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Minor Identities for Quasi-Determinants and Quantum Determinants

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Abstract: We present several identities involving quasi-minors of noncommutative generic matrices. These identities are specialized to quantum matrices, yielding q-analogues of various classical determinantal formulas.

1. Introduction

A common feature of the algebraic constructions which originated from the quantum inverse scattering method is the systematic use of matrices T with noncommutative entries, obeying a relation of the form

$$RT_1T_2 = T_2T_1R ,$$

where the R-matrix is a solution of the Yang–Baxter equation [13, 20, 35]. The entries of the monodromy matrix T may be regarded as the generators of an associative algebra subject to the above relation. Many interesting examples of algebras arise in this way. Among them are $A_q(GL_n)$, the quantized algebra of functions on GL_n [35], the quantized universal enveloping algebra $U_q(gl_n)$ [13, 20, 35], the Yangian $Y(gl_n)$ [13, 33, 27] and the quantized Yangian $Y_q(gl_n)$ [8]. In each of these cases, an appropriate concept of *quantum determinant* can be defined [22, 21, 35] which is of fundamental importance in the description of the center of these algebras and their representation theory. For example the Drinfeld generators [14] of the Yangian $Y(gl_n)$ are given by some *quantum minors* of the T-matrix. These generators can be used to construct the Gelfand–Zetlin bases for certain irreducible representations of $Y(gl_n)$ [30, 26]. Moreover, it is shown in [30] that the Gelfand–Zetlin formulas for $U_q(gl_n)$ follow from certain algebraic identities satisfied by quantum minors of the T-matrix corresponding to the quantized Yangian $Y_q(gl_n)$. Another application of quantum determinants is the construction of a q-deformation of the coordinate ring of the Grassmannian and the flag manifold, whose basis consists in products of quantum minors of the T-matrix associated with the algebra $A_q[GL_n]$ [23, 38]. In this case, the quadratic relations satisfied by quantum minors can be used to establish an analogue of the classical straightening formula [6]. These examples suggest that it is an important task to explore these various quantum determinants and to investigate the algebraic relations between their minors.

In fact, the problem of defining the determinant of a matrix with noncommutative entries is an old one and can be traced back to Cayley [7]. An example of great significance in the classical representation theory is Capelli's determinant [5, 40, 19, 29, 32]. In the forties, Dieudonné proposed his famous definition of the determinant of a matrix over a noncommutative skew-field [12] which was subsequently used and extended by Sato and Kashiwara in the context of the theory of pseudo-differential operators [36]. Another interesting construction is that of Berezin who defined an analogue of the determinant for supermatrices [1]. However, it is only recently that Gelfand and Retakh initiated a completely different approach, introducing the *quasi-determinants* of a matrix with noncommutative entries [16, 17].

The most striking facts about quasi-determinants are the following: (1) a $n \times n$ matrix $A = (a_{ij})$ admits not only one but (in general) n^2 quasi-determinants related by the so-called homological relations; (2) the quasi-determinants of A are not polynomials but noncommutative rational functions of the a_{ij}; (3) in contrast to the Capelli determinant or to the various quantum determinants which only make sense for very particular matrices with entries obeying some specific commutation rules, quasi-determinants are defined for matrices with formal noncommutative entries, and can therefore be specialized to any matrix; (4) the Capelli determinant, the Dieudonné determinant, the Berezin determinant and the quantum determinants of $A_q[GL_n]$ and $Y(gl_n)$ can be expressed in a uniform way as products of commuting quasi-determinants.

The aim of this article is to demonstrate that the quasi-determinants of Gelfand and Retakh can be applied successfully to the important problem of describing the algebraic relations satisfied by the quantum minors of a monodromy matrix. To this end, we first investigate identities satisfied by *quasi-minors* of the generic noncommutative matrix, and then derive quantum determinantal identities by specializing them to a T-matrix. For simplicity, we only consider in this paper the T-matrix of the generators of the quantum group $A_q(GL_n)$, but the same technique applies also to the case of $Y(gl_n)$.

We emphasize that from our point of view, the generic quasi-minors identities are perhaps more important than their specializations to a given monodromy matrix. Indeed, they lend themselves to other applications, as illustrated by [15] where the same identities are used for studying noncommutative symmetric functions, Padé approximants and orthogonal polynomials. Noncommutative Padé approximants and orthogonal polynomials appear for instance in Quantum Field theory where they are used for computing rational approximations of perturbation series [2, 18].

The paper is organized as follows. Section 2.1 introduces the *free field*, which is the natural algebraic setting for dealing with quasi-determinants. Section 2.2 provides a self-contained introduction to quasi-determinants and their basic properties. It happens that quasi-determinants are closely related to the representation aspect of automata theory initiated by Schützenberger (see [3, 37]). The presentation we give here is influenced by this point of view. We describe then in Sect. 2.3 noncommutative analogues of several classical theorems, including Jacobi's theorem, Cayley's law of complementaries, Muir's law of extensible minors, Sylvester's theorem, Bazin's theorem and Schweins' series. Finally, these results are specialized in Sect. 3 to quantum minors of $A_q(GL_n)$, yielding quantum analogues of the same theorems.

2. Quasi-determinants

2.1. The Free Field. Let A be a set of noncommutative indeterminates. We denote by $\mathbf{Q}\langle A\rangle$ the free associative algebra generated by A over $\mathbf{Q}$. We wish to imbed $\mathbf{Q}\langle A\rangle$ in a field, called its universal field of fractions, or *free field.* In other words, the problem is to extend to noncommutative polynomials the classical construction of the field of fractions of a ring of commutative polynomials. There are different equivalent definitions of the free field due to Amitsur, Bergman and Cohn. Cohn's approach, which relies on the resolution of linear systems with coefficients in $\mathbf{Q}\langle A\rangle$, is the most closely related to the definition of quasi-determinants [9, 10]. We shall recall his construction in the case where $\mathbf{Q}$ is the ground field, the general case (where $\mathbf{Q}$ is replaced by an arbitrary field) being essentially the same.

From a categorical viewpoint, the problem may be formulated as follows. A $\mathbf{Q}\langle A\rangle$*-field* is a (skew) field K equipped with some ring morphism φ_K from $\mathbf{Q}\langle A\rangle$ into K such that K is the least field containing the image of φ_K. A *specialization* between two $\mathbf{Q}\langle A\rangle$-fields K, L is a ring homomorphism φ from a subring K_0 of K to L such that any element of K_0 which is not in the kernel of φ has an inverse in K_0. The class of fields does not form a category, for it is not possible to define a notion of morphism of fields due to the zero inverting problem. However, one can show that $\mathbf{Q}\langle A\rangle$-fields equipped with specializations form a category. Moreover, there is an initial object in this category which is exactly the so-called free field $\mathbf{Q}\{\!\!\{A\}\!\!\}$,

$$\begin{tikzcd} \mathbf{Q}\langle A\rangle \arrow[r, "\iota"] \arrow[dr, "\varphi_K"'] & \mathbf{Q}\{\!\!\{A\}\!\!\} \arrow[d, "\overline{\varphi_K}"] \\ & K \end{tikzcd}$$

In other words, for every $\mathbf{Q}\langle A\rangle$-field K, there is a unique specialization $\overline{\varphi_K}$ from $\mathbf{Q}\{\!\!\{A\}\!\!\}$ to K that extends φ_K.

More concretely, here is how Cohn constructs $\mathbf{Q}\{\!\!\{A\}\!\!\}$. A $n \times n$ matrix M with entries in $\mathbf{Q}\langle A\rangle$ is called a *full* matrix if it cannot be written as a product of an $n \times r$ by an $r \times n$ matrix where $r < n$. M is called *linear* if its entries have degree $\leqq 1$. Let Σ be the set of full linear matrices, and for each $n \times n$ matrix $M = (m_{ij})$ in Σ, take a set of n^2 symbols, arranged as an $n \times n$ matrix $M' = (m'_{ij})$. Define a ring by the presentation consisting of all the elements of $\mathbf{Q}\langle A\rangle$, as well as all the m'_{ij} as generators, and as defining relations take all the relations

$$M\,M' = M'\,M = I_n$$

for each M in Σ. This ring is none other than the free field $\mathbf{Q}\{\!\!\{A\}\!\!\}$. Using this construction one can show that any element x of $\mathbf{Q}\{\!\!\{A\}\!\!\}$ can be represented as

$$x = I\,M^{-1}T\,, \tag{1}$$

where $I = (1, 0, \ldots, 0)$ considered as an n-dimensional row vector, T is some column vector of $\mathbf{Q}^n$ and M is some element of Σ. This means that every element x of $\mathbf{Q}\{\!\!\{A\}\!\!\}$ is the first component of the solution X of some linear system of the form $MX = T$. Cohn and Reutenauer have recently shown the unicity (up to linear isomorphisms) of the representation of an element $x \in \mathbf{Q}\{\!\!\{A\}\!\!\}$ under the form (1) when the dimension n is minimal [11].

