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Toeplitz Algebras and Rieffel Deformations

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Abstract: We establish a representation theorem for Toeplitz operators on the Segal–Bargmann (Fock) space of $\mathbf{C}^n$ whose "symbols" have uniform radial limits. As an application of this result, we show that Toeplitz algebras on the open ball in $\mathbf{C}^n$ are "strict deformation quantizations", in the sense of M. Rieffel, of the continuous functions on the corresponding closed ball.

1. Introduction

In [R], Rieffel proposed a general scheme for producing "strict deformation quantizations" of C^*-algebras with $\mathbf{R}^{2n}$ action. His scheme is modelled on classical Weyl quantization. As one example, Rieffel showed, following earlier work of Sheu [S], that the Toeplitz algebra $\tau(\mathbf{D})$ on the unit disc $\mathbf{D}$ arises from his scheme as a strict deformation quantization of the sup norm algebra $C(\mathbf{D})$ of continuous functions on the closed unit disc. In this note, we extend Rieffel's analysis to show that the Toeplitz algebra $\tau(\mathbf{B}_{2n})$ of the unit ball $\mathbf{B}_{2n}$ (in $\mathbf{C}^n$) is a strict deformation quantization of the algebra $C(\bar{\mathbf{B}}_{2n})$ of continuous functions on the closed unit ball.

Let $\mathbf{C}^n$ be the vector space of n-tuples of complex numbers with elements $z = (z_1, \ldots, z_n)$ and the usual norm $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$. We denote by $\mathbf{B}_{2n}$ the (real) $2n$-dimensional open unit ball in $\mathbf{C}^n$, $\mathbf{B}_{2n} = \{z \in \mathbf{C}^n : |z| < 1\}$, and write $S^{2n-1} = \{z \in \mathbf{C}^n : |z| = 1\}$ for the unit sphere with $\bar{\mathbf{B}}_{2n} = \mathbf{B}_{2n} \cup S^{2n-1}$.

In what follows, we consider three related Hilbert spaces of functions on $\mathbf{C}^n$. The first is the Bergmann space of Lebesgue volume (dv)-square-integrable holomorphic functions on the open unit ball $\mathbf{B}_{2n}$, $H^2(\mathbf{B}_{2n})$. The next, is the space of Lebesgue surface area $(d\sigma)$-square-integrable functions on the unit sphere S^{2n-1} which extend to be holomorphic in $\mathbf{B}_{2n}$, $H^2(S^{2n-1})$. Finally we have the Segal-Bargmann space $H^2(\mathbf{C}^n)$ of entire functions on $\mathbf{C}^n$ which are square integrable with respect to the Gaussian measure $d\mu(z) = e^{-|z|^2/2}(2\pi)^{-n}dv(z)$. Here dv and $d\sigma$ are normalized by $v(\mathbf{B}_{2n}) = \pi^n/n!$ and $\sigma(S^{2n-1}) = 2\pi^n/(n-1)!$.

These spaces have the common feature that an orthonormal basis for each can be constructed in the form

$$a_k z^k \ ,$$

where $k = (k_1, \ldots, k_n)$ and k_j are integers, $k_j \geqq 0$. Here a_k is some complex scalar and

$$z^k \equiv z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$$

is the standard monomial. Of course, the weights a_k vary, depending on the space of functions. It is known [C1, BC1] that on $H^2(\mathbf{B}_{2n})$, we have the orthonormal basis

$$\tilde{e}_k = (\pi^n)^{-1/2} \left\{ \frac{(|k| + n)!}{k!} \right\}^{1/2} z^k \ ,$$

while on $H^2(\mathbf{C}^n)$, we have the orthonormal basis

$$e_k = \left(2^{|k|} k!\right)^{-1/2} z^k \ ,$$

where $|k| \equiv k_1 + k_2 + \cdots + k_n$ and $k! \equiv k_1! k_2! \ldots k_n!$.

Our key technical result is that the canonical isometry V from $H^2(\mathbf{B}_{2n})$ to $H^2(\mathbf{C}^n)$ defined by

$$V \tilde{e}_k = e_k$$

induces a representation in $\tau(\mathbf{B}_{2n})$ of Toeplitz operators on $H^2(\mathbf{C}^n)$ whose "symbols" have uniform radial limits. A related result, on Toeplitz operators whose symbols vary in the radial direction only, was obtained in [G, Theorem 10.1]. See also [H].

2. Representation of Toeplitz Operators on $H^2(\mathbf{C}^n)$

In [C1, BC1], Toeplitz operators on $H^2(S^{2n-1})$, $H^2(\mathbf{B}_{2n})$, $H^2(\mathbf{C}^n)$ are defined and studied. For f a bounded measurable function on the underlying space, the Toeplitz operator $T_f(\tilde{T}_f)$ is given by

$$T_f g = P(f \cdot g) \ ,$$

where P is the orthogonal projection from L^2 onto the corresponding H^2 space. There are natural isometries from $H^2(S^{2n-1})$ to $H^2(\mathbf{B}_{2n})$ and from $H^2(\mathbf{B}_{2n})$ onto $H^2(\mathbf{C}^n)$ which map $\tilde{e}_k \to e_k$. In [C1, Theorem 1], it was shown that the natural isometry from $H^2(S^{2n-1})$ to $H^2(\mathbf{B}_{2n})$ "intertwines" Toeplitz operators in a suitably weak sense. Here, we consider the corresponding problem for $H^2(\mathbf{C}^n)$.

For f a bounded measurable function on $\mathbf{B}_{2n}$, we write $\tilde{T}_f$ for the Toeplitz operator on $H^2(\mathbf{B}_{2n})$. Similarly, for f bounded measurable on $\mathbf{C}^n$, we write T_f for the Toeplitz operator on $H^2(\mathbf{C}^n)$.

