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# Chern–Simons Theory, Coloured-Oriented Braids and Link Invariants

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**Abstract:** A method to obtain explicit and complete topological solution of SU(2) Chern–Simons theory on $S^3$ is developed. To this effect the necessary aspects of the theory of coloured-oriented braids and duality properties of conformal blocks for the correlators of $SU(2)_k$ Wess–Zumino conformal field theory are presented. A large class of representations of the generators of the groupoid of coloured-oriented braids are obtained. These provide a whole lot of new link invariants of which Jones polynomials are the simplest examples. These new invariants are explicitly calculated as illustrations for knots up to eight crossings and two-component multicoloured links up to seven crossings.

## 1. Introduction

Topological quantum field theories provide a bridge between quantum physics on one hand and geometry and topology of low dimensional manifolds on the other [1]. The functional integral formulation of such quantum field theories provides a framework to study this relationship. In particular, a class of topological field theories which are related to knot theory have attracted a good deal of attention in recent times. This started with the seminal work of Witten who not only put the Jones polynomials [2] in a field theoretic setting, but also presented a general field theoretic framework in which knot theory could be studied in an arbitrary three-manifold [3].

In $SU(2)$ Chern–Simons gauge theory, the expectation value of Wilson link operators with doublet representation placed on all the component knots yields Jones polynomials. Two variable generalization of these polynomials, the so-called HOMFLY polynomials [4], are obtained as the expectation value of Wilson link operators with $N$ dimensional representation on all the component knots in an $SU(N)$ Chern–Simons theory. In fact Witten [3] has shown that the expectation values of such link operators obey the same Alexander–Conway skein relation as those by Jones and HOMFLY polynomials respectively. These relations can be recursively solved to obtain these polynomials for an arbitrary link. Placing arbitrary representations on the component knots, corresponding generalizations of Alexander–Conway relations can also be obtained [5, 6]. But unfortunately these relations can not be solved recursively to obtain the link invariants. Therefore there is a need to develop methods which would allow direct calculations of expectation values of Wilson link operators with arbitrary representations living on the component knots. In refs. 6, an attempt was made to develop one such method. This allowed us to obtain invariants for links that can be constructed from braids made of up to four strands. However, links related to braids with larger number of strands still stayed elusive. Another interesting method based on the construction of knot operators has also been developed [7]. It allows readily calculation of invariants for torus knots in an elegant way. However, the scope of this method also appears to be limited and it cannot be applied to obtain invariants for other knots.

In this paper we shall present a general and simple method of obtaining the expectation value for an arbitrary Wilson link operator in an $SU(2)$ Chern–Simons gauge theory on $S^3$. The method can be generalised to other compact non-abelian gauge groups as well as to manifolds other than $S^3$.

The $SU(2)$ Chern–Simons action is given by:

$$kS=\frac{k}{4\pi}\int_{S^3}\mathrm{tr}\left(AdA+\frac{2}{3}A^3\right), \tag{1.1}$$

where $A$ is a matrix valued connection one-form of the gauge group $SU(2)$. The topological operators of this topological field theory are given in terms of Wilson loop (knot) operators:

$$W_j[C]=\mathrm{tr}_j P\exp\oint_C A \tag{1.2}$$

for an oriented knot $C$ carrying spin $j$ representation. These operators are independent of the metric of the three-manifold. For a link $L$ made up of oriented component knots $C_1$, $C_2, \ldots, C_s$ carrying spin $j_1, j_2, \ldots j_s$ representations respectively, we have the Wilson link operator defined as

$$W_{j_1 j_2\ldots j_s}[L]=\prod_{l=1}^{s} W_{j_l}[C_l]\,. \tag{1.3}$$

We are interested in the functional averages of these operators:

$$V_{j_1 j_2\ldots j_s}[L]=Z^{-1}\int_{S^3}[dA]\,W_{j_1 j_2\ldots j_s}[L]\,e^{ikS}\,, \tag{1.4}$$

$$Z=\int_{S^3}[dA]e^{ikS}\,. \tag{1.5}$$

Here both the integrands in the functional integrals as well as the measure are metric independent [8]. Therefore, these expectation values depend only on the isotopy type of the oriented link $L$ and the set of representations $j_1, j_2 \ldots j_s$ associated with the component knots. The partition function above is given by [3]:

$$Z=\sqrt{2/(k+2)}\sin(\pi/(k+2))\,.$$

The method of calculating the functional averages (1.4) developed in the present paper makes use of two important ingredients:

