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Non-Compact Quantum Groups Associated with Abelian Subgroups

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Abstract: Let G be a Lie group. For any Abelian subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G, and any $r \in \mathfrak{h} \wedge \mathfrak{h}$, the difference of the left and right translates of r gives a compatible Poisson bracket on G. We show how to construct the corresponding quantum group, in the C^*-algebra setting. The main tool used is the general deformation quantization construction developed earlier by the author for actions of vector groups on C^*-algebras.

Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, let H be a closed connected Abelian subgroup of G with Lie algebra $\mathfrak{h}$, and let $r \in \mathfrak{h} \wedge \mathfrak{h}$. Given this data, there is a simple way to equip G with a compatible Poisson bracket in Drinfeld's sense [D1, D3]. But according to Drinfeld's outlook, compatible Poisson brackets give precisely the "directions" in which one can hope to quantize a Lie group to obtain a quantum group. We will show here that, for the data given above, there is a natural way to carry out this quantization so as to construct the corresponding quantum group, within the framework of C^*-algebras. This quantization will be a *preferred* quantization in the sense of [GS1, GS2, Gq], meaning that the comultiplication will be unchanged – only the pointwise product of functions on G will be changed, into a non-commutative product. In the monograph [Rf2] I developed a general construction for the deformation quantization of any C^*-algebra in the direction of any Poisson bracket coming from an action of $\mathbb{R}^d$ for $d \geqq 2$. It is this construction which we will employ here to produce our quantum groups.

In [Rf4] I have described the details of this construction for the case in which G is compact, where there are some very substantial simplifications. This provided some new examples of compact quantum groups. Here we will see that our construction yields some reasonably interesting non-compact quantum groups which do not seem to have been constructed before. Since non-compact quantum groups have been

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difficult to construct [Wr4, Wr5, Wr6, Wr7, WZ, PW, BS, B1, B2, Rf1, Vn1, Z], any new examples seem valuable at this point.

But our construction also seems to be a significant corner of the general theory of quantum groups. As suggested in [Rf4], our construction can also be used to deform quantum groups themselves into new quantum groups. The details for compact quantum groups have been worked out by S. Wang [Wa3]. (See also Sect. 14 of [GS1].) But at present, in contrast to the compact case [Wr8, Wa1, Wa2, Vn3], no definitive set of axioms seems yet to be known for what one should mean by a non-compact quantum group within the C^*-algebra framework (e.g. from which one can derive the existence of a Haar measure – but see [Vn4, BS]). Thus it is not presently clear how best to formulate the statement of a general process of deforming a non-compact quantum group, and so we will not attempt to treat this situation here.

A significant virtue of our construction is that we are able to directly deform the pointwise product on a Fréchet algebra of functions which includes many smooth functions. For most quantum groups which have been constructed up to now one does not know how to do this. Indeed, even for the fundamental example of quantum $SU(2)$ [Wr2, VS], it seems not to be known how to find a suitable Fréchet algebra of functions on which the pointwise product can be deformed – the algebra of representative functions which one usually employs seems unlikely to be complete for any topology nicely related to the smooth structure of the Lie group $SU(2)$.

Another virtue of the general quantization construction developed in [Rf2], and so of the construction given here, is that, in a suitable sense discussed in part in [Sh], it preserves the symplectic leaves of the associated Poisson bracket. Recently Sheu [Sh] has made the striking observation that it is impossible to construct a preferred deformation quantization of $SU(2)$ which preserves the symplectic leaves. This indicates an interesting potential obstruction to using possible variations of our construction to obtain other quantum groups.

There is an interesting obstacle to our construction. For many cases the range of the comultiplication will not consist of norm-continuous vectors for the action of H which our construction employs. This causes technical difficulties. We are able to overcome this obstacle, but only at the cost of letting the comultiplication lose some contact with the smooth structure of G. Elsewhere [Rf6] we have shown that the range of the comultiplication will consist of norm-continuous vectors for our action of H exactly when H is what we call a SIN-subgroup of G. In this special case we can maintain direct contact with the smooth structure, and show that there is a Fréchet quantum group underlying our C^*-quantum group. We plan to treat this matter elsewhere [Rf7].

As discussed in the introduction of [Rf4], our construction is related to the twisting construction first introduced by Drinfeld. We will not repeat here the discussion of the literature given in [Rf4]. We will just mention the following related papers which have come to our attention more recently. A recent preprint of Zakrzewski [Z] is closely related to the present work in ways which will be indicated below. The paper [AST] contains a purely algebraic analogue for GL_n of our construction. (See especially their Theorem 4.) Purely algebraic versions also play a role in [TV1, TV2] and antecedent papers they reference. A more abstract twisting construction, taking place at the level of the group Fourier algebra, using unitary dual cocycles, has been introduced recently by Landstad and Raeburn [LR]. This construction has been employed very recently by Landstad [L] to construct quantum groups which largely coincide with the ones constructed here. His techniques, while very much related to ours, are formulated fairly differently. Because he works initially with the Fourier algebra and the group von Neumann algebra (in ways related to the dual quantum groups of Sect. 6 of [Z]), he does not face the technical obstacle involving smooth functions which is alluded to above. Thus his definition of the comultiplication is quite clean. But the Fourier algebra is a somewhat abstract object, which makes little contact with the smooth structure of the underlying Lie groups. Thus Landstad's approach has the disadvantage that it will probably be awkward to use it directly for considerations of non-commutative differential geometry (though his avoidance of smooth functions permits him to also treat groups which are not connected Lie groups). Landstad provides a rich collection of specific examples, some of which go somewhat beyond the considerations given here. Much of Landstad's article is concerned with obtaining more concrete descriptions of his quantum groups in terms of function algebras more closely tailored to his various classes of examples. Even more recently, the relationship between Landstad's approach and the theory of Kac algebras has been nicely elucidated in [V].

In the first section of this paper we give the Lie algebra background for our results, and in the second we construct the quantum spaces for our quantum groups. The comultiplication for our quantum groups is constructed in Sect. 3, and shown to be coassociative in Sect. 4. Important information about the range of the comultiplication is obtained in Sect. 5. In Sect. 6 we briefly examine the coidentity and coinverse (antipode) for our quantum groups.

1. The Lie Bialgebra Background

In this section we briefly review certain well-known [D1, D3, Tk1, Tk2, Z] Lie bialgebra considerations which lie behind our construction. These considerations provide motivation for what follows, but we will not use specific facts from this section (though we will use some of the notation introduced here).

We denote by ad the adjoint representation of $\mathfrak{g}$ on itself, as well as the corresponding representation on the space $\mathfrak{g} \wedge \mathfrak{g}$ of anti-symmetric tensors. We denote by ad^* the coadjoint representation of $\mathfrak{g}$ on the vector-space dual $\mathfrak{g}^*$. By a 1-cocycle for ad with values in $\mathfrak{g} \wedge \mathfrak{g}$ we mean a linear map ϕ from $\mathfrak{g}$ to $\mathfrak{g} \wedge \mathfrak{g}$ satisfying

$$\phi([X, Y]) = \mathrm{ad}_X(\phi(Y)) - \mathrm{ad}_Y(\phi(X))$$

for $X, Y \in \mathfrak{g}$. The pair $(\mathfrak{g}, \phi)$ is called a Lie bialgebra [D1, D3, Tk1, Tk2] if ϕ is a 1-cocycle such that the dual map, ϕ^*, from $\mathfrak{g}^* \wedge \mathfrak{g}^*$ to $\mathfrak{g}^*$ is a Lie algebra product on $\mathfrak{g}^*$, that is, satisfies the Jacobi identity. Drinfeld [D1, D3] showed that the Lie bialgebra structures on $\mathfrak{g}$ are in bijection with the compatible Poisson brackets on the simply-connected Lie group G corresponding to $\mathfrak{g}$, that is, the brackets for which the group product is a Poisson map. Drinfeld indicated that the compatible Poisson brackets give the "directions" in which we can hope to quantize the Lie group into a quantum group. At the level of universal enveloping algebras $\mathscr{U}(\mathfrak{g})$, Reshetikhin [Re] has indicated how to carry out this quantization. But at an analytical level we seem to be far from understanding how to carry this out in general, although there have been some notable successes, described in some of the papers listed in the introduction.