Key Lemma 1. *The operators $T_{z_j/|z|} - V\tilde{T}_{z_j}V^{-1}$ are compact for all $j, 1 \leqq j \leqq n$.*

Proof. By symmetry, it suffices to consider $j = 1$. Direct calculation shows that

$$\tilde{T}_{z_1} \tilde{e}_k = \beta_k \tilde{e}_{k+\delta_1} \ ,$$

$$T_{z_1/|z|} e_k = \alpha_k e_{k+\delta_1} \ ,$$

where $\delta_1 = (1,0,0,\ldots,0)$. It follows that

$$T_{z_1/|z|} - V\tilde{T}_{z_1}V^{-1} = S_{\delta_1}D \ ,$$

where

$$S_{\delta_1}e_k = e_{k+\delta_1}$$

and

$$De_k = (\alpha_k - \beta_k)e_k \ .$$

Thus, it will suffice to check that D is compact, i.e. that for arbitrary $\varepsilon > 0$,

$$|\alpha_k - \beta_k| < \varepsilon$$

for k outside of some finite set of multi-indices F_ε.

We need very precise estimates on $\alpha_k - \beta_k$. It is not hard to check (as in [C1]) that

$$\beta_k = (k_1 + 1)^{1/2}(|k| + n + 1)^{-1/2} \ .$$

The calculation of a useable value of α_k is more complicated. Direct calculation shows that

$$\alpha_k = \{2(k_1 + 1)\}^{-1/2}(2^{|k|}k!)^{-1}\mathscr{I} \ ,$$

where

$$\mathscr{I} = \int_0^\infty \cdots \int_0^\infty \frac{r_1^{2(k_1+1)+1} r_2^{2k_2+1} \cdots r_n^{2k_n+1}}{\sqrt{r_1^2 + \cdots + r_n^2}} e^{-(r_1^2+\cdots+r_n^2)/2} dr_1 \ldots dr_n \ .$$

Making a change of variables in the first two coordinates to polar form, and proceeding inductively, we obtain

$$\mathscr{I} = \int_0^\infty s^{2(|k|+n)} e^{-s^2/2} ds \cdot \prod_{m=1}^{n-1} \int_0^{\pi/2} \cos^{2(k_1+\cdots+k_m+m)+1}\theta \ \sin^{2k_{m+1}+1}\theta \ d\theta \ .$$

It is a standard calculation [BC1] that

$$\int_0^\infty s^{2(|k|+n)} e^{-s^2/2} ds = \frac{\{2(|k|+n)\}!\sqrt{\pi}}{2^{|k|+n}(|k|+n)!\sqrt{2}} \ .$$

A beautiful classical result of Euler [WW] is that

$$\int_0^{\pi/2} \cos^{2m_1+1}\theta \ \sin^{2m_2+1}\theta \ d\theta = \frac{1}{2} \frac{m_1!m_2!}{(m_1 + m_2 + 1)!} \ .$$

It follows that

$$\prod_{m=1}^{n-1} \int_0^{\pi/2} \cos^{2(k_1+\cdots+k_m+m)+1}\theta \ \sin^{2k_{m+1}+1}\theta \ d\theta = \frac{1}{2^{n-1}} \frac{k!(k_1+1)}{(|k|+n)!} \ .$$

Putting the pieces together, we have

$$\alpha_k = \sqrt{\pi}(k_1 + 1)^{1/2} \frac{\{2(|k|+n)\}!}{4^{|k|+n}[(|k|+n)!]^2} \ .$$

To complete our analysis, we need Stirling's Formula in the form [WW]

$$m! = m^{m+1/2}e^{-m}e^{\theta(m)/12m}\sqrt{2\pi} ,$$

where $0 < \theta(m) < 1$. This gives

$$\alpha_k = (k_1 + 1)^{1/2}(|k| + n)^{-1/2}e^{\delta(|k|)/(|k|+n)} ,$$

where $|\delta(|k|)| \leqq 1/6$. Thus, we have

$$\alpha_k - \beta_k = (k_1 + 1)^{1/2}\{(|k| + n)^{-1/2}e^{\delta(|k|)/(|k|+n)} - (|k| + n + 1)^{-1/2}\}$$

with $|\delta(|k|)| \leqq 1/6$. Using $e^{-x} \geqq 1 - x$ for $x \geqq 0$, we see that

$$\alpha_k - \beta_k \geqq 0$$

and, using $e^x \leqq 1 + 3x$ for $0 \leqq x \leqq 1$, we can check that

$$\alpha_k - \beta_k \leqq (|k| + n)^{-1} .$$

This allows us to conclude that D is compact.

We also have

Lemma 2. *If p is any polynomial in $z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n$ which is homogeneous of degree k, then*

$$T_{|z|^{-k}p} - V\tilde{T}_pV^{-1}$$

is compact.

Proof. The functions $z_j/|z|$ are ESV in the sense of [BC2,Theorem 3]. Note that by [BC2, Theorem 11],

$$T_fT_g - T_{fg}$$

is compact for f, g in ESV and ESV is a $*$-algebra under the usual pointwise operations on functions. It follows from [C1, Theorem 1] and Lemma 1 that the desired result holds.

For g in the sup-norm algebra $C(S^{2n-1})$ of continuous complex-valued functions on S^{2n-1}, we define

$$\hat{g}(z) = g(z/|z|)$$

on $\mathbf{C}^n\backslash\{0\}$. Note that for $p(z)$ a homogeneous polynomial in $z_1,\dots,z_n,\bar{z}_1,\dots,\bar{z}_n$ of degree l,

$$\hat{p}(z) = |z|^{-l}p(z)$$

for all z in $\mathbf{C}^n\backslash\{0\}$. It is known that $\hat{g}$ is in ESV of [BC2].

We write $\tau(\mathbf{B}_{2n})$ for the C^*-algebra generated by all $\tilde{T}_f$ with f continuous on $\bar{\mathbf{B}}_{2n} = \mathbf{B}_{2n} \cup S^{2n-1}$. This algebra was studied in [C1] and [V].

We will use the definitions of [BDF] without much discussion. Recall that an exact sequence of C^*-algebras

$$0 \to \mathscr{K} \to \mathscr{A} \to C(X) \to 0 ,$$

where $\mathscr{K}$ is the full algebra of compact operators and $C(X)$ is the sup-norm algebra of all continuous complex-valued functions on the compact, separable metric space

X, defines an element of $Ext(X)$. For a Hilbert space H, let $B(H)$ denote the collection of all bounded operators on this space. Let π denote the quotient map from $B(H)$ to the Calkin algebra $B(H)/\kappa$. It is well-known that $\tau(\mathbf{B}_{2n})$ is an element of $\mathrm{Ext}(S^{2n-1})$ [C1, V]. Indeed this element is represented by the $*$-isomorphism

$$\tau(f) = \pi(\tilde{T}_{f_e})$$

from $C(S^{2n-1})$ into the Calkin algebra $B(H^2(\mathbf{B}_{2n}))/\kappa$, where f_e is *any* continuous extension of f to $\bar{\mathbf{B}}_{2n}$.