There is another interesting construction of the free field based on Malcev–Neumann series. This method provides a series expansion for every element of $\mathbf{Q}\langle\!\langle A\rangle\!\rangle$. We first recall a general construction related to ordered groups introduced independently by Malcev and Neumann [25,31].

Let $\leqq$ be some total order on a group G compatible with the group structure, which means that

$$g_1 \leqq h_1\,,\; g_2 \leqq h_2 \Longrightarrow g_1g_2 \leqq h_1h_2$$

for any $g_1,g_2,h_1,h_2 \in G$. A *Malcev–Neumann series* is a formal series over G whose support is well-ordered with respect to $\leqq$. Malcev–Neumann series can therefore be multiplied, and one can show that they form a field denoted by $\mathbf{Q}_M[[G]]$.

Consider now the free group $F(A)$ constructed over A. There are several classical ways of totally ordering $F(A)$, based on the fact that the successive quotients of the lower central series of $F(A)$ are free abelian groups [34]. Hence one can consider the Malcev–Neumann series field $\mathbf{Q}_M[[F(A)]]$. One can show that the subfield of $\mathbf{Q}_M[[F(A)]]$ generated by the group algebra $\mathbf{Q}[F(A)]$ is always (independently of the order $\leqq$ chosen on $F(A)$) isomorphic to $\mathbf{Q}\langle\!\langle A\rangle\!\rangle$.

Consider for instance the element $(ab-ba)^{-1}$ of $\mathbf{Q}\langle\!\langle a,b\rangle\!\rangle$. Choose an order on $F(a,b)$ such that $ba \leqq ab$. Then

$$1 \geqq bab^{-1}a^{-1} \geqq (bab^{-1}a^{-1})^2 \geqq \cdots,$$

and the expansion of $(ab-ba)^{-1}$ as a Malcev–Neumann series is

$$(ab-ba)^{-1} = (ab)^{-1}(1-bab^{-1}a^{-1})^{-1} = (ab)^{-1}\sum_{n=0}^{+\infty}(bab^{-1}a^{-1})^n\,.$$

2.2. Definition of Quasi-Determinants. We let now $A=\{a_{ij},\ 1\leqq i,j\leqq n\}$ be an alphabet of n^2 letters. The matrix $(a_{ij})_{1\leqq i,j\leqq n}$, also denoted by A, is called the *generic matrix* of order n. It is a full linear matrix, as well as all its submatrices. Therefore, all square submatrices of A are invertible in $\mathbf{Q}\langle\!\langle A\rangle\!\rangle$. Throughout the paper we shall use the following notation for submatrices. For P, Q subsets of $\{1,\dots,n\}$, we let A_{PQ} denote the submatrix whose row indices belong to P and column indices to Q, and $A^{PQ}=A_{\overline{P}\overline{Q}}$, where $\overline{P}=\{1,\dots,n\}\backslash P$ and $\overline{Q}=\{1,\dots,n\}\backslash Q$ are the set complements of P and Q. Consider a block decomposition of A

$$A = \begin{array}{c} \\ P \\ Q \end{array}\!\!\begin{array}{c} \begin{array}{cc} R & S \end{array} \\ \begin{pmatrix} A_{PR} & A_{PS} \\ A_{QR} & A_{QS} \end{pmatrix} \end{array},$$

and the corresponding decomposition of $A^{-1}=B=(b_{ij})$,

$$A^{-1}=B= \begin{array}{c} \\ R \\ S \end{array}\!\!\begin{array}{c} \begin{array}{cc} P & Q \end{array} \\ \begin{pmatrix} B_{RP} & B_{RQ} \\ B_{SP} & B_{SQ} \end{pmatrix} \end{array}.$$

Here we suppose that $|P|=|R|$ and $|Q|=|S|$ so that A_{PR}, A_{QS}, B_{RP}, B_{SQ} are square matrices. By block multiplication we get the classical relations

$$B_{RP} = (A_{PR} - A_{PS}A_{QS}^{-1}A_{QR})^{-1} , \tag{2}$$

$$B_{RQ} = -A_{PR}^{-1}A_{PS}(A_{QS} - A_{QR}A_{PR}^{-1}A_{PS})^{-1} , \tag{3}$$

$$B_{SP} = -A_{QS}^{-1}A_{QR}(A_{PR} - A_{PS}A_{QS}^{-1}A_{QR})^{-1} , \tag{4}$$

$$B_{SQ} = (A_{QS} - A_{QR}A_{PR}^{-1}A_{PS})^{-1} . \tag{5}$$

In particular, taking $P = \{p\}$, $R = \{r\}$, one obtains that the entries of the inverse of A are given by the recursive formula

$$b_{rp} = (a_{pr} - A_{pS}(A_{QS})^{-1}A_{Qr})^{-1} , \tag{6}$$

where $Q = \{1,\ldots,n\}\backslash\{p\}$ and $S = \{1,\ldots,n\}\backslash\{r\}$. This leads to the following definition.

Definition 2.1 (Gelfand, Retakh; [16]). *Let A^{pq} denote the matrix obtained from A by deleting the p^{th} row and the q^{th} column. Let also $\xi_{pq} = (a_{p1},\ldots,\hat{a}_{pq},\ldots,a_{pn})$ and $\eta_{pq} = (a_{1q},\ldots,\hat{a}_{pq},\ldots,a_{nq})$. The quasi-determinant $|A|_{pq}$ of index pq of the generic matrix A is the element of $\mathbf{Q}\langle A\rangle$ defined by*

$$|A|_{pq} = a_{pq} - \xi_{pq}(A^{pq})^{-1}\eta_{pq} = a_{pq} - \sum_{i\neq p, j\neq q} a_{pj}((A^{pq})^{-1})_{ji}a_{iq} , \tag{7}$$

where ξ_{pq} is considered as a row vector and η_{pq} as a column vector.

It is sometimes convenient to use the following more explicit notation;

$$|A|_{pq} = \left| \begin{matrix} a_{11} & \ldots & a_{1q} & \ldots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{p1} & \ldots & \boxed{a_{pq}} & \ldots & a_{pn} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \ldots & a_{nq} & \ldots & a_{nn} \end{matrix} \right| .$$

For instance, for $n = 2$ there are four quasi-determinants

$$\left| \begin{matrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right| = a_{11} - a_{12}\,a_{22}^{-1}a_{21} , \quad \left| \begin{matrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{matrix} \right| = a_{12} - a_{11}a_{21}^{-1}a_{22} ,$$

$$\left| \begin{matrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{matrix} \right| = a_{21} - a_{22}\,a_{12}^{-1}a_{11} , \quad \left| \begin{matrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{matrix} \right| = a_{22} - a_{21}a_{11}^{-1}a_{12} .$$

The quasi-determinants $|M|_{pq}$ of a matrix $M = (m_{ij})$ with entries in a given field K are obtained by applying the specialization $a_{ij} \to m_{ij}$ to the rational expressions $|A|_{pq}$. Some of them may fail to be defined. A sufficient condition for $|M|_{pq}$ to be well-defined is that M^{pq} is invertible in K. It follows from (6) and (7) that when K is a commutative field, $|M|_{pq} = (-1)^{p+q}\det M/\det M^{pq}$. Thus quasi-determinants may be regarded as noncommutative analogues of the ratio of a determinant to one of its principal minors.

By construction the quasi-determinants of the generic matrix A are the inverses of the entries of $B = A^{-1}$:

$$b_{ij}^{-1} = |A|_{ji}, \;\; i, j = 1,\ldots,n . \tag{8}$$

Thus we can rewrite (7) as

$$|A|_{pq} = a_{pq} - \sum_{i \neq p, j \neq q} a_{pj}(|A^{pq}|_{ij})^{-1} a_{iq} \,, \tag{9}$$

which can be regarded as a recursive definition of $|A|_{pq}$.

We now recall from [16, 17] how quasi-determinants behave under elementary operations on rows and columns.