(i) The first ingredient is the intimate relationship that Chern–Simons theory on a three-manifold with boundary has with corresponding Wess–Zumino conformal field theory on that boundary [3, 9, 7]. Consider a 3-manifold with a number of two dimensional boundaries $\Sigma^{(1)}, \Sigma^{(2)}, \ldots \Sigma^{(r)}$. Each of these boundaries, $\Sigma^{(i)}$ may have a number of Wilson lines carrying spins $j_l^{(i)}$, $l=1, 2, \ldots$ . ending or beginning at some points (punctures) $P_l^{(i)}$ on them. Following Witten [3], we associate with each boundary $\Sigma^{(i)}$ a Hilbert space $\mathscr{H}^{(i)}$. The Chern–Simons functional integral over such a three-manifold is then given as a state in the tensor product of these Hilbert spaces, $\otimes^r \mathscr{H}^{(i)}$. The operator formalism developed in ref. 7, gives an explicit representation of these states as well as determines the form of the inner products of vectors belonging to these Hilbert spaces. The conformal blocks of the $SU(2)_k$ Wess–Zumino field theory on these boundaries $\Sigma^{(i)}$ with punctures $P_l^{(i)}, l=1, 2, \ldots$ carrying spins $j_l^{(i)}$ determine the properties of the associated Hilbert spaces $\mathscr{H}^{(i)}$. In fact, these provide a basis for these Hilbert spaces $\mathscr{H}^{(i)}$. There are more than one possible bases. These different bases are related by duality of the correlators of the Wess–Zumino conformal field theory. We shall need to write down these duality matrices explicitly for our discussion here.

(ii) The second input we shall need is the close connection knots and links have with braids. Theory of braids, first developed by Artin, is generally studied for identical strands [10, 11]. What we need for our purpose here is instead a theory of coloured and oriented braids. The individual strands are coloured by putting different $SU(2)$ spins on them. The necessary aspects of the theory of such braids will be developed here. In particular a theorem due to Birman [11] relating links to plats of braids will be restated for coloured-oriented braids. This theorem along with the duality properties of conformal blocks of correlators in $SU(2)_k$ Wess–Zumino conformal field theory on an $S^2$ then will allow us to present an explicit and complete solution of $SU(2)$ Chern–Simons gauge theory on $S^3$. Alternatively, a theorem due to Alexander relating closure of braids to links can also be stated for coloured-oriented braids. This theorem also provides an equivalent method of solving Chern–Simons gauge theory.

The knot invariants have also been extensively studied from the point of view of exactly solvable models in statistical mechanics [12, 14]. Wadati et al. have exploited the intimate connection between exactly solvable lattice models with knot invariants to obtain a general method for constructing such invariant polynomials [14]. Besides these, knot invariants have also been studied from the point of view of quantum groups [15].

This paper is organised as follows. In Sect. 2, we shall write down the duality matrices relating two convenient complete sets of conformal blocks for the $SU(2)_k$ Wess–Zumino conformal field theory on an $S^2$ with $2m$ ($m=2, 3, \ldots$) punctures carrying arbitrary $SU(2)$ spins. Next, in Sect. 3, the required aspects of the theory of coloured-oriented braids will be developed. A theorem due to Birman [11] relating oriented links with plats will be restated for plats of coloured-oriented braids. Alexander's theorem [16] relating closure of braids with oriented links can also be restated for coloured-oriented links. In Sect. 4, a class of representations of the generators of braid groupoids will be presented. These ingredients then will allow us to write down the complete and explicit solution of $SU(2)$ Chern–Simons theory on $S^3$. This will be presented in terms of a theorem which gives the expectation values of Wilson link operators (1.4) in terms of properties of the plat of a corresponding coloured-oriented braid. This main theorem and the sketch of its proof has already been announced in ref. 17. Here we are presenting the details of the proof as well as discussing some other implication of the theorem. For example, a corresponding theorem which alternatively yields the link invariants in terms of closure of oriented-coloured braids will also be present in Sect. 4. Next in Sect. 5, we shall illustrate how the main theorem can be used to write down the link invariants. This we do by discussing a multicoloured three component link, the Borromean rings. In Sect. 6, a few concluding remarks will be made. Appendix I will contain explicit formulae for the duality matrices in terms of $q$-Racah coefficients of $SU(2)_q$ needed in the main text. Invariants for knots up to eight crossings and multicoloured two component links up to seven crossings as given in the tables of Rolfsen [18] will be listed in Appendix II.