We now have our main technical result.

Representation Theorem 1. *For g in $C(S^{2n-1})$,*

$$V^{-1}T_{\hat{g}}V - \tilde{T}_{g_e}$$

is compact for g_e any continuous extension of g to $\bar{\mathbf{B}}_{2n}$.

Proof. This is immediate from Lemma 2 above and [C1,Theorem 1]. We simply choose a sequence of polynomials $\{p_k\}$ so that

$$p_k|_{S^{2n-1}} \to g$$

uniformly. It follows that

$$V^{-1}T_{\hat{p}_k}V \to V^{-1}T_{\hat{g}}V$$

in norm. By Lemma 2, $V^{-1}T_{\hat{g}}V$ is in $\tau(\mathbf{B}_{2n})$. Moreover,

$$\pi(\tilde{T}_{p_k}) \to \pi(V^{-1}T_{\hat{g}}V)$$

and

$$\pi(\tilde{T}_{p_k}) \to \pi(\tilde{T}_{g_e})$$

in norm, and the desired result follows.

We now have, for M_r the full algebra of $r \times r$ matrices and matrix Toeplitz operators defined in the obvious way:

Corollary 1. *For g in $C(S^{2n-1}) \otimes M_r, T_{\hat{g}}$ is Fredholm if and only if g is invertible-valued. If g is invertible-valued, then*

$$\mathrm{index}(T_{\hat{g}}) = (-1)^n \text{mapping degree}(g)\ .$$

Proof. Immediate from Theorem 1 above and [V,Theorem 1.5].

For $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$, we write $t_1(z) = z_1$ and

$$t_j(z) = \begin{pmatrix} t_{j-1}(z) & -\bar{z}_jI \\ z_jI & t^*_{j-1}(z) \end{pmatrix},$$

where I is the $2^{j-2} \times 2^{j-2}$ identity matrix and $2 \leqq j \leqq n$. Then the $2^{n-1} \times 2^{n-1}$ matrix function $t_n(z)$ is unitary on the unit sphere S^{2n-1} and generates $K^1(S^{2n-1})$ [V]. Moreover, the entries of $t_n(z)$ are either 0 or polynomials of degree one in $\{z_j, \bar{z}_j : j = 1,2,\ldots,n\}$. It follows that

$$t(z) \equiv |z|^{-1}t_n(z)$$

is a homogeneous function of degree 0 and

$$t|_{S^{2n-1}} \equiv t_n|_{S^{2n-1}} .$$

Corollary 2. *The operator T_t on $H^2(\mathbf{C}^n) \otimes M_{2n-1}$ is Fredholm with*

$$\operatorname{index}(T_t) = (-1)^n .$$

Proof. By Theorem 5.1 of [V], we have $\tilde{T}_{t_n}$ Fredholm with $\operatorname{index}(\tilde{T}_{t_n}) = (-1)^n$. It follows immediately from Theorem 1 that T_t is also Fredholm, with

$$\operatorname{index}(T_t) = \operatorname{index}(\tilde{T}_{t_n}) = (-1)^n .$$

3. Rieffel Deformations

In the interest of completeness, we provide a brief discussion of certian aspects of the Rieffel construction which are central to this application.

Suppose that A is a C^*-algebra on which a vector space V of real dimension $2n$ acts via a group of automorphisms $\alpha = \{\alpha_x : x \in V\}$. Assume that V is equipped with the usual topology which makes it a topological vector space and that the action of α is strongly continuous. That is, for every $a \in A, x \mapsto \alpha_x(a)$ is a continuous map from V to A. Suppose that J is a skew-symmetric operator on V. Rieffel showed in [R] that given such data $\{A, V, \alpha, J\}$, one can always produce a new C^*-algebra A_J by deforming the original product on a smooth subalgebra of A. The C^*-algebra A_J is constructed in the following way.

Let $\mathscr{S}^A$ denote the collection of A-valued functions f on V which, together with its partial derivations of all orders, rapidly decrease to 0 at infinity. For $f \in \mathscr{S}^A$, define

$$\|f\|_2 = \left\| \int_V f(x)^* f(x) dx \right\|^{1/2} .$$

Let A^∞ denote the collection of elements $a \in A$ such that the A-valued function $x \mapsto \alpha_x(a)$ is C^∞ on V. Each $a \in A^\infty$ gives rise to an operator

$$(L_a f)(x) = (2\pi)^{-(\dim V)/2} \int_V \int_V \alpha_{x+Ju}(a) f(x+v) e^{iu \cdot v} dv du$$

on $\mathscr{S}^A$. It is easy to check that for any $a, b \in A^\infty$, we have

$$L_a L_b = L_{a \times_J b} ,$$

where

$$a \times_J b = (2\pi)^{-(\dim V)/2} \int_V \int_V \alpha_{Ju}(a) \alpha_v(b) e^{iu \cdot v} dv du .$$

The above is known as an oscillatory integral and its convergence for $a, b \in A^\infty$ was shown in [R]. Rieffel also showed in [R] that

$$\|a\|_J = \|L_a\| = \sup\{\|L_a f\|_2 : f \in \mathscr{S}^A, \|f\|_2 = 1\}$$

is a C^*-norm on A^∞. Therefore with the product $\times_J$ and the norm $\| \cdot \|_J, A^\infty$ becomes a pre-C^*-algebra. The C^*-algebra A_J, i.e., the Rieffel deformation of A, is defined to be the completion of A^∞ with respect to the norm $\| \cdot \|_J$. We may, of course, also regard A_J as the completion of $\{L_a : a \in A^\infty\}$ with respect to the operator norm on $\mathscr{S}^A$.