Proposition 2.2. *A permutation of the rows or columns of a quasi-determinant does not change its value.*

Proof. Let $\sigma \in S_n$ and let P_σ be the associated permutation matrix. Then we have

$$|P_\sigma A P_\sigma^{-1}|_{pq}^{-1} = ((P_\sigma A P_\sigma^{-1})^{-1})_{qp} = ((P_\sigma A^{-1} P_\sigma^{-1})_{qp} = (A^{-1})_{\sigma(q)\sigma(p)} = |A|_{\sigma(p)\sigma(q)}^{-1} \,.$$

□

For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & \boxed{a_{31}} & a_{33} \end{vmatrix} .$$

Proposition 2.3. *If the matrix C is obtained from the matrix A by multiplying the p^{th} row on the left by λ, then*

$$|C|_{kq} = \begin{cases} \lambda |A|_{pq} & \text{for} \quad k = p \,, \\ |A|_{kq} & \text{for} \quad k \neq p \,. \end{cases}$$

Similarly, if the matrix C is obtained from the matrix A by multiplying the q^{th} column on the right by μ, then

$$|C|_{pl} = \begin{cases} |A|_{pq}\mu & \text{for} \quad l = q \,, \\ |A|_{pl} & \text{for} \quad l \neq q \,. \end{cases}$$

Finally, if the matrix D is obtained from A by adding to some row (resp. column) of A its k^{th} row (resp. column), then $|D|_{pq} = |A|_{pq}$ for every $p \neq k$ (resp. $q \neq k$).

Proof. The two first properties follow from (9) by induction on n. Let D be obtained from A by adding its k^{th} row to its l^{th} row, and set $M = I_n + E_{lk}$, where E_{lk} denotes the matrix whose unique non-zero entry is the lk^{th} entry equal to 1. Then $D = MA$, and

$$|D|_{pq}^{-1} = (D^{-1})_{qp} = (A^{-1}M^{-1})_{qp} = (A^{-1})_{qp} = |A|_{pq}^{-1}$$

for every $p \neq k$, since multiplying a matrix by M on the right modifies only its k^{th} column. □

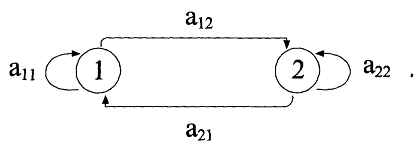
A major difference between quasi-determinants and determinants is that quasi-determinants are not polynomials but rational functions of the entries of the matrix. However, formal power series expansions of quasi-determinants can be obtained, which are conveniently described in terms of graphs. To this end, we introduce the field automorphism ω defined by setting

$$\omega(a_{ij}) = \begin{cases} 1 - a_{ii} & \text{if } i = j \\ -a_{ij} & \text{if } i \neq j \end{cases}$$

for $1 \leqq i, j \leqq n$ [10]. This involution maps the generic matrix A on $I - A$, and its inverse on the *star* of the matrix A,

$$A^* = (I - A)^{-1} = \sum_{i=0}^{+\infty} A^i \,.$$

The star operation is a basic tool of automata theory [3], and ω establishes a correspondence between quasi-determinants and formal power series associated with automatas. In our case, it is useful to associate with A the automaton $\mathscr{A}$ whose transition matrix is A. In other words, $\mathscr{A}$ is the complete oriented graph constructed over $\{1, \ldots, n\}$, the edge from i to j being labelled by a_{ij}. Thus, for $n = 2$, the automaton $\mathscr{A}$ is



Denote by $\mathscr{P}_{ij}$ the set of words labelling a path in $\mathscr{A}$ going from i to j, *i.e.* the set of words of the form $w = a_{ik_1}\, a_{k_1 k_2}\, a_{k_2 k_3} \ldots a_{k_{r-1} j}$. A *simple path* is a path such that $k_s \neq i, j$ for every s. We denote by $\mathscr{S}\mathscr{P}_{ij}$ the set of words labelling simple paths from i to j. It is clear that the entry of index ij of A^* is equal to

$$(A^*)_{ij} = \sum_{w \in \mathscr{P}_{ij}} w \,,$$

or equivalently,

$$|I - A|_{ij}^{-1} = \sum_{w \in \mathscr{P}_{ji}} w \,.$$

Using natural decompositions of these sets of paths, we arrive at the classical formula

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} (a_{11} + a_{12}a_{22}^*a_{21})^* & a_{11}^*a_{12}(a_{22} + a_{21}a_{11}^*a_{12})^* \\ {}^{*}_{22}a_{21}(a_{11} + a_{12}a_{22}^*a_{21})^* & (a_{22} + a_{21}a_{11}^*a_{12})^* \end{pmatrix}, \qquad (10)$$

where the star of a series s in the a_{ij} with zero constant coefficient is defined by

$$s^* = \sum_{n \geqq 0} s^n \,.$$

For instance, the equality of the entries of index 11 in (10) amounts to the decomposition of paths from 1 to 1 into sequences of paths going from 1 to 1 without using 1 as an intermediate state. We note that (10) is to be seen as the image under ω of (2), (3), (4), (5), the noncommutative entries a_{ij} of (10) being interpreted as the blocks A_{PR} of some block decomposition of the matrix A. Similarly, one has

$$|I - A|_{ii} = 1 - \sum_{\mathscr{S}\mathscr{P}ii} w \,. \qquad (11)$$

For example,

$$\begin{vmatrix} \boxed{1-a_{11}} & -a_{12} \\ -a_{21} & 1-a_{22} \end{vmatrix} = 1 - a_{11} - \sum_{p \geqq 0} a_{12}\, a_{22}^p\, a_{21} \ .$$

The graphical interpretation of formal power series expansions of quasi-determinants is an important question. It has been studied at length by Gelfand and Retakh, and we refer the reader to [17] for many other results.

2.3. Minors Identities for Quasi-Determinants. In this section, we give noncommutative analogues of several classical theorems. The reader is referred to [24] for a review of these theorems in the commutative case. We adopt the following convention for indexing *quasi-minors*, that is, quasi-determinants of submatrices. If a_{ij} is an entry of some submatrix A^{PQ} or A_{PQ}, we denote by $|A^{PQ}|_{ij}$ or $|A_{PQ}|_{ij}$ the quasi-minor of this submatrix in which a_{ij} is boxed.

2.3.1. Jacobi's Ratio Theorem. In the commutative case, Jacobi's ratio theorem [39,4] states that each minor of the inverse matrix A^{-1} is equal, up to a sign factor, to the ratio of the corresponding complementary minor of the transpose of A to $\det A$. This generalizes the well-known expression of the entries of A^{-1} as ratios of principal minors of A to $\det A$. The corresponding statement in the noncommutative case is even more natural.

Theorem 2.4 (Gelfand, Retakh; [16]). *Let A be the generic matrix of order n, let B be its inverse and let $(\{i\},L,P)$ and $(\{j\},M,Q)$ be two partitions of $\{1,2,\ldots,n\}$ such that $|L| = |M|$ and $|P| = |Q|$. Then there holds:*

$$|B_{M\cup\{j\},L\cup\{i\}}|_{ji} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}^{-1} \ .$$

Proof. Using appropriate permutation matrices allows to reduce the proof to the case $i = j$, $L = M$ and $P = Q$. Set $R = P \cup \{i\}$. Formula (2) yields

$$(A_{RR})^{-1} = B_{RR} - B_{RL}(B_{LL})^{-1}B_{LR} \ .$$

Now, considering the entry of index ii of the matrices on both sides, we find

$$|A_{RR}|_{ii}^{-1} = b_{ii} - \sum_{k,l \in L} b_{ik} |B_{LL}|_{lk}^{-1} b_{li} = |B_{L\cup\{i\},L\cup\{i\}}|_{ii} \ .$$

□

For example, take $n = 5$, $i = 3$, $j = 4$, $L = \{1,2\}$, $M = \{1,3\}$, $P = \{4,5\}$ and $Q = \{2,5\}$. Theorem 2.4 shows that

$$\begin{vmatrix} a_{32} & \boxed{a_{34}} & a_{35} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & \boxed{b_{43}} \end{vmatrix}^{-1} \ .$$

2.3.2. Cayley's Law of Complementaries. In the commutative case, Cayley's law of complementaries assumes the following form. Let I be an identity between minors of the generic matrix A. If every minor is replaced by its complement in A (multiplied by a suitable power of $\det A$), a new identity I^C is obtained, which is said to result from I by application of the law of complementaries [28, 4]. In the noncommutative case, we have the following analogue of this law.