Among the cocycles are the coboundaries, that is, the maps ϕ for which there is an $r \in \mathfrak{g} \wedge \mathfrak{g}$ such that $\phi(X) = \mathrm{ad}_X(r)$ for all $X \in \mathfrak{g}$. Then ϕ^* is a Lie algebra product on $\mathfrak{g}^*$ exactly if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \tag{1.1}$$

is an ad-invariant element in the 3-fold tensor product of $\mathcal{U}(\mathfrak{g})$ with itself [D2, Tk1, Tk2], where $r_{12} = r \otimes 1$, etc. The corresponding Lie bialgebra is then said to be a coboundry Lie bialgebra. If (1.1) is in fact $= 0$, then this equation is the classical Yang–Baxter equation, and the Lie bialgebra is said to be triangular. The Lie bracket on $\mathfrak{g}^*$ is determined by

$$\langle X, [\mu, v]\rangle = \langle \mathrm{ad}_X(r), \mu \otimes v\rangle$$

for $\mu, v \in \mathfrak{g}^*$, where the brackets denote the duality between $\mathfrak{g}$ and $\mathfrak{g}^*$. It is easy to express the corresponding compatible Poisson bracket on G in this case. Let λ and ρ denote left and right translation on G. Then $x \mapsto \lambda_x(r)$ and $x \mapsto \rho_x(r)$ can be shown [D2, Tk1, Tk2] to be Poisson brackets on G (in the triangular case), and

$$x \mapsto \lambda_x(r) - \rho_x(r) \tag{1.2}$$

turns out to be the desired compatible Poisson bracket.

Now let $\mathfrak{h}$ be an Abelian subalgebra of $\mathfrak{g}$. Then *any* $r \in \mathfrak{h} \wedge \mathfrak{h}$ satisfies the classical Yang–Baxter equation. The corresponding triangular Lie bialgebras are the ones which lie behind the considerations of this paper. Let $\tilde{r}$ denote the linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ determined by

$$\langle \tilde{r}(\mu), v\rangle = \langle r, \mu \otimes v\rangle .$$

Note that $\tilde{r}(\mu) \in \mathfrak{h}$ for all $\mu \in \mathfrak{g}^*$. It is not difficult to show that the Lie bracket on $\mathfrak{g}^*$ for the corresponding ϕ is defined by

$$[\mu, v] = \mathrm{ad}^*_{\tilde{r}(\mu)}(v) - \mathrm{ad}^*_{\tilde{r}(v)}(\mu) .$$

(This fits well with the corresponding Lie group structure on the Poisson dual given in Eq. 14 of [Z].) From this it is easily shown that $[\mathfrak{g}^*, \mathfrak{g}^*] \subseteq \mathfrak{h}^\perp$ and that $[\mathfrak{h}^\perp, \mathfrak{g}^*] = \{0\}$, so that $\mathfrak{g}^*$ is a 2-step solvable Lie algebra.

It is also not difficult to show that $\mathrm{tr}(\mathrm{ad}_\mu) = \mathrm{tr}(\mathrm{ad}_{\tilde{r}(\mu)})$. Recall [Vr] that the Lie group G will be unimodular exactly if $\mathrm{tr}(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$. Then we see that if G is unimodular, so will be the Lie group corresponding to $\mathfrak{g}^*$ (the Poisson dual to G). In particular, for compact G as in [Rf4], the corresponding Poisson duals will always be unimodular, as mentioned there.

2. The Quantized Space

Let G, $\mathfrak{g}$, H, $\mathfrak{h}$ and r be as above. We can assume that H is closed (connected), for if it is not, we can replace it by its closure, enlarging $\mathfrak{h}$ accordingly, but leaving r unchanged. We denote the exponential map from $\mathfrak{h}$ onto $H \subseteq G$ by η. Let $V = \mathfrak{h} \oplus \mathfrak{h}$, and let $A = C_\infty(G)$, the C^*-algebra of continuous complex-valued functions on G which vanish at infinity, with sup-norm $\| \ \|_\infty$. To have some uniformity with the notation of [Rf2, Rf4] we will denote elements of $\mathfrak{h}$ by letters such as s, t, u, v. Define an action, α, of V on A by

$$(\alpha_{(s,u)}f)(x) = f(\eta(-s)x\eta(u))$$

for $(s,u) \in V$ and $x \in G$. (The minus sign in $-s$ has little significance here – we include it only in anticipation of the discussion in [Rf7] where we need to relate α to actions of some non-commutative subgroups of G, and also in anticipation of perhaps someday being able to treat cases in which H itself is not Abelian. But see the conjecture at the end of Sect. 1 of [Z].) On $\mathfrak{h}$, and so on V, we fix an arbitrary positive definite inner product, which we denote by a dot. Then for any skew-symmetric operator J on V we are exactly in position to apply the main construction of [Rf2] to produce the deformed C^*-algebra A_J. To orient the reader, we mention that at the level of smooth functions the deformed product $\times_J$ is defined by

$$(f \times_J g)(x) = \iint (\alpha_{J(s,u)}f)(x)(\alpha_{(t,v)}g)(x)e(s \cdot t + u \cdot v)\,. \tag{2.1}$$

Here, as later, $e(r) = e^{2\pi ir}$ for $r \in \mathbf{R}$, we omit the ds, etc., and this integral must be interpreted as an oscillatory integral [Rf2].

The algebra A_J is a suitable quantization for the *space* G, in the direction of a Poisson bracket coming from J and α (see [Rf2]). But without further hypotheses there is no reason to expect it to relate well to the group structure of G. If we examine the formula (1.2) for the compatible Poisson bracket on G, together with the definition of α as a combination of left and right translation, we see that we need to have J of the form $K \oplus (-K)$, where K is a skew-symmetric operator on $\mathfrak{h}$. (The minus sign here is crucial.) Thus K is our version in the present context of the earlier $r \in \mathfrak{h} \wedge \mathfrak{h}$. More specifically, the dot product gives an isomorphism from $\mathfrak{h}^*$ to $\mathfrak{h}$, and if we follow this by K we obtain a map from $\mathfrak{h}^*$ to $\mathfrak{h}$ which will be the $\tilde{r}$ of the previous section. Alternatively, we could from the beginning take K to be a skew map from $\mathfrak{h}^*$ to $\mathfrak{h}$, so that J goes from V^* to V, and proceed as in Sect. 1 of [Rf5].

3. The Construction of the Deformed Comultiplication

Let Δ denote the usual comultiplication for A, defined for $f \in A$ by

$$(\Delta f)(x,y) = f(xy)\,.$$

If G is not compact, Δf will not vanish at infinity, but rather will be in the algebra $C_b(G \times G)$ of bounded continuous functions on $G \times G$. Let $A \otimes A$ denote the completed C^*-tensor product, so $A \otimes A = C_\infty(G \times G)$. Then $C_b(G \times G)$ is exactly the multiplier algebra, $M(A \otimes A)$, of $A \otimes A$. (See [Pe] for multiplier algebras.) Thus Δ maps A to $M(A \otimes A)$. What we would like to show is that, when we deform the product on A to obtain A_J, we do not need to deform Δ, but rather that Δ, at the level of functions, determines an algebra homomorphism, Δ_J, from A_J to $M(A_J \otimes A_J)$, which will be the comultiplication for our quantum group. The calculation done early in Sect. 2 of [Rf4] which shows that at least at the level of functions this works for compact groups, holds here in general only at the heuristic level (see also [Z]). We will not pursue this here since a stronger approach is needed anyway if we are to extend to the C^*-algebras. Now $A_J \otimes A_J$ can be obtained from $A \otimes A$ by using the action $\alpha \otimes \alpha$ of $V \oplus V$ (Proposition 2.1 of [Rf4]). To obtain Δ_J we need to consider $\alpha \otimes \alpha$ as an action on $M(A \otimes A) = C_b(G \times G)$ in the evident way. Now in general $\alpha \otimes \alpha$ will not be a strongly continuous action on $C_b(G \times G)$. Closer examination of the situation [Rf6] then reveals an interesting obstacle, namely that, even for $f \in C_c^\infty(G)$, in general Δf will not be a norm-continuous vector for the action $\alpha \otimes \alpha$ (and derivatives of Δf need not be bounded). This causes difficulties when one tries to manipulate oscillatory integrals such as (2.1) to show that Δ determines an algebra homomorphism Δ_J with the desired properties. Thus we will need to take a somewhat more complicated route.

