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Realization of $U_q(so(N))$ within the Differential Algebra on R_q^N

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Abstract. We realize the Hopf algebra $U_{q^{-1}}(so(N))$ as an algebra of differential operators on the quantum Euclidean space $\mathbf{R}_q^N$. The generators are suitable q-deformed analogs of the angular momentum components on ordinary $\mathbf{R}^N$. The algebra Fun($\mathbf{R}_q^N$) of functions on $\mathbf{R}_q^N$ splits into a direct sum of irreducible vector representations of $U_{q^{-1}}(so(N))$; the latter are explicitly constructed as highest weight representations.

1. Introduction

One of the most appealing facts explaining the present interest for quantum groups [1] is perhaps the idea that they can be used to generalize the ordinary notion of space(time) symmetry. This generalization is tightly coupled to a radical modification of the ordinary notion of space(time) itself, and can be performed through the introduction of a pair consisting of a quantum group and the associated quantum space [2, 3].

The structure of a quantum group and of the corresponding quantum space on which it coacts are intimately interrelated [2]. The differential calculus on the quantum space [4] is built so as to extend the covariant coaction of the quantum group to derivatives. Here we consider the N-dimensional quantum Euclidean space $\mathbf{R}_q^N$ and $SO_q(N)$ as the corresponding quantum group; the Minkowski space and the Lorentz algebra could also be considered, and we will deal with them elsewhere [5].

In absence of deformations, a function of the space coordinates is mapped under an infinitesimal $SO(N)$ transformation of the coordinates to a new one which can be obtained through the action of some differential operators, the angular momentum components. In other words the algebra Fun($\mathbf{R}^N$) of functions on $\mathbf{R}^N$ is the base space of a reducible representation of $so(N)$, which we can call the regular (vector) representation of $so(N)$. It is interesting to ask whether an analog of this fact occurs in the deformed case; in proper language, whether Fun($\mathbf{R}_q^N$) can be considered as a left (or right) module of the universal enveloping algebra $U_q(so(N))$, the latter being realized as some subalgebra U_q^N of the algebra of differential operators Diff($\mathbf{R}_q^N$) on $\mathbf{R}_q^N$.

In this paper we give a positive answer to this question. The result mimics the classical (i.e. $q=1$) one: starting from the only $2N$ objects $\{x^i, \partial_j\}$ (the coordinates and derivatives, i.e. the generators of Diff($\mathbf{R}_q^N$)) with already fixed commutation and derivation relations, we end up with a very economic way of realizing $U_{q^{-1}}(so(N))$ and its regular vector representation (the fact that in this way we realize $U_{q^{-1}}(so(N))$ rather than $U_q(SO(N))$ is due to the choice that our differential operators act from the left as usual, rather than from the right). In this framework, the real structure of Diff($\mathbf{R}_q^N$) induces the real structure of $U_{q^{-1}}(so(N))$. This is the subject of this work.

What is more, this approach makes inhomogeneous extensions of $U_{q^{-1}}(so(N))$ and the study of the corresponding representation spaces immediately at hand, without introducing any new generator: it essentially suffices to add derivatives to the generators of $U_{q^{-1}}(so(N))$ to find a realization of the q-deformed universal enveloping algebra of the Euclidean algebra in N dimensions [6], containing $U_{q^{-1}}(so(N))$ as a subalgebra. In fact this method was used in [7] to find the q-deformed Poincaré Hopf algebra. In both cases the inhomogeneous Hopf algebra contains the homogeneous one as a Hopf subalgebra, and we expect it to be the dual of a inhomogeneous q-group constructed as a semidirect product in the sense of [8].

The plan of the work is as follows. In Sect. 2 we give preliminaries on the quantum Euclidean space $\mathbf{R}_q^N$ and the differential algebra Diff($\mathbf{R}_q^N$) on it. In Sect. 3 we define a subalgebra $U_q^N \subset$ Diff($\mathbf{R}_q^N$) by requiring that its elements commute with scalars, introduce two different sets of generators for it and study the commutation relations of the second set. In Sect. 4 we find the commutation relations of these generators with the coordinates and derivatives and derive the natural Hopf algebra structure associated to U_q^N (thought of as algebra of differential operators on Fun($\mathbf{R}_q^N$)); the Hopf algebra U_q^N is then identified with $U_{q^{-1}}(so(N))$. In Sect. 5 we find that the q-deformed homogeneous symmetric spaces are the bases spaces of the irreducible representations of U_q^N in Fun($\mathbf{R}_q^N$), and we show that they can be explicitly constructed as highest weight representations. When $q \in \mathbf{R}^+$ the representation are unitary and the hermitian conjugation coincides with the complex conjugation in Diff($\mathbf{R}_q^N$).

We will treat by a unified notation odd and even N's whenever it is possible, and n will be related to N by the formulae $N=2n+1$ and $N=2n$ respectively. We will assume that q is not a root of unity. Finally, we will often use the shorthand notation $[A,B]_a := AB - aBA$ ($\Rightarrow [\cdot,\cdot]_1 = [\cdot,\cdot]$).

2. Preliminaries

In this section we recollect some basic definitions and relations characterizing the algebra Fun($\mathbf{R}_q^N$) ($O_q^N(\mathbf{C})$ in the notation of [2]) of functions on the quantum euclidean space $\mathbf{R}_q^N$, $N \geq 3$, (which is generated by the noncommuting coordinates $x = (x^i)$), the ring Diff($\mathbf{R}_q^N$) of differential operators on $\mathbf{R}_q^N$, the quantum group $SO_q(N)$. In the first part we give a general overview of this matter; in Subsects. 2.1, 2.2 we collect some more explicit formulae which we will use in the following sections for explicit computations. In particular, in Subsect. 2.2 we report on a very useful transformation [9] from the $SO_q(N)$-covariant generators x, ∂ of Diff($\mathbf{R}_q^N$) to completely decoupled ones. As in [9], index $i = -n, -n+1, \ldots, -1, 0, 1, \ldots n$ if $N = 2n+1$, and $i = -n, -n+1, \ldots, -1, 1, \ldots n$ if $N = 2n$. For further details we refer the reader to [10, 9, 11].

The braid matrix $\hat{R}_q := \|\hat{R}^{ij}_{hk}\|$ for the quantum group $SO_q(N)$ is explicitly given by

$$\hat{R}_q = q \sum_{i \neq -i} e^i_i \otimes e^i_i + \sum_{\substack{i \neq j, -j \\ \text{or } i=j=0}} e^j_i \otimes e^i_j + q^{-1} \sum_{i \neq -i} e^{-i}_i \otimes e^i_{-i} \tag{1}$$

$$+ (q - q^{-1}) \left[\sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} q^{-\varrho_i + \varrho_j} e^{-j}_i \otimes e^j_{-i} \right], \tag{2}$$

where $(e^i_j)^h_k := \delta^{ih}\delta_{jk}$. $\hat{R}_q$ is symmetric: $\hat{R}^t = \hat{R}$.

The q-deformed metric matrix $C := \|C_{ij}\|$ is explicitly given by

$$C_{ij} := q^{-\varrho_i}\delta_{i,-j}, \tag{3}$$

where

$$(\varrho_i) := \begin{cases} \left(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2} - n\right) & \text{if } N = 2n+1 \\ (n-1, n-2, \ldots, 0, 0, \ldots, 1-n) & \text{if } N = 2n. \end{cases} \tag{4}$$

Notice that $N = 2 - 2\varrho_n$ both for even and odd N. C is not symmetric and coincides with its inverse: $C^{-1} = C$. Indices are raised and lowered through the metric matrix C, for instance

$$a_i = C_{ij}a^j, \qquad a^i = C^{ij}a_j. \tag{5}$$

Both C and $\hat{R}$ depend on q and are real for $q \in \mathbf{R}$. $\hat{R}$ admits the very useful decomposition

$$\hat{R}_q = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{1-N}\mathcal{P}_1. \tag{6}$$

$\mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1$ are the projection operators onto the three eigenspaces of $\hat{R}$ $\left(\text{the latter have respectively dimensions } \frac{N(N+1)}{2} - 1, \frac{N(N-1)}{2}, 1\right)$: they project the tensor product $x \otimes x$ of the fundamental corepresentation x of $SO_q(N)$ into the corresponding irreducible corepresentations [the symmetric modulo trace, antisymmetric and trace, namely the q-deformed versions of the corresponding ones of $SO(N)$].

The projector $\mathcal{P}_1$ is related to the metric matrix C by $\mathcal{P}_{1\,hk}^{\,ij} = \frac{C^{ij}C_{hk}}{Q_N}$; the factor Q_N is defined by $Q_N := C^{ij}C_{ij}$. $\hat{R}^{\pm 1}, C$ satisfy the relations

$$[f(\hat{R}), P \cdot (C \otimes C)] = 0, \qquad f(\hat{R}_{12})\hat{R}^{\pm 1}_{23}\hat{R}^{\pm 1}_{12} = \hat{R}^{\pm 1}_{23}\hat{R}^{\pm 1}_{12} f(\hat{R}_{23}) \tag{7}$$

(P is the permutator: $P^{ij}_{hk} := \delta^i_k \delta^j_h$ and f is any rational function); in particular this holds for $f(\hat{R}) = \hat{R}^{\pm 1}, \mathcal{P}_A, \mathcal{P}_S, \mathcal{P}_1$.

Let us recall that the unital algebra $\mathrm{Diff}(\mathbf{R}^N_q)$ of differential operators on the real quantum euclidean plane $\mathbf{R}^N_q$ is defined as the space of formal series in the (ordered) powers of the $\{x^i\}$, $\{\partial_i\}$ variables, modulo the commutation relations

$$\mathcal{P}_{A\,hk}^{\;ij} x^h x^k = 0, \qquad \mathcal{P}_{A\,hk}^{\;ij} \partial^h \partial^k = 0. \tag{8}$$

and the derivation relations

$$\partial_i x^j = \delta^j_i + q\hat{R}^{jh}_{ik} x^k \partial_h. \tag{9}$$

The subalgebra Fun($\mathbf{R}_q^N$) of "functions" on $\mathbf{R}_q^N$ is generated by $\{x^i\}$ only. Below we will give the explicit form of these relations.

For any function $f(x) \in$ Fun($\mathbf{R}_q^N$) $\partial_i f$ can be expressed in the form

$$\partial_i f = f_i + f_i^j \partial_j \,, \qquad f_i, f_i^j \in \mathrm{Fun}(\mathbf{R}_q^N) \tag{10}$$

(with f_i, f_j^i uniquely determined) upon using the derivation relations (9) to move step by step the derivatives to the right of each x^l variable of each term of the power expansion of f, as far as the extreme right. We denote f_i by $\partial_i f|$. This defines the action of ∂_i as a differential operator $\partial_i : f \in$ Fun($\mathbf{R}_q^N$) $\rightarrow \partial_i f| \in$ Fun($\mathbf{R}_q^N$): we will say that $\partial_i f|$ is the "evalaution" of ∂_i on f. For instance:

$$\partial_i \mathbf{1}| = 0 \,, \qquad \partial^i x^j| = C^{ij} \,, \qquad \partial^i x^j x^k| = C^{ij} x^k + q\hat{R}^{-1}{}^{ij}_{hl} x^h C^{lk} \,. \tag{11}$$

By its very definition, ∂_i satisfies the generalized Leibnitz rule:

$$\partial_i(fg) = \partial_i f|g + \mathcal{O}_i^j f|\partial_j g| \,, \qquad f, g \in \mathrm{Fun}(\mathbf{R}_q^N) \,, \qquad \mathcal{O}^j \in \mathrm{Diff}(\mathbf{R}_q^N) \tag{12}$$

($\mathcal{O}^j f| = f_i^j$). Any $D \in$ Diff($\mathbf{R}_q^N$) can be considered as a differential operator on Fun($\mathbf{R}_q^N$) by defining its evaluation in a similar way; a corresponding Leibnitz rule will be associated to it. In Sect. 4 we will consider as differential operators the angular momentum components.

If $q \in \mathbf{R}$ one can introduce an antilinear involutive antihomomorphism $*$:

$$*^2 = id \,, \qquad (AB)^* = B^* A^* \tag{13}$$

on Diff($\mathbf{R}_q^N$). On the basic variables x^i* is defined by

$$(x^i)^* = x^j C_{ij} \,, \tag{14}$$

whereas the complex conjugates of the derivatives ∂^i are not combinations of the derivatives themselves. It is useful to introduce barred derivatives $\bar{\partial}^i$ through

$$(\partial^i)^* = -q^{-N} \bar{\partial}^j C_{ji} \,. \tag{15}$$

They satisfy relation (8) and the analog of (9) with $q, \hat{R}$ replaced by $q^{-1}, \hat{R}^{-1}$. These $\bar{\partial}$ derivatives can be expressed as functions of x, ∂ [11], see formula (29).