Theorem 2.5. *Let I be an identity between quasi-minors of the generic matrix A of order n. If every quasi-minor $|A_{L,M}|_{ij}$ involved in I is replaced by $|A_{\overline{M}\cup\{j\},\overline{L}\cup\{i\}}|_{ji}^{-1}$, where $\overline{L}=\{1,\ldots,n\}\backslash L$ and $\overline{M}=\{1,\ldots,n\}\backslash M$, there results a new identity I^C.*

Proof. Applying identity I to A^{-1} gives identity I^C by means of Theorem 2.4. □

For example, let $n=3$ and let I be the identity:

$$a_{13}^{-1}\begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{13}^{-1}a_{12} - a_{33}^{-1}a_{32}\ .$$

By means of the law of complementaries, one can deduce from I the new identity I^C:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix}^{-1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1}$$
$$- \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1}\ .$$

2.3.3. Muir's Law of Extensible Minors. Let us first recall Muir's law of extensible minors in the commutative case [28, 4]. Let D be a square matrix of order $n+p$, let $A=D_{P,Q}$, $C=D^{P,Q}$, where P, Q are two subsets of $\{1,\ldots,n+p\}$ of cardinality n and let I be an identity between minors of A. When every minor $|A_{L,M}|$ involved in I is replaced by its *extension* $|D_{L\cup\overline{P},M\cup\overline{Q}}|$ (multiplied by a suitable power of the *pivot* $|C|$ if the obtained identity is not homogeneous), a new identity I^E is obtained, which is called an *extensional* of I. A similar rule holds in the noncommutative case.

Theorem 2.6. *Let D be the generic matrix of order $n+p$, let $A=D_{P,Q}$, where P, Q are subsets of $\{1,\ldots,n+p\}$ of cardinality n and let I be an identity between quasi-minors of A. If every quasi-minor $|A_{L,M}|_{ij}$ involved in I is replaced by its extension $|D_{L\cup\overline{P},M\cup\overline{Q}}|_{ij}$, a new identity I^E is obtained which is called an extensional of I. The submatrix $D_{\overline{P},\overline{Q}}$ is called the pivot of the extension.*

Proof. As shown by Muir, Theorem 2.6 results from two successive applications of Theorem 2.5. Indeed, a first application of the law of complementaries to identity I transforms it into another identity I^C between quasi-minors of A. But quasi-minors of A may be seen as quasi-minors of D and identity I^C may be seen as an identity between quasi-minors of D. A new application to I^C of the law of complementaries, but taking now the complements relatively to D, yields identity I^E. □

As an illustration, consider the following identity which results from (8):

$$a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} + a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{-1} = 0\ . \tag{12}$$

An extensional of (12) that illustrates Theorem 1.3 of [17] is:

$$
\begin{vmatrix} \boxed{a_{11}} & a_{14} & a_{15} \\ a_{41} & a_{44} & a_{45} \\ a_{51} & a_{54} & a_{55} \end{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \boxed{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1}
$$

$$
+\begin{vmatrix} \boxed{a_{12}} & a_{14} & a_{15} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & \boxed{a_{22}} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1}
$$

$$
+\begin{vmatrix} \boxed{a_{13}} & a_{14} & a_{15} \\ a_{43} & a_{44} & a_{45} \\ a_{53} & a_{54} & a_{55} \end{vmatrix}
\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & \boxed{a_{23}} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}^{-1} = 0 .
$$

Another example is given by the so-called *homological relations* [16]. Start with the trivial identity

$$
a_{il}^{-1} \begin{vmatrix} a_{kl} & a_{kj} \\ a_{il} & \boxed{a_{ij}} \end{vmatrix} = -a_{kl}^{-1} \begin{vmatrix} a_{kl} & \boxed{a_{kj}} \\ a_{il} & a_{ij} \end{vmatrix} ,
$$

and extend it using the pivot $A^{ki,lj}$. The following relation arises:

$$
(|A^{kj}|_{il})^{-1}|A|_{ij} = -(|A^{ij}|_{kl})^{-1}|A|_{kj} , \tag{13}
$$

which relates the two quasi-determinants $|A|_{ij}$ and $|A|_{kj}$ via quasi-minors of lower rank. Arguing similarly, one also obtains

$$
|A|_{ij}(|A^{il}|_{kj})^{-1} = -|A|_{il}(|A^{ij}|_{kl})^{-1} . \tag{14}
$$

The homological relations prove to be a fundamental tool for dealing with quasi-determinants. For example, they lead immediately to the following analogue of the classical expansion of a determinant by a row or column

$$
|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj}(|A^{pq}|_{kj})^{-1}|A^{pj}|_{kq} , \tag{15}
$$

$$
|A|_{pq} = a_{pq} - \sum_{i \neq p} |A^{iq}|_{pl}(|A^{pq}|_{il})^{-1}a_{iq} , \tag{16}
$$

for any $k \neq p$ and $l \neq q$. Indeed, it follows from (8) that

$$
1 = \sum_{j=1}^{n} a_{p,j}|A|_{pj}^{-1} .
$$

The row expansion (15) is obtained by multiplying this last equation from the right by $|A|_{pq}$, and then using (14). An explicit example of (15), where $n = p = q = 4$, is the following:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix} = a_{44} - a_{43} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix}$$

$$-a_{42} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix} - a_{41} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \boxed{a_{31}} & a_{32} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & \boxed{a_{34}} \end{vmatrix}.$$

2.3.4. Sylvester's theorem. Another important application of Muir's law of extensible minors is the noncommutative version of Sylvester's theorem. As in the commutative case, it can be obtained by applying Theorem 2.6 to the complete expansion of a quasi-determinant.

Theorem 2.7 (Gelfand, Retakh; [16]). *Let A be the generic matrix of order n and let P, Q be two subsets of $\{1,\dots,n\}$ of cardinality k. For $i \in \overline{P}$ and $j \in \overline{Q}$, we set $c_{ij} = |A_{P\cup\{i\},Q\cup\{j\}}|_{ij}$ and form the matrix $C = (c_{ij})_{i\in\overline{P},j\in\overline{Q}}$ of order $n-k$. Then one has*

$$|A|_{lm} = |B|_{lm}$$

for every $l \in \overline{P}$ and $m \in \overline{Q}$.

Let us take $n = 3, P = Q = \{3\}$ and $l = m = 1$. Applying Muir's law to the expansion of $|A_{\{1,2\},\{1,2\}}|_{11}$:

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}\, a_{22}^{-1}\, a_{21} \,,$$

we get the identity

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \boxed{a_{11}} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - \begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} \begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \boxed{\begin{vmatrix} \boxed{a_{11}} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}} & \begin{vmatrix} \boxed{a_{12}} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} \boxed{a_{21}} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}$$

which is the simplest instance of Sylvester's theorem for quasi-determinants.

We note that Sylvester's theorem furnishes a recursive method for evaluating quasi-determinants since it allows to reduce the computation of a quasi-determinant of order n to the computation of a quasi-determinant of order $n-1$ whose $(n-1)^2$ entries are quasi-determinants of order 2. As one can check, this leads to a cubic algorithm for computing quasi-determinants.

We presented here Sylvester's theorem as a simple consequence of Theorem 2.6. It can also be directly deduced from relation (2).

2.3.5. Bazin's theorem. Sylvester's theorem takes the form of a relation between (quasi-)minors of a square matrix. Bazin's theorem deals with maximal (quasi-) minors of a rectangular matrix. In fact, in both commutative and noncommutative cases, these theorems are equivalent and each one may be deduced from the other by specialization to a suitable matrix.

Given a $n\times 2n$ matrix A and a subset P of $\{1,\dots,2n\}$, we denote for short by A_P the submatrix $A_{\{1,\dots n\},P}$.

Theorem 2.8. *Let A be the generic matrix of order n by $2n$. Fix an integer m in $\{1,\dots,n\}$. For $1\leqq i,j\leqq n$, set $d_{ij}=|A_{\{j,n+1,\dots,n+i-1,n+i+1,\dots,2n\}}|_{mj}$ and form the matrix $D=(d_{ij})_{1\leqq i,j\leqq n}$. Then,*

$$|D|_{kl}=|A_{\{n+1,\dots,2n\}}|_{m,n+k}|A_{\{1,\dots,l-1,l+1,\dots,n,n+k\}}|_{m,n+k}^{-1}|A_{\{1,\dots,n\}}|_{ml}$$

for any integers k, l in $\{1,\dots,n\}$.