We also notice another feature with which we must deal. The algebra $A_J \otimes A_J$ can be defined in terms of the action $\alpha \otimes \alpha$ of $V \times V$ on $A \otimes A$. But A_J is defined in terms of the action α of V. Since different groups are acting here, it does not make sense to ask whether Δ is equivariant, and so we cannot directly invoke functoriality to produce Δ_J. This too means that we need to take a somewhat more complicated route, which we now begin to describe.

We will let $G \times_H G$ denote the quotient of $G \times G$ by the equivalence relation $(xw, y) \sim (x, wy)$ for $w \in H$. This comes from an evident free and proper action of H on $G \times G$, and so $G \times_H G$ is a locally compact Hausdorff space for the quotient topology. Note also that the product on G gives a well-defined continuous map from $G \times_H G$ onto G.

The quotient map from $G \times G$ onto $G \times_H G$, and the map from $G \times_H G$ onto G just indicated, are not in general proper (in the sense that preimages of compact sets are compact). Thus they do not give homomorphisms between the corresponding algebras of functions vanishing at infinity. Rather they give "morphisms", where we recall [Wr1, Vl, Wr4] that a morphism from a C^*-algebra A to a C^*-algebra B is a homomorphism, say θ, from A into the multiplier algebra $M(B)$ of B such that $B = \theta(A)B$ (closed span of products), or equivalently, such that for one, and hence for any, bounded approximate identity $\{e_\lambda\}$ for A we have $\theta(e_\lambda)b \to b$ for all $b \in B$. Because of our need to work on subalgebras where group actions are strongly continuous, we remark that to construct a morphism from A to B it suffices to construct a homomorphism, satisfying the above condition on approximate identities, from A into any C^*-algebra, M, containing B as an essential ideal (i.e. such that if $FB = 0$ for $F \in M$, then $F = 0$), since we will have an identification $M \subseteq M(B)$.

Although, as noted, Δ is not equivariant in a convenient way, our basic tool will be the fact that our deformation process preserves equivariant morphisms. The ingredients for proving this are mostly contained in Proposition 5.10 of [Rf2], and that proposition would have been more felicitously formulated as exactly the fact that deformation preserves morphisms. But it wasn't, so we state this here, and indicate how the proof follows from Proposition 5.10.

3.1. Theorem. *Let A and B be C^*-algebras, and let α and β be (strongly continuous) actions of a vector group V on A and B respectively. Let J be any skew-symmetric operator on V. Let $\bar{\beta}$ denote the extension of β to an action on $M(B)$ (which need not be strongly continuous). Let $\theta : A \to M(B)$ be a morphism from A to B which is equivariant for α and $\bar{\beta}$. Then θ determines a well-defined homomorphism, θ_J, from A_J to $M(B_J)$, which is a morphism (still equivariant) from A_J to B_J.*

Proof. Let M denote the C^*-subalgebra of $M(B)$ consisting of norm-continuous vectors for $\bar{\beta}$, and let $D = \theta(A)$. Because θ is equivariant and α is strongly continuous on A, it is easily seen that $D \subseteq M$. Since $\bar{\beta}$ is strongly continuous on M, the functoriality of deformation (Theorem 5.7 of [Rf2]) gives us a homomorphism θ_J of A_J into M_J, which is still equivariant by Theorem 5.12 of [Rf2]. Since $\bar{\beta}$ carries D into itself, the functoriality together with Proposition 5.8 of [Rf2] shows that θ_J factors as a surjection from A_J onto D_J followed by an inclusion of D_J into M_J.

Note that B is an essential ideal of M, so that B_J is an essential ideal of M_J by Proposition 5.9 of [Rf2]. Thus $M_J \subseteq M(B_J)$. Since θ is a morphism, D contains an approximate identity with the properties related to B noted earlier. Then D_J will contain an approximate identity with the same properties related to B_J, by Proposition 5.10 of [Rf2]. It follows that θ_J is a morphism. □

We recall [Wr1, Vl, Wr4] that morphisms can be composed to give morphisms. To be specific, let θ be a morphism from B to C. Then θ lifts [I, Vl, Wr1] to a unique unital homomorphism, $\bar{\theta}$, from $M(B)$ to $M(C)$. If φ is a morphism from A to B, and so a homomorphism from A to $M(B)$, then $\bar{\theta} \circ \varphi$ is a homomorphism from A to $M(C)$ which one can easily check is a morphism to C. We will denote composition of morphisms by juxtaposition, so $\theta\varphi$ in this case.

3.2. Proposition. *Let α, β and γ be actions of a vector group V on C^*-algebras A, B and C, with extensions $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ to the multiplier algebras. Let J be a skew-symmetric operator on V. Let φ be a morphism from A to B which is equivariant for α and $\bar{\beta}$, and let θ be a morphism from B to C which is equivariant for β and $\bar{\gamma}$. Then $\bar{\theta} \circ \varphi$ is equivariant for α and $\bar{\gamma}$, and we have*

$$(\theta\varphi)_J = \theta_J\varphi_J \,.$$

Proof. One readily checks that $\bar{\theta}$ is equivariant for $\bar{\beta}$ and $\bar{\gamma}$, so that $\bar{\theta} \circ \varphi$ is equivariant for α and $\bar{\gamma}$. Thus $(\bar{\theta} \circ \varphi)_J \;\; (= (\theta\varphi)_J)$ is a well-defined morphism by Theorem 3.1. Let M and N denote the subalgebras of norm-continuous vectors in $M(B)$ and $M(C)$ for $\bar{\beta}$ and $\bar{\gamma}$ respectively. Then φ carries A into M, and $\bar{\theta}$ carries M into N. Thus at the level of the smooth subalgebras we clearly have $(\bar{\theta} \circ \varphi)_J = \bar{\theta}_J \circ \varphi_J$. By continuity this holds at the C^*-algebra level. But from Theorem 3.1 we see that this means exactly that $(\theta\varphi)_J = \theta_J\varphi_J$. □

From now on we will often omit the bars on $\bar{\theta}$, $\bar{\beta}$, etc., when this should not cause confusion.

Now note that the comultiplication Δ can be factored as $\Delta = \Xi\Gamma$. Here Γ is the morphism from $A = C_\infty(G)$ to $B = C_\infty(G \times_H G)$ defined by

$$(\Gamma f)((x, y)^\sim) = f(xy) \,,$$

where $x, y \in G$ and $(x, y)^\sim$ denotes their equivalence class in $G \times_H G$; and Ξ denotes the injective morphism from B to $A \otimes A = C_\infty(G \times G)$ defined by

$$(\Xi F)(x, y) = F((x, y)^\sim) \,.$$

This factorization of Δ corresponds to the factorization through $G \times_H G$ indicated earlier for the product on G. We will often write $F(x, y)$ instead of $F((x, y)^\sim)$, etc., when this should not cause confusion.

Define a (strongly continuous) action, β, of V on B by

$$(\beta_{(s,u)}F)(x,y) = F(\eta(-s)x, y\eta(u))\,.$$

Clearly Γ is equivariant for the actions α and $\bar{\beta}$. It follows from Theorem 3.1 that Γ_J is well-defined as a morphism from A_J to B_J.

Now consider the action $\alpha \otimes \alpha$ of $V \times V$ on $A \otimes A$. By means of Ξ we can identify B with a subalgebra of $M(A \otimes A)$. Let γ be $\alpha \otimes \alpha$ but viewed as an action on B. A little computation shows that it carries B into itself exactly because H is Abelian. Furthermore, γ is strongly continuous, because it comes from an evident action of V on $G \times_H G$. Clearly Ξ is equivariant for γ and $\alpha \otimes \alpha$. Let $L = J \oplus J$ on $V \times V$. It follows from Theorem 3.1 that Ξ_L is a well-defined morphism from B_L to $(A \otimes A)_L$, and it is injective by Proposition 5.8 of [Rf2].