By definition a scalar $I(x, \partial) \in$ Diff($\mathbf{R}_q^N$) transforms trivially under the coaction associated to the quantum group of symmetry $SO_q(N, \mathbf{R})$ [2]. Any scalar polynomial $I(x, \partial) \in$ Diff($\mathbf{R}_q^N$) of degree $2p$ in x, ∂ is a combination of terms of the form

$$I = (\eta_{\varepsilon_1})^{i_1} (\eta_{\varepsilon_2})^{i_2} \ldots (\eta_{\varepsilon_p})^{i_p} (\eta_{\varepsilon'_p})_{i_p} \ldots (\eta_{\varepsilon'_2})_{i_2} (\eta_{\varepsilon'_1})_{i_1} \,, \tag{16}$$

where $\varepsilon_i, \varepsilon'_j = +, -, \eta_+ := x$ and $\eta_- := \partial$. From here we see that no polynomial of odd degree in η_ε^i can be a scalar. One can show that any scalar polynomial $I(x, \partial)$ can be expressed as an ordered polynomial in two particular scalar variables (see for instance Appendix C of [12]), namely the square length $x \cdot x$ and the laplacian $\partial \cdot \partial$, which are defined in formulae (20) below.

We will use two types of q-deformed integers:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad (n)_q := \frac{q^n - 1}{q - 1} \,; \tag{17}$$

both $[n]_q$ and $(n)_q$ go to n when $n \rightarrow 1$.

2.1. Some Explicit Formulae in Terms of x, ∂ Generators

For any "vectors" $a := (a^i)$, $b := (b^i)$ let us define

$$(a * b)_j := \sum_{l=1}^{j} a^{-l} b_{-l} + \begin{cases} \frac{q}{q+1} a^0 b_0 & \text{if } N \text{ odd} \\ 0 & \text{if } N \text{ even} \end{cases}, \qquad 0 \le j \le n, \tag{18}$$

$$\mathscr{B}^j(a,b) := a^j b^{-j} - a^{-j} b^j - (q^2-1) q^{-\varrho_j - 2} (a * b)_{j-1}, \qquad j \ge 1, \tag{19}$$

$$(1 + q^{-2\varrho_j})(a \cdot b)_j := \sum_{l=-j}^{j} a^l b_l, \qquad 0 < j \le n \tag{20}$$

(when this causes no confusion we will also use the notation $a \cdot b := (a \cdot b)_n$). Then it is easy to verify that

$$(a \cdot b)_j = (a * b)_j + \frac{\sum_{l=1}^{j} \mathscr{B}^l(a,b) q^{-\varrho_l}}{1 + q^{-2r_j}}. \tag{21}$$

Note that the preceding four formulae make sense for any $n \ge j$ and do not formally depend on n.

Relations (8), (9) defining Diff($\mathbf{R}_q^N$) amount respectively to

$$x^i x^j = q x^j x^i, \qquad \partial^i \partial^j = q \partial^j \partial^i, \qquad i < j, \tag{22}$$

$$\mathscr{B}^i(x,x) = 0, \qquad \mathscr{B}^i(\partial,\partial) = 0, \qquad i = 1, 2, \ldots, n, \tag{23}$$

and

$$\begin{cases} \partial_k x^j = q x^j \partial_k - (q^2-1) q^{-\varrho_j - \varrho_k} x^{-k} \partial_{-j}, & j < -k,\ j \neq k \\ \partial_k x^j = q x^j \partial_k & j > -k, j \neq k \\ \partial_{-k} x^k = x^k \partial_{-k}, & k \neq 0 \\ \partial_i x^i = 1 + q^2 x^i \partial_i + (q^2-1) \sum_{j>i} x^j \partial_j, & i > 0 \\ \partial_i x^i = 1 + q^2 x^i \partial_i + (q^2-1) \sum_{j>i} x^j \partial_j & \\ \quad - q^{-2\varrho_i} (q^2-1) x^{-i} \partial_{-i}, & i < 0 \\ \partial_0 x^0 = 1 + q x^0 \partial_0 + (q^2-1) \sum_{j>0} x^j \partial_j, & (\text{only for } N \text{ odd}). \end{cases} \tag{24}$$

Here are some useful formulae (sum over l is understood):

$$\partial^i (x \cdot x)_n = q^{2\varrho_n} x^i + q^2 (x \cdot x)_n \partial^i, \qquad (\partial \cdot \partial)_n x^i = q^{2\varrho_n} \partial^i + q^2 x^i (\partial \cdot \partial)_n, \tag{25}$$

$$\begin{aligned} (x^l \partial_l) x^i &= x^i + q^2 x^i x^l \partial_l + (1 - q^2)(x \cdot x)_n \partial^i, \\ \partial^i (x^l \partial_l) &= \partial^i + q^2 (x^l \partial_l) \partial^i + (1 - q^2) x^i (\partial \cdot \partial)_n. \end{aligned} \tag{26}$$

In [9, 11] the dilatation operator Λ_n

$$\Lambda_n(x,\partial) := 1 + (q^2-1) x^i \partial_i + q^{N-2} (q^2-1)^2 (x \cdot x)(\partial \cdot \partial) \tag{27}$$

was introduced; it fulfills the relations

$$\Lambda_n x^i = q^2 x^i \Lambda_n, \qquad \Lambda_n \partial^i = q^{-2} \partial^i \Lambda_n. \tag{28}$$

Then one can prove [11] that

$$\bar{\partial}^k = \Lambda_n^{-1} [\partial^k + q^{N-2} (q^2-1) x^k (\partial \cdot \partial)]. \tag{29}$$

In the sequel we will also need the operator

$$\mathscr{B}_n := 1 + q^{N-2}(q^2-1)(x\cdot\partial). \tag{30}$$

It is easy to show that it is the only operator of degree one in $x^i\partial^j$ satisfying the relations

$$\mathscr{B}_n(x\cdot x) = q^2(x\cdot x)B\,, \qquad \mathscr{B}_n(\partial\cdot\partial) = q^{-2}(\partial\cdot\partial)B_n\,. \tag{31}$$

Under complex conjugation

$$\mathscr{B}_n^* = q^{-N}\mathscr{B}\Lambda_n^{-1}, \qquad \Lambda_n^* = q^{-2N}\Lambda_n^{-1} \quad \text{if} \quad q\in\mathbf{R}^+. \tag{32}$$

In the sequel we will drop the index n in $\mathscr{B}_n, \Lambda_n$ when this causes no confusion.

2.2. *Decoupled Generators of* Diff($\mathbf{R}_g^N$)

In [9] it is shown that there exists a natural embedding of Diff($\mathbf{R}_q^{N-2}$) into Diff($\mathbf{R}_q^N$). In next sections we will see that it naturally induces an embedding of $U_q(so(N-2))$ into $U_q(so(N))$. We just need to do the change of generators of Diff($\mathbf{R}_q^N$) $(x^i,\partial^j)\to(X^i,D^j)$ $(|i|,|j|\le n)$, with

$$\begin{cases} x^i = \mu_n^{1/2}X^i\,, \quad \partial_i = \mu_n^{1/2}D_i\,, \quad |i|<n \\ x^n = X^n\,, \quad \partial_n = D_n \\ x^{-n} = \Lambda_n^{1/2}\mu_n^{\frac{-1}{2}}X^{-n} - q^{-2-\varrho_n}(q^2-1)(X\cdot X)_{n-1}D_n \\ \partial_{-n} = q^{-1}\Lambda_n^{1/2}\mu_n^{\frac{-1}{2}}D_{-n} - q^{-2-\varrho_n}(q^2-1)X^n(D\cdot D)_{n-1} \end{cases} \tag{33}$$

and

$$\mu_n := \mu(X^n,D_n) := D_nX^n - X^nD_n = 1+(q^2-1)X^nD_n\,. \tag{34}$$

Then the variables X^i, D^j, $(|i|,|j|\le n-1)$ satisfy the commutation and derivation relations (8), (9) for Diff($\mathbf{R}_q^{N-2}$), whereas

$$\begin{gathered} [X^{\pm n},X^i]=0\,, \quad [X^{\pm n},D_i]=0\,, \quad [D_{\pm n},X^i]=0\,, \quad [D_{\pm n},D_i]=0\,, \\ [D_{\pm n},X^{\mp n}]=0\,, \quad [D_n,D_{-n}]=0\,, \quad [X^n,X^{-n}]=0\,, \end{gathered} \tag{35}$$

and

$$D_nX^n = 1+q^2X^nD_n\,, \qquad D_{-n}X^{-n} = 1+q^{-2}X^{-n}D_{-n}\,. \tag{36}$$

As a direct consequence of the previous relations, μ_n commutes with all the X,D variables, except X^n, D_n themselves:

$$\mu_nX^n = q^2X^n\mu_n\,, \qquad \mu_nD_n = q^{-2}D_n\mu_n\,. \tag{37}$$

The dilatation operator Λ_n in terms of X^i, D^j variables reads

$$\Lambda_n(x,\partial) = \Lambda_{n-1}(X,D)\mu_n\mu_{-n}\,,$$

where $\mu_{-n} := (D_{-n}X^{-n} - X^{-n}D_{-n})^{-1}$ and $\Lambda_{n-1}(X,D)$ depends only on X^i, D_j $(|i|,|j|\le n-1)$ as dictated by formula (27) (after the replacement $n\to n-1$).

For odd N it is convenient to start the chain of embeddings from the "differential algebra of the quantum line" Diff($\mathbf{R}_q^1$) generated by x^0,∂_0 satisfying the relation

$$\partial_0x^0 = 1+qx^0\partial_0\,. \tag{38}$$

For N even, it is convenient to start the chain from the differential algebra Diff($\mathbf{R}_q^2$) of two commuting quantum lines; it is generated by the four variables $x^{\pm 1}, \partial_{\pm 1}$ all commuting with each-other, except for the relations

$$\partial_{\pm 1} x^{\pm 1} = 1 + q^2 x^{\pm 1} \partial_{\pm 1} \,. \tag{39}$$

Summing up, we have the two chains of embeddings

$$\mathrm{Diff}(\mathbf{R}_q^h) \hookrightarrow \mathrm{Diff}(\mathbf{R}_q^{h+2}) \hookrightarrow \mathrm{Diff}(\mathbf{R}_q^{h+4}) \hookrightarrow \dots \,, \tag{40}$$

where $h = \begin{cases} 1 & \text{for odd } N\text{'s} \\ 2 & \text{for even } N\text{'s} \end{cases}$.

From the abovementioned embeddings, it trivially follows the important

Proposition 1.

$$\begin{aligned} & F(x^i, \partial_j) = 0 \quad in \quad \mathrm{Diff}(\mathbf{R}_q^{N-2}) \\ & \Rightarrow F(X^i(x^h, \partial_k), D_j(x^h, \partial_k)) = 0 \quad in \quad \mathrm{Diff}(\mathbf{R}_q^N) \,, \end{aligned} \tag{41}$$

with $|i|, |j| \leq n-1$, $|h|, |k| \leq n$. In the LHS *x^i, ∂_j are the (x, ∂)-type generators for* Diff($\mathbf{R}_q^{N-2}$, *in the* RHS *X^i, D_j and x^h, ∂_k are respectively X, D- and (x, ∂)-type generators for* Diff($\mathbf{R}_q^N$)*, and F for our purposes will be some polynomial function in the variables $x, \partial, \mu_{n-1}^{\pm 1/2} \Lambda_{n-1}^{\pm 1/2}$.*

Let us introduce variables $\chi^i, \mathscr{D}_i, i \in \mathbf{Z}$, such that

$$\mathscr{D}_i \chi^i = 1 + a \chi^i \mathscr{D}_i \,, \qquad a = \begin{cases} q^2 & \text{if } i > 0 \text{ or } N \text{ even and } i = -1 \\ q & \text{if } i = 0 \\ q^{-2} & \text{otherwise} \end{cases} \,, \tag{42}$$

$$[\eta, \xi] = 0 \quad \text{if} \quad \eta = \chi^i, \mathscr{D}_i \,, \qquad \xi = \chi^j, \mathscr{D}_j \quad \text{with} \quad i \neq j \,. \tag{43}$$

By iterating the transformation (33) one arrives precisely at generators of Diff($\mathbf{R}_q^N$) of the type $\chi^i, \mathscr{D}_i$ with $|i| \leq n$ (and $i \neq 0$ when N is even), by identifying

$$X^{\pm n} = \chi^{\pm n} \,, \qquad D_{\pm n} = \mathscr{D}_{\pm n} \,, \qquad X^{n-1} = \chi^{n-1} \,, \qquad D_{n-1} = \mathscr{D}_{n-1} \cdots \,. \tag{44}$$