Proof. Let us consider the $2n\times 2n$ matrix C defined by

$$C=\begin{pmatrix} A_{\{1,\dots,n\}} & A_{\{n+1,\dots,2n\}} \\ 0_n & I_n \end{pmatrix},$$

where I_n and 0_n denote respectively the unit and zero matrix of order n. Applying Sylvester's theorem to this matrix with $C_{\{1,\dots,n\},\{n+1,\dots,2n\}}$ as pivot, we get

$$|C|_{n+k,l}=\left|\left(\begin{vmatrix} A_j & A_{\{n+1,\dots,2n\}} \\ \boxed{0} & u_{n+i} \end{vmatrix}\right)_{1\leqq i,j\leqq n}\right|_{kl},$$

where u_i denotes for every integer i the row vector whose only non-zero entry is the i^{th} entry equal to 1. Expanding by its last row each quasi-determinant involved in the above identity and using Proposition 2.3, we obtain

$$|C|_{n+k,l}=-|A_{\{n+1,\dots,2n\}}|_{m,n+k}^{-1}|D|_{kl}\,. \tag{17}$$

On the other hand, it follows from (3) that

$$|C|_{n+k,l}^{-1}=-(A_{\{1,\dots,n\}}^{-1}A_{\{n+1,\dots,2n\}})_{l,n+k}=-\sum_{j=1}^{n}|A_{\{1,\dots,n\}}|_{j,l}^{-1}a_{j,n+k}\,.$$

Using Definition 2.1, this relation can be written in the form

$$|C|_{n+k,l}^{-1}=\begin{vmatrix} A_{\{1,\dots,n\}} & A_{n+k} \\ u_l & \boxed{0} \end{vmatrix}.$$

Expanding now this quasi-determinant by the last row, we get the identity

$$|C|_{n+k,l}^{-1}=-|A_{1,\dots,n}|_{ml}^{-1}\;|A_{\{1,\dots,l-1,l+1,\dots,n,n+k\}}|_{m,n+k}\,,$$

and we conclude by comparing to relation (17). □

Example 2.9. Let $n=3$ and $k=l=m=1$. We adopt more appropriate notations, writing for short $|2\boxed{4}5|$ instead of $|M_{\{2,4,5\}}|_{14}$. Bazin's identity reads

$$\begin{vmatrix} \boxed{|\boxed{1}56|} & |\boxed{2}56| & |\boxed{3}56| \\ |\boxed{1}46| & |\boxed{2}46| & |\boxed{3}46| \\ |\boxed{1}45| & |\boxed{2}45| & |\boxed{3}45| \end{vmatrix}=|\boxed{4}56|\;|23\boxed{4}|^{-1}\;|\boxed{1}23|\,.$$

2.3.6. Schweins' series. "Schweins found an important series, in 1825, for the quotient of two n-rowed determinants which differ only in one column. This series is of great use in many branches of algebra and analysis, and many interesting cases arise by treating one column as a column of the unit matrix" [39]. Here is an example of Schweins' commutative series.

$$\frac{(abcd)_{1234}}{(abce)_{1234}} = \frac{(abc)_{123}(abed)_{1234}}{(abe)_{123}(abce)_{1234}} + \frac{(ab)_{12}(aed)_{123}}{(ae)_{12}(abe)_{123}} + \frac{a_1(ed)_{12}}{e_1(ae)_{12}} + \frac{d_1}{e_1}, \qquad (18)$$

where for instance $(aed)_{123}$ denotes the determinant $\begin{vmatrix} a_1 & e_1 & d_1 \\ a_2 & e_2 & d_2 \\ a_3 & e_3 & d_3 \end{vmatrix}$.

Schweins' series is still valid in the noncommutative case. Keeping the notations of 2.3.5, we first note that according to the homological relations one has for a 3×6 matrix A, say,

$$|A_{123}|_{13}^{-1}\,|A_{124}|_{14} = |A_{123}|_{23}^{-1}\,|A_{124}|_{24} = |A_{123}|_{33}^{-1}\,|A_{124}|_{34}\,.$$

This common value will be denoted for short by $|12\boxed{3}|^{-1}|12\boxed{4}|$. We can now state Schweins' series for quasi-determinants. For convenience, we limit ourselves to the case of quasi-determinants of order 3 and 4, the general case being easily induced from these.

Theorem 2.10. *The maximal quasi-minors of a* 3×6 *generic matrix satisfy the relation*

$$|12\boxed{3}|^{-1}|12\boxed{4}| = |12\boxed{3}|^{-1}|12\boxed{5}|\;|23\boxed{5}|^{-1}|23\boxed{4}| +$$

$$|25\boxed{3}|^{-1}|25\boxed{6}|\;|35\boxed{6}|^{-1}|35\boxed{4}| + |56\boxed{3}|^{-1}|56\boxed{4}|\,.$$

The maximal quasi-minors of a 4×8 *generic matrix satisfy the relation*

$$|123\boxed{4}|^{-1}|123\boxed{5}| = |123\boxed{4}|^{-1}|123\boxed{6}|\;|234\boxed{6}|^{-1}|234\boxed{5}|$$

$$+|236\boxed{4}|^{-1}|236\boxed{7}|\;|346\boxed{7}|^{-1}|346\boxed{5}| + |367\boxed{4}|^{-1}|367\boxed{8}|\;|467\boxed{8}|^{-1}|467\boxed{5}|$$

$$+|678\boxed{4}|^{-1}|678\boxed{5}|\,.$$

Proof. Let us take again the notations of Example 2.9. Applying Bazin's theorem for $n = 2$ to the matrix (4513), we get:

$$|\boxed{4}1| - |\boxed{5}1|\;|\boxed{5}3|^{-1}|\boxed{4}3| = \begin{vmatrix} |\boxed{4}3| & |\boxed{5}3| \\ \boxed{|\boxed{4}1|} & |\boxed{5}1| \end{vmatrix} = |1\boxed{3}|\;|5\boxed{3}|^{-1}|\boxed{4}5|\,.$$

Multiplying from the left by $|1\boxed{3}|^{-1}$ and using Muir's law, one obtains the relation

$$|12\boxed{3}|^{-1}|12\boxed{4}| = |12\boxed{3}|^{-1}|12\boxed{5}|\;|23\boxed{5}|^{-1}|23\boxed{4}| + |25\boxed{3}|^{-1}|25\boxed{4}|\,.$$

Schweins' series for order 3 results from two applications of this lemma. The general case is similar. □

As noted by Turnbull, interesting corollaries are obtained by specialization to a particular matrix some columns of which are columns of the unit matrix. Let us mention the following, which for convenience is stated for order 3 and 4 only.

Theorem 2.11. *The quasi-minors of a* 3×4 *generic matrix satisfy the relation*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix}^{-1} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix} + a_{13}^{-1} a_{14} \, .$$

The quasi-minors of a 4×5 *generic matrix satisfy the relation*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{35} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{45}} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & \boxed{a_{44}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & \boxed{a_{45}} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix}^{-1}$$

$$\begin{vmatrix} a_{11} & a_{14} & a_{15} \\ a_{21} & a_{24} & a_{25} \\ a_{31} & a_{34} & \boxed{a_{35}} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & \boxed{a_{24}} \end{vmatrix}^{-1} \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & \boxed{a_{25}} \end{vmatrix} + a_{14}^{-1} a_{15} \, .$$

Proof. Let us prove the first relation, the general case being similar. We specialize Theorem 2.10 to the 3×6 matrix

$$M = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & a_{11} & a_{12} \\ 1 & 0 & a_{23} & a_{24} & a_{21} & a_{22} \\ 0 & 1 & a_{33} & a_{34} & a_{31} & a_{32} \end{pmatrix}.$$

Using the homological relations (14), we get

$$|12\boxed{3}|^{-1}|12\boxed{5}|\;|23\boxed{5}|^{-1}|23\boxed{4}| = a_{13}^{-1} a_{11} \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & \boxed{a_{21}} \end{vmatrix}^{-1} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix}$$

$$= - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix}^{-1} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & \boxed{a_{24}} \end{vmatrix} .$$

Similarly, we also find that

$$|25\boxed{3}|^{-1}|25\boxed{6}|\;|35\boxed{6}|^{-1}|35\boxed{4}| = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & \boxed{a_{34}} \end{vmatrix},$$

and the claim follows. □

Theorem 2.11 is the true analogue for quasi-determinants of (18) and it even looks more natural this way.