We must now see how all this relates to A_J. By Proposition 2.1 of [Rf4], applied just as in Corollary 2.2 of [Rf4], we see that $(A \otimes A)_L = A_J \otimes A_J$. (The tensor product for which Proposition 2.1 of [Rf4] is stated is the minimal tensor product, but A_J is nuclear by Theorem 4.1 of [Rf3], so all C^*-tensor products agree here.) Thus we can view Ξ_L as an injective morphism from B_L to $A_J \otimes A_J$.

Let $W = V \times V$, and let $W_0 = \{(0,u,u,0) : u \in \mathfrak{h}\}$, a subspace of W. Because of the definition of B, it is clear that W_0 is in the kernel of γ as an action on B. Let P denote the projection of W onto the orthogonal complement of W_0, so that $\gamma = \gamma \circ P$ as actions on B. Then by Theorem 8.11 of [Rf2] we have $B_L = B_{PLP}$ under the evident identification of smooth element. But a quick calculation shows that

$$PLP(s,u,t,v) = (Ks, 0, 0, -Kv)\,.$$

In particular, if we let Q denote the orthogonal projection onto $W_1 = \mathfrak{h} \times \{0\} \times \{0\} \times \mathfrak{h}$, we see that $Q(PLP)Q = PLP$. We are thus in a position to apply Theorem 8.7 of [Rf2] to the restrictions of γ and PLP to W_1. But these restrictions, under the evident identifications, are just the β and J of earlier. Thus Theorem 8.7 of [Rf2] tells us, for the evident meaning of the superscripts, that:

3.3. Lemma. *With notation as above,* $B_L^{\gamma} = B_J^{\beta}$.

Here the sign $=$, rather than just $\cong$, is appropriate, since, as seen from [Rf2] and Proposition 1.1 of [Rf4], the subalgebra of γ-smooth vectors in B will be a dense subalgebra of both sides, the C^*-norms on both sides will coincide on this dense subalgebra, and each side can be viewed as just the completion of this dense subalgebra for this norm.

It follows from Lemma 3.3 that we can view Γ_J as a morphism from A_J to B_L. Thus it can be composed with Ξ_L.

3.4. Definition. *We let* Δ_J *denote the morphism*

$$\Delta_J = \Xi_L \Gamma_J$$

from A_J *to* $A_J \otimes A_J$.

It is this Δ_J which we will show is a comultiplication for A_J. It is easily seen that if H is a SIN-subgroup [Rf6] for G so that the range of Δ is contained in the algebra of norm-continuous vectors for $\alpha \otimes \alpha$, then for smooth f the deformed product of $\Delta_J f$, as defined above, with smooth functions in $A \otimes A$ will be given by the expected oscillatory integral. But in the more general case it seems difficult to give a direct definition of the product of $\Delta_J f$ with smooth functions in $A \otimes A$. For example, if $f, g, h \in C_c^\infty(G)$, it is not in general clear whether the element $(\Delta_J f) \times_J (g \otimes h)$ of the C^*-algebra $A_J \otimes A_J$ can be represented by a function on $G \times G$, much less one which is smooth in some sense. Note that if $K = 0$ then all our products are just the pointwise products, but if $F \in (A \otimes A)^\infty$ for $\alpha \otimes \alpha$ and if $f \in A^\infty$, it will not be true in general that $(\Delta f)F \in (A \otimes A)^\infty$.

This loss of control over the smooth structure will make it awkward to use Δ_J when considering the non-commutative differential geometry of A_J as a quantum group, unless G is a SIN-subgroup.

Nevertheless, let us examine briefly what Definition 3.4 says about how to compute Δ_J. Let $f \in A^\infty$, and let $F \in (A \otimes A)^\infty$ where as above this means smooth for $\alpha \otimes \alpha$. How, operationally, do we compute $\Delta_J(f) \times_L F$? Suppose we are given $\varepsilon > 0$. We want an algorithm for finding a function in $(A \otimes A)^\infty$ which we know is within $\varepsilon \|f\|_J \|F\|_L$ of the element $\Delta_J(f) \times_L F$ in the C^*-algebra $A_J \otimes A_J$. We proceed as follows. By Proposition 2.17 of [Rf2] we can choose in $B^{\infty\gamma}$ (i.e. smooth for γ) a bounded approximate identity $\{E_n\}$ for the C^*-algebra B which is also a bounded approximate identity for the Fréchet algebra $B^{\infty\gamma}$, and hence also for $B^{\infty\beta}$. By Proposition 4.13 of [Rf2] the E_n's will then also form a bounded approximate identity for the C^*-algebra $B_L = B_J$. Since Ξ_L is a morphism, we can find an n such that

$$\|\Xi_L(E_n) \times_L F - F\|_L \leqq (\varepsilon/2)\|F\|_L ,$$

where the product is in $M((A \otimes A)_L)$. Because the C^*-norm on $(A \otimes A)_L$ is dominated by suitable Fréchet seminorms (Proposition 4.10 of [Rf2]), one can effectively choose E_n by examining various derivatives of E_n and F.

Now at the C^*-algebra level $\Delta_J(f) \times_L F$ will be the limit of the $\Xi_L(\Gamma_J(f) \times_J E_m) \times_L F$ as m increases. Quantitatively, we will have

$$\begin{aligned}\|\Delta_J(f) \times_L F - \Xi_L(\Gamma_J(f) \times_J E_m) \times_L F\|_L &= \|\Delta_J(f) \times_L (F - \Xi_L(E_m) \times_L F)\|_L \\ &\leqq \|f\|_J \|F - \Xi_L(E_m) \times_L F\|_L .\end{aligned}$$

For the E_n chosen earlier, this is $\leqq (\varepsilon/2)\|f\|_J \|F\|_L$. Now $\Gamma_J(f) \times_J E_m$ will be a well-defined function in $B^{\infty\beta}$, but it may not be in $B^{\infty\gamma}$. (Consider $K = 0$.) However, we can approximate it in B_L by elements of $B^{\infty\gamma}$, for example by smoothing it in the standard way. (E.g. see the proof of Lemma 5.7 below.) That is, we can find $\varphi_f \in B^{\infty\gamma}$ such that

$$\|\Gamma_J(f) \times_J E_m - \varphi_f\|_L < (\varepsilon/2)\|f\|_J .$$

Again this can be done effectively by examining suitable derivatives. Then $\Xi(\varphi_f)$ will be a function in $(M(A \otimes A))^\infty$ (i.e. smooth for $\alpha \otimes \alpha$), so that $\Xi(\varphi_f) \times_L F$ can be computed by the oscillatory integrals used in [Rf2], and is a function in $(A \otimes A)^\infty$. This function meets our requirements, since by the estimates above

$$\|\Delta_J(f) \times_L F - \Xi(\varphi_f) \times_L F\|_L \leqq \varepsilon \|f\|_J \|F\|_L .$$

4. The Coassociativity of Δ_J

This section is devoted to proving:

4.1. Theorem. *With notation as above, Δ_J is coassociative, that is,*

$$(\Delta_J \otimes I)\Delta_J = (I \otimes \Delta_J)\Delta_J \ . \tag{4.2}$$

Here I denotes the identity map on A_J. Equation (4.2) must be interpreted in terms of the composition of morphisms. Thus, $\Delta_J \otimes I$ is easily seen to be a morphism from $A_J \otimes A_J$ to $A_J \otimes A_J \otimes A_J$, and so lifts to a unique homomorphism, $(\Delta_J \otimes I)^{-}$, between the multiplier algebras. Then Eq. (4.2) means that

$$(\Delta_J \otimes I)^{-} \circ \Delta_J = (I \otimes \Delta_J)^{-} \circ \Delta_J \ , \tag{4.3}$$

an ordinary composition of homomorphisms.

The verification of (4.3) must use the definition of Δ_J as $\Xi_L \Gamma_J$, and so will require several steps. In view of the definition, it is natural enough in connection with $\Delta_J \otimes I$ to consider the algebra $B \otimes A = C_\infty(G \times_H G \times G)$. On this algebra we have the actions $\gamma \otimes \alpha$ and $\beta \otimes \alpha$. From Proposition 2.1 of [Rf4] together with Lemma 3.3 above we immediately obtain

$$(B \otimes A)^{\gamma \otimes \alpha}_{L \oplus J} = (B \otimes A)^{\beta \otimes \alpha}_{L} \ . \tag{4.4}$$

A similar result holds for $A \otimes B$.