## 2. Duality of Correlators in $SU(2)_k$ Wess–Zumino Conformal Field Theory

To develop the solution for $SU(2)$ Chern–Simons theory on $S^3$, we need to make use of duality properties [19] of correlators of $SU(2)_k$ Wess–Zumino conformal field theory on an $S^2$. We now list these properties.

Four-point correlators for primary fields with spins $j_1, j_2, j_3$ and $j_4$ (such that these combine into an $SU(2)$ singlet) can be represented in three equivalent ways. Two such ways are given by Figs. 1(a) and (b). In the first, each of pairs of spins $j_1, j_2$ and $j_3, j_4$ is combined into common spin $j$ representation according to the fusion rules of the $SU(2)_k$ Wess–Zumino model. Then these two spin $j$ representations combine to give singlets. For sufficiently large values of $k$, allowed values of $j$ are those given by group theory: $\max(|j_1-j_2|, |j_3-j_4|) \leqq j \leqq \min(j_1+j_2, j_3+j_4)$. In the second equivalent representation for the four-point correlators spins $(j_2, j_3)$ and $(j_1, j_4)$ are first combined into common intermediate spin $l$ representation and then two spin $l$ representations yield singlets, with $\max(|j_2-j_3|, |j_1-j_4|) \leqq l \leqq \min(j_2+j_3, j_1+j_4)$ for sufficiently large $k$. These two sets of linearly independent but equivalent representations will be called $\phi_j(j_1j_2j_3j_4)$ and $\phi'_l(j_1j_2j_3j_4)$ respectively. These are related to each other by duality:

$$\phi_j(j_1j_2j_3j_4) = \sum_l a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} \phi'_l(j_1j_2j_3j_4)\,, \tag{2.1}$$

where the duality matrices $a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$ are given [15, 19, 6] in terms of $q$-Racah coefficients for $SU(2)_q$. We have listed these and some of their useful properties explicitly in Appendix I. This fact that these two bases are related by $q$-Racah coefficients is not surprising. The representation theory of integrable representations of $SU(2)_k$ Wess–Zumino field theory is the same as that of $SU(2)_q$ with the deformation parameter as $q = \exp(-2\pi i/(k+2))$.

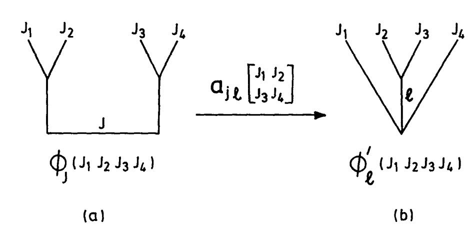


**Fig. 1.** Two ways of combining four spins into singlets

The duality transformation (2.1) can be successively applied to obtain duality properties of higher correlators. In particular, we shall be interested in the two equivalent sets of correlators for $2m$ primary fields with spin assignments $j_1, j_2, \ldots, j_{2m}$, $\phi_{(p;r)}(j_1 j_2 \ldots j_{2m})$ and $\phi'_{(q;s)}(j_1 j_2 \ldots j_{2m})$ as shown in Fig. 2(a) and (b) respectively. Here indices $(p)$ and $(r)$ collectively represent the spins $(p_0 p_1 \ldots p_{m-1})$ and $(r_1 r_2 \ldots r_{m-3})$ on the internal lines respectively as shown in Fig. 2(a). Similarly, $(q)=(q_0 q_1 \ldots q_{m-1})$ and $(s)=(s_1 s_2 \ldots s_{m-3})$ in Fig. 2(b). These two figures represent two equivalent ways of combining spins $j_1, j_2 \ldots j_{2m}$ into singlets and are related by duality. This fact we now present in the form of a theorem:

**Theorem 1.** *The correlators for* $2m$ *primary fields with spins* $j_1, j_2 \ldots j_{2m}$, $(m \geqq 2)$ *in* $SU(2)_k$ *Wess–Zumino conformal field theory on an* $S^2$ *as shown in Figs.* 2(*a*) *and* (*b*) *are related to each other by*

$$\phi_{(p;r)}(j_1 j_2 \ldots j_{2m}) = \sum_{(q;s)} a_{(p;r)(q;s)} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} \phi'_{(q;s)}(j_1 j_2 \ldots j_{2m}), \tag{2.2}$$