We generalize the definition (34) in the following way:

$$\begin{cases} (\mu_{\pm i})^{\pm 1} := \mathscr{D}_{\pm i} \chi^{\pm i} - \chi^{\pm i} \mathscr{D}_{\pm i} = 1 + (q^{\pm 2} - 1) \chi^{\pm i} \mathscr{D}_{\pm i} \\ \quad i > 0, \text{ except when } N \text{ even and } i = 1; \\ \mu_{\pm 1} := \mathscr{D}_{\pm 1} \chi^{\pm 1} - \chi^{\pm 1} \mathscr{D}_{\pm 1} = 1 + (q^2 - 1) \chi^{\pm 1} \mathscr{D}_{\pm 1} \\ \quad \text{when } N \text{ even;} \\ (\mu_0)^{1/2} := \mathscr{D}_0 \chi^0 - \chi^0 \mathscr{D}_0 = 1 + (q - 1) \chi^0 \mathscr{D}_0 = \mathscr{B}_0 \\ \quad \text{when } N \text{ odd.} \end{cases} \tag{45}$$

Consequently,

$$[\mu_i, \mu_j] = 0 \,,$$

$$\mu_i \chi^j = \chi^j \mu_i \cdot \begin{cases} q^2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \,, \qquad \mu_i \mathscr{D}_j = \mathscr{D}_j \mu_i \cdot \begin{cases} q^{-2} & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \,, \tag{46}$$

and $\Lambda_n = \prod_{i=-n}^{n} \mu_i$. In terms of X, D and $\chi, \mathscr{D}$ variables the square length $x \cdot x$ and the laplacian $\partial \cdot \partial$ take respectively the forms

$$\begin{aligned} x \cdot x &= \Lambda_n^{1/2} \mu_n^{-1/2} X^n X^{-n} q^{\varrho_n} + q^{-2} (X \cdot X)_{n-1} \\ &= \sum_{i=1}^{n} \Lambda_i^{1/2} \mu_i^{-1/2} \chi^i \chi^{-i} q^{\varrho_i - 2(n-i)} + \begin{cases} 0 & \text{if } N = 2n \\ \dfrac{q^{-2n+1}}{q+1} \chi^0 \chi^0 & \text{if } N = 2n+1 \end{cases}, \end{aligned} \tag{47}$$

$$\begin{aligned} \partial \cdot \partial &= \Lambda_n^{1/2} \mu_n^{-1/2} D^n D^{-n} q^{\varrho_n - 1} + q^{-2} (D \cdot D)_{n-1} \\ &= \sum_{i=1}^{n} \Lambda_i^{1/2} \mu_i^{-1/2} \mathscr{D}_i \mathscr{D}_{-i} q^{\varrho_i - 1 - 2(n-i)} \\ &\quad + \begin{cases} 0 & \text{if } N = 2n \\ \frac{q^{-2n+1}}{q+1} \mathscr{D}_0 \mathscr{D}_0 & \text{if } N = 2n+1 \,. \end{cases} \end{aligned} \tag{48}$$

3. The U.E.A. of the Angular Momentum on R_q^N

Inspired by the classical (i.e. $q = 1$) case, we give the following

Definition. *The universal enveloping algebra U_q^N of the angular momentum on $\mathbf{R}_q^N$ is the subalgebra of* Diff($\mathbf{R}_q^N$) *whose elements commute with any scalar $I(x, \partial) \in$* Diff($\mathbf{R}_q^N$). *Since any such I can be expressed as a function of the laplacian and of the square length $x \cdot x$, $\partial \cdot \partial$, our definition amounts to*

$$U_q^N := \{u \in \mathrm{Diff}(\mathbf{R}_q^N) : [u, x \cdot x] = 0 = [u, \partial \cdot \partial]\} \,. \tag{49}$$

In the next two subsections we consider two sets of generators of U_q^N (actually we will prove in Appendix B that any $u \in U_q^N$ can be expressed as a function of them). The generators of the first set transform in the same way as the products $x^i x^j$ under the coaction, since (up to a scalar) they are q-antisymmetrized products of x, ∂ variables, but have rather complicated commutation relations; nevertheless Casimirs have very compact expressions in terms of them. The generators of the second set have a quite simple form in terms of $\chi, \mathscr{D}$ variables and are much more useful for practical purposes, since they have simple commutation relations and are directly connected with the Cartan-Weyl generators of $U_{q^{-1}}(so(N))$.

3. The set of generators $\{L^{ij}, B\}$

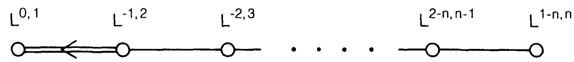
Keeping the classical case in mind, where the angular momentum components are antisymmetrized products $x^i \partial^j - x^j \partial^i$ of coordinates and derivatives, we try with the q-deformed antisymmetrized products

$$\mathscr{L}^{ij} := \mathscr{P}_{A\ hk}^{\ ij} x^h \partial^k = -q^{-2} \mathscr{P}_{A\ hk}^{\ ij} \partial^h x^k \,. \tag{50}$$

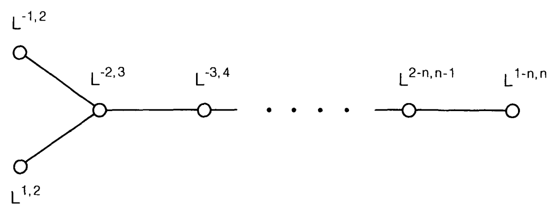
From relations (25), (8) it follows that

$$\mathscr{L}^{ij} x \cdot x = q^2 x \cdot x \mathscr{L}^{ij} , \qquad \mathscr{L}^{ij} \partial \cdot \partial = q^{-2} \partial \cdot \partial \mathscr{L}^{ij} . \tag{51}$$

Series B_n:



Series D_n.



This implies that $\mathscr{L}^{ij}$ commutes only with scalars having natural dimension $d = 0$. This shortcoming can be cured by introducing a scalar $S \in \mathrm{Diff}(\mathbf{R}_q^N)$ with natural dimension $d = 0$ and such that $Sx\cdot x = q^{-2}x\cdot xS$, $S\partial\cdot\partial = q^2\partial\cdot\partial S$; then by defining $L^{ij} := \mathscr{L}^{ij}S$ we get

$$[L^{ij}, I] = 0\,; \tag{52}$$

L^{ij} are therefore candidates to the role of angular momentum components. The simplest choice is to take $S = \Lambda^{-1/2}$, as we did in [12], and will be adopted in the sequel.

Starting from commutation relations for the $\mathscr{L}^{ij}$'s we get corresponding relations for the L^{ij}'s by multiplying them by a suitable power of $\Lambda^{-1/2}$. In fact, it is clear that the former must be homogeneous in $\mathscr{L}$'s to be consistent with (51). Nevertheless, commutation relations including factors such as $\mathscr{L}^{ij}\mathscr{L}^{-j,l}$ cannot be of this form. In fact, performing the derivations $\partial^{-j}x^j$ according to rules (9) one lowers by 1 the degree in $x\partial$ of some terms; this can be taken into account only by considering homogeneous relations both in $\mathscr{L}^{ij}$'s and $\mathscr{B}$ ($\mathscr{B}$ was defined in (30)), since $\mathscr{B}$ is the only other 1st degree polynomial in $x^i\partial_j$ with the same scaling law (51) as $\mathscr{L}^{hk}$. Summing up, we expect homogeneous commutation relations in the L^{ij}'s and $B := \mathscr{B}\Lambda^{-1/2}$. B is not really an independent generator, as we will see below. Therefore, the alternative choice $S := \mathscr{B}^{-1}$ (as considered in [13]) would yield the same algebra.

Remark 1. When $q = 1$, $\mathscr{B} = 1 = \Lambda$ and L^{ij} reduce to the classical "angular momentum" components, i.e. to generators of $U(so(N))$ (note that they are expressed as functions of the non-real coordinates x^i of $\mathbf{R^N}$ and of the corresponding derivatives). In this limit one can take as generators of the Cartan subalgebra the $L^{i,-i}$'s, as ladder operators corresponding to positive (resp. negative) roots the L^{jk}'s with $|j| < |k|$ and $k \geq 0$ (resp. $k \leq 0$), as ladder operators corresponding to simple roots the $L^{1-i,i}$'s together with $L^{j,2}$ $\left(i = 2,\ldots,n, \text{ and } j = \begin{cases} 0 & \text{if } N = 2n+1 \\ 1 & \text{if } N = 2n \end{cases}\right)$. A Chevalley basis is formed by the set of triples $\{(L^{1-i,i}, L^{-i,i-1}, L^{i,-i} - L^{i-1,1-i}), i = 1,\ldots,n\}$ if $N = 2n+1$ (here $L^{0,0} = 0$) and $\{(L^{1,2}, L^{-2,-1}, L^{2,-2} + L^{1,-1}), (L^{1-i,i}, L^{-i,i-1}, L^{i,-i} - L^{i-1,1-i}), i = 2,\ldots,n\}$ if $N = 2n$. The correspondence with spots in the Dynkin diagrams of the classical series $\mathbf{B_n}, \mathbf{D}_n$ is shown in Fig. 1.

Remark 2. One could work with $\bar{\partial}$ instead of ∂ derivatives and define $\bar{L}^{ij} := \mathscr{P}_A{}^{ij}_{hk} x^h \bar{\partial}^k \Lambda^{1/2}$, $\bar{B}_n := \bar{\mathscr{B}}_n \Lambda^{1/2}$, where $\bar{\mathscr{B}}_n := 1 + (q^{-2} - 1)(x \cdot \bar{\partial})$. But using formulae (15), (28), (32) one shows that $q^{-1}\bar{L}^{ij} = qL^{ij}$. In the language of [12] this means that the angular momentum in the barred and unbarred representation essentially coincide.

Instead of the N linearly dependent operators $\mathscr{L}^{-i,i}$ one can use their n linearly independent combinations

$$\mathscr{L}^i := \mathscr{A}^i(x, \partial), \qquad i = 1, 2, \ldots, n. \tag{53}$$

As for the operators $\mathscr{L}^{ij}$, $i \neq -j$, for simplicity we will renormalize them as follows:

$$\begin{aligned} \mathscr{L}^{ij} &:= (1+q^2)\mathscr{P}_A{}^{ij}_{hk} x^i \partial^j = (x^i \partial^j - q x^j \partial^i), \quad i < j, \\ \mathscr{L}^{ij} &= -q \mathscr{L}^{ji}, \quad i > j. \end{aligned} \tag{54}$$

The scalar $(L \cdot L)_n := L^{ij} L_{ji}$ commutes with any L^{ij} and reduces (up to a factor) to the classical square angular momentum when $q = 1$. We will call this casimir the (q-deformed) square angular momentum. Higher order Casimirs can be obtained by forming nontrivial independent scalars out of j-th powers ($j > 2$) of the L's,

$$(\underbrace{L \cdot L \cdot \ldots L}_{j \text{ times}})_n := L^{i_1 i_2} L_{i_2}{}^{i_3} L_{i_3}{}^{i_4} \ldots L_{i_j, i_1}, \tag{55}$$

for the same values of j as in the classical case.

Proposition 2. *The following important relation connects $\Lambda, \mathscr{B}$ and $\mathscr{L} \cdot \mathscr{L}$:*

$$\Lambda_n = (\mathscr{B}_n)^2 - \frac{(q^2-1)(q^2-q^{-2})}{(1+q^{2\varrho_n})(1+q^{-2\varrho_n-2})} (\mathscr{L} \cdot \mathscr{L})_n . \tag{56}$$

Proof. Using formulae (7), (6), (8) one can easily show [12] that

$$(\mathscr{L} \cdot \mathscr{L})_n = \alpha_N(q) x^i \partial_i + \beta_N(q) x^i x^j \partial_j \partial_i + \gamma_N(q) (x^i x_i)(\partial^i \partial_i), \tag{57}$$

where

$$\alpha_N(q) := \frac{(q^{2-\frac{N}{2}} + q^{\frac{N}{2}-2})(q^{1-N} - q^{N-1})}{(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1})(q^{-2} - q^2)} \tag{58}$$

$$\beta_N := \frac{q^3 + q^{N-1}}{(1+q^{2-N})(q+q^{-1})}, \qquad \gamma_N := -\frac{(q^{5-N}+q)(1+q^{-N})}{(1+q^{2-N})^2(q+q^{-1})}. \tag{59}$$

Performing derivations in $\mathscr{B}^2$ according to formula (26) we realize that the RHS of formula (56) gives Λ as defined in formula (27). □

As a consequence, B^2 is not an independent generator, as anticipated, but depends on $L \cdot L$.

