integrable systems, Euclidean Lie algebras, and curves. Adv. Math. **38**, 267–317 (1980)

[A-vM1] Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional W_p-Gravity. Commun. Math. Phys. **147**, 25–56 (1992)

[A-vM2] Adler, M., van Moerbeke, P.: Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials. To appear in Duke Math J. (1995)

[A-vM3] Adler, M., van Moerbeke, P.: Two-matrix integrals and two-Toda symmetries. To appear

[A-Sh-vM] Adler, M., Shiota, T., van Moerbeke, P.: From the w_∞-algebra to its central extension: a τ-function approach. To appear in Physics Letters A. (1994)

[A-K] Aoyama, S., Kodama, Y.: A generalized Sato equation and the W_∞ algebra. Phys. Lett. **B278** 56–62 (1992)

[BX] Bonora, L., Xiong, C.S.: Matrix models without scaling limit. SISSA reprint

[B-K] Brézin, E., Kazakov, V.A.: Exactly solvable field theories of closed strings. Phys. Lett. **236B**, 144–150 (1990)

[DJKM] Date, E., Jimbo, M., Kashiwara, M., Miwa, T.: Transformation groups for soliton equations. In: Proc. RIMS Symp. Nonlinear integrable systems – Classical theory and quantum theory (Kyoto 1981), Singapore: World Scientific, 1983, pp. 39–119

[D1] Dickey, L.: Soliton equations and integrable systems. Singapore: World Scientific, 1991

[D2] Dickey, L.: Lectures on classical W-algebras. Cortona, Italy 1993

[D3] Dickey, L.A.: On additional symmetries of the KP hierarchy and Sato's Bäcklund transformation. Commun. Math. Phys.

[DG] Duistermaat, J.J., Grünbaum, F.A.: Differential equations in the spectral parameter. Commun. Math. Phys. **103**, 177–240 (1986)

[Fa] Fastré, J.: Boson-correspondence for W-algebras, Bäcklund–Darboux transformations and the equation $[L, P] = L^n$. University of Louvain, doctoral dissertation (1993)

[Fu] Fuchssteiner, B.: Mastersymmetries, Higher Order Time-Dependent Symmetries and Conserved Densities of Nonlinear Evolution Equations. Prog. Theor. Phys. **70**, No. 6, 1508–1522 (1983)

[FKN] Fukuma, M., Kawai, H., Nakayama, R.: Infinite dimensional Grassmannian structure of two-dimensional Quantum gravity. Commun. Math. Phys. **143**, 371–403 (1992)

[GMO] Gerasimov, A., Marshakov, A., Mironov, A., Morozov, A., Orlov, A.: Matrix models of Two-dimensional gravity and Toda theory. Nucl. Phys. **B357**, 565–618 (1991)

[Gr-OS] Grinevich, P.G., Orlov, A.Yu., Schulman, E.I.: On the Symmetries of the integrable system. In: Modern development of the Soliton theory, ed. A. Fokas, V.E. Zakharov, (1992)

[Gr-O] Grinevich, P.G., Orlov, A.Yu.: Flag spaces in KP theory and Virasoro action on DetD_j and Segal-Wilson tau-function. In: Modern problems of Quantum Field theory, Belavin, Zamolodchikov, Klimuk (eds.) Berlin, Heidelberg, New York: Springer, 1989

[Ka] Kac, V.G.: Infinite-dimensional Lie algebras. 3rd edition, Cambridge: Cambridge Univ. Press, 1990

[K-R] Kac, V.G., Raina, A.K.: Bombay lectures on highest weight representations of infinite dimensional Lie Algebras. Adv. Series Math. Phys. vol. 2, 1987

[K-S] Kac, V.G., Schwarz, A.: Geometric interpretation of partition function of 2D gravity. Phys. Lett. **257B**, 329–334 (1991)

[K] Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Commun. Math. Phys. **147**, 1–23 (1992)

[M-Z] Magri, F., Zubelli, J.P.: Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV. Commun. Math. Phys. **141** (2), 329–352, (1991)

[Me] Mehta, M.L.: A method of integration over matrix variables. Commun. Math. Phys. **79**, 327–340 (1981)

[N] Niedermaier, M.: $W_{1+\alpha}$ as a symmetry of GL_α-orbits, preprint 1993

[O-Fa] Oevel, W., Falck, M.: Master Symmetries for Finite-Dimensional Integrable Systems: the Calogero–Moser system. Prog. Theor. Phys. **75**, 1328–1341 (1986)

[O-Fu] Oevel, W., Fuchssteiner, B.: Explicit formulas for symmetries and conservation laws of the Kadomtsev-Petviashvili equation. Phys. Lett. **88A**, 323-327 (1982)

[O-F-B] Oevel, G., Fuchssteiner, B., Blaszak, M.: Action-Angle Representation of Multisolitons by Potentials of Mastersymmetries. Prog. Theo. Phys. **83**, N^0 3, 395–413 (1990)

[O] Orlov, A.Y.: Vertex operator, $\bar{\partial}$-problem, symmetries, variational identities and Hamiltonian formalism for 2+1 integrable systems. In: Proc. Kiev Intern. Workshop "Plasma Theory and Non-linear and Turbulent Processes in Physics", Baryakhtar (ed.) Singapore: World Scientific, 1988

[O-Sc] Orlov, A.Y., Schulman, E.I.: Additional Symmetries for Integrable and Conformal Algebra Representation. Lett. Math. Phys. **12**, 171-179 (1986)

[Sa1] Sato, M.: Soliton equations and the universal Grassmann manifold (by Noumi in Japanese). Math. Lect. Note Ser. No. **18**, Sophia University, Tokyo, 1984

[Sa2] Sato, M.: The KP Hierarchy and Infinite-Dimensional Grassmann Manifolds. Proceedings of Symposia in Pure Mathematics, Vol. **149** Part 1, (1989) pp. 51–66

[S-S] Sato, M., and Sato, Y.: Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds. Lect. notes in Num. Appl. Anal. Vol. **5**, 259–271 (1982)

[Sc] Schwarz, A.: On solutions to the string equation. Mod. Phys. Lett. **A 6** 2713–2725 (1991)

[Sh] Shiota, T.: Characterization of Jacobian varieties in terms of soliton equations. Invent. Math. **83**, 333–382 (1986)

[T-T] Takasaki, K., Takebe, T.: Integrable hierarchies and dispersionless limit. Reviews in Mathematical Physics, to appear

[U-T] Ueno, K., Takasaki, K.: Toda Lattice Hierarchy. Adv. Studies in Pure Math., Vol. **4**, 1–95 (1984)

[v1] van de Leur, J.: The $W_{1+\alpha}(gl_s)$-symmetries of the s-component KP-hierarchy, preprint 1994

[v2] van de Leur, J.: The Adler–Shiota–van Moerbeke formula for the BKP-hierarchy, preprint 1994

[vM-M] van Moerbeke, P., Mumford, D.: The spectrum of difference operators and algebraic curves. Acta Math. **143**, 93–154 (1979)

[vM] van Moerbeke, P.: Integrable foundations of string theory. CIMPA Summer School at Sophia-Antipolis (June 1991), Lectures on Integrable systems, Ed.: O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach, World Scientific, 1994, pp. 163–267

[W2] Witten, E.: On the Kontsevich Model and other Models of Two Dimensional Gravity. IASSNS-HEP-91/24 (6/1991), Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. **1, 2** (New York, 1991), 176–216, World Sci. Publishing, River Edge, NJ 1992

Communicated by A. Jaffe