Given $\{A, V, \alpha, J\}$ a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ can be constructed on A^∞ as follows. Fix a basis $x_1,\ldots,x_d$ for V so that J is represented by a skew-symmetric matrix (J_{jk}) with respect to this basis. Let $X_1,\ldots,X_d$ be the basis dual to $x_1,\ldots,x_d$ in the Lie algebra L of V. Accordingly, we have the infinitesimal generators $\alpha_{X_1},\ldots,\alpha_{X_d}$ of the automorphism group α. That is, for any $a \in A^\infty$,

$$\alpha_{X_j}(a) = \lim_{t\to 0}\frac{1}{t}(\alpha_{tx_j}(a) - a)\ ,$$

$1 \leqq j \leqq d$. Then

$$\{a,b\} = \sum_{1\leqq j,k\leqq d} J_{jk}\alpha_{X_j}(a)\alpha_{X_k}(b), \quad a,b \in A^\infty\ ,$$

defines a Poisson bracket on A^∞.

For $\hbar$ a real parameter, $\hbar J$ is also a skew-symmetric operator on V. Therefore we also have the deformed product $\times_{\hbar J}$ and the norm $\|\cdot\|_{\hbar J}$ on A^∞. It was shown in [R] that the family $\{(A^\infty, \times_{\hbar J}, \|\cdot\|_{\hbar J}) : 0 < |\hbar| \leqq 1\}$ forms a *strict deformation quantization of* A^∞ *in the direction of* $\{\,\cdot\,,\,\cdot\,\}$ in the following sense:

(1) For every $a \in A^\infty$, the map $\hbar \mapsto \|a\|_{\hbar J}$ is continuous.
(2) For every pair, $a,b \in A^\infty$,

$$\lim_{\hbar\to 0}\left\|\frac{1}{i\hbar}(a \times_{\hbar J} b - ab) - \{a,b\}\right\|_{\hbar J} = 0\ .$$

It turns out that the deformed algebra A_J can be quite interesting even when A itself is rather ordinary. For example, if one takes $A = A(\mathbf{R}^2)$ to be the collection of continuous functions f on $\mathbf{C} = \mathbf{R}^2$ such that the limit

$$\lim_{R\to+\infty} f(Rz)$$

exists uniformly on the circle $\{z \in \mathbf{C} : |z| = 1\}, \alpha$ to be the natural translation of $\mathbf{R}^2$, and $J(x,y) = (y,-x)$, then, as was shown in [R], $A_J = A_J(\mathbf{R}^2)$ is isomorphic to the Toeplitz algebra on the Bergmann space of the unit disc. The main purpose of this paper is to show that the analogous result holds for $\mathbf{C}^n = \mathbf{R}^{2n}$.

We now make precise the connection to Rieffel's construction of $A_J(\mathbf{R}^{2n})$. To ease notation, we will simply write A_J instead of $A_J(\mathbf{R}^{2n})$ for the rest of the section.

From now on we let A be the collection of continuous functions f on $\mathbf{R}^{2n}$ which have the property that the radial limit

$$f_{\text{radial}}(w) = \lim_{R\to+\infty} f(Rw)$$

exists uniformly on the sphere S^{2n-1}. The vector space $V = \mathbf{R}^{2n}$ acts on A by the natural translation. Let J be the standard simplectic operator on $\mathbf{R}^{2n}$. That is,

$$J(x_1,y_1,\ldots,x_n,y_n) = (y_1,-x_1,\ldots,y_n,-x_n)\ .$$

Let A^∞ denote the collection of $f \in A$ such that $z \mapsto f(.+z)$ is a C^∞-map from $\mathbf{R}^{2n}$ into A. Accordingly, $\mathscr{S}^A$ consists of A-valued, smooth functions on $\mathbf{R}^{2n}$ which, together with all their derivatives, are rapidly decreasing. By Rieffel's construction, each $f \in A^\infty$ gives rise to an operator L_f on $\mathcal{S}^A$:

$$(L_f g)(x,z) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} \int_{\mathbf{R}^{2n}} f(z+x+Ju)g(x+v,z)e^{iu\cdot v}dudv \ .$$

(For each $x \in \mathbf{R}^{2n}, g(x, \cdot\,)$ denotes the value of g at x, which is an element in A.) The C^*-algebra A_J is defined to be the completion of $\{L_f : f \in A^\infty\}$ with respect to the norm $\|f\|_J = \|L_f\| = \sup\{\|L_f g\|_{\mathcal{S}A} : g \in \mathcal{S}^A, \|g\|_{\mathcal{S}A} = 1\}$. Let $\mathcal{S}(\mathbf{R}^{2n})$ denote the collection of smooth, rapidly decreasing functions on $\mathbf{R}^{2n}$. For each $f \in A^\infty$ and each $z \in \mathbf{R}^{2n}$, we can define a Weyl operator

$$(W_f^z \eta)(x) = (2\pi)^{-n} \int_{\mathbf{R}^{2n}} \int_{\mathbf{R}^{2n}} f(z+x+Ju)\eta(x+v)e^{iu\cdot v}dudv$$

on $\mathcal{S}(\mathbf{R}^{2n})$. W_f^x extends to a bounded operator on $L^2(\mathbf{R}^{2n})$. Indeed because $\mathcal{S}(\mathbf{R}^{2n})$ is actually a subset of $\mathcal{S}^A$, by the definition of the norm on $\mathcal{S}^A$ and the definition of the L^2-norm, it is obvious that $\|W_f^z\| \leqq \|L_f\|$. The norm $\|W_f^z\|$ is independent of z. This can be seen in the following way.

Define the unitary operator $(U_z\xi)(x) = \xi(x+z)$ on $L^2(\mathbf{R}^{2n})$. Then it is straightforward to verify that

$$U_z W_f^0 U_{-z} = W_f^z \ .$$

This equality also implies that

$$\|W_f^z\| = \|L_f\| \ .$$

Indeed by the definition of the norm on $\mathcal{S}^A$, we have

$$\begin{aligned}\|L_f g\|_{\mathcal{S}A}^2 &= \sup_{z\in\mathbf{R}^{2n}} \int_{\mathbf{R}^{2n}} \left|(2\pi)^{-n} \int_{\mathbf{R}^{2n}} \int_{\mathbf{R}^{2n}} f(z+x+Ju)g(x+v,z)e^{iu\cdot v}dudv\right|^2 dx \\ &= \sup_{z\in\mathbf{R}^{2n}} \|W_f^z g_z\|_2^2 \leqq \|W_f^0\| \sup_{z\in\mathbf{R}^{2n}} \|g_z\|_2^2 = \|W_f^0\|\|g\|_{\mathcal{S}A}^2 \ ,\end{aligned}$$

where g_z denotes the element $g_z(x) = g(x,z)$ in $\mathcal{S}(\mathbf{R}^{2n})$.