When $q \in \mathbf{R}$ from formulae (14), (15), (28), (32) it follows that under complex conjugation

$$(L^{i,j})^* = q^{\varrho_i + \varrho_j} L^{-j,-i}; \tag{60}$$

this implies in particular that $L \cdot L$, the other casimirs and the L^i's are real. Moreover, it is easy to show that all the L^i's commute with each other, as the $\mathscr{L}^i$'s do.

The basic commutation relations between L^{ij}, B are quadratic in these variables but rather complicated and we won't give them here.

3.2. The Set of Generators $\{\mathbf{L}^{ij}, k^i\}_{i\neq j,-j}$

On the contrary, the new generators defined below admit very simple commutation relations, allowing a straightforward proof of the isomorphism $U_q^N \approx U_{q^{-1}}(so(N))$. It is convenient to use $\chi, \mathcal{D}$ variables to define and study them. The definitions of $\mathbf{L}^{ij}, \mathbf{k}^i$ involve only $\chi^l, \mathcal{D}_m$ variables with $|l|, |m| \leq J := \max\{|i|, |j|\}$, so that in terms of these variables it makes sense for any $n \geq J$. Hence, it trivially follows the embedding $U_q^N \hookrightarrow U_q^{N+2}$, since, as we will show in Appendix B, $\mathbf{L}^{ij}, \mathbf{k}^i$ generate U_q^N.

Proposition 3. *The elements*

$$\mathbf{k}^i := \mu_i \mu_{-i}^{-1} \in \mathrm{Diff}(\mathbf{R}_q^N), \qquad 0 < i \leq n \tag{61}$$

belong to U_q^N and commute with each other.

Proof. The thesis is a trivial consequence of formulae (46), (47), (48). □

We will call the subalgebra generated by $\mathbf{k}^i$ the "Cartan subalgebra" $H_q^N \subset U_q^N$.

In Appendix A we show that the elements $\mathbf{k}_i \in U_q^N$ can be expressed as functions of B, L^{ij}.

Now we define the generators $\mathbf{L}^{ij} \in U_q^N$, which correspond to roots. Since the generators of U_q^{N-2} belong also to U_q^N (in the sense of the abovementioned embedding), we can stick to the definition of the new generators, i.e. the ones belonging to $(U_q^N - U_q^{N-2})$. For this purpose it is convenient to use the X, D variables of $\mathrm{Diff}(\mathbf{R}_q^N)$.

Definition.

$$\left\{\begin{array}{l} \mathbf{L}^{ln} := q^{-2}\Lambda_n^{-1/2}\mu_{-n}[D^l, (X\cdot X)_{n-1}]D^n - \mu_n^{-1/2}X^n D^l\,, \\ \quad \text{(positive roots)} \\ \mathbf{L}^{-nl} := q^{-1}\Lambda_n^{-1/2}\mu_{-n}X^{-n}[(D\cdot D)_{n-1}, X^l] - \mu_n^{-1/2}X^l D^{-n}\,, \\ \quad \text{(negative roots)} \end{array}\right. \quad |l| < n\,. \tag{62}$$

In particular, it is easy to show that the complete list of generators corresponding to simple roots of U_q^N (i.e. the ones with indices as prescribed in Remark 1) in terms of $\chi, \mathcal{D}$ variables reads

$$\left\{\begin{array}{l} \mathbf{L}^{1-k,k} := \mu_k^{-1/2}[q^{2\varrho_k}(\mu_{-k}\mu_{k-1})^{1/2}\chi^{1-k}\mathcal{D}^k - \chi^k\mathcal{D}^{1-k}] \\ \quad j \leq k \leq n,\ j = \left\{\begin{array}{ll} 2 & \text{if } N = 2n+1 \\ 3 & \text{if } N = 2n \end{array}\right. \\ \mathbf{L}^{01} := (\mu_1)^{-1/2}[q^{-2}(\mu_{-1})^{1/2}\chi^0\mathcal{D}^1 - \chi^1\mathcal{D}^0] \\ \quad \text{if } N = 2n+1\,, \\ \mathbf{L}^{\pm 1,2} := \mu_2^{-1/2}[q^{-2}(\mu_{-2}\mu_{\mp 1})^{1/2}(\mu_{\pm 1})^{-1/2}\chi^{\pm 1}\mathcal{D}^2 - \chi^2\mathcal{D}^{\pm 1}] \\ \quad \text{if } N = 2n; \end{array}\right. \tag{63}$$

the list of corresponding negative Chevalley partners is given by

$$\begin{cases} \mathbf{L}^{-k,k-1} := \mu_k^{-1/2}[q^{2\varrho_k-1}(\mu_{-k}\mu_{k-1})^{1/2}\chi^{-k}\mathscr{D}^{k-1} - \chi^{k-1}\mathscr{D}^{-k}] \\ \quad 2 \le k \le n\,, \\ \mathbf{L}^{-1,0} = (\mu_1)^{-1/2}[\mu_{-1}^{1/2}\chi^{-1}\mathscr{D}^0 - \chi^0\mathscr{D}^{-1}] \\ \quad \text{if } N = 2n+1\,, \\ \mathbf{L}^{-2,\pm 1} := \mu_2^{-1/2}[q^{-1}(\mu_{-2}\mu_{\pm 1})^{1/2}(\mu_{\mp 1})^{-1/2}\chi^{-2}\mathscr{D}^{\pm 1} - \chi^{\pm 1}\mathscr{D}^{-2}] \\ \quad \text{if } N = 2n\,. \end{cases} \tag{64}$$

Note that when $N = 2n+1$ $L^{0\pm 1} = (\mathbf{k}^1)^{1/2}\mathbf{L}^{0\pm 1}$, when $N = 2n$ $L^{\pm 1,2} = (\mathbf{k}^2)^{1/2}(\mathbf{k}^1)^{\pm 1/2}\mathbf{L}^{\pm 1,2}$, $L^{-2,\pm 1} = q(\mathbf{k}^2)^{1/2}(\mathbf{k}^1)^{\mp 1/2}\mathbf{L}^{-2,\pm 1}$. In Appendix A we show that the simple roots and their Chevalley partners are functions of L^{ij}, B.

Proposition 4. $\mathbf{L}^{ln}, \mathbf{L}^{-n,l} \in U_q^N$.

Proof. In terms of X, D variables, formulae (47), (46), (42) yield

$$\begin{aligned} [\mathbf{L}^{ln}, (x\cdot x)_n] &= [q^{-2}\Lambda_n^{-1/2}\mu_{-n}[D^l,(X\cdot X)_{n-1}]D^n, \Lambda_n^{1/2}\mu_{-n}^{-1/2}X^nX^{-n}q^{\varrho_n}] \\ &\quad - [\mu_n^{-1/2}X^nD^l, q^{-2}(X\cdot X)_{n-1}] \\ &= q^{\varrho_n-2}\mu_{-n}\mu_n^{-1/2}[D^l,(X\cdot X)_{n-1}]\,[D^n, X^{-n}]X^n \\ &\quad - q^{-2}\mu_n^{-1/2}X^n[D^l,(X\cdot X)_{n-1}] = 0\,, \end{aligned} \tag{65}$$

and formulae (48), (46), (42) yield

$$\begin{aligned} [\mathbf{L}^{ln}, (\partial\cdot\partial)_n] &= -q^{\varrho_n}\mu_n^{-1}\Lambda_n^{1/2}[X^n, D^{-n}]_{q^{-2}}D^lD^n \\ &\quad + q^{-4}\Lambda_n^{-1/2}\mu_{-n}D^n[[D^l,(X\cdot X)_{n-1}],(D\cdot D)_{n-1}]_{q^{-2}} \\ &= q^{2\varrho_n-2}\mu_n^{-1}\Lambda_n^{1/2}D^lD^n \\ &\quad - q^{-6}\Lambda_n^{-1/2}\mu_{-n}D^n\left[D^l, \Lambda_{n-1}\frac{q^{4+2\varrho_n}}{q^2-1}\right] = 0 \end{aligned} \tag{66}$$

$\left(\text{here we have used the identity } [\partial\cdot\partial, x\cdot x]_{q^2} = \frac{q^{2+2\varrho_n}}{q^2-1}(\Lambda_n - q^{2\varrho_n-2})\right)$; namely $\mathbf{L}^{ln} \in U_q^N$. Similarly one proves that $\mathbf{L}^{-nl} \in U_q^N$. □

Lemma 1.

$$[\mathbf{L}^{hn}, \partial_n]_q = \partial^h\,, \qquad [\mathbf{L}^{-n,h}, x^n]_{q^{-1}} = -q^{\varrho_n}x^h\,, \qquad |h| < n\,, \tag{67}$$

$$[\partial^{n-1}, \mathbf{L}^{1-n,n}]_{q^{-1}} = q^{\varrho_n - 1}\partial^n\,, \qquad [\mathbf{L}^{-n,n-1}, x^{1-n}]_{q^{-1}} = q^{\varrho_l}x^{-n}\,, \qquad n > 1\,, \tag{68}$$

$$[\partial^0, \mathbf{L}^{01}] = q^{-1}\partial^1\,, \qquad [\mathbf{L}^{-10}, x^0] = x^{-1}\,, \qquad \text{if } N = 3\,. \tag{69}$$

Proof. For the proof see Proposition 11 of the next section and the remark following it. □

The following proposition allows to construct all the roots starting from the Chevalley ones.

Proposition 5. *The following relations hold in* Diff($\mathbf{R}_q^N$)*:*

$$[\mathbf{L}^{-jl}, \mathbf{L}^{-lk}]_q = q^{\varrho_l}\mathbf{L}^{-j,k}, \quad [\mathbf{L}^{-kl}, \mathbf{L}^{-l,j}]_q = q^{\varrho_l+1}\mathbf{L}^{-k,j}, \quad n \ge k > l > j \ge \begin{cases} 0 & \text{if } N = 2n+1 \\ -1 & \text{if } N = 2n \end{cases}, \tag{70}$$

$$[\mathbf{L}^{l-1,k}, \mathbf{L}^{1-l,l}]_{q^{-1}} = q^{\varrho_l - 1}\mathbf{L}^{lk}, \quad [\mathbf{L}^{-l,l-1}, \mathbf{L}^{-k,1-l}]_{q^{-1}} = q^{\varrho_l}\mathbf{L}^{-k,-l}, \quad 2 \le l < k \le n \tag{71}$$

$$[\mathbf{L}^{0k}, \mathbf{L}^{01}] = q^{-1}\mathbf{L}^{1k}, \quad [\mathbf{L}^{-10}, \mathbf{L}^{-k0}] = \mathbf{L}^{-k,-1}, \quad 1 < k \le n \quad \text{if} \quad N = 2n+1. \tag{72}$$

Proof. As an example we prove Eq. $(70)_1$. First consider the case $n = k$. We note that

$$[\mathbf{L}^{-jl}, \mathbf{L}^{-lk}]_q = q^{-2}\Lambda_k^{-1/2}\mu_{-k}[[\mathbf{L}^{-j,l}, D^{-l}]_q, (X\cdot X)_{k-1}]D^k - \mu_k^{-1/2}X^k[\mathbf{L}^{-j,l}, D^{-l}]_q, \tag{73}$$

as $[(X\cdot X)_{k-1}, \mathbf{L}^{-j,l}] = 0$. But

$$[\mathbf{L}^{-j,l}, D^{-l}]_q = D^{-j}q^{\varrho_l} \tag{74}$$

as a consequence of the preceding lemma and Proposition 1, therefore the RHS of Eq. $(70)_1$ gives $q^{\varrho_l}\mathbf{L}^{-jk}$. Applying Proposition 1 $(n-k)$ times we prove formula $(70)_1$ in the general case. The proofs of the other equations are similar. □

Proposition 6. *When* $q \in \mathbf{R}$,

$$(\mathbf{k}^i)^* = \mathbf{k}^i, \quad (\mathbf{L}^{1-k,k})^* = q^{-2}\mathbf{L}^{-k,k-1}, \quad k \ge 2, \quad \begin{cases} (\mathbf{L}^{01})^* = q^{-3/2}\mathbf{L}^{-10} & \text{if } N = 2n+1 \\ (\mathbf{L}^{12})^* = q^{-2}\mathbf{L}^{-2,-1} & \text{if } N = 2n \end{cases}. \tag{75}$$

Proof. The thesis can be proved by writing these $\mathbf{k}, \mathbf{L}$ generators in terms of the B, L ones as shown in Appendix A and by using the conjugation relations (60). □

The following three propositions give the basic commutation relations among the Chevalley generators. More relations for the other roots can be obtained from these ones using the relations of Proposition 5. In the following two propositions we assume that $k \ge \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases}$

Proposition 7.