Our approach to verifying (4.3) involves relating matters to the space $G \times_H G \times_H G$. The following commutative diagram of morphisms will guide our progress:

$$\begin{tikzcd}
C_\infty(G) \arrow[rr, "\Delta"] \arrow[dr, "\Gamma"] & & C_\infty(G \times G) \arrow[rr, "\Delta \otimes I"] \arrow[dr, "\Gamma \otimes I"] & & C_\infty(G \times G \times G) \\
& C_\infty(G \times_H G) \arrow[ur, "\Xi"] \arrow[dr, "\Gamma \otimes \hat{I}"'] & & C_\infty(G \times_H G \times G) \arrow[ur, "\Xi \otimes I"] & \\
& & C_\infty(G \times_H G \times_H G) \arrow[ur, "\hat{I} \otimes \Xi"'] & &
\end{tikzcd}$$

We must examine how various actions and deformations relate to this diagram.

Let $C = C_\infty(G \times_H G \times_H G)$. On C we have the evident actions which we will denote, with only small abuse of notation, by $\alpha \otimes \alpha \otimes \alpha = \gamma \otimes \alpha = \alpha \otimes \gamma$, and $\beta \otimes \alpha$. It is fairly clear from Lemma 3.3 that we will have

$$C^{\gamma \otimes \alpha}_{L \oplus J} = C^{\beta \otimes \alpha}_{L} \ ,$$

and indeed arguments just as in the proof of Lemma 3.3 verify this. But by the same arguments we also obtain

$$C^{\beta \otimes \alpha}_{L} = C^{\beta}_{J} \ , \tag{4.5}$$

where now β must be viewed as given by

$$(\beta_{(s,u)}\Phi)(x, y, z) = \Phi(\eta(-s)x, y, z\eta(u))$$

for $\Phi \in C$. With this understanding, we thus have

$$C^{\gamma \otimes \alpha}_{L \oplus J} = C^{\beta}_{J} \ . \tag{4.6}$$

Now in view of the definition of Δ_J, the left-hand side of (4.3) will involve

$$(\Gamma_J \otimes I)^- \circ \Xi_L \, , \tag{4.7}$$

which is a morphism from B_L to $B_L \otimes A_J$. But Γ_J was initially defined as a morphism to B_J, and so we can view (4.7) as a morphism to $B_J \otimes A_J$. Then notice that both B_L and $B_J \otimes A_J$ $(= (B \otimes A)_L)$, as well as the intermediate algebra $(A \otimes A)_L$ where the composition takes place, are all defined in terms of actions of $V \times V$. So it now makes sense to inquire about equivariance. And indeed, examination quickly shows that Ξ and $\Gamma \otimes I$ are equivariant for these actions. It follows from Proposition 3.2 that

$$(\Gamma_J \otimes I)^- \circ \Xi_L = ((\Gamma \otimes I)^- \circ \Xi)_L \tag{4.8}$$

as a morphism to $B_J \otimes A_J$.

Now let $\Gamma \otimes \hat{I}$ denote the morphism from B to C defined by

$$((\Gamma \otimes \hat{I})F)(x, y, z) = F(xy, z)$$

(involving equivalence classes). Then $\Gamma \otimes \hat{I}$ is clearly equivariant for $\alpha \otimes \alpha$ and $\beta \otimes \alpha$. Let $\hat{I} \otimes \Xi$ denote the morphism from C to $B \otimes A$ defined by

$$((\hat{I} \otimes \Xi)\Phi)((x, y)^\sim, z) = \Phi((x, y, z)^\sim) \, .$$

It is easily seen that $\hat{I} \otimes \Xi$ is equivariant for $\beta \otimes \alpha$ on both C and $B \otimes A$. Furthermore,

$$(((\Gamma \otimes I)^- \circ \Xi)F)(x, y, z) = (\Xi F)(xy, z) = F(xy, z)$$

for $F \in B$, while

$$((\hat{I} \otimes \Xi)^- \circ (\Gamma \otimes \hat{I})F)(x, y, z) = ((\Gamma \otimes \hat{I})F)(x, y, z) = F(xy, z) \, .$$

Thus as morphisms,

$$(\Gamma \otimes I)\Xi = (\hat{I} \otimes \Xi)(\Gamma \otimes \hat{I}) \, .$$

From Proposition 3.2 it follows that

$$((\Gamma \otimes I)\Xi)_L = ((\hat{I} \otimes \Xi)(\Gamma \otimes \hat{I}))_L = (\hat{I} \otimes \Xi)_L(\Gamma \otimes \hat{I})_L \, .$$

Putting all of this together, we obtain

$$(\Gamma_J \otimes I)\Xi_L = (\hat{I} \otimes \Xi)_L(\hat{\Gamma} \otimes I)_L \, . \tag{4.9}$$

In exactly the same way we have

$$(I \otimes \Gamma_J)\Xi_L = (\Xi \otimes \hat{I})_L(\hat{I} \otimes \Gamma)_L \tag{4.10}$$

for the evident meaning of $\Xi \otimes \hat{I}$ and $\hat{I} \otimes \Gamma$.

Let us now see how these relations help us to prove coassociativity. We have

$$(\Delta_J \otimes I)\Delta_J = (\Xi_L \otimes I)(\Gamma_J \otimes I)\Xi_L\Gamma_J \, ,$$

which from (4.9) is

$$= (\Xi_L \otimes I)(\hat{I} \otimes \Xi)_L(\hat{\Gamma} \otimes I)_L\Gamma_J \, . \tag{4.11}$$

But $B_L = B_J$ by Lemma 3.3, and $C_L = C_J$ by (4.5), and so $(\Gamma \otimes \hat{I})_L = (\Gamma \otimes \hat{I})_J$ since they agree on a dense subalgebra of smooth elements. We are using here the fact that $\Gamma \otimes \hat{I}$ is equivariant for β on B and C. Then by Proposition 3.2,

$$(\Gamma \otimes \hat{I})_J \Gamma_J = ((\Gamma \otimes \hat{I})\Gamma)_J \, .$$

In much the same way $\hat{I} \otimes \Xi$ is clearly equivariant for $\gamma \otimes \alpha$ on C and $B \otimes A$, so from (4.4) and (4.6) we obtain $(\hat{I} \otimes \Xi)_L = (\hat{I} \otimes \Xi)_{L \oplus J}$. Furthermore, a moment's thought shows that $\Xi_L \otimes I = (\Xi \otimes I)_{L \oplus J}$ on $(B \otimes A)_{L \oplus J}$. Consequently by Proposition 3.2 we have

$$(\Xi_L \otimes I)(\hat{I} \otimes \Xi)_L = ((\Xi \otimes I)(\hat{I} \otimes \Xi))_{L \oplus J} \, .$$

Putting these equalities into (4.11), we find that

$$(\Delta_J \otimes I)\Delta_J = ((\Xi \otimes I)(\hat{I} \otimes \Xi))_{L \oplus J}((\Gamma \otimes \hat{I})\Gamma)_J \, , \tag{4.12}$$

where the composition of morphisms is taking place at $C_{L \oplus J} = C_J$.

In exactly the same way we find that

$$(I \otimes \Delta_J)\Delta_J = ((I \otimes \Xi)(\Xi \otimes \hat{I}))_{L \oplus J}((\hat{I} \otimes \Gamma)\Gamma)_J \, . \tag{4.13}$$

But a trivial computation shows that

$$(\Xi \otimes I)(\hat{I} \otimes \Xi) = (I \otimes \Xi)(\Xi \otimes \hat{I}) \, , \tag{4.14}$$

with both sides sending $\Phi \in C$ to the function $\Phi((x, y, z)^{\sim})$ on $G \times G \times G$. Another trivial computation shows that

$$(\Gamma \otimes \hat{I})\Gamma = (\hat{I} \otimes \Gamma)\Gamma \, , \tag{4.15}$$

with both sides sending $f \in A$ to the function $f(xyz)$ on $G \times_H G \times_H G$. (Note that this is the place where we use the full associativity of the group product on G.) By means of (4.14) and (4.15) we see that (4.12) and (4.13) coincide. This completes the proof of Theorem 4.1.