3. Quantum Determinants

In this section we specialize the previous theorems on quasi-determinants to the matrix of generators of the q-deformation $A_q(GL_n)$ of the algebra of functions on GL_n.

3.1. Definitions and notations. We recall fundamental facts on the algebras $A_q(Mat_n)$ and $A_q(GL_n)$ [35]. The algebra $A_q(Mat_n)$ is the associative algebra over $\mathbf{C}[q, q^{-1}]$ generated by n^2 letters t_{ij}, $i, j = 1, \ldots, n$ subject to the relations (written in matrix form)

$$RT_1 T_2 = T_2 T_1 R \,, \tag{19}$$

where

$$R = q^{-1} \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{1 \leqq i \neq j \leqq n} e_{ii} \otimes e_{jj} + (q^{-1} - q) \sum_{1 \leqq i < j \leqq n} e_{ij} \otimes e_{ji} \,.$$

Here, $T = (t_{ij})$, $T_1 = T \otimes I$, $T_2 = I \otimes T$, and e_{ij}'s are the matrix units. More explicitly, the relations obeyed by the symbols t_{ij}'s can be written

$$\begin{gathered} t_{ik}\, t_{il} = q^{-1} t_{il}\, t_{ik} \quad \text{for } k < l, \quad t_{ik}\, t_{jk} = q^{-1} t_{jk}\, t_{ik} \quad \text{for } i < j \,, \\ t_{il}\, t_{jk} = t_{jk}\, t_{il} \quad \text{for } i < j,\ k < l \,, \\ t_{ik}\, t_{jl} - t_{jl}\, t_{ik} = (q^{-1} - q)\, t_{il}\, t_{jk} \quad \text{for } i < j,\ k < l \,. \end{gathered}$$

The algebra $A_q(Mat_n)$ has a bialgebra structure whose comultiplication Δ and counit ε are given by

$$\begin{gathered} \Delta(T) = T \otimes T \quad \text{i.e.} \quad \Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj} \quad i, j = 1, \ldots, n \,, \\ \varepsilon(T) = I \quad \text{i.e.} \quad \varepsilon(t_{ij}) = \delta_{ij} \quad i, j = 1, \ldots, n \,. \end{gathered}$$

The *quantum determinant* of T is the element of $A_q(Mat_n)$ defined by

$$\det_q = \det_q T = \sum_{\sigma \in S_n} (-q)^{-\ell(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)} \,,$$

where S_n is the symmetric group on $\{1, \ldots, n\}$ and $\ell(\sigma)$ denotes the length of the permutation σ. The quantum determinant of T belongs to the center of $A_q(Mat_n)$ and is a group-like element, namely $\Delta(\det_q T) = \det_q T \otimes \det_q T$. More generally, for $P = \{i_1 < \cdots < i_k\}$ and $Q = \{j_1 < \cdots < j_k\}$ one defines the *quantum minor* of the submatrix T_{PQ} as

$$\det_q T_{PQ} = \sum_{\sigma \in S_k} (-q)^{-\ell(\sigma)} t_{i_1 j_{\sigma(1)}} \ldots t_{i_k j_{\sigma(k)}} \,.$$

In particular, the *quantum comatrix* $C(T) = (c_{ij})$ is defined by

$$c_{ij} = (-q)^{j-i} \det_q (T^{ji}) \quad i, j = 1, \ldots, n \,.$$

Then, one has

$$T\, C(T) = C(T)\, T = \det_q T \,.I_n \,, \tag{20}$$

which amounts to the expansion of $\det_q T$ by one of its rows or columns. This leads to the definition of the algebra $A_q(GL_n)$ as the localization $A_q(Mat_n)[\det_q^{-1}]$ of $A_q(Mat_n)$. The algebra $A_q(GL_n)$ is a Hopf algebra whose coproduct and counit are defined as above, and whose antipode is the anti-automorphism given by

$$S(T) = \det_q^{-1} C(T), \quad \text{i.e.} \quad S(t_{ij}) = \det_q^{-1} c_{ij}, \quad i, j = 1, \ldots, n ,$$
$$S(\det_q) = \det_q^{-1} .$$

In other words

$$T\,S(T) = S(T)\,T = I_n , \tag{21}$$

and $S(T) = (s_{ij})$ is the inverse matrix of T. Finally, we remark that since S is an anti-automorphism, the entries of $S(T)$ and $C(T)$ obey the same commutation rules as those of T with q replaced by q^{-1}.

3.2. Quantum determinants and quasi-determinants. We now consider the connection between quantum determinants and quasi-determinants. As recalled in Sect. 2.2, quasi-determinants are noncommutative analogues of the ratio of a determinant to one of its principal minors. Thus if the entries t_{ij} of a matrix T belong to a commutative field, one has the following expression of $\det T$ in terms of quasi-determinants

$$\begin{vmatrix} t_{11} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{vmatrix} = \begin{vmatrix} \boxed{t_{11}} & t_{12} & \ldots & t_{1n} \\ t_{21} & t_{22} & \ldots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \ldots & t_{nn} \end{vmatrix} \begin{vmatrix} \boxed{t_{22}} & \ldots & t_{2n} \\ \vdots & \ddots & \vdots \\ t_{n2} & \ldots & t_{nn} \end{vmatrix} \ldots \begin{vmatrix} \boxed{t_{n-1\,n-1}} & t_{n-1\,n} \\ t_{n\,n-1} & t_{nn} \end{vmatrix} t_{nn} .$$

The following theorem provides an analogue of this formula for quantum determinants.

Theorem 3.1. (Gelfand, Retakh; [16]). *Let $T = (t_{ij})_{1 \leqq i,j \leqq n}$ be the matrix of generators of $A_q(GL_n)$. In the field of fractions of $A_q(GL_n)$, one has*

$$\det_q T = |T|_{11} |T^{11}|_{22} \ldots t_{nn}$$

and the quasi-minors in the right-hand side commute all together. More generally, let $\sigma = i_1 \ldots i_n$ and $\tau = j_1 \ldots j_n$ be two permutations of S_n. There holds

$$\det_q T = (-q)^{\ell(\sigma) - \ell(\tau)} |T|_{i_1 j_1} |T^{i_1 j_1}|_{i_2 j_2} \ldots t_{i_n j_n} \tag{22}$$

and the quasi-minors in the right-hand side commute all together.

Proof. We first note that $A_q(GL_n)$ is an Ore ring and therefore has a field of fractions. By (8), the quasi-determinants of T are the inverses of the entries of $S(T) = (\det_q T)^{-1} C(T)$. Hence we have

$$\det_q T = (-q)^{i-j} |T|_{ij} \det_q T^{ij} = (-q)^{i-j} \det_q T^{ij}\, |T|_{ij} , \tag{23}$$

and (22) follows by induction on n. Let us now prove that the quasi-determinants involved in (22) commute all together. For simplicity, we only argue in the case $i_1 = j_1 = 1, \ldots, i_n = j_n = n$. By induction on n, it is enough to prove that $|T|_{11}$ commutes with $|T^{11}|_{22}, \ldots, t_{nn}$. Using relation (23), it suffices to show that $\det_q T^{11}$

commutes with $\det_q T^{\{1,\ldots,i\},\{1,\ldots,i\}}$ for $1 \leqq i \leqq n-1$, which follows from the fact that $\det_q T^{11}$ commutes with $t_{ij}, i, j = 2, \ldots, n$. □

Thus, for $n = 2$, we have

$$\det_q T = (t_{11} - t_{12}t_{22}^{-1}t_{21})\,t_{22} = (-q)^{-1}\,(t_{12} - t_{11}t_{21}^{-1}t_{22})\,t_{21}$$

$$= (-q)\,(t_{21} - t_{22}t_{12}^{-1}t_{11})\,t_{12} = (t_{22} - t_{21}t_{11}^{-1}t_{12})\,t_{11}\,.$$

Note that the parameter q no longer appears in the first and fourth expression.

3.3. Minor identities for quantum determinants. In this section, we derive quantum analogues of several classical determinantal formulas. We shall sometimes use the following terminology. A $k \times k$ matrix $M = (m_{ij})$ with entries in $A_q(GL_n)$ is called a $k \times k$ *quantum matrix* if its entries m_{ij} obey the same commutation rules as the generators t_{ij} of $A_q(GL_k)$. More generally, a rectangular matrix M is said to be a quantum matrix if all its 2×2 submatrices are quantum matrices.