$$[\mathbf{k}^i, \mathbf{L}^{\pm(1-k),\pm k}]_a = 0, \quad a = \begin{cases} q^{\pm 2} & \textit{if } i = k \le n \\ q^{\mp 2} & \textit{if } i = k-1 \\ 1 & \textit{otherwise} \end{cases}, \qquad [\mathbf{k}^i, \mathbf{L}^{\pm 1,\pm 2}]_a = 0, \quad a = \begin{cases} q^{\pm 2} & \textit{if } i = 1,2 \\ 1 & \textit{otherwise} \end{cases}. \tag{76}$$

Proof. A trivial consequence of formula (46) and of the definition of $\mathbf{L}, k$'s $\square$

Proposition 8. *(Commutation relations between positive and negative simple roots).*

$$[\mathbf{L}^{1-m,m}, \mathbf{L}^{-k,k-1}]_a = 0\,, \qquad a = \begin{cases} q^{-1} & m-1=k \\ 1 & \textit{if } m-1>k \end{cases}, \tag{77}$$

$$[\mathbf{L}^{-m,m-1}, \mathbf{L}^{1-k,k}]_a = 0\,, \qquad a = \begin{cases} q & \textit{if } m-1=k \\ 1 & \textit{if } m-1>k \end{cases}, \tag{78}$$

$$[\mathbf{L}^{12}, \mathbf{L}^{-2,1}] = 0\,, \qquad [\mathbf{L}^{-1,2}, \mathbf{L}^{-2,-1}] = 0\,, \quad \textit{if} \quad N=2n\,, \tag{79}$$

$$\begin{cases} [\mathbf{L}^{1-m,m}, \mathbf{L}^{-m,m-1}]_{q^2} = q^{1+2\varrho_m}\,\dfrac{1-\mathbf{k}^{m-1}(\mathbf{k}^m)^{-1}}{q-q^{-1}}\,, \quad 2\le m\le n \\ [\mathbf{L}^{01}, \mathbf{L}^{-1,0}]_q = q^{-1/2}\,\dfrac{1-(\mathbf{k}^1)^{-1}}{q-q^{-1}} \qquad \textit{if } N=2n+1 \\ [\mathbf{L}^{12}, \mathbf{L}^{-2,-1}]_{q^2} = q^{-1}\,\dfrac{1-(\mathbf{k}^2\mathbf{k}^1)^{-1}}{q-q^{-1}} \qquad \textit{if } N=2n\,. \end{cases} \tag{80}$$

Proof. Use Eqs. (42), (46) and perform explicit computations. $\square$

Proposition 9 *(Serre relations).*

$$[\mathbf{L}^{1-m,m}, \mathbf{L}^{1-k,k}] = 0\,, \qquad [\mathbf{L}^{-m,m-1}, \mathbf{L}^{-k,k-1}] = 0\,,$$
$$m,k>0,\ |m-k|>1\,, \tag{81}$$

$$[\mathbf{L}^{12}, \mathbf{L}^{1-j,j}] = 0\,, \qquad [\mathbf{L}^{-2,-1}, \mathbf{L}^{-j,j-1}] = 0\,,$$
$$j = 2,4,5,\ldots,n,\ N=2n\,, \tag{82}$$

$$[\mathbf{L}^{1+j-m,m-j}, \mathbf{L}^{2-m,m}]_a = 0 = [\mathbf{L}^{-m,m-2}, \mathbf{L}^{j-m,m-j-1}]_a\,,$$
$$a = \begin{cases} q & \textit{if } j=0 \\ q^{-1} & \textit{if } j=1 \end{cases}, \; m\ge 3\,, \tag{83}$$

$$\begin{cases} [\mathbf{L}^{01}, \mathbf{L}^{12}]_{q^{-1}} = 0 \\ \mathbf{L}^{-1,2}, \mathbf{L}^{02}]_q = 0 \end{cases}, \quad \begin{cases} [\mathbf{L}^{-2,-1}, \mathbf{L}^{-1,0}]_{q^{-1}} = 0 \\ [\mathbf{L}^{-2,0}, \mathbf{L}^{-2,1}]_q = 0 \end{cases}, \quad \textit{if} \quad N=2n+1\,, \tag{84}$$

$$\begin{cases} [\mathbf{L}^{12}, \mathbf{L}^{13}]_{q^{-1}} = 0 \\ [\mathbf{L}^{-23}, \mathbf{L}^{13}]_q = 0 \end{cases}, \quad \begin{cases} [\mathbf{L}^{-3,-1}, \mathbf{L}^{-2,-1}]_{q^{-1}} = 0 \\ [\mathbf{L}^{-3,-1}, \mathbf{L}^{-3,2}]_q = 0 \end{cases}, \quad \textit{if} \quad N=2n\,. \tag{85}$$

Proof. Use the definitions (62), commutation and derivation relations for the X, D variables, Eq. (37) and perform explicit computations. $\square$

We collect below all the basic commutation relations characterizing U_q^3, U_q^4. Their algebras read respectively

$$\begin{cases} [(\mathbf{k}^1)^{1/2}, L^{01}]_q = 0 \\ [(\mathbf{k}^1)^{1/2}, L^{-10}]_{q^{-1}} = 0 \\ [\mathbf{L}^{01}, \mathbf{L}^{-10}]_q = q^{-1/2}\,\dfrac{1-(\mathbf{k}^1)^{-1}}{q-q^{-1}} \end{cases} \tag{86}$$

and

$$\begin{cases} [(\mathbf{k}^1\mathbf{k}^2)^{1/2}, \mathbf{L}^{12}]_{q^2} = 0 \\ [(\mathbf{k}^1\mathbf{k}^2)^{1/2}, \mathbf{L}^{-2,-1}]_{q^{-2}} = 0 \\ [\mathbf{L}^{12}, \mathbf{L}^{-2,-1}]_{q^2} = q^{-1}\dfrac{1-\mathbf{k}^1\mathbf{k}^2}{q-q^{-1}} \end{cases}, \qquad \begin{cases} [((\mathbf{k}^1)^{-1}\mathbf{k}^2)^{1/2}, \mathbf{L}^{-1,2}]_{q^2} = 0 \\ [((\mathbf{k}^1)^{-1}\mathbf{k}^2)^{1/2}, \mathbf{L}^{-2,1}]_{q^{-2}} = 0 \\ [\mathbf{L}^{-12}, \mathbf{L}^{-2,1}]_{q^2} = q^{-1}\dfrac{1-(\mathbf{k}^1)^{-1}\mathbf{k}^2}{q-q^{-1}} \end{cases}, \tag{87}$$

$$[L, L'] = 0, \qquad L = \mathbf{L}^{12}, \mathbf{L}^{-2,-1}, (\mathbf{k}^1\mathbf{k}^2), \qquad L' = \mathbf{L}^{-12}, \mathbf{L}^{-21}, (\mathbf{k}^1)^{-1}\mathbf{k}^2. \tag{88}$$

We see that U_q^4 is the direct sum of two (commuting) identical algebras, (the ones in the L and L' generators respectively). This is no surprise, since it preludes to the relation $U_q^4 \approx U_q(so(4)) \approx U_q(su(2)) \otimes U_q(su(2))$, which we will prove in Sect. 4.

4. The Hopf Algebra Structure of U_q^N and its Identification

In this section we show that U_q^N is an Hopf algebra, more precisely that it is isomorphic to $U_{q^{-1}}(so(N))$.

A natural bialgebra structure can be associated to U_q^N for the reason that its elements satisfy some Leibnitz rule when acting as differential operators on Fun($\mathbf{R}_q^N$). A matched antipode can be found in a straightforward way, so that U_q^N acquires a Hopf algebra structure. As for the mentioned isomorphism, we will prove it by constructing an invertible transformation from the generators of U_q^N to those of $U_{q^{-1}}(so(N))$, in such a way that the commutation relations, coproduct, counit, antipode of U_q^N are mapped into the ones of $U_{q^{-1}}(so(N))$.

This means that the Hopf algebra $U_{q^{-1}}(so(N)$ admits a representation on all of Fun($\mathbf{R}_q^N$).

The Hopf algebra $U_q(so(N))$ [1,2] is generated by X_i^+, X_i^-, H_i $(i = 1, \ldots, n)$ satisfying the commutation relations

$$\begin{cases} [H_i, H_j] = 0, \qquad [H_i, X_j^{\pm}] = \pm(\alpha_i, \alpha_j)X_i^{\pm}, \\ [X_i^+, X_j^-] = \delta_{ij}\dfrac{q^{H_i} - q^{-H_i}}{q-q^{-1}} \\ \sum\limits_{t=1}^{m_{ij}} (-1)^t \begin{bmatrix} m_{ij} \\ t \end{bmatrix}_{q^i} (X_i^{\pm})^t X_j^{\pm} (X_i^{\pm})^{m_{ij}-t} = 0, \qquad i \neq j, \end{cases} \tag{89}$$

where

$$q^i = q^{(\alpha_i,\alpha_i)}, \qquad m_{ij} = 1 - \frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}, \qquad \begin{bmatrix} m \\ t \end{bmatrix}_q := \frac{[m]_q}{[t]_q[m-t]_q}, \tag{90}$$

and the $(n \times n)$ matrix of scalar products between the simple roots α_i is given by

$$\|b_{ij}\| := \|(\alpha_i, \alpha_j)\| = \begin{Vmatrix} 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{Vmatrix} \tag{91}$$

if $N = 2n+1$ and by

$$\|b_{ij}\| := \|(\alpha_i, \alpha_j)\| = \left\| \begin{array}{ccccccccc} 2 & & -1 & & & & & & \\ & 2 & -1 & & & & & & \\ -1 & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{array} \right\| \tag{92}$$

if $N = 2n$. Moreover, when $q \in \mathbf{R}$ they also satisfy the adjointness relations

$$H_i^\dagger = H_i, \qquad (X_i^+)^\dagger = X_i^- . \tag{93}$$

The coproduct, counit, antipode $\Phi_q, \varepsilon, \sigma_q$ are defined respectively by

$$\Phi_q(H_i) = \mathbf{1} \otimes H_i + H_i \otimes \mathbf{1}, \qquad \Phi_q(X_i^\pm) = X_i^\pm \otimes q^{-\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes X_i^\pm, \tag{94}$$

$$\varepsilon(X_i^\pm) = 0, \qquad \varepsilon(H_i) = 0, \tag{95}$$

$$\sigma_q(H_i) = -H_i, \qquad \sigma_q(X_i^\pm) = -q^{-\frac{(\alpha_i,\alpha_i)}{2}} X_i^\pm, \tag{96}$$

on the generators and extended as algebra homomorphisms/antihomomorphisms.

Now we show that there exist closed commutation relations between the generators of U_q^N and the coordinates x^i.

Proposition 10.

$$[\mathbf{k}^i, x^h]_a = 0, \qquad a = \begin{cases} q^2 & \text{if } h = i > 0 \\ q^{-2} & \text{if } h = -i < 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{in } \mathrm{Diff}(\mathbf{R}_q^N). \tag{97}$$

Proof. One has just to write x^h as functions of $\chi^j, \mathscr{D}_j$ and use relations (46). □

As for the commutation relations between roots $\mathbf{L}$'s and x^i, we write down only the ones involving simple roots and their opposite (the other ones can be obtained in the same way or using Proposition 5).