It is straightforward to verify that for $f_1, f_2 \in A^\infty$,

$$W_{f_1}^z W_{f_2}^z = W_{f_1\times_J f_2}^z \ ,$$

where $f_1 \times_J f_2$ is the deformation product described earlier. Hence for every $z \in \mathbf{R}^{2n}$, the map

$$\pi_z : L_f \mapsto W_f^z$$

extends to a C^*-algebra isomorphism from A_J onto a subalgebra of $B(L^2(\mathbf{R}^{2n}))$. We will next show that each $\pi_z(A_J)$ is isomorphic to a C^*-algebra of pseudo-differential operators of order zero on $L^2(\mathbf{R}^n)$. Obviously it suffices to do this for the case $z = 0$.

For each $j = 1, \ldots, n$, define

$$\begin{aligned}(M_j^1 f)(s_1, t_1, \ldots, s_n, t_n) &= s_j f(s_1, t_1, \ldots, s_n, t_n) \ , \\ (M_j^2 f)(s_1, t_1, \ldots, s_n, t_n) &= t_j f(s_1, t_1, \ldots, s_n, t_n) \ , \\ (\partial_j^1 f)(s_1, t_1, \ldots, s_n, t_n) &= -i\frac{\partial}{\partial s_j} f(s_1, t_1, \ldots, s_n, t_n) \ ,\end{aligned}$$

and

$$(\partial_j^2 f)(s_1, t_1, \ldots, s_n, t_n) = -i\frac{\partial}{\partial t_j} f(s_1, t_1, \ldots, s_n, t_n) .$$

Furthermore, we define

$$D_j^1 = M_j^1 - \partial_j^2$$

and

$$D_j^2 = M_j^2 + \partial_j^1 ,$$

$j = 1, \ldots, n$. We have the commutation relations

$$D_j^1 D_j^2 - D_j^2 D_j^1 = 2i \tag{1}$$

for all j and

$$D_j^p D_k^q - D_k^q D_j^p = 0 \tag{2}$$

for all $j \neq k$ and $p,\ q = 1,\ 2$. Let e_j^1 (resp. e_j^2) be the vector in $\mathbf{R}^{2n}$ whose $(2j-1)^{\mathrm{st}}$ (resp. $2j^{\mathrm{th}}$) coordinate is 1 and whose other coordinates are 0. Then it is straightforward to verify that

$$(\exp(is_j D_j^1)\eta)(x) = \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(is_j(x_{2j-1} + u_{2j}))\eta(x + v_{2j}e_j^2)\exp(iu_{2j}v_{2j})du_{2j}dv_{2j}$$

and

$$(\exp(it_j D_j^2)\eta)(x)$$
$$= \frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(it_j(x_{2j} - u_{2j-1}))\eta(x + v_{2j-1}e_j^1)\exp(iu_{2j-1}v_{2j-1})du_{2j-1}dv_{2j-1}$$

for every $\eta \in \mathscr{S}(\mathbf{R}^{2n})$. By the commutation relation (1), we have

$$\begin{aligned}
&(\exp(is_j D_j^1)\exp(it_j D_j^2)\eta)(x) \\
&= \exp(is_j D_j^1)\exp(it_j M_j^2)\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(-it_j u_{2j-1})\eta(x + v_{2j-1}e_j^1) \\
&\quad \times \exp(iu_{2j-1}v_{2j-1})du_{2j-1}dv_{2j-1} \\
&= \exp(-is_j t_j)\exp(it_j M_j^2)\exp(is_j D_j^1)\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(-it_j u_{2j-1})\eta(x + v_{2j-1}e_j^1) \\
&\quad \times \exp(iu_{2j-1}v_{2j-1})du_{2j-1}dv_{2j-1} \\
&= \exp(-is_j t_j)\exp(it_j x_{2j})(2\pi)^{-2}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(is_j(x_{2j-1} + u_{2j}))\exp(-it_j u_{2j-1}) \\
&\quad \times \eta(x + v_{2j-1}e_j^1 + v_{2j}e_j^2)\exp\ (i[u_{2j-1}v_{2j-1} + u_{2j}v_{2j}])du_{2j-1}dv_{2j-1}du_{2j}dv_{2j} \\
&= \exp(-is_j t_j)(2\pi)^{-2}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} \exp(i(s_j e_j^1 + t_j e_j^2) \cdot (x + J(u_{2j-1}e_j^1 + u_{2j}e_j^2))) \\
&\quad \times \eta(x + v_{2j-1}e_j^1 + v_{2j}e_j^2)\exp(i(u_{2j-1}e_j^1 + u_{2j}e_j^2) \\
&\quad \cdot (v_{2j-1}e_j^1 + v_{2j}e_j^2))du_{2j-1}dv_{2j-1}du_{2j}dv_{2j}.
\end{aligned}$$

Suppose that $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$. By the commutation relation (2) and the above identity, we have

$$\left(\prod_{j=1}^{n} \exp(is_jD_j^1)\exp(it_jD_j^2)\eta\right)(x)$$
$$= e^{-is \cdot t}(2\pi)^{-n} \int_{\mathbf{R}^{2n}}\int_{\mathbf{R}^{2n}} e^{i(As+Bt) \cdot (x+Ju)}\eta(x+v)e^{iu \cdot v}dudv\,.$$

Here, $A,\ B : \mathbf{R}^n \to \mathbf{R}^{2n}$ are the linear transformations defined by the formulas

$$A(s_1, \ldots, s_n) = (s_1, 0, \ldots, s_n, 0)$$

and

$$B(t_1, \ldots, t_n) = (0, t_1, \ldots, 0, t_n)\,.$$

If $b \in \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)(\cong \mathscr{S}(\mathbf{R}^{2n}))$, then

$$\left(\left[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)\prod_{j=1}^{n} \exp(is_jD_j^1)\exp\ (it_jD_j^2)dsdt\right]\eta\right)(x)$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^{2n}}\int_{\mathbf{R}^{2n}}\left[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)e^{i((As+Bt) \cdot (x+Ju)-s \cdot t)}dsdt\right]\eta(x+v)e^{iu \cdot v}dudv\,.$$