*where the duality matrices are given as products of the basic duality coefficients for the four-point correlators* (2.1) *as*

$$a_{(p;r)(q;s)} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} = \sum_{t_1 t_2 \ldots t_{m-2}} \prod_{i=1}^{m-2} \left( a_{t_i p_i} \begin{bmatrix} r_{i-1} & j_{2i+1} \\ j_{2i+2} & r_i \end{bmatrix} a_{t_i s_{i-1}} \begin{bmatrix} t_{i-1} & q_i \\ s_i & j_{2m} \end{bmatrix} \right)$$
$$\times \prod_{l=0}^{m-2} a_{r_l q_{l+1}} \begin{bmatrix} t_l & j_{2l+2} \\ j_{2l+3} & t_{l+1} \end{bmatrix}. \tag{2.3}$$

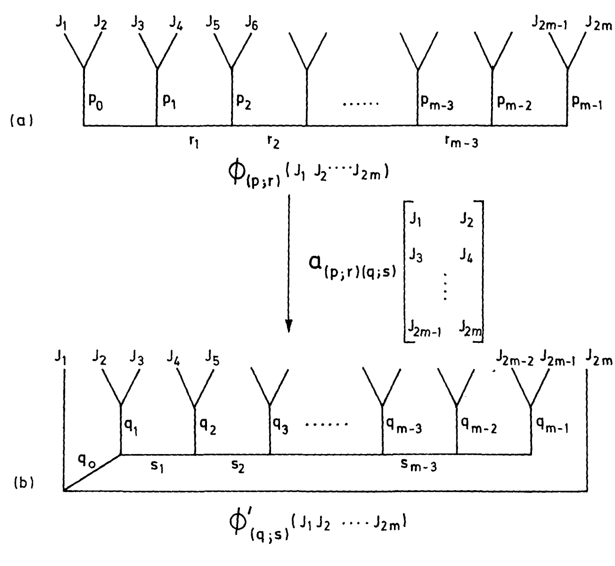


**Fig. 2.** Two ways of combining $2m$ spins into singlets

Here $r_0 \equiv p_0$, $r_{m-2} \equiv p_{m-1}$, $t_0 \equiv j_1$, $t_{m-1} \equiv j_{2m}$, $s_0 \equiv q_0$ and $s_{m-2} \equiv q_{m-1}$ and spins $\vec{j}_1 + \vec{j}_2 + \ldots + \vec{j}_{2m-1} = \vec{j}_{2m}$ and the spins meeting at trivalent points in Fig. 2 satisfy the fusion rules of the $SU(2)_k$ conformal field theory.

Using the properties of the matrices $a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$ as given in Appendix I, we can readily see that the duality matrices (2.3) satisfy the following orthogonality property:

$$\sum_{(p;r)} a_{(p;r)(q;s)}\begin{bmatrix} j_1 & j_2 \\ & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} a_{(p;r)(q';s')}\begin{bmatrix} j_1 & j_2 \\ & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} = \delta_{(q,q')}\delta_{(s,s')} \,. \tag{2.4}$$

The proof of Theorem 1 is rather straightforward. It can be developed by applying the duality transformation (2.1) successively on the $2m$-point correlators. For example, 6-point correlators, represented by $\phi_{(p_0p_1p_2)}$ and $\phi'_{(q_0q_1q_2)}$, are related by a sequence of four duality transformations each involving four spins at a time as shown in Fig. 3. Thus

$$\phi_{(p_0p_1p_2)}(j_1 \ldots j_6) = \sum_{q_0q_1q_2} a_{(p_0p_1p_2)(q_0q_1q_2)}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ j_5 & j_6 \end{bmatrix} \phi'_{(q_0q_1q_2)}(j_1 \ldots j_6)\,, \tag{2.5}$$

where

$$a_{(p_0p_1p_2)(q_0q_1q_2)}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ j_5 & j_6 \end{bmatrix} = \sum_{t_1} a_{t_1p_1}\begin{bmatrix} p_0 & j_3 \\ j_4 & p_2 \end{bmatrix} a_{p_0q_1}\begin{bmatrix} j_1 & j_2 \\ j_3 & t_1 \end{bmatrix} \times a_{p_2q_2}\begin{bmatrix} t_1 & j_4 \\ j_5 & j_6 \end{bmatrix} a_{t_1q_0}\begin{bmatrix} j_1 & q_1 \\ q_2 & j_6 \end{bmatrix}. \tag{2.6}$$