Proposition 11. *Let* $m \geq \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases}$. *Then*

$$[\mathbf{L}^{1-m,m}, x^i]_a = 0, \qquad a = \begin{cases} 1 & \text{if } |i| < m-1 \text{ or } |i| > m \\ q^{-1} & \text{if } i = 1-m, m \end{cases}, \tag{98}$$

$$[\mathbf{L}^{-m,m-1}, x^i]_a = 0, \qquad a = \begin{cases} 1 & \text{if } |i| - 1 \text{ or } |i| > m \\ q & \text{if } i = -m, m-1 \end{cases}, \tag{99}$$

$$[\mathbf{L}^{1-m,m}, x^{m-1}]_q = -q^{\varrho_m} x^m, \qquad [\mathbf{L}^{1-m,m}, x^{-m}]_q = q^{\varrho_m} x^{1-m}, \tag{100}$$

$$[\mathbf{L}^{-m,m-1}, x^m]_{q^{-1}} = -q^{\varrho_m} x^{m-1}, \qquad [\mathbf{L}^{-m,m-1}, x^{1-m}]_{q^{-1}} = q^{\varrho_m} x^{-m}, \tag{101}$$

$$[\mathbf{L}^{01}, x^0] = -q^{-1} x^1, \qquad [\mathbf{L}^{-1,0}, x^0] = x^{-1}, \quad \text{if} \quad N = 2n+1, \tag{102}$$

$$\begin{cases} [\mathbf{L}^{12}, x^1]_{q^{-1}} = 0 \\ [\mathbf{L}^{12}, x^2]_{q^{-1}} = 0 \\ [\mathbf{L}^{12}, x^{-1}]_q = -q^{-1}x^2 \\ [\mathbf{L}^{12}, x^{-2}]_q = q^{-1}x^1 \end{cases}, \quad \begin{cases} [\mathbf{L}^{-2-1}, x^{-1}]_q = 0 \\ [\mathbf{L}^{-2-1}, x^{-2}]_q = 0 \\ [\mathbf{L}^{-2-1}, x^1]_{q^{-1}} = q^{-1}x^{-2} \\ [\mathbf{L}^{-2-1}, x^2]_{q^{-1}} = -q^{-1}x^{-1} \end{cases} \quad \text{if} \quad N = 2n \,. \tag{103}$$

Proof. One has just to write x^h as functions of $\chi^j, \mathscr{D}_j$ and use relations (42), (46). $\square$

The commutation relations of $\mathbf{k}^i, \mathbf{L}^{ij}$'s with $\partial_i, \bar{\partial}_i$ are the same, since $\partial_i \propto [\partial \cdot \partial, x_i]_{q^2}$, $\bar{\partial} \propto [\bar{\partial} \cdot \bar{\partial}, x_i]_{q^{-2}}$ and the $\mathbf{k}, \mathbf{L}$'s commute with scalars. The knowledge of the latter commutation relations will allow us to construct the inhomogeneous extension of $U_q^N \approx U_{q^{-1}}(so(N))$, i.e. the universal enveloping algebra of the quantum Euclidean group, adding derivatives as new generators [6].

We can consider $\mathbf{L}^{ij}$'s, $\mathbf{k}^i$'s as differential operators on Fun($\mathbf{R}_q^N$) in the same way as we did in Sect. 2 with ∂_i. The commutation relations (97)–(103) allow us to define iteratively their evaluations and Leibnitz rules starting from

$$\mathbf{k}^1 \mathbf{1}| = 1 \,, \qquad \mathbf{L}^{ij} \mathbf{1}| = 0 \tag{104}$$

($\mathbf{1}$ denotes the unit of Fun($\mathbf{R}_q^N$)). For instance, by applying $\mathbf{k}^i$ to x^h, using Eq. (97) and the previous relation we find

$$\mathbf{k}^i x^h| = x^h \cdot \begin{cases} q^{\pm 2} & \text{if } \pm h = i > 0 \\ 1 & \text{otherwise} \end{cases}; \tag{105}$$

by applying $\mathbf{k}^i$ to $x^h \cdot g$ ($g \in$ Fun($\mathbf{R}_q^N$)) and using again Eq. (97) we find

$$\mathbf{k}^i (f \cdot g)| = \mathbf{k}^i f | \mathbf{k}^i g| \tag{106}$$

for $f = x^h$ first, and then by recurrence for any $f \in$ Fun($\mathbf{R}_q^N$). The latter relation is the Leibnitz rule for $\mathbf{k}^i$. Equations (104), (105), (106) are equivalent to (97), (104) and determine the evaluation of $\mathbf{k}^i$ on all of Fun($\mathbf{R}_q^N$). Similarly the Leibnitz rule for the simple roots is determined to be

$$\mathbf{L}^{1-m,m}(f \cdot g)| = \mathbf{L}^{1-m,m} f | g + (k^{m-1}(k^m)^{-1})^{1/2} f | \mathbf{L}^{1-m,m} g| \,, \qquad m \geq \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases}, \tag{107}$$

$$\mathbf{L}^{12}(f \cdot g)| = \mathbf{L}^{12} f | g + (\mathbf{k}^1 \mathbf{k}^2)^{-1/2} f | \mathbf{L}^{12} g| \quad \text{if} \quad N = 2n \tag{108}$$

($f, g \in$ Fun($\mathbf{R}_q^N$)), and the same formulae hold by replacing each simple root by its negative partner.

More abstractly, the above formulae define: 1) a counit $\varepsilon : U_q^N \to \mathbf{C}$, by setting $\varepsilon(u) := \pi(u\mathbf{1}|)$, $u \in U_q^N$ and $\pi(\alpha \mathbf{1}) := \alpha \, \forall \alpha \in \mathbf{C}$, implying that ε is an homomorphism which on the generators $\mathbf{k}^i, \mathbf{L}^{ij}$ takes the form

$$\varepsilon(\mathbf{L}^{ij}) = 0 \,, \qquad \varepsilon(\mathbf{k}^i) = 1 \,, \tag{109}$$

2) a coassociative coproduct $\phi: U_q^N \to U_q^N \otimes U_q^N$ which on the generators $\mathbf{k}^i, \mathbf{L}^{ij}$ takes the form

$$\phi(\mathbf{k}^i) = \mathbf{k}^i \otimes \mathbf{k}^i , \tag{110}$$

$$\begin{cases} \phi(\mathbf{L}^{1-m,m}) = \mathbf{L}^{1-m,m} \otimes \mathbf{1}' + (\mathbf{k}^{m-1}(k^m)^{-1})^{1/2} \otimes \mathbf{L}^{1-m,m} \\ \phi(\mathbf{L}^{-m,m-1}) = \mathbf{L}^{-m,m-1} \otimes \mathbf{1}' + (k^{m-1}(k^m)^{-1})^{1/2} \otimes \mathbf{L}^{-m,m-1} \end{cases}, \quad m \geq \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases}, \tag{111}$$

$$\begin{cases} \phi(\mathbf{L}^{12}) = \mathbf{L}^{12} \otimes \mathbf{1}' + (\mathbf{k}^1\mathbf{k}^2)^{-1/2} \otimes \mathbf{L}^{12} \\ \phi(\mathbf{L}^{-2,-1}) = \mathbf{L}^{-2,-1} \otimes \mathbf{1}' + (\mathbf{k}^1\mathbf{k}^2)^{-1/2} \otimes \mathbf{L}^{-2,-1} \end{cases}, \quad \text{if} \quad N = 2n \tag{112}$$

($\mathbf{1}'$ here denotes the unit of Diff($\mathbf{R}_q^N$), which acts as the identity when considered as an operator on Fun($\mathbf{R}_q^N$), and $\mathbf{k}^0 \equiv \mathbf{1}'$), and is extended to all of U_q^N as an homomorphism. ε, ϕ are matched so as to form a bialgebra; in particular the coassociativity of ϕ follows from the associativity of the Leibnitz rule, which in turn is a consequence of the associativity of Diff($\mathbf{R}_q^N$).

An antipode σ which is matched with ϕ, ε, (i.e. satisfies all the required axioms) is found by first imposing the two basic axioms

$$m \circ (\sigma \otimes id) \circ \phi = m \circ (id \otimes \sigma) \circ \phi = i \circ \varepsilon \tag{113}$$

on the generators of U_q^N, and then by extending it as an antihomomorphism; here m denotes the multiplication in U_q^N and i is the canonical injection $i: \mathbf{C} \to U_q^N$. Computations are straightforward:

$$\sigma(\mathbf{k}^i) = (\mathbf{k}^i)^{-1} , \tag{114}$$

$$\begin{cases} \sigma(\mathbf{L}^{1-m,m}) = -(\mathbf{k}^m(\mathbf{k}^{m-1})^{-1})^{1/2}\mathbf{L}^{1-m,m} \\ \sigma(\mathbf{L}^{-m,m-1}) = -(\mathbf{k}^m(\mathbf{k}^{m-1})^{-1})^{1/2}\mathbf{L}^{-m,m-1} \end{cases}, \quad m \geq \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases} \tag{115}$$

$$\begin{cases} \sigma(\mathbf{L}^{12}) = -(\mathbf{k}^1\mathbf{k}^2)^{-1/2}\mathbf{L}^{12} \\ \sigma(\mathbf{L}^{-2,-1}) = -(\mathbf{k}^1\mathbf{k}^2)^{-1/2}\mathbf{L}^{-2,-1} \end{cases} \quad \text{if} \quad N = 2n . \tag{116}$$

Finally, when $q \in \mathbf{R}$ it is straightforward to check that the complex conjugation $*$ (the antilinear involutive antihomomorphism defined in Sect. 3, which acts on the basic generators as shown in formula (75)) is compatible with the Hopf algebra structure of U_q^N, so that U_q^N gets a $*$-Hopf algebra.

Now it is easy to identify the Hopf algebra U_q^N.

Proposition 12. *All the relations characterizing the* $(*)$*-Hopf algebra* $U_{q^{-1}}(so(N))$ *are mapped into the ones characterizing the* $(*)$*-Hopf algebra* U_q^N *through the transformation of generators*

$$[\mathbf{k}^i(\mathbf{k}^{i-1})^{-1}]^{1/2} = q^{H_i} , \qquad \mathbf{L}^{1-i,i} = q^{\varrho_i - 3/2} X_i^+ q^{-\frac{H_i}{2}} ,$$

$$\mathbf{L}^{-i,i-1} = q^{\varrho_i + 3/2} X_i^- q^{-\frac{H_i}{2}} ,$$

$\left(i \geq \begin{cases} 1 & \text{if } N = 2n+1 \\ 2 & \text{if } N = 2n \end{cases}, \mathbf{k}^0 \equiv 1\right)$, *and*

$$\begin{aligned} [\mathbf{k}^2(\mathbf{k}^1)]^{1/2} = q^{H_1}\,, \qquad \mathbf{L}^{1,2} = q^{-5/2} X_1^+ q^{-\frac{H_1}{2}}\,, \\ \mathbf{L}^{-2,-1} = q^{1/2} X_1^- q^{-\frac{H_1}{2}}\,, \qquad N = 2n\,. \end{aligned} \tag{117}$$

after setting $\Phi_{q^{-1}} = \phi$, $\sigma_{q^{-1}} = \sigma$ (or, alternatively, $\tau \circ \Phi_q = \phi$, $\sigma_q = \sigma$, τ being the permutation operator). In other words $U_q^N \approx U_{q^{-1}}(so(N))$.

Proof. Straightforward computations. □

Note that if we had defined the elements of U_q^N as differential operators acting on Fun($\mathbf{R}_q^N$) from the right (instead of from the left), we would have got the isomorphism $U_q^N \approx U_q(so(N))$.

As a concluding remark, the final lesson we learn is that the product in Fun($\mathbf{R}_q^N$) realizes the tensor product of representations of $U_{q^{-1}}(so(N))$, the Leibnitz rule satisfied by the differential operators of U_q^N realizes the corresponding coproduct, and the real structure of Diff($\mathbf{R}_q^N$) realizes the real structure of $U_{q^{-1}}(so(N))$.

5. Representations

Let us now look at U_q^N as an operator algebra over Fun($\mathbf{R}_q^N$). In other words we consider "evaluations" of its elements on Fun($\mathbf{R}_q^N$) as defined in the previous section. We look for its irreducible representations. Since $L \cdot L$ commutes with any L^{ij}, it is proportional to the identity matrix on the base space W of each of them.

As a first remark, we note that any W must consist of polynomials of fixed degree in x, as any $u \in U_q^N$ is a power series in the products $x^i \partial_j$. Of course, the degree of these polynomials must be the same, say k, also after factoring out all powers of $x \cdot x$, since $[u, x \cdot x] = 0$. One can easily realize (see [12]) that the subspace of Fun($\mathbf{R}_q^N$) satisfying these two requirements is

$$W_k := \mathrm{Span}_{\mathbf{C}}[\mathscr{P}_{k,S}{}^{j_1 \ldots j_k}_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}]\,, \qquad k \in \mathbf{N}\,, \tag{118}$$

and that W_k is an eigenspace of $L \cdot L$. Here $\mathscr{P}_{k,S}$ denotes the (q-deformed) k-symmetric (modulo trace) projector, which can be defined through

$$\mathscr{P}_{k,S} \mathscr{P}_{A\, i,(i+1)} = 0 = \mathscr{P}_{k,S} \mathscr{P}_{1\, i,(i+1)}\,, \qquad 1 \leq i \leq k-1\,, \tag{119}$$

where $\mathscr{P}_{A\, i,(i+1)} = (\otimes \mathbf{1})^{i-1} \otimes \mathscr{P}_A \otimes (\otimes \mathbf{1})^{n-i-1}$, etc. Hence $W \subset W_k$.

In particular the fundamental (vector) representation W_1 is spanned by the N independent vectors x^i.

Below we are going to see that the representations of U_q^N in W_k's are irreducible and of highest weight type. When $q = 1$ they reduce to the vector representations of $so(N)$.