5. The Range of the Comultiplication

There is a further property which one requires of a comultiplication in the non-compact case. In our setting this property states that

$$(\Delta_J(A_J))(1 \otimes A_J) = A_J \otimes A_J = (\Delta_J(A_J))(A_J \otimes 1) \tag{5.1}$$

(closed span of products). Properties of this type were probably first discussed extensively by Vallin [Vl], though he does not require quite this much. Property 5.1 has recently become one of the key axioms in a general definition of compact quantum groups, and seems likely to play a corresponding role in the non-compact case. See [Wr8, Vn3, BS, Wa1, Wa2]. Here we will treat explicitly only the first of the equalities in 5.1, since the treatment of the second is very similar. These equalities have two aspects, namely a containment and a density. The main difficulty is the fact that $\Delta_J(A_J)$ is not an $\alpha \otimes \alpha$-invariant subalgebra. Note also that the expressions in (5.1) do not have especially attractive algebraic properties.

Let $D = C_\infty(H\backslash G)$. The map $(x, y)^\sim \mapsto Hy = \tilde{y}$ gives an injective morphism from D to B. Note that D is then carried into itself by both β and γ (extended to $M(B)$), and that these actions are strongly continuous on D. Because large subspaces of the domains of these actions act trivially on D, we find from arguments as in the proof of Lemma 3.3 that

$$D_L = D_J, \tag{5.2}$$

and that by Proposition 1.11 of [Rf2] the product on this algebra is given by

$$(\varphi \times_J \psi)(y) = \int \varphi(y\eta(-Ku))\psi(y\eta(v))e(u \cdot v)$$

for φ, ψ smooth in D. This is just the deformed product for the action of H by right translation on D. The corresponding morphism from D_J to B_J is injective by Proposition 5.8 of [Rf2], and in what follows we will usually identify D_J with the corresponding subalgebra of $M(B_J) = M(B_L)$.

The usefulness of D for our present purposes stems from the following observations. We of course have $A \otimes D = C_\infty(G \times (H\backslash G))$. But the map $(x, y)^\sim \mapsto (xy, \tilde{y})$ is easily seen to be a homeomorphism from $G \times_H G$ onto $G \times (H\backslash G)$. We let Ω denote the corresponding algebra isomorphism from $A \otimes D$ onto B, defined for $\Phi \in A \otimes D$ by

$$(\Omega\Phi)(x, y) = \Phi(xy, \tilde{y}) .$$

Notice the relationship with the formula for the fundamental multiplicative unitary operator [BS] given in the first equation after Proposition 2.5 of [Rf4]. Notice also that for $f \in A$ and $\varphi \in D$ we have

$$(\Omega(f \otimes \varphi))(x, y) = (\Gamma f)(x, y)\varphi(y) = ((\Gamma f)(1 \otimes \varphi))(x, y) .$$

Thus $(\Gamma A)(1 \otimes D) = \Omega(A \otimes D)$, where on the left we mean as usual the closed span of products. Since Ω is an isomorphism, we obtain in particular:

5.3. Lemma. *With notation as above, $(\Gamma A)(1 \otimes D) = B$.*

We seek a similar result for the deformed algebras. Let M denote the subalgebra of $M(B)$ on which β is strongly continuous. Notice that ΓA and $1 \otimes D$, as subalgebras of $M(B)$, are carried into themselves by β, and that in fact they are contained in M (and the inclusions are morphisms to B). We now see that the following proposition is pertinent:

5.4. Proposition. *Let α be an action of a vector group V on a C^*-algebra M, and let J be a skew-symmetric operator on V. Let B, C and D be α-invariant C^*-subalgebras of M such that $CD \subseteq B$. Then inside M_J we have*

$$C_J D_J \subseteq B_J .$$

Proof. We clearly have $C^\infty D^\infty \subseteq B^\infty$, where the notation means α-smooth vectors. Then for $c \in C^\infty$, $d \in D^\infty$ the function $(u, v) \mapsto \alpha_{Ju}(c)\alpha_v(d)$ on $V \times V$ has values in B^∞. By the definition of $\times_J$ it follows that at the level of Fréchet algebras $c \times_J d \in B^\infty$. But this will then also be true at the C^*-algebra level. Continuity then gives the desired result. □

Since Γ is equivariant for α and β, we have $(\Gamma A)_J = \Gamma_J(A_J)$, and so from Proposition 5.4 we immediately conclude that

$$(\Gamma_J(A_J)) \times_J (1 \otimes D_J) \subseteq B_J .$$

We would like to show that we actually have equality here. This requires a more specific analysis. Let α also denote the action of V on $D \subseteq M(A)$, so that

$$(\alpha_{(s,u)}\varphi)(\tilde{x}) = \varphi(\tilde{x}\eta(u)) .$$

Then we have the corresponding outer tensor product action, $\alpha \otimes \alpha$, of $V \otimes V$ on $A \otimes D$, and the corresponding inner tensor product action, $\alpha \boxtimes \alpha$, of V on $A \otimes D$. A simple calculation shows that Ω is equivariant for $\alpha \boxtimes \alpha$ and β. Consequently $(A \otimes D)_J^{\alpha \boxtimes \alpha} \cong B_J^\beta$ via Ω_J. Now at the level of multiplier algebras we have

$$\Omega(f \otimes 1) = \Gamma f \text{ for } f \in A ,$$
$$\Omega(1 \otimes \varphi) = 1 \otimes \varphi \text{ for } \varphi \in D .$$

Thus to show that $(\Gamma_J(A_J)) \times_J (1 \otimes D_J) = B_J$ it suffices to show that within $M((A \otimes D)_J)$ we have $A_J \times_J D_J = (A \otimes D)_J$.

One might expect this to be a particular case of Proposition 2.1 of [Rf4], but it is not. To see what the issue is, let α and β be actions of V on any two C^*-algebras A and B, and let J and K be any skew-symmetric operators on V. Then we have the outer tensor product action $\alpha \otimes \beta$ of $V \times V$ on $A \otimes B$, and Proposition 2.1 of [Rf4] tells us that $(A \otimes B)_{J \oplus K} \cong A_J \otimes B_K$. Note in particular that $A_J \otimes 1$ and $1 \otimes B_K$ commute within $M((A \otimes B)_{J \oplus K})$.

However in our present situation we do not expect $\Gamma(A_J)$ and $1 \otimes D_J$ to commute. What is going on here is that if, in the general situation just above, we restrict $\alpha \otimes \beta$ to a subspace Z of $V \times V$, and if we let L be a general skew-symmetric operator on Z, then $(A \otimes B)_L$ will not in general have a tensor product decomposition as algebras, and, within its multiplier algebra, A_L and B_L will not in general commute. In particular, all this will apply when $A = B$ and $\beta = \alpha$, but where we use the inner tensor product action $\alpha \boxtimes \alpha$. Thus what we need (and will also need later) is:

5.5. Theorem. *Let α and β be actions of vector groups V and W on C^*-algebras A and B respectively, and let $\otimes$ be a C^*-tensor product such that $\alpha \otimes \beta$ gives an action of $V \times W$ on $A \otimes B$ (which will then be strongly continuous). Let Z be a subspace of $V \times W$, and let γ be the restriction of $\alpha \otimes \beta$ to Z. Let L be a skew-symmetric operator on Z. Then within $M((A \otimes B)_L^\gamma)$ we have*

$$(A \otimes B)_L^\gamma = (A_L^\gamma \otimes 1) \times_L (1 \otimes B_L^\gamma)$$

(closed span of products).

Proof. Note that γ gives evident actions on A and B, viewed as the subalgebras $A \otimes 1$ and $1 \otimes B$ of $M(A \otimes B)$. Thus the above assertion makes sense. Let π and ρ denote the canonical projections of $V \times W$ onto V and W respectively, restricted to Z. Let A^∞ and B^∞ denote the smooth subalgebras for α and β respectively. Denote by $\otimes_a$ the algebraic tensor product. Thus $A^\infty \otimes_a B^\infty$ is contained in the smooth subalgebra $(A \otimes B)^{\infty\gamma}$ for γ. For $a \in A^\infty$ and $b \in B^\infty$ we have

$$(a \otimes 1) \times_L (1 \otimes b) = \int \gamma_{Lp}(a \otimes 1)\gamma_q(1 \otimes b)e(p \cdot q) ,$$

where p and q range over Z. This is

$$= \int \alpha_{\pi L p}(a) \otimes \beta_{\rho q}(b) e(p \cdot q) = \int (\alpha \otimes \beta)_{(\pi L p, \rho q)}(a \otimes b) e(p \cdot q) .$$

This suggests that on the smooth subalgebra, $(A \otimes B)^\infty$, for $\alpha \otimes \beta$, with its Fréchet topology, we define an operator T_L by

$$T_L(c) = \int (\alpha \otimes \beta)_{(\pi L p, \rho q)}(c) e(p \cdot q) .$$

Note that this is well-defined by 1.4 of [Rf2].