In other words, if we define

$$(\Phi b)(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)e^{i((As+Bt) \cdot x-s \cdot t)}dsdt\,, \tag{3}$$

then

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)\prod_{j=1}^{n} \exp(is_jD_j^1)\exp(it_jD_j^2)dsdt = W_{\Phi b}^0\,.$$

Suppose now that $a, b \in \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$. Then, by (1) and (2), we have

$$W^0_{(\Phi a)\times_J(\Phi b)} = W^0_{\Phi a}W^0_{\Phi b}$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s,t)\left[\prod_{j=1}^{n}\exp(is'_jD_j^1)\exp(it'_jD_j^2)\right.$$
$$\left.\times \exp(is_jD_j^1)\exp(it_jD_j^2)\right]ds'dt'dsdt$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s,t)\left[\prod_{j=1}^{n}\exp(is'_jD_j^1)\exp(is_jD_j^1)\ \exp(it'_jD_j^2)\exp(2is_jt'_j)\right.$$
$$\left.\times \exp(it_jD_j^2)\right]ds'dt'dsdt$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s,t)e^{2is \cdot t'}\left[\prod_{j=1}^{n}\exp(i(s'_j+s_j)D_j^1)\right.$$
$$\left.\exp(i(t'_j+t_j)D_j^2)\right]ds'dt'dsdt$$

$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \left[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s-s',t-t')e^{2i(s-s')\cdot t'}ds'dt'\right]$$
$$\times \left[\prod_{j=1}^{n} \exp(is_jD_j^1)\exp(it_jD_j^2)\right]dsdt$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (a*b)(s,t)\left[\prod_{j=1}^{n} \exp(is_jD_j^1)\exp(it_jD_j^2)\right]dsdt\ ,$$

where

$$(a*b)(s,t) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s-s',t-t')e^{2i(s-s')\cdot t'}ds'dt'\ .$$

Hence

$$(\Phi a)\times_J(\Phi b) = \Phi(a*b)\ .$$

We will now establish the relations between W_j^x and the Weyl operators on $L^2(\mathbf{R}^n)$. Following [R], we define a Weyl operator on $L^2(\mathbf{R}^n)$ with symbol function $\alpha \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$ by the formula

$$(\Psi_\alpha\xi)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \alpha(x+y,v)e^{i(x-y)\cdot v}\xi(y)dydv$$

for $\xi \in \mathcal{S}(\mathbf{R}^n)$. Also recall that for $\alpha,\beta \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$, the Weyl symbol calculus $\alpha\times_W\beta$ is defined by the relation

$$\Psi_{\alpha\times_W\beta} = \Psi_\alpha\Psi_\beta\ .$$

This relation is still valid if one of α, β is only a function in A^∞.

It follows from (1) and (2) that there is a unitary operator $Y : L^2(\mathbf{R}^{2n}) \to L^2(\mathbf{R}^{2n})$ such that

$$YD_j^1Y^* = 2M_j^1 \quad \text{and} \quad YD_j^2Y^* = \partial_j^1\ . \tag{4}$$

(Because of (2), one only needs to prove this in the case $n = 1$. But in this case such a Y can be constructed explicitly. See, for example, [X].) We can identify M_j^1 with $m_j \otimes 1$ and ∂_j^1 with $d_j \otimes 1$, where m_j and d_j are the operators

$$(m_jf)(x_1,\ldots,x_n) = x_jf(x_1,\ldots,x_n)$$

and

$$(d_jf)(x_1,\ldots,x_n) = -i\frac{\partial}{\partial x_j}f(x_1,\ldots,x_n)$$

on $L^2(\mathbf{R}^n)$. Thus, we have shown that for any $b \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^n)$,

$$YW_{\Phi b}^0Y^* = \left[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)\prod_{j=1}^{n} \exp\ (2is_jm_j)\exp(it_jd_j)dsdt\right]\otimes 1\ . \tag{5}$$

Since

$$b(s,t) = \exp(is\cdot t)(2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (\Phi b)(Au+Bv)e^{-i[s\cdot u+t\cdot v]}dudv\ ,$$

for any $\xi \in \mathscr{S}(\mathbf{R}^n)$, we have

$$\left(\left[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)\prod_{j=1}^{n} \exp\,(2is_jm_j)\exp(it_jd_j)dsdt\right]\xi\right)(x)$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} b(s,t)e^{2is\cdot x}\xi(x+t)dsdt$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{2is\cdot x}[e^{is\cdot t}$$
$$\times \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\Phi b)(Au+Bv)e^{-i[s\cdot u+t\cdot v]}dudv]\xi(x+t)dsdt$$

$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\Phi b)(Au+Bv)e^{-i[s\cdot(u-x-t)+(t-x)\cdot v]}\xi(t)dudvdsdt$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\Phi b)(A(u+t+x)+Bv)e^{-i[s\cdot u+(t-x)\cdot v]}\xi(t)dudvdsdt$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\Phi b)(A(t+x)+Bv)e^{i(x-t)\cdot v}\xi(t)dvdt$$
$$= (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(T\Phi b)(t+x,\,v)e^{i(x-t)\cdot v}\xi(t)dvdt\ .$$

Here, we use the notation

$$(TF)(u,v) = F(Au+Bv),\ \ u,\ \ v\in\mathbf{R}^n\ ,$$

for functions F defined on $\mathbf{R}^{2n}$ ([1]). Hence it follows from this calculation and (5) that

$$YW^0_{\Phi b}Y^* = \Psi_{T\Phi b}\otimes 1\ . \tag{6}$$

It follows from (4) and the commutation relations (1) and (2) that, for any $a,b\in\mathscr{S}(\mathbf{R}^n\times\mathbf{R}^n)$,

$$\Psi_{(T\Phi a)\times_W(T\Phi b)} = \Psi_{T\Phi a}\Psi_{T\Phi b}$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} a(s',t')b(s,t)$$
$$\times\left[\prod_{j=1}^{n}\exp(2is'_jm_j)\exp(it'_jd_j)\exp(2is_jm_j)\exp(it_jd_j)\right]ds'dt'dsdt$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(a*b)(s,t)\left[\prod_{j=1}^{n}\exp(2is_jm_j)\exp(it_jd_j)\right]dsdt$$
$$= \Psi_{T\Phi(a*b)} = \Psi_{T[(\Phi a)\times_J(\Phi b)]}\ .$$