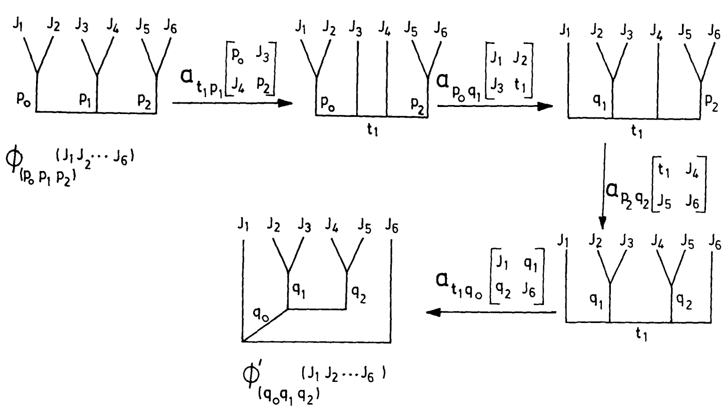

**Fig. 3.** Duality transformation of 6-point correlators

Similarly for 8-point correlators $\phi_{(p_0 p_1 p_2 p_3; r_1)}(j_1 \,..\, j_8)$ and $\phi'_{(q_0 q_1 q_2 q_3; s_1)}(j_1 \,..\, j_8)$, which are related by a sequence of seven four-point duality transformations as shown in Fig. 4, we have

$$\phi_{(p_0 \,...\, p_3; r_1)}(j_1 \,...\, j_8) = \sum_{(q;s)} a_{(p_0 \,...\, p_3; r_1)(q_0 \,...\, q_3; s_1)} \begin{bmatrix} j_1 & j_2 \\ & \vdots \\ j_7 & j_8 \end{bmatrix} \phi'_{(q_0 \,...\, q_3; s_1)}(j_1 \,...\, j_8)$$

with

$$a_{(p_0 \,...\, p_3; r_1)(q_0 \,...\, q_3; s_1)} \begin{bmatrix} j_1 & j_2 \\ \vdots & \\ j_7 & j_8 \end{bmatrix} = \sum_{t_1 t_2} a_{t_2 p_2} \begin{bmatrix} r_1 & j_5 \\ j_6 & p_3 \end{bmatrix} a_{t_1 p_1} \begin{bmatrix} p_0 & j_3 \\ j_4 & r_1 \end{bmatrix} a_{p_0 q_1} \begin{bmatrix} j_1 & j_2 \\ j_3 & t_1 \end{bmatrix}$$
$$\times a_{r_1 q_2} \begin{bmatrix} t_1 & j_4 \\ j_5 & t_2 \end{bmatrix} a_{p_3 q_3} \begin{bmatrix} t_2 & j_6 \\ j_7 & j_8 \end{bmatrix}$$
$$\times a_{t_2 s_1} \begin{bmatrix} t_1 & q_2 \\ q_3 & j_8 \end{bmatrix} a_{t_1 q_0} \begin{bmatrix} j_1 & q_1 \\ s_1 & j_8 \end{bmatrix}. \qquad (2.7)$$

Clearly, in this manner Theorem 1 for arbitrary $2m$-point correlators follows.