As "ladder operators" corresponding to positive, negative, simple roots we take the ones indicated in Remark 1 for the case $q = 1$. Correspondingly,

Proposition 13. *The highest (respectively lowest) weight eigenvector is the vector* $u_k^n := (x^n)^k$ *(respectively* $(x^{-n})^k$*).* W_k *is generated by iterated application of negative (*resp. *positive) ladder operators and is an eigenspace of* $L \cdot L$ *with eigenvalue*

$$l_{k,N}^2 = [k]_q[k+N-2]_q \frac{(q^{\varrho_n+1}+q^{-\varrho_n-1})}{(q+q^{-1})(q^{\varrho_n}+q^{-\varrho_n})}\,. \tag{120}$$

Proof. Using the derivation rules (42) it is straightforward to show that all positive ladder operators $\mathbf{L}^{jk}$ annihilate $(x^n)^k$. Moreover, it is easy to show that this vector is an eigenvector of $L \cdot L$ (with eigenvalue l_k^2) and therefore belongs to W_k. This follows from the fact that it is an eigenvector of $x^l \partial_l$ (with eigenvalue $(k)_{q^2}$) and from formulae (56), (30). As already noted, the application of negative ladder operators then yields a space $W \subset W_k$. As known, $W = W_k$ when $q = 1$; but $\dim(W), \dim(W_k)$ are constant with q, therefore $W = W_k \, \forall q$. Similar one proves that $(x^{-n})^k$ is the lowest weight eigenvector. □

Let us consider the space of homogeneous polynomials of degree k

$$M_k := \mathrm{Span}_{\mathbf{C}}[x^{i_1} \ldots x^{i_k}]\,. \tag{121}$$

As a consequence of the definition of W_l, we are able to decompose M_k into irreducible representations of U_q^N (see [12]), just as in the case $q = 1$:

$$M_k = \bigoplus_{0 \le m \le \frac{k}{2}} W_{k-2m}(x \cdot x)^m\,. \tag{122}$$

Recall that $\dim(M_k) = \begin{pmatrix} N+k-1 \\ N-1 \end{pmatrix}$, therefore this formula allows to recursively find $\dim(W_k)$: $\dim(W_k) = \dim(M_k) - \dim(M_{k-2})$. The formula

$$\mathrm{Fun}(\mathbf{R}_q^N) = \bigoplus_{l=0}^{\infty} M_l = \bigoplus_{l=0}^{\infty} \bigoplus_{0 \le m \le \frac{l}{2}} W_{l-2m}(x \cdot x)^m\,, \tag{123}$$

gives the formal decomposition of $\mathrm{Fun}(\mathbf{R}_q^N)$ into irreducible vector representations of $U_{q^{-1}}(so(N))$. All of them are involved (infinitely many times), and therefore $\mathrm{Fun}(\mathbf{R}_q^N)$ can be called the base space of the "regular" vector representation of $U_{q^{-1}}(so(N))$, in analogy with the classical case.

When $q \in \mathbf{R}$, starting from the prescriptions $(u_k^n, u_k^n) := 1$, $u^\dagger := u^* \, \forall u \in U_q^N$, and using the commutation relations of U_q^N one can define an inner product $(\cdot,\cdot)$ in all of W_k.

Proposition 14. *The inner product* $(\cdot,\cdot)$ *is positive definite,* i.e. *the representations* W_k *are unitary (when* $q \in \mathbf{R}^+$*)* w.r.t. *it.*

Proof. In [12] (or [14]) the integration $\int$ over $\mathbf{R}_q^N$ satisfying Stoke's theorem was defined. According to it

$$(f,g) = c_k \int dV f^* g \varrho(x \cdot x)\,, \qquad f,g \in W_k\,. \tag{124}$$

Indeed, integrating by parts "border terms" vanish, and therefore taking the adjoint $u^\dagger$ of u w.r.t. this inner product amounts to taking its complex conjugate. In this formula

$\varrho(x \cdot x)$ denotes a "rapidly decreasing function" function of the square length such as the q-deformed gaussian $\exp_q(-ax \cdot x)$ and the normalization factor c_k is chosen so that $u_k^n, u_k^n) = 1$. But we have proved in [12, Lemma 7.3] that $(\cdot,\cdot)$ is positive definite. $\square$

According to the theory of representations of $U_{q^{-1}}(so(N))$, when $q \in \mathbf{R}$ the Cartan subalgebra generators H_i make up a complete set of commuting observables in W_k, $\forall k \geq 0$. The highest weight associated to W_k is the n-ple $(0, 0, \dots, 0, k)$ of eigenvalues of the n-ple of operators $(H_1, H_2, \dots, H_n)$ on u_k^n.

According to the commutation relations (76), a basis E_k of W_k consisting of eigenvectors of $(H_1, H_2, \dots, H_n)$ is obtained by considering all the independent vectors obtained by applying negative root operators to u_k^n.

For instance, in the case $N = 3$ the dimension of W_k is $2k + 1$ and

$$E_k := \{u_{k,h} := (\mathbf{L}^{-10})^{-k-h} u_k^1, h = -k, -k+1, \dots, k\}. \tag{125}$$

For any monomial $M(k, \{i\}) := x^{i_1} x^{i_2} \dots x^{i_k}$ define $t(M) := i_1 + i_2 + \dots i_k$. Looking at formulae (42) we realize that the effect of the action of $\mathbf{L}^{0\pm1}$ on any monomial $M_{\{i\}}$ is to give a combination of monomials M' with $t(M') = t(M) \pm 1$. Therefore $u_{k,h}$ is a combination of monomials M with $t(M) = h$.

The functions $(x \cdot x)^{\frac{-k}{2}} u(u \in E_k)$ will be said q-deformed spherical functions of degree k, since they reduce to the classical ones in the limit $q = 1$, when we express $x^i (x \cdot x)^{\frac{-1}{2}}$ in terms of angular coordinates.

6. Appendix A

In this appendix we show how to express the generators $k^i, \mathbf{L}^{ij}$ as functions of L^{ij}, B_n. One can easily check that this map is invertible.

We first introduce some useful combinations F of the L^i, B variables introduced in Subsect. 3.1.

Let us iteratively define objects $\mathscr{F}_n^l \in \mathrm{Diff}(\mathbf{R}_q^N)$, $\left(N = \begin{cases} 2n+1 & \text{for odd } N \\ 2n & \text{for even } N \end{cases} \text{as usually}\right)$ by

$$\mathscr{F}_l^{l+1} := \mathscr{B}_l\,, \qquad l \geq 0, \forall N \geq 2\,; \qquad \mathscr{F}_1^{-1} = \mu_{-1} \quad \text{if} \quad N = 2 \tag{126}$$

($\mathscr{B}_0 \equiv 1$ when $N = 2$),

$$\begin{aligned} \mathscr{F}_n^{l+1}(x, \partial) &:= \mu_n \mathscr{F}_{n-1}^{l+1}(X, D)\,, \qquad n > l \geq 0\,; \\ \mathscr{F}_n^{-1}(x, \partial) &:= \mu_n \mathscr{F}_{n-1}^{-1}(X, D) \quad \text{if} \quad N = 2n\,. \end{aligned} \tag{127}$$

Let $F_n^l := \mathscr{F}_n^l \Lambda^{-1/2}$. One easily checks that $F_n^l \in U_q^N$, more precisely

$$F_n^{l+1} = B_n + \frac{q^2 - 1}{1 + q^{-2\varrho_n}} \left[\sum_{j=l+1}^{n} L^j q^{-\varrho_j} - (q^2 - 1) \frac{(n-l)}{1 + q^{2\varrho_l}} \sum_{j=1}^{l} L^j q^{-\varrho_j} \right], \tag{128}$$

$$\begin{aligned} F_n^{-1} &= B_n + \frac{q^2 - 1}{1 + q^{-2\varrho_n}} \sum_{j=1}^{n} L^j q^{-\varrho_j} + (1 - q^2) L^1 \\ &= F_n^1 + (1 - q^2) L^1 = F_n^2 - \frac{q^2 - 1}{2} L^1\,. \end{aligned} \tag{129}$$

Next we define

$$K_n^{i+1} := (\Lambda_n)^{-1}\Lambda_i(\mu_{i+1})^2\dots(\mu_n)^2 \in \mathrm{Diff}(\mathbf{R}_q^N)\,, \qquad 0 \le i \le n-1\,, \tag{130}$$

and observe that

Proposition 15. *K_n^i's belong to U_q^N and*

$$K_n^{i+1} = (F_n^{i+1})^2 - \frac{q^2-1}{q^{2\varrho_i}+1}\,\frac{q^2-q^{-2}}{1+q^{-2\varrho_i-2}}\,(L\cdot L)_i\,. \tag{131}$$

Proof. From the definitions of Λ_l, μ_l the first part of the proposition immediately follows. Relation (131) is a consequence of formula (56) and of the definitions of the F's. □

Formulae (128), (129), (131) allow to express $\mathbf{k}_i$ as functions of L^{ij}, B after noting that

$$\mathbf{k}^i = K_n^i(K_n^{i+1})^{-1}\,, \qquad (K_n^{n+1} \equiv 1)\,. \tag{132}$$

As for the $\mathbf{L}$'s, we find the

Proposition 16.

$$\begin{cases} \mathbf{L}^{1-k,k} = (K_n^k)^{-1}\left[F_n^{k-1}L^{1-k,k} - \dfrac{q^2-1}{q^2+q^{-2-2\varrho_k}}\displaystyle\sum_{l=2-k}^{l=k-2} L^{1-k,l}L_l{}^k\right] \\ \mathbf{L}^{-k,k-1} = q^{-1}(K_n^k)^{-1}\left[L^{-k,k-1}F_n^{k-1} - \dfrac{q^2-1}{q^2+q^{-2-2\varrho_k}}\displaystyle\sum_{l=2-k}^{l=k-2} L^{-k,l}L_l{}^{k-1}\right] \\ 2 \le k \le n \end{cases} \tag{133}$$

and

$$\begin{cases} \mathbf{L}^{0\pm1} = (F_n^1)^{-1}L^{0\pm1} & \text{if } N = 2n+1 \\ \mathbf{L}^{\pm(1,2)} = (F_n^1)^{-1}L^{\pm(1,2)} & \text{if } N = 2n \end{cases}\,. \tag{134}$$

Proof. As an example we prove Eq. $(133)_1$. As usual, it is sufficient to prove the claim when $k = n$, and then use Proposition 1 to extend it to $n > k$. Inverting relation $(33)_4$ we get $D^n = q\Lambda^{-1/2}\mu_n^{1/2}[\partial^n + q^{-2-2\varrho_n}(q^2-1)X^n(D\cdot D)_{n-1}]$. Replacing this expression in the definition (62) of $\mathbf{L}^{1-n,n}$ and using the definition (27) for Λ_{n-1} we easily find

$$\begin{aligned} \Lambda_n\mathbf{L}^{1-n,n} &= \mu_{-n}\mu_n^{1/2}\{[D^{1-n},(X\cdot X)_{n-1}]\partial^n \\ &\quad - [D^{1-n},(1+q^{-2-2\varrho_n})(X\cdot D)_{n-1}]X^n\} \\ &= \mu_{-n}\mu_n^{1/2}\{\mu_{n-1}(q^{2\varrho_n+2}X^{1-n}\partial^n - X^nD^{1-n}) \\ &\quad + (1-q^{-2})\,[(X\cdot X)_{n-2}D^{1-n}\partial^n + X^{1-n}(D\cdot D)_{n-2}X^n] \\ &\quad - (q^2-1)\,(1+q^{-2\varrho_n-4})\,(X\cdot D)_{n-2}]X^nD^{1-n}\}\,; \end{aligned} \tag{135}$$

on the other hand, using the normalization (54) for $\mathscr{L}^{ij}$,

$$\frac{\sum_{l=2-n}^{n-2} \mathscr{L}^{1-n,l}\mathscr{L}_l^{\ n}}{1+q^{-2\varrho_n-4}} = (\partial\cdot x)_{n-2}x^{1-n}\partial^n - (x\cdot x)_{n-2}\partial^{1-n}\partial^n$$
$$- x^{1-n}(\partial\cdot\partial)_{n-2}x^n + q^2(x\cdot\partial)_{n-2}\partial^{1-n}x^n$$
$$= \mu_n^{3/2}\Big\{(X\cdot D)_{n-2}(q^2D^{1-n}X^n + q^{-2-2\varrho_n}X^{1-n}\partial^n)$$
$$+ \frac{q^2-q^{2\varrho_n+4}}{q^2-1}\,\mu_{n-1}X^{1-n}\partial^n$$
$$- (X\cdot X)_{n-2}D^{1-n}\partial^n - X^{1-n}(D\cdot D)_{n-2}X^n\Big\} \tag{136}$$

and

$$\mathscr{F}_n^{n-1}\mathscr{L}^{1-n,n} = \mu_n^{3/2}[1+(q^2-1)(X^{n-1}D_{n-1} + q^{-2\varrho_n-4}(X\cdot D)_{n-2}]$$
$$\times (X^{1-n}\partial^n - X^nD^{1-n}). \tag{137}$$

From the preceding three formulae we find that

$$\Lambda_n\mathbf{L}^{1-n,n} = \mu_{-n}(\mu_n)^{-1}\left[\mathscr{F}_n^{n-1}\mathscr{L}^{1-n,n} - \frac{q^2-1}{q^2+q^{-2-2\varrho_n}}\sum_{l=2-n}^{l=n-2}\mathscr{L}^{1-n,l}\mathscr{L}_l^{\ n}\right], \tag{138}$$

which is equivalent to the claim upon use of formula (132). □

Note that $K_n^1 = (F_n^1)^2$ both for odd and even N, and $(K_n^2)^2 = K_n^1(F^{-1})^2$ when $N=2n$. All K_n^i go to 1 in the limit $q\to 1$. Moreover, for $N=3F_1^1 = (\mathbf{k}^1)^{1/2}$ and for $N=4F_2^1 = (\mathbf{k}^1\mathbf{k}^2)^{1/2}$ $F_2^{-1} = (\mathbf{k}^1)^{-1/2}(\mathbf{k}^2)^{1/2}$.