That is,

$$(T\Phi a)\times_W(T\Phi b) = T[(\Phi a)\times_J(\Phi b)]\ .$$

---

[1] It may seem that the introduction of T does nothing but create inconvenience. After all, why don't we simply identify $F(u,v)$ and $F(Au+Bv)$? What is implicit in such an identification of functions, however, is an identification of $\mathbf{R}^n\times\mathbf{R}^n$ with $\mathbf{R}^{2n}$ through a rearrangement of coordinates. There are $(2n)!$ ways to do so. The presence of T serves as a reminder of our particular rearrangement of coordinates which was actually dictated by the choice of the operator J.

If $f \in A^\infty$ (technically, elements of A^∞ are functions on $\mathbf{R}^{2n}$, not functions of the form $F(u,v)$, u, $v \in \mathbf{R}^n$), then for any $b \in \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$, we also have

$$(Tf) \times_W (T\Phi b) = T[f \times_J (\Phi b)]$$

and

$$\Psi_{(Tf)\times_W(T\Phi b)} = \Psi_{Tf}\Psi_{T\Phi b} \ .$$

Hence it follows from (6) that

$$\begin{aligned} YW^0_f W^0_{\Phi b} Y^* &= YW^0_{f\times_J(\Phi b)}Y^* = \Psi_{T[f\times_J(\Phi b)]} \otimes 1 \\ &= \Psi_{(Tf)\times_W(T\Phi b)} \otimes 1 = [\Psi_{Tf}\Psi_{T\Phi b}] \otimes 1 \end{aligned}$$

for any $f \in A^\infty$ and $b \in \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$. By choosing a sequence $\{b_n\} \subset \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$ such that $\{\Psi_{T\Phi b_n}\}$ converges to the identity operator strongly, we may conclude that

$$YW^0_f Y^* = \Psi_{Tf} \otimes 1 \ .$$

It follows from Rieffel's construction that $\{L_f : f \in A^\infty\}$ is dense in A_J. Hence the map

$$\pi_J : L_f \mapsto \Psi_{Tf}, \ f \in A^\infty \ ,$$

extends to a C^*-algebra isomorphism from A_J into $B(L^2(\mathbf{R}^n))$.

For f a bounded function (or matrix of such functions) on $\mathbf{C}^n$, we recall that the *Bargmann isometry* [B; F, p. 40; BC3]

$$B : L^2(\mathbf{R}^n, dv) \to H^2(C^n, d\mu)$$

has the property that

$$B^{-1}T_f B = W_\beta \ ,$$

where

$$\beta(\xi,x) = \tilde{f}(x - i\xi) = \frac{1}{\pi^n}\int_{\mathbf{C}^n} f(w)e^{-|w-(x-i\xi)|^2}dv(w), \quad x, \ \xi \in \mathbf{R}^n \ ,$$

is defined to be the solution of the heat equation at time $t = 1/4$ with initial-value f, and W_β is the Weyl operator on $L^2(\mathbf{R}^n, dv)$ given by

$$(W_\beta g)(x) = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \beta\left(\xi, \frac{x+y}{2}\right)e^{i(x-y)\cdot\xi}g(y)dyd\xi$$

[F, p. 141; BC3].

The definition of the Weyl operator W_β with the symbol function β is slightly different from the definition of Ψ_β. However the two sets of pseudo-differential operators $\{\Psi_{Tf} : f \in A^\infty\}$ and $\{W_{Tf} : f \in A^\infty\}$ are identical and, therefore, generate the same C^*-algebra. This fact can be seen from a transformation on A. For any $\lambda > 0$, define the linear operator S_λ on $\mathbf{R}^{2n}$ by the formula $S_\lambda(x_1,\ldots,x_n,x_{n+1},\ldots,x_{2n}) = (x_{n+1},\ldots,x_{2n},\lambda x_1,\ldots,\lambda x_n)$. Also define $(S_{\lambda *}f)(x) = f(S_\lambda x)$ for $f \in A$. If we denote $(\alpha_z(f))(x) = f(x+z)$, then we have $\alpha_z(S_{\lambda *}f) = S_{\lambda *}\alpha_{S_\lambda z}(f)$. Hence if $z \mapsto \alpha_z(f)$ is an A-valued C^∞-function, then so are $z \mapsto \alpha_{S_\lambda z}(f)$ and $z \mapsto S_{\lambda *}\alpha_{S_\lambda z}(f)$. In other words, the operator $S_{\lambda *}$ maps A^∞ to itself. For any $f \in A^\infty$, we have

$$\Psi_{TS_{1/2*}f} = W_{Tf} \quad \text{and} \quad W_{TS_{2*}f} = \Psi_{Tf} \ . \tag{7}$$

Let I denote the ideal $C_0(\mathbf{R}^{2n})$ in A. We have an exact sequence

$$0 \to I \to A \to C(S^{2n-1}) \to 0 \ .$$

It is clear that I is invariant under the action of $\mathbf{R}^{2n}$ and that the induced action of $\mathbf{R}^{2n}$ on the quotient algebra $C(S^{2n-1})$ is trivial. Hence by Theorem 7.7 of [R], we have an induced exact sequence

$$0 \to I_J \to A_J \to C(S^{2n-1}) \to 0 \ .$$

If π denotes the quotient map from A_J onto $C(S^{2n-1})$, then, by Rieffel's construction,

$$\pi(L_f) = f_{\text{radial}}$$

for every $f \in A^\infty$. We have shown that π_J is an isomorphism from A_J onto the C^*-algebra generated by $\{\Psi_{Tf} : f \in A^\infty\}$. Because $\pi_J(L_f + I_J) = \Psi_{Tf} + \pi_J(I_J)$, we see that the image of Ψ_{Tf} under the quotient map $\pi_J(A_J) \to \pi_J(A_J)/\pi_J(I_J)$ is also f_{radial} for every $f \in A^\infty$.