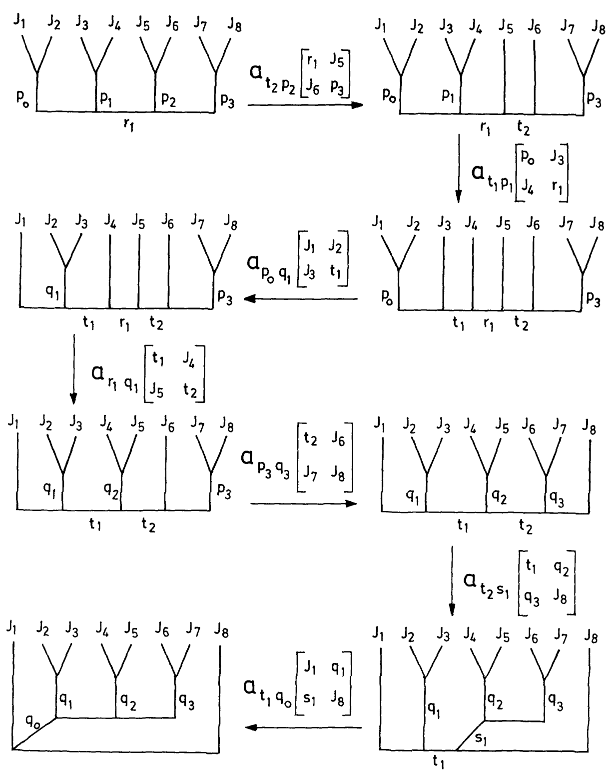


**Fig. 4.** Duality transformation of 8-point correlators

These duality properties will be made use of in Sect. 4 to obtain the solution of $SU(2)$ Chern–Simons theory. But before we do that, we need to discuss the other ingredient necessary for our purpose. As stated earlier, this has to do with the theory of coloured-oriented braids.

## 3. Coloured-Oriented Braids

An $n$-braid is a collection of non-intersecting strands connecting $n$ points on a horizontal plane to $n$ points on another horizontal plane directly below the first set of $n$ points. The strands are not allowed to go back upwards at any point in their travel. The braid may be projected onto a plane with the two horizontal planes collapsing to two parallel rigid rods. The over-crossings and under-crossings of the strands are to be clearly marked. When all the strands are identical, we have ordinary braids. The theory of such braids is well developed [10, 11]. However, for our purpose here we need to orient the individual strands and further distinguish them by putting different colours on them. We shall represent different colours by different $SU(2)$ spins. Examples of such braids are drawn in Fig. 5. These braids, unlike braids made from identical strands, have a more general structure than a group. These instead form a groupoid [20]. Now we shall develop some necessary elements of the theory of groupoid of such coloured-oriented braids.

A general $n$-strand coloured-oriented braid will be specified by giving $n$ assignments $\hat{j}_i=(j_i, \varepsilon_i)$, $i=1, 2, \ldots n$ representing the spin $j_i$ and orientation $\varepsilon_i$ ($\varepsilon_i=\pm 1$ for the $i^{\text{th}}$ strand going into or away from the rod) on the $n$ points on the upper rod and another set of $n$ spin-orientation assignments $\hat{l}_i=(l_i, \eta_i)$ on $n$ points on the lower rod as shown in Fig. 6. For a spin-orientation assignment $\hat{j}_i=(j_i, \varepsilon_i)$, we define a conjugate assignment as $\hat{j}_i^*=(j_i, -\varepsilon_i)$. Then the assignments $(\hat{l}_i)$ are just a permutation of $(\hat{j}_i^*)$. The shaded box in the middle of the figure represents a weaving pattern with various strands going over and under each other. Such a braid will be represented by the symbol $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_2 \ldots \hat{j}_n \\ \hat{l}_1 & \hat{l}_2 \ldots \hat{l}_n\end{pmatrix}$.

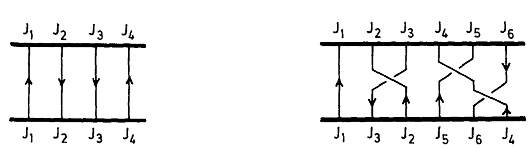


**Fig. 5.** Examples of coloured-oriented braids

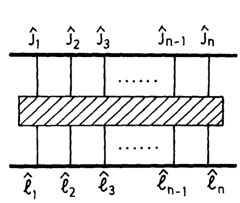


**Fig. 6.** An oriented-coloured $n$-braid $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_2 \ldots \hat{j}_n \\ \hat{l}_1 & \hat{l}_2 \ldots \hat{l}_n\end{pmatrix}$

*Composition*: Unlike usual braids made from identical strands, the composition for two arbitrary coloured braids is not always defined. Two such braids $\mathscr{B}^{(1)}\begin{pmatrix} \hat{j}_1 & \hat{j}_2 \dots \hat{j}_n \\ \hat{j}'_1 & \hat{j}'_2 \dots \hat{j}'_n \end{pmatrix}$ and $\mathscr{B}^{(2)}\begin{pmatrix} \hat{l}_1 & \hat{l}_2 \dots \hat{l}_n \\ \hat{l}'_1 & \hat{l}'_2 \dots \hat{l}'_n \end{pmatrix}$ can be composed only if the spin-orientations at the merged rods match, that is, the composition $\mathscr{B}^{(1)}\mathscr{B}^{(2)}$ is defined only if $\hat{j}'_i = \hat{l}^*_i$ and composition $\mathscr{B}^{(2)}\mathscr{B}^{(1)}$ only if $\hat{j}^*_i = \hat{l}'_i$.