7. Appendix B

Define

$$\begin{cases} \hat{\mathbf{L}}^{in} := X^iD^n - q^{-2-2\varrho_n}\mu_n^{1/2}\Lambda_n^{-1/2}[X^i,(D\cdot D)_{n-1}]X^n \\ \hat{\mathbf{L}}^{-n,i} := X^{-n}D^i - q^{-3-2\varrho_n}\Lambda_N^{-1/2}\mu_n^{-1/2}\mu_{-n}[D^i,(X\cdot X)_{n-1}]D^{-n} \end{cases} \quad |i|<n\,. \tag{139}$$

Lemma 2. $\hat{\mathbf{L}}^{in}, \hat{\mathbf{L}}^{-n,i} \in U_q^N$ *and can be easily expressed as simple functions of the* $\mathbf{L}, \mathbf{k}$*'s.*

Since $[\mathbf{k}^n,\chi^j]_a = 0 = [\mathbf{k}^n,\mathscr{D}_j]_b$ with some a,b, we can introduce a grading $p\in\mathbf{Z}$ in $\mathrm{Diff}(\mathbf{R}_q^N)$ and decompose the latter as follows:

$$\mathrm{Diff}(\mathbf{R}_q^N) = \bigoplus_{p\in\mathbf{Z}}\mathrm{Diff}^p \quad \text{where} \quad \mathbf{k}^n\,\mathrm{Diff}^p := q^{2p}\,\mathrm{Diff}^p\,\mathbf{k}^n\,; \tag{140}$$

note that for each monomial $M(\chi,\mathscr{D}) := (\chi^n)^l(\chi^{-n})(\mathscr{D}_n)^s(\mathscr{D}_{-n})^r$

$$p(M) = l+r-m-s\,. \tag{141}$$

Decomposition (140) induces the decomposition $U_q^N = \bigoplus_{p\in\mathbf{Z}} U_q^N \cap \mathrm{Diff}^p$.

Now we can sketch the proof of the main theorem of this appendix.

Proposition 17.

$$u \in U_q^N \Rightarrow u = u(\mathbf{k}^i, jk)\,, \qquad i = 1, \ldots, n,\ |j|, |k| \leq n\,. \tag{142}$$

Moreover

$$f \in \mathrm{Diff}(\mathbf{R}_q^N)\colon \left[f, \begin{Bmatrix} x \cdot x \\ \partial \cdot \partial \end{Bmatrix}\right] = 0 \Rightarrow f = \sum_l u_l \begin{Bmatrix} f_l(x) \\ f_l(\partial) \end{Bmatrix}, \qquad u_l \in U_q^N\,. \tag{143}$$

Sketch of the Proof. As a preliminary remark, let us recall that $[\Lambda_n, u] = 0$, namely u has natural dimension zero. Our proof will be by induction in n. It is easy to prove that $U_q^1 = \mathbf{1} \cdot \mathbf{C}$, and that U_q^2 is generated by $\mathbf{k}^1$. Now assume that the thesis is true for U_q^{N-2}.

The most general $u \in \mathrm{Diff}(\mathbf{R}_q^N)$ can be written in the form

$$\begin{aligned} u = \sum_{l,m=0}^{\infty} &\{(\chi^{-n})^l (\mu_n^{-1/2}\chi^n)^m v_{l,m}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \\ &+ (\mu_n^{-1/2}\mathscr{D}_n)^l (\mathscr{D}_{-n})^m v_{-l,-m}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \\ &+ (\mathscr{D}_{-n})^l (\mu_n^{-1/2}\chi^n)^m v_{-l,m}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \\ &+ (\chi^{-n})^l (\mu_n^{-1/2}\mathscr{D}_n)^m v_{l,-m}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) + |j| < n\,. \end{aligned} \tag{144}$$

In fact the dependence on powers of $\chi^{\pm n}\mathscr{D}_{\pm n}$ can be reabsorbed into the dependence on $\mu_{\pm n}$.

It is easy to realize that, if we impose the constraint that the natural dimension $d(u)$ of u is zero, formula (144) can be rewritten in the form

$$\begin{aligned} u = \sum_{\{l_i, l'_i\}} &(\mathbf{L}^{-n,1-n})^{l_{1-n}} \ldots (\mathbf{L}^{-n,n-1})^{l_{n-1}} (\mathbf{L}^{1-n,n})^{l'_{1-n}} \ldots (\mathbf{L}^{n-1,n})^{l'_{n-1}} \\ &\times \sum_{p=0}^{\infty} \left[\sum_{h=0}^{p} [(\mu_n^{-1/2}\chi^n)^h (\mathscr{D}_{-n})^{p-h} v^{p,h}_{\{l_i, l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \right. \\ &\left. + (\mu_n^{-1/2}\mathscr{D}_n)^h (\chi^{-n})^{p-h} v^{-p,-h}_{\{l_i, l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j)] \right], \end{aligned} \tag{145}$$

where $i = 1 - n, \ldots, n - 1$. We sketch the procedure which leads to this result. For each $\mu_n^{-1/2}\chi^n$ or χ^{-n} (respectively $\mathscr{D}^{-n}$ or $\mu_n^{-1/2}\mathscr{D}^n$) we can extract out of the corresponding coefficient function v a D^i (respectively a X^i) variable (since $d(u) = 0$) and replace the LHS's of the following identities by the RHS's (see the definitions (62)):

$$\begin{aligned} \mu_n^{-1/2}\chi^n D^i &= q^{-2}\Lambda_{n-1}\mathbf{k}_n^{-1/2}[D^i, (X \cdot X)_{n-1}]\mathscr{D}^n - \mathbf{L}^{in}\,, \\ \chi^{-n} D^i &= q^{-2}\Lambda_{n-1}\mathbf{k}_n^{-1/2}[D^i, (X \cdot X)_{n-1}]\mathscr{D}^n - \hat{\mathbf{L}}^{in}\,, \end{aligned} \tag{146}$$

$$\begin{aligned} \mu_n^{-1/2}\mathscr{D}^{-n} X^i &= q^{-1}\Lambda_{n-1}\mathbf{k}_n^{-1/2}[(D \cdot D)_{n-1}, X^i]\chi^{-n} - \mathbf{L}^{-n,i}\,, \\ \chi^{-n} D^i &= q^{-2}\Lambda_{n-1}\mathbf{k}_n^{-1/2}[D^i, (X \cdot X)_{n-1}]\mathscr{D}^n - \hat{\mathbf{L}}^{in}\,. \end{aligned} \tag{147}$$

Then each factor $\chi^n\mathscr{D}_n, \chi^{-n}\mathscr{D}_{-n}$ can be reabsorbed into the μ_n, μ_{-n}-dependence of the coefficient functions v's. Finally, we arrive at (145) using the result of Lemma 2 and the commutation relations of Sect. 3, which allow us to reorder all $\mathbf{L}, \mathbf{k}$'s according to the ordering shown in that formula.

Now we impose the conditions $[u, x \cdot x] = 0 = [u, \partial \cdot \partial]$ explicitly. They reduce to

$$\begin{cases} \left[\sum_{h=0}^{p} (\mu_n^{-1/2}\chi^n)^h (\mathscr{D}_{-n})^{p-h} v^{p,h}_{\{l_i,l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j), \begin{Bmatrix} x \cdot x \\ \partial \cdot \partial \end{Bmatrix} \right] = 0\,, \\ \left[\sum_{h=0}^{p} (\mu_n^{-1/2}\mathscr{D}_n)^h (\chi^{-n})^{p-h} v^{-p,-h}_{\{l_i,l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j), \begin{Bmatrix} x \cdot x \\ \partial \cdot \partial \end{Bmatrix} \right] = 0\,. \end{cases} \tag{148}$$

In fact the powers of $\mathbf{L}$'s appearing in formula (145) belong to a Poincaré basis of U_q^N, therefore are independent, and their coefficient functions can be split into components belonging to different subspaces Diff^p (140). Using a procedure which, for the sake of brevity, we describe only in the case $p = 1$, it is easy to show that from the latter equations it follows decompositions of the type

$$\begin{cases} \sum_{h=0}^{p} (\mu_n^{-1/2}\chi^n)^h (\mathscr{D}_{-n})^{p-h} v^{p,h}_{\{l_i,l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \\ \quad = \sum_{\{i_1,\dots i_p\}} f_{\{i_1,\dots i_p\}}(\mathbf{k}^n) u_{\{i_1,\dots i_p\}} \mathbf{L}^{i_1,n} \dots \mathbf{L}^{i_p,n}\,, \\ \sum_{h=0}^{p} (\mu_n^{-1/2} D_n)^h (\chi^{-n})^{p-h} v^{-p,-h}_{\{l_i,l'_i\}}(\mu_n, \mu_{-n}, \chi^j, \mathscr{D}_j) \\ \quad = \sum_{\{i_1,\dots i_p\}} f_{\{i_1,\dots i_p\}}(\mathbf{k}^n) u_{\{i_1,\dots i_p\}} \mathbf{L}^{-n,i_1} \dots \mathbf{L}^{-n,i_p}\,, \end{cases} \tag{149}$$

$u_{\{i_1,\dots i_p\}} \in U_q^{N-2}$, which completes the proof of formula (142). When $p = 1$, upon use of formulae (42), (46), it is easy to verify that the LHS's of Eqs. (148) are combination of $(\mu_n^{-1/2}\chi^n)^2\chi^{-n}, \mathscr{D}_{-n}, \mu_n^{-1/2}\chi^n$ and $\mu_n^{-1/2}\mathscr{D}_n(\mathscr{D}_{-n})^2$, $\mathscr{D}_{-n})^2, \mathscr{D}_{-n}, \chi^n$ respectively, and that setting their coefficients equal to zero amounts to

$$\begin{gathered} v^m = v^m(\mathbf{k}^n, \chi^j, \mathscr{D}_j)\,, \quad m = 0, 1,\ |j| < n\,, \\ [v^0, (X \cdot X)_{n-1}] = 0 = [v^1, (D \cdot D)_{n-1}]\,, \end{gathered} \tag{150}$$

$$v^0 = -q^{-2-2\varrho_n} \Lambda_{n-1}^{-1/2} (\mathbf{k}^n)^{-1/2} [v^1, (X \cdot X)_{n-1}]\,. \tag{151}$$

Hence

$$0 = [[v^1, (X \cdot X)_{n-1}], (X \cdot X)_{n-1}]_{q^2} = [[v^1, (X \cdot X)_{n-1}]_{q^2}, (X \cdot X)_{n-1}]\,, \tag{152}$$

implying upon use of recursion hypothesis (143), formula (25) and of relations $d(v^1) = 1$,

$$[(D \cdot D)_{n-1}, (X \cdot X)_{n-1}]_{q^2}(q^2 - 1) = q^{4+2\varrho_n}(\Lambda_{n-1} - q^{2\varrho_n})\,, \tag{153}$$

the equation

$$\begin{gathered} [v^1, (X \cdot X)_{n-1}]_{q^2} = u_i X^i\,, \\ u_i \in U_q^{N-2} \Rightarrow v^1 \propto \left[v^1, \frac{q^{4+2\varrho_n}(\Lambda_{n-1} - q^{2\varrho_n})}{q^2 - 1} \right]_{q^2} \propto u_i D^i\,. \end{gathered} \tag{154}$$

This yields $v^1(\mu_n^{-1/2}\chi^n) + v^0\mathscr{D}_{-n} \propto u_i \mathbf{L}^{in}$, as claimed.

The proof of (143) can be given recursively by constraining the general expansion (144) in a similar way. □

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