The ideal $\pi_J(I_J)$, which is generated by $\{\Psi_{Tf} : f \in I^\infty\}$, is the collection of compact operators on $L^2(\mathbf{R}^n)$. In fact, if $\alpha \in \mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$, then obviously Ψ_α is a compact operator. From this it is easy to see that $\pi_J(I_J)$ at least contains all the compact operators on $L^2(\mathbf{R}^n)$. Now, because I_J is the completion of I^∞ with respect to the J-norm, to establish that every operator in $\pi_J(I_J)$ is compact, it suffices to verify that $\{L_f : f \in \mathscr{S}(\mathbf{R}^{2n})\}$ is dense in $\{L_f : f \in I^\infty\}$. For this purpose, we fix a C^∞-function $0 \leqq \eta \leqq 1$ on $[0, \infty)$ such that $\eta = 1$ on $[0,1]$ and such that $\eta = 0$ on $[2, \infty)$. For each $k \in \mathbf{Z}_+$, define $\eta_k(t) = 1$ if $|t| \leqq k$ and $\eta_k(t) = \eta(|t| - k)$ if $|t| > k$. Let

$$\xi_k(t_1, \ldots, t_{2n}) = \eta_k(t_1) \ldots \eta_k(t_{2n}) \ .$$

Then straightforward differentiation shows that for any $f \in I^\infty$, any mixed partial derivative of $(1 - \xi_k)f$ of arbitrary order tends to zero uniformly on $\mathbf{R}^{2n}$ as $k \to \infty$. By Proposition 4.10 of [R], this means

$$\lim_{k \to \infty} \|L_{\xi_k f} - L_f\| = 0 \ .$$

Hence $\{L_f : f \in \mathscr{S}(\mathbf{R}^{2n})\}$ is dense in $\{L_f : f \in I^\infty\}$ and $\pi_J(I_J)$ consists of the compact operators on $L^2(\mathbf{R}^n)$.

4. Main Result

We have, for Rieffel's algebra $A_J(\mathbf{R}^{2n})$ discussed in Sect. 3,

Theorem 2. *The C^*-algebras $A_J(\mathbf{R}^{2n})$ and $\tau(\mathbf{B}_{2n})$ are $*$-isomorphic via $\mathrm{ad}_V^{-1} \circ \mathrm{ad}_B \circ \pi_J$. Here, as usual, ad_U denotes the conjugation by U.*

Proof. Recall that every element in $\tau(\mathbf{B}_{2n})$ is the sum of a Toeplitz operator whose symbol is continuous on $\mathbf{B}_{2n}$ and a compact operator. We also recall that for f in $C(S^{2n-1})$,

$$B^{-1} T_{\hat{f}} B = W_{\alpha_f} \ ,$$

where W_β was defined earlier as was the Bargmann isometry B and

$$\alpha_f(\xi, x) = \tilde{\tilde{f}}(x - i\xi) .$$

We consider the C^*-algebra Weyl(S^{2n-1}) generated by the full algebra $\mathscr{K}$ of compact operators on $L^2(\mathbf{R}^n)$ and the operators

$$\{W_{\alpha_f} : f \in C(S^{2n-1})\} .$$

Using Theorem 1, it is not hard to check directly that conjugation by the unitary operator

$$B^{-1}V : H^2(\mathbf{B}_{2n}) \to L^2(\mathbf{R}^n)$$

implements an isomorphism from Weyl(S^{2n-1}) on $L^2(\mathbf{R}^n)$ onto $\tau(\mathbf{B}_{2n})$ on $H^2(\mathbf{B}_{2n})$.

In Sect. 3, we checked that Rieffel's algebra $A_J(\mathbf{R}^{2n})$ is $*$-isomorphic via π_J to the C^*-algebra $\pi_J(A_J(\mathbf{R}^{2n}))$ on $L^2(\mathbf{R}^n)$ generated by the operators

$$(\Psi_{T\gamma}g)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (T\gamma)(x + y, \ \xi) e^{i(x-y)\cdot\xi} g(y) dy d\xi ,$$

where γ is in A^∞ with (uniform) radial limit function γ_{radial} in $C(S^{2n-1})$. Moreover, comparing [R] and Sect. 3, $\pi(\Psi_{T\gamma}) = \gamma_{\text{radial}}$.

Because α_f is convolution of $\hat{f}$ with a Gaussian, it is easy to check that $T^{-1}\alpha_f$ is in A^∞ of [R] for arbitrary f in $C(S^{2n-1})$. This fact was already observed for $n = 1$ in [R]. Using Eq. (7) of Sect. 3 and the fact that

$$(\alpha_f)_{\text{radial}}(x, \xi) = f(\xi, -x) ,$$

it is easy to check that Weyl$(S^{2n-1}) = \pi_J(A_J(\mathbf{R}^{2n}))$. Hence, $A_J(\mathbf{R}^{2n})$ and $\tau(\mathbf{B}_{2n})$ are $*$-isomorphic.

5. Problems and Remarks

It would be of some interest to know if other standard Toeplitz algebras arise as strict deformation quantizations of commutative algebras. In this connection, we should mention [BLU] where an "intrinsic" Toeplitz quantization on bounded symmetric domains is described following earlier work of [KL] and [C2]. The Toeplitz quantization of [KL, C2], [BLU] satisfies a weaker version of the strict deformation conditions required by [R].

Problem 1. Can the Toeplitz algebra on the polydisc, $\tau(\mathbf{D} \times \mathbf{D})$, be realized as a strict deformation quantization of $C(\mathbf{D} \times \mathbf{D})$,

While we have exhibited a $*$-isomorphism

$$\mu : A_J(\mathbf{R}^{2n}) \to \tau(\mathbf{B}_{2n}) ,$$

it is not obvious precisely what elements of $\tau(\mathbf{B}_{2n})$ are in $\mu\{A_J^\infty(\mathbf{R}^{2n})\}$.

Problem 2. Can $\mu\{A_J^\infty(\mathbf{R}^{2n})\}$ be precisely identified?

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