*Generators*: An arbitrary coloured-oriented braid such as one shown in Fig. 6 can be generated by applying a set generators on the trivial (no entanglement) braids $I\begin{pmatrix} \hat{j}_1 & \hat{j}_2 \dots \hat{j}_n \\ \hat{j}^*_1 & \hat{j}^*_2 \dots \hat{j}^*_n \end{pmatrix}$ shown in Fig. 7. Unlike the case of usual ordinary braids, here we have more than one "identity" braid due to the different values of spin-orientation assignments $\hat{j}_1, \hat{j}_2, \dots \hat{j}_n$ placed on the strands. The set of $n-1$ generators $B_l$, $l=1, 2 \dots n-1$ are represented in Fig. 7. By convention we twist the strands by half-units from below keeping the points on the upper rod fixed. Thus the generator $B_l$ introduces from below a half-twist in the anti-clockwise direction in the $l^{\text{th}}$ and $(l+1)^{\text{th}}$ strands. Like in the case of usual ordinary braids, the generators of coloured-oriented braids satisfy two defining relations:

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}$$

$$B_i B_j = B_j B_i \qquad |i-j| \geqq 2 \,. \tag{3.1}$$

These relations are depicted diagrammatically in Figs. 8(a) and (b) respectively. We shall present a whole class of new representations of these generators in the next Sect. 4. These in turn will finally lead to new link invariants.

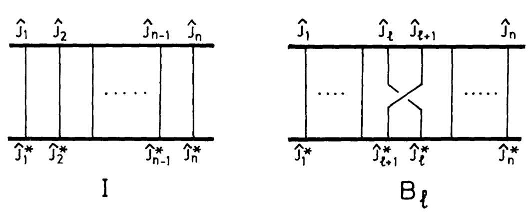


**Fig. 7.** Identity braids $I\begin{pmatrix} \hat{j}_1 & \hat{j}_2 \dots \hat{j}_n \\ \hat{j}^*_1 & \hat{j}^*_2 \dots \hat{j}^*_n \end{pmatrix}$ and braid generators $B_l\begin{pmatrix} \hat{j}_1 \dots & \hat{j}_l & \hat{j}_{l+1} \dots \hat{j}_n \\ \hat{j}^*_1 \dots \hat{j}^*_{l+1} & & \hat{j}^*_l \dots \hat{j}^*_n \end{pmatrix}$

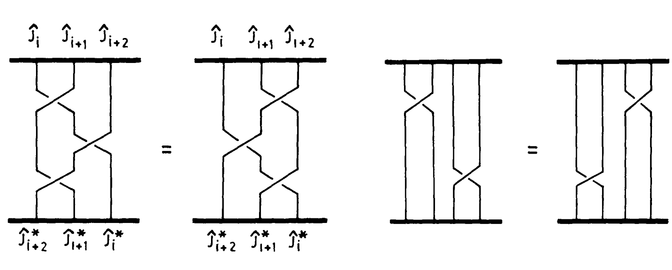

**Fig. 8.** Relations among braid generators

*Platting of an oriented-coloured braid*: Like in usual case of braids [11], we may introduce the concept of platting of a coloured-oriented braid. Consider a coloured-oriented braid with an even number of strands with spin-orientation assignments as given by $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_1^* & \hat{j}_2 & \hat{j}_2^* & \dots & \hat{j}_m & \hat{j}_m^* \\ \hat{l}_1 & \hat{l}_1^* & \hat{l}_2 & \hat{l}_2^* & \dots & \hat{l}_m & \hat{l}_m^*\end{pmatrix}$. The platting then constitutes of pair wise joining of successive strands $(2i-1, 2i)$, $i=1, 2, 3, \dots m$ from above and below as shown in Fig. 9. Such a construction obviously can be defined only for braids made of even number of strands with above given specific spin-orientation assignments. There is a theorem due to Birman which relates oriented links to plats of ordinary even braids [11]. This theorem can obviously also be stated in terms of coloured-oriented braids of our present interest. Thus we state

**Theorem 2.** *A coloured-oriented link can be represented by a plat constructed from an oriented-coloured braid* $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_1^* & \dots & \hat{j}_m & \hat{j}_m^* \\ \hat{l}_1 & \hat{l}_1^* & \dots & \hat{l}_m & \hat{l}_m^*\end{pmatrix}$.

Clearly, platting of these braids does not provide a unique representation of a given knot or link.