3.3.1. Jacobi's ratio theorem. Recall that Jacobi's theorem states that each minor of the inverse matrix A^{-1} is equal, up to a sign factor, to the ratio of the corresponding complementary minor of the transpose of A to $\det A$. For quantum determinants, we have the following analogue.

Theorem 3.2. *Let $P = \{i_1 < \cdots < i_k\}$, $Q = \{j_1 < \cdots < j_k\}$ be two subsets of $\{1,\ldots,n\}$, and $\overline{P} = \{i_{k+1} < \cdots < i_n\}$, $\overline{Q} = \{j_{k+1} < \cdots < j_n\}$, be their set complements. Set $\sigma = i_1 \ldots i_n$ and $\tau = j_1 \ldots j_n$. Then,*

$$\det_{q^{-1}} S(T)_{P,Q} = (-q)^{\ell(\tau)-\ell(\sigma)} \det_q T^{Q,P}\ (\det_q T)^{-1}\,.$$

Proof. Express $\det_{q^{-1}} S(T)_{P,Q}$ as a product of quasi-determinants by means of Theorem 3.1 and apply Jacobi's Theorem 2.4 to each of them. The result then follows from a second application of Theorem 3.1. □

For instance, if $n = 5$, $P = \{1, 3, 4\}$ and $Q = \{1, 2, 3\}$, we have

$$\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{31} & s_{32} & s_{33} \\ s_{41} & s_{42} & s_{43} \end{vmatrix}_{q^{-1}} = (-q)^{-2} \begin{vmatrix} t_{42} & t_{45} \\ t_{52} & t_{55} \end{vmatrix}_q (\det_q T)^{-1}\,.$$

3.3.2. Cayley's law of complementaries. For quantum determinants, we obtain the following analogue of Cayley's law of complementaries (see Sect. 2.3.2).

Theorem 3.3. *Let I be a polynomial identity with coefficients in $\mathbf{C}[q, q^{-1}]$ between quantum minors of the matrix T of generators of $A_q(GL_n)$. If each minor $\det_q T_{P,Q}$ involved in I is replaced by its complement $\det_q T^{P,Q}$ multiplied by $(\det_q T)^{-1}$ and if, in addition, the substitution $q \to q^{-1}$ is made in the coefficients of I, there results a new identity I^C.*

Proof. The entries of the matrix $S(T)^t$ obey the same commutation rules as those of T with q replaced by q^{-1}. Therefore, replacing in I each minor $\det_q T_{P,Q}$ by $\det_{q^{-1}} S(T)^t_{P,Q}$ and substituting q^{-1} to q in the coefficients, one obtains a polynomial identity between quantum minors of $S(T)^t$. Identity I^C then results from Theorem 3.2. □

For example, take $n = 4$ and consider the following identity (see Proposition 3.6)

$$\begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}_q t_{23} = q^{-1} t_{23} \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}_q . \tag{24}$$

Applying Cayley's law to (24), we get

$$\begin{vmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{31} & t_{32} & t_{34} \\ t_{41} & t_{42} & t_{44} \end{vmatrix}_q = q \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{31} & t_{32} & t_{34} \\ t_{41} & t_{42} & t_{44} \end{vmatrix}_q \begin{vmatrix} t_{33} & t_{34} \\ t_{43} & t_{44} \end{vmatrix}_q .$$

3.3.3. Muir's law of extensible minors. From Cayley's law 3.3 is deduced the following quantum analogue of Muir's law. The proof is similar to the one of Theorem 2.6.

Theorem 3.4. *Let $n \leqq m$ be two integers, and P, Q be two subsets of $\{1,\ldots,m\}$ of cardinality n. We consider the imbedding of $A_q(GL_n)$ in $A_q(GL_m)$ obtained by identifying the matrix T_n of generators of $A_q(GL_n)$ to the submatrix $(T_m)_{PQ}$ of T_m. Let I be a polynomial identity with coefficients in $\mathbf{C}[q, q^{-1}]$ between quantum minors of T_n. When every quantum minor $\det_q (T_n)_{L,M}$ involved in I is replaced by its extension $\det_q (T_m)_{L\cup\overline{P},M\cup\overline{Q}}$ (multiplied by a suitable power of the pivot $\det_q (T_m)_{\overline{P},\overline{Q}}$ if the identity is not homogeneous), a new identity I^E is obtained, which is called an extensional of I.*

As an illustration, take $n = 2$, $m = 4$, $P = \{2, 4\}$, $Q = \{2, 3\}$ and consider the identity

$$t_{22}\, t_{23} = q^{-1} t_{23}\, t_{22} .$$

Applying Muir's law, we get

$$\begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{13} & t_{14} \\ t_{21} & t_{23} & t_{24} \\ t_{31} & t_{33} & t_{34} \end{vmatrix}_q = q^{-1} \begin{vmatrix} t_{11} & t_{13} & t_{14} \\ t_{21} & t_{23} & t_{24} \\ t_{31} & t_{33} & t_{34} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{vmatrix}_q .$$

3.3.4. Sylvester's theorem. An important consequence of Muir's law is the quantum analogue of Sylvester's theorem.

Theorem 3.5. *Let $\det_q T_{P,Q}$ be a quantum $k \times k$ minor of the matrix T of generators of $A_q(GL_n)$. For $i \in \overline{P}$ and $j \in \overline{Q}$, set $u_{ij} = \det_q T_{P\cup\{i\},Q\cup\{j\}}$ and let U denote the $(n-k) \times (n-k)$ matrix $(u_{ij})_{i\in\overline{P},j\in\overline{Q}}$. Then U is a quantum matrix, and there holds*

$$\det_q U = \det_q T \ (\det_q T_{P,Q})^{n-k-1} .$$

Proof. The commutation rules for the entries of U follow from Muir's law. Also, applying Muir's law to the complete expansion of the quantum minor $\det_q T^{P,Q}$ yields Sylvester's theorem, the term, $(\det_q T_{P,Q})^{n-k-1}$ in the right-hand side being an homogeneity factor. □

Thus, let $n = 3,\ k = 1,\ P = Q = \{3\}$. Sylvester's theorem reads

$$\begin{vmatrix} \begin{vmatrix} t_{11} & t_{13} \\ t_{31} & t_{33} \end{vmatrix}_q & \begin{vmatrix} t_{12} & t_{13} \\ t_{32} & t_{33} \end{vmatrix}_q \\ \begin{vmatrix} t_{22} & t_{23} \\ t_{32} & t_{33} \end{vmatrix}_q & \begin{vmatrix} t_{21} & t_{23} \\ t_{31} & t_{33} \end{vmatrix}_q \end{vmatrix}_q = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}_q t_{33} .$$

Sylvester's quantum theorem can also be directly deduced from its noncommutative analogue (Theorem 2.7), using the same method as in the proof of Theorem 3.2.

3.3.5. Bazin's theorem. The proof of Bazin's theorem for quantum determinants requires two lemmas of independent interest, which provide sufficient conditions for certain quantum minors to commute up to a power of q.

Throughout this section, we fix two integers $n < m$, and we consider quantum minors of the matrix T of generators of $A_q(GL_m)$. Let $1 \leqq j_1 < \cdots < j_n \leqq m$. The quantum minor $\det_q T_{\{1,\ldots,n\},\{j_1,\ldots,j_n\}}$ is written for short $[j_1,\ldots,j_n]_q$.