5.6. Lemma. *The operator T_L is continuous for the Fréchet topology of $(A \otimes B)^\infty$, and is invertible, with inverse T_{-L}, which is continuous.*

Proof. The semi-norms which define the Fréchet topology (see [Rf2]) are defined in terms of derivatives with respect to the action $\alpha \otimes \beta$, and these derivatives commute with the action. It then follows rapidly from 1.4 of [Rf2] that T_L is continuous, as is T_{-L} for the same reasons.

Examination of the integrals for $T_{-L}T_L$ suggests that we consider

$$\int (\alpha \otimes \beta)_{(-\pi L s, \rho t)}(\alpha \otimes \beta)_{(\pi L p, \rho q)}(c) e(p \cdot q + s \cdot t) ,$$

where variables of integration range over Z. Application of 1.4 of [Rf2] shows that this is well-defined. If we argue as near the beginning of the proof of Theorem 2.14 of [Rf2], we find that this integral is indeed equal to $T_{-L}(T_L c)$. But we can rewrite the integral as

$$\int (\alpha \otimes \beta)_{(\pi L(p-s),\ \rho(q+t))}(c) e(p \cdot q + s \cdot t) ,$$

and this is seen to be equal to c by means of the change of variables $p \mapsto p + s$, $t \mapsto t - q$, justified as at the end of the proof of Theorem 2.14 of [Rf2]. By interchanging L and $-L$ we see that we also have $T_L(T_{-L}c) = c$. □

We now need the following fact (which is also needed to make the proof of Proposition 2.1 of [Rf4] complete):

5.7. Lemma. *Let α and β be (strongly continuous) actions of Lie groups G and H on Banach spaces A and B. Let $\otimes$ denote some Banach cross-norm such that $\alpha \otimes \beta$ gives an action of $G \times H$ on $A \otimes B$. Then $A^\infty \otimes_a B^\infty$ is dense in $(A \otimes B)^\infty$ for the Fréchet topology.*

Proof. This is just a variation on the proof of Proposition 1.1 of [Rf4]. Suppose we are given $c \in (A \otimes B)^\infty$, $\varepsilon > 0$, and $X \in \mathscr{U}(\mathfrak{g})$ and $Y \in \mathscr{U}(\mathfrak{h})$. The strong continuity of $\alpha \otimes \beta$ on the Fréchet space $(A \otimes B)^\infty$ (see the appendix of [Sc]) tells us that we can find $\varphi \in C_c^\infty(G)$ and $\psi \in C_c^\infty(H)$ such that

$$\|c - (\alpha_\varphi \otimes \beta_\psi)(c)\|_{X \otimes Y} < \varepsilon/2 .$$

The proof then continues along the same lines as that of Proposition 1.1 of [Rf4]. □

We now complete the proof of Theorem 5.5. Since $A^\infty \otimes_a B^\infty$ is dense in $(A \otimes B)^\infty$, and T_L is continuous with continuous inverse, it follows that $T_L(A^\infty \otimes_a B^\infty)$ is dense in $(A \otimes B)^\infty$. Now $(A \otimes B)^\infty$ is dense in $A \otimes B$ as usual, and the inclusion is equivariant for γ. It follows from Proposition 1.1 of [Rf4] that $(A \otimes B)^\infty$ is dense in $(A \otimes B)^{\infty\gamma}$ for its Fréchet topology. Consequently $T_L(A^\infty \otimes_a B^\infty)$ is dense in $(A \otimes B)^{\infty\gamma}$. But T_L was defined exactly so that for $a \in A^\infty$ and $b \in B^\infty$ we have

$$T_L(a \otimes b) = (a \otimes 1) \times_L (1 \otimes b) .$$

(In fact, T_L is related to the "braided" mathematics of [GM, M], and in particular to the map in the second paragraph of Sect. 4 of [GRZ]. Our $(A \otimes B)_L^\gamma$ can be viewed as a "twisted" tensor product.) It follows that the algebraic linear span $(A^\infty \otimes 1) \times_L (1 \otimes B^\infty)$ is dense in $(A \otimes B)^{\infty\gamma}$. But $(A \otimes B)^{\infty\gamma}$ is dense in $(A \otimes B)_L$ by definition. The assertion of the theorem follows immediately. □

When we apply this general theorem to our earlier particular situation, we obtain:

5.8. Corollary. *With our earlier specific notation,*

$$\Gamma_J(A_J) \times_J (1 \otimes D_J) = B_J .$$

Now we have seen in Lemma 3.3 that $B_J = B_L$. The morphism Ξ_L is injective by Proposition 5.8 of [Rf2], and through it we will now identify B_L with its image under Ξ_L, so with a subalgebra of $M((A \otimes A)_L)$.

5.9. Proposition. *With notation as above,*

$$B_L \times_L (1 \otimes A_J) = (A \otimes A)_L .$$

Proof. One can easily verify that the map $(x, y) \mapsto ((x, y)^\sim, y)$ from $G \times G$ to $(G \times_H G) \times G$ is proper, that is, preimages of compact acts are compact. It is also injective. Consequently, it gives an actual homomorphism, $\mathscr{X}$, from $B \otimes A$ onto $A \otimes A$. We have used earlier the actions $\gamma \otimes \alpha$ on $B \otimes A$ and $\alpha \otimes \alpha$ on $A \otimes A$. These are actions of different groups, so $\mathscr{X}$ cannot be equivariant for them. But let Z be the subspace $\{(u, v, v) : u, v \in V\}$ of $V \times V \times V$, and identify Z with $V \times V$ in the evident way. Let δ denote the restriction of $\gamma \otimes \alpha$ to Z. (Note that δ is not quite an inner tensor product of actions.) Then it is easily verified that $\mathscr{X}$ is equivariant for δ on $B \otimes A$ and $\alpha \otimes \alpha$ on $A \otimes A$. Consequently, by Theorem 5.7 of [Rf2] we obtain a homomorphism, $\mathscr{X}_L$, of $(B \times A)_L^\delta$ to $(A \otimes A)_L$, which is surjective by Proposition 5.8 of [Rf2].

Now $B \otimes 1$ and $1 \otimes A$ can be viewed as subalgebras of $M(B \otimes A)$ which are carried into themselves by δ, and on which δ is strongly continuous. By arguments such as those used earlier, it is easily seen that $(B \otimes 1)_L^\delta = B_L^\gamma \otimes 1$, and $(1 \otimes A)_L^\delta = 1 \otimes A_J^\alpha$. But for F smooth in B and f smooth in A we have

$$\begin{aligned}\mathscr{X}_L((F \otimes 1) \times_L (1 \otimes f))(x, y) &= ((F \otimes 1) \times_L (1 \otimes f))((x, y)^\sim, y) \\ &= \int F(x, \eta(-Ks)y\eta(-Ku))f(\eta(-t)y\eta(v))e(s \cdot t + u \cdot v) \\ &= (F \times_L (1 \otimes f))(x, y) ,\end{aligned}$$

where we have used Proposition 1.11 of [Rf2] to reduce the integral. Thus at the algebraic level before taking closures we have

$$B_L \times_L (1 \otimes A_J) = \mathscr{X}_L((B \otimes 1)_L \times_L (1 \otimes A)_L) .$$

Since $\mathscr{X}_L$ is surjective, we see that the proof of Proposition 5.9 will be complete if we know that $(B\otimes 1)_L \times_L (1\otimes A)_L$ is dense in $(A\otimes B)_L^{\delta}$. But this follows immediately from Theorem 5.5. □

5.10. Theorem. *With notation as above*

$$(\Delta_J(A_J)) \times_L (1\otimes A_J) = A_J \otimes A_J = (\Delta_J(A_J)) \times_L (A_J\otimes 1)$$

(closed span of products).