Lemma 3.6. *Consider an increasing sequence of integers $1 \leqq j_1 < \cdots < j_n < k \leqq m$. The following commutation relation holds for all $i \leqq n$:*

$$[j_1 \ldots j_n]_q\, t_{ik} = q^{-1} t_{ik}\, [j_1 \ldots j_n]_q .$$

Proof. We may suppose that $j_s = s$ for $s = 1,\ldots,n$ and $k = n+1$. The proof is by induction on n. For $n = 1$, $t_{11}\, t_{12} = q^{-1}\, t_{12}\, t_{11}$ is one of the defining relations of $A_q(GL_m)$. Assume that the commutation relation holds for $n-1$. Expand the quantum minor $[1 \ldots n]_q$ by its last row:

$$[1 \ldots n]_q = \sum_{s=1}^{n} (-q)^{s-n}\, t_{ns}\, [1 \ldots \hat{s} \ldots n]_q .$$

For $i \leqq n-1$, t_{in+1} commute with t_{ns} and

$$[1 \ldots \hat{s} \ldots n]_q\, t_{i\,n+1} = q^{-1}\, t_{i\,n+1}\, [1 \ldots \hat{s} \ldots n]_q$$

by induction. Therefore $[1 \ldots n]_q\, t_{i\,n+1} = q^{-1}\, t_{i\,n+1}\, [1 \ldots n]_q$ for $i \leqq n-1$. In the remaining case $i = n$, we may use Cayley's law 3.3. Indeed, applying Cayley's law to the relation

$$[1 \ldots \hat{s} \ldots n]_q\, t_{1\,n+1} = q^{-1}\, t_{1\,n+1}\, [1 \ldots \hat{s} \ldots n]_q$$

regarded as an identity between minors of the matrix T of generators of $A_q(GL_{n+1})$, we get:

$$t_{n+1\,n+1} \det_q T_{\{2,\ldots,n+1\},\{1,\ldots n\}} = q \det_q T_{\{2,\ldots,n+1\},\{1,\ldots n\}}\, t_{n+1\,n+1}$$

which is equivalent to

$$t_{n\,n+1}\, [1 \ldots n]_q = q\, [1 \ldots n]_q\, t_{n\,n+1}$$

by translation on the row indices. □

An immediate corollary of Lemma 3.6 is

Lemma 3.7. *Consider two increasing sequences of integers $1 \leqq j_1 < \cdots < j_n \leqq m$ and $1 \leqq k_1 < \cdots < k_n \leqq m$ and suppose that for some $s \in \{0,\ldots,n\}$, one has $k_s < j_1 < j_n < k_{s+1}$. Then,*

$$[j_1 \ldots j_n]_q\, [k_1 \ldots k_n]_q = q^{2s-n}\, [k_1 \ldots k_n]_q\, [j_1 \ldots j_n]_q .$$

Thus, for $n=2$, $m=4$, we have

$$[12]_q\,[34]_q = q^{-2}\,[34]_q\,[12]_q, \quad [14]_q\,[23]_q = [23]_q\,[14]_q\,.$$

We can now state Bazin's theorem for quantum determinants.

Theorem 3.8. *Let $J=\{j_1<\cdots<j_n\}$ and $K=\{k_1<\cdots<k_n\}$ be two subsets of $\{1,\ldots m\}$ such that $j_n<k_1$. Then the entries of the matrix $B_n=(b_{st})_{1\leqq s,t\leqq n}$ defined by*

$$b_{st} = [j_t,\ (K\setminus k_s)]_q \quad \textit{for } 1\leqq s,\,t\leqq n\,,$$

obey the same commutation rules as the generators of $A_q(GL_n)$ and we have

$$\det_q B_n = q^{\binom{n}{2}}\,[j_1\ldots j_n]_q\,[k_1\ldots k_n]_q^{n-1}\,.$$

Proof. The proof is by induction on $n\geqq 2$. For $n=2$, one can check by means of Plucker relations for quantum determinants (described for example in [38]) that the entries of

$$B_2 = \begin{pmatrix} [j_1\,k_2]_q & [j_2\,k_2]_q \\ [j_1\,k_1]_q & [j_2\,k_1]_q \end{pmatrix}$$

obey the same commutation rules as the generators of $A_q(GL_2)$, and that

$$\det_q B_2 = q\ [j_1\,j_2]_q\,[k_1\,k_2]_q\,.$$

Using Muir's law 3.4, it follows that every 2×2 submatrix of B_n is a quantum matrix, and therefore that B_n is itself a quantum $n\times n$ matrix for every $n\geqq 2$. Assume now that

$$\det_q B_{n-1} = q^{\binom{n-1}{2}}\ [j_1\ldots j_{n-1}]_q\,[k_1\ldots k_{n-1}]_q^{n-2}$$

for all sequences J and K of cardinality $n-1$ satisfying the hypothesis of Theorem 3.8. From Theorem 3.1, it results that

$$\det_q B_n = |B_n|_{nn}\,\det_q B_n^{nn}\,.$$

Now Muir's law and the induction hypothesis show that

$$\det_q B_n^{nn} = q^{\binom{n-1}{2}}\ [j_1 j_2\ldots j_{n-1}k_n]_q\,[k_1k_2\ldots k_n]_q^{n-2}\,.$$

On the other hand, expanding all entries of $|B_n|_{nn}$ according to Theorem 3.1, and applying Bazin's theorem for quasi-determinants, one obtains

$$|B_n|_{nn} = [k_1\ldots k_n]_q\,[j_1j_2\ldots j_{n-1}k_n]_q^{-1}\,[j_1\ldots j_n]_q\,.$$

The claim follows now from Muir's law, which shows that

$$[j_1\ldots j_n]_q\,[j_1j_2\ldots j_{n-1}k_n]_q = q^{-1}[j_1j_2\ldots j_{n-1}k_n]_q\,[j_1\ldots j_n]_q\,,$$

and from Proposition 3.7. □

As an illustration, take $n=3$, $J=\{1,2,3\}$ and $K=\{4,5,6\}$. Then, Bazin's theorem reads

$$\begin{vmatrix} [145]_q & [245]_q & [345]_q \\ [146]_q & [246]_q & [346]_q \\ [156]_q & [256]_q & [356]_q \end{vmatrix}_q = q^3\ [123]_q\,[456]_q^2\,.$$

3.3.6. Schweins' series. Using Theorem 3.1, one readily deduces from Schweins' series for quasi-determinants (Theorems 2.10, 2.11) the following quantum analogues. Here again identities are stated for quantum determinants of order 3 and 4 only, the general case being easily understood from these. The notations for quantum minors are those introduced in Sect. 3.3.5.

Theorem 3.9. *The maximal minors of a* 3×6 *quantum matrix satisfy the relation*

$$[123]_q^{-1}[124]_q = [123]_q^{-1}[125]_q[235]_q^{-1}[234]_q + [253]_q^{-1}[256]_q[356]_q^{-1}[354]_q + [563]_q^{-1}[564]_q \,.$$

The maximal minors of a 4×8 *quantum matrix satisfy the relation*

$$[1234]_q^{-1}[1235]_q = [1234]_q^{-1}[1236]_q[2346]_q^{-1}[2345]_q + [2364]_q^{-1}[2367]_q[3467]_q^{-1}[3465]_q + [3674]_q^{-1}[3678]_q[4678]_q^{-1}[4675]_q + [6784]_q^{-1}[6785]_q \,.$$

Theorem 3.10. *The minors of a* 3×4 *quantum matrix satisfy the relation*

$$\begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}_q^{-1} \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{vmatrix}_q = \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}_q^{-1} \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{13} & t_{14} \\ t_{21} & t_{23} & t_{24} \\ t_{31} & t_{33} & t_{34} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{13} \\ t_{21} & t_{23} \end{vmatrix}_q^{-1} + \begin{vmatrix} t_{11} & t_{13} \\ t_{21} & t_{23} \end{vmatrix}_q^{-1} t_{11} \begin{vmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{vmatrix}_q t_{13}^{-1} + t_{13}^{-1} t_{14} \,.$$

The minors of a 4×5 *quantum matrix satisfy the relation*

$$\begin{vmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{vmatrix}_q^{-1} \begin{vmatrix} t_{11} & t_{12} & t_{13} & t_{15} \\ t_{21} & t_{22} & t_{23} & t_{25} \\ t_{31} & t_{32} & t_{33} & t_{35} \\ t_{41} & t_{42} & t_{43} & t_{45} \end{vmatrix}_q = \begin{vmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{vmatrix}_q^{-1} \begin{vmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{12} & t_{14} & t_{15} \\ t_{21} & t_{22} & t_{24} & t_{25} \\ t_{31} & t_{32} & t_{34} & t_{35} \\ t_{41} & t_{42} & t_{44} & t_{45} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{vmatrix}_q^{-1} + \begin{vmatrix} t_{11} & t_{12} & t_{14} \\ t_{21} & t_{22} & t_{24} \\ t_{31} & t_{32} & t_{34} \end{vmatrix}_q^{-1} \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{14} & t_{15} \\ t_{21} & t_{24} & t_{25} \\ t_{31} & t_{34} & t_{35} \end{vmatrix}_q \begin{vmatrix} t_{11} & t_{14} \\ t_{21} & t_{24} \end{vmatrix}_q^{-1} + \begin{vmatrix} t_{11} & t_{14} \\ t_{21} & t_{24} \end{vmatrix}_q^{-1} t_{11} \begin{vmatrix} t_{14} & t_{15} \\ t_{24} & t_{25} \end{vmatrix}_q t_{14}^{-1} + t_{14}^{-1} t_{15} \,.$$

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