Proof. Since the evident map from D to $M(A)$ is a morphism, we have $A_J = D_J A_J$ by Theorem 3.1. But $D_J = D_L$ as seen above. Thus

$$\begin{aligned}\Delta_J(A_J) \times_L (1\otimes A_J) &= \Delta_J(A_J) \times_L ((1\otimes D_J) \times_L (1\otimes A_J)) \\ &= (\Delta_J(A_J) \times_L (1\otimes D_L)) \times_L (1\otimes A_J)\,. \qquad (5.11)\end{aligned}$$

Now the morphism which sends D as $1\otimes D$ to $M(A\otimes A)$, sending $\varphi\in D$ to $(x,y)\mapsto \varphi(\tilde{y})$, clearly factors through the morphism Ξ from B to $A\otimes A$, namely by sending φ to $(x,y)^{\sim}\mapsto\varphi(\tilde{y})$. This factorization is clearly equivariant for $\alpha\otimes\alpha$ and its restrictions to B and $(1\otimes D)$. By Proposition 3.2 this factorization holds for Ξ_L. If we identify D_L with its image in $M(B_L)$, this says that

$$\Xi_L(F) \times_L (1\otimes\varphi) = \Xi_L(F\times_L (1\otimes\varphi))$$

for $F\in B_L$ and $\varphi\in D_L$. From this we see that (5.11) is equal to

$$(\Xi_L(\Gamma_J(A_J) \times_J (1\otimes D_J))) \times_L (1\otimes A_J)\,,$$

which by Corollary 5.8

$$= \Xi_L(B_J) \times_L (1\otimes A_J) = B_L \times_L (1\otimes A_J)\,,$$

which by Proposition 5.9

$$= (A\otimes A)_L = A_J\otimes A_J\,.$$

The proof of the other equality is similar. □

6. The Coidentity and Coinverse

Let π denote the restriction map from $A = C_\infty(G)$ onto $E = C_\infty(H)$. Now α clearly defines also an action on E, and π is equivariant. Thus π determines a surjection from A_J onto E_J by Proposition 5.8 of [Rf2]. But H is Abelian, and so the action on E can be expressed as $(\alpha_{(s,u)}f)(x) = f(\eta(-s+u)x)$. From this it is easily seen, as discussed before Proposition 2.4 of [Rf4], or much as in the proofs of Lemmas 3.3 and 5.6, that the deformation of E is trivial, i.e. $E_J = E$. Thus π_J is a surjection from A_J onto E. The map $\pi_J\otimes\pi_J$ is onto $E\otimes E$, and so lifts to a map between the corresponding multiplier algebras. We want to compare $(\pi_J\otimes\pi_J)\Delta_J$ and $\Delta^E\pi_J$, where Δ^E is just the usual comultiplication for $E = C_\infty(H)$.

In view of the definition of Δ_J, we should examine

$$(\pi_J\otimes\pi_J)\Xi_L = (\pi\otimes\pi)_L\Xi_L = ((\pi\otimes\pi)\Xi)_L\,.$$

Since $B_L = B_J$, and $E_J = E$ so that $(E \otimes E)_L = E \otimes E$, we will have

$$((\pi \otimes \pi)\Xi)_L = ((\pi \otimes \pi)\Xi)_J$$

at the level of smooth elements, and so at the C^*-algebra level. Thus

$$(\pi_J \otimes \pi_J)\Delta_J = ((\pi \otimes \pi)\Xi\Gamma)_J = ((\pi \otimes \pi)\Delta)_J = (\Delta^E\pi)_J = \Delta^E\pi_J \ .$$

Once we have discussed the coidentity and coinverse for A_J, it will be clear that they also are compatible with π_J, so we will have obtained:

6.1. Proposition. *The group H (qua the Hopf algebra E) is a subgroup of the quantum group A_J (that is, E is a quotient Hopf algebra of A_J).*

Let ε^0 denote evaluation of elements of E at the identity element of H, so that ε^0 is the coidentity on E. Let $\varepsilon = \varepsilon^0 \circ \pi_J$. Then ε is a continuous multiplicative linear functional on A_J.

6.2. Proposition. *The functional ε on A_J is a coidentity for Δ_J, that is,*

$$(\varepsilon \otimes id)\Delta_J = id = (id \otimes \varepsilon)\Delta_J \ .$$

Proof. We prove the first equality, the proof of the second being similar. Of course

$$(\varepsilon \otimes \mathrm{id})\Delta_J = (\varepsilon^0 \otimes \mathrm{id})(\pi_J \otimes \mathrm{id})\Xi_L\Gamma_J \ .$$

But $\pi_J \otimes \mathrm{id} = (\pi \otimes \mathrm{id})_L$ in the evident sense, while $(\pi \otimes \mathrm{id})_L\Xi_L = ((\pi \otimes \mathrm{id})\Xi)_L$. Now $B_L = B_J$ by Lemma 3.3, and it is easily seen from above that $(E \otimes A)_L = (E \otimes A)_J$ for $\mathrm{id} \otimes \alpha$. Consequently $(\pi \otimes \mathrm{id})_L\Xi_L = ((\pi \otimes \mathrm{id})\Xi)_J$. Since $\varepsilon^0 \otimes \mathrm{id}$ is really $\varepsilon^0 \otimes \mathrm{id}_J$, which is $(\varepsilon^0 \otimes \mathrm{id})_J$ for $\mathrm{id} \otimes \alpha$, we have

$$(\varepsilon \otimes \mathrm{id})\Delta_J = ((\varepsilon^0 \otimes \mathrm{id})(\pi \otimes \mathrm{id})\Delta)_J \ .$$

But $(\varepsilon^0 \otimes \mathrm{id})(\pi \otimes \mathrm{id})\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$. $\square$

We now turn to consideration of the coinverse (or antipode). Just as with Δ and ε, it does not need to be deformed at the level of functions. We denote it by S, and it is defined on functions by $(Sf)(x) = f(x^{-1})$. By exactly the arguments on p. 477 of [Rf4], but applied to $A = C_\infty(G)$ rather than $C(G)$, one sees that S determines an anti-automorphism of A_J (which we will still denote by S). Clearly $S^2 = I$. By chasing through the definition of Δ_J it is easily seen that

$$(S \otimes S)\Delta_J = \sigma\Delta_J S \ ,$$

where σ is the "flip", determined by $\sigma(a \otimes b) = b \otimes a$. This is the first property usually required of a coinverse.

In the purely algebraic setting, the main property required for S to be a coinverse is that

$$m_J(I \otimes S)\Delta_J = \iota\varepsilon = m_J(S \otimes I)\Delta_J \ , \tag{6.3}$$

where m_J is the product on A_J, and ι is the unital homomorphism from $\mathbb{C}$ to A_J. But, as is well-known, and mentioned in [Rf4], this causes serious difficulties in the C^*-algebra framework because m_J will not in general be continuous (or everywhere defined) on $A_J \otimes A_J$. (For one way around this, using operator spaces beyond C^*-algebras, see [ER].)

In the C^*-algebra context, where we cannot use (6.3), the usual route is to introduce Haar measure, and then require that S satisfy suitable properties with respect to Haar measure, as discussed towards the end of Sect. 4 of [Rf4]. For our present situation, the Haar measure on A_J will just be the Haar measure on G, suitably extended. It will be an unbounded weight on A_J. The main work which is needed to establish this is to extend Theorem 4.1 of [Rf4], which deals with invariant states, to the case of semi-invariant unbounded traces. This is quite technical. So it seems best to deal with this, and its relation to S, elsewhere. Since the Haar measure is needed for the discussion of the fundamental multiplicative unitary of [BS], we defer this too. (We take this opportunity to mention that near the end of the proof of Theorem 4.1 of [Rf4], the argument that if μ is faithful then so is μ_J, is, as written, correct only for tracial states. But it is easily repaired by replacing the K there by the space of those $c \in A_J$ for which $\mu_J(c^* \times_J c) = 0$. This is only a left ideal in A_J, but that causes no difficulty for the proof. Also, in the discussion of representations in Sect. 5 of [Rf4], some additional argument, along the lines given in [Wr3], is needed to see that the representations for the group G give representations for the quantum group $C(G)_J$ which are non-degenerate.)

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