*Closure of an oriented-coloured braid*: In addition to platting, we may also define the closure of a coloured-oriented braid. For an $m$-strand braid with spin-orientation assignments as in $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_m \\ \hat{j}_1^* & \hat{j}_2^* & \dots & \hat{j}_m^*\end{pmatrix}$, the closure of the braid is obtained by joining the top end of each string to the same position on the bottom of the braid as shown in Fig. 10. Clearly, closure is defined only if the spin-orientation assignments are mutually conjugate at the same positions on the upper and lower rods. Now

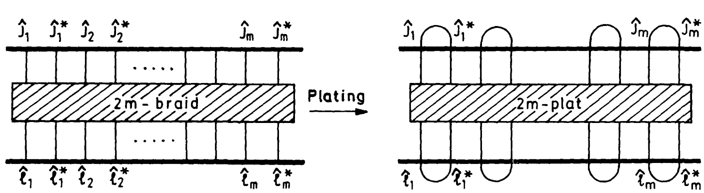


**Fig. 9.** Platting of a coloured-oriented braid

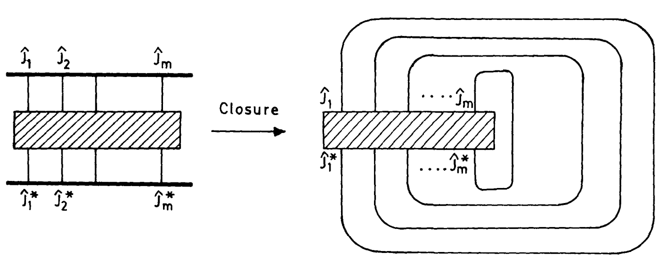


**Fig. 10.** Closure of a coloured-oriented braid

there is a theorem due to Alexander [16] which relates oriented links with closure of ordinary braids. This theorem can as well be stated for our coloured-oriented braids:

**Theorem 3.** *A coloured-oriented link can be represented, though not uniquely, by the closure of an oriented-coloured braid* $\mathscr{B}\begin{pmatrix} \hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_m \\ \hat{j}_1^* & \hat{j}_2^* & \dots & \hat{j}_m^* \end{pmatrix}$.

In the following we shall see that Theorem 1 with Theorem 2 or 3 provide a complete solution to $SU(2)$ Chern–Simons gauge theory on an $S^3$.

## 4. Link Invariants from $SU(2)$ Chern–Simons Theory

To develop a method of calculating the expectation value of an arbitrary Wilson link operator (1.4), consider an $S^3$ with two three-balls removed from it. This is a manifold with two boundaries, each an $S^2$. Let us place $2m$ ($m=2,3\dots$) unbraided Wilson lines with spins $j_1, j_2, \dots j_{2m}$ (such that all these spins make an $SU(2)$ singlet) going from one boundary to the other as shown in Fig. 11. Thus we have put an "identity" braid $I\begin{pmatrix} \hat{j}_1^* & \hat{j}_2^* & \dots & \hat{j}_{2m}^* \\ \hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_{2m} \end{pmatrix}$ inside the manifold. An arbitrary coloured-oriented braid can be generated from this identity by applying the half-twist (braiding) generators $B_1, B_2, \dots B_{2m-1}$ on the lower boundary. As discussed in Sect. 1, the Chern–Simons functional integral over this manifold can be represented by a state in the tensor product of vector spaces, $\mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)}$, associated with the two boundaries, $\Sigma^{(1)}$ and $\Sigma^{(2)}$. Convenient basis vectors for these vector spaces can be taken to correspond to the conformal blocks (Eq. (2.2)), $\phi_{(p;r)}(j_1 j_2 \dots j_{2m})$ or equivalently $\phi'_{(q;s)}(j_1 j_2 \dots j_{2m})$ as shown in Fig. 2 for the $2m$-point correlators of the corresponding $SU(2)_k$ Wess–Zumino conformal field theory. We shall represent these bases for each vector space as $|\phi_{(p;r)}(\hat{j}_1 \hat{j}_2 \dots \hat{j}_{2m})\rangle$ and $|\phi'_{(q;s)}(\hat{j}_1 \hat{j}_2 \dots \hat{j}_{2m})\rangle$ respectively. For dual vector spaces associated with boundaries with opposite orientation, we have the dual bases $\langle \phi_{(p;r)}(\hat{j}_1 \dots \hat{j}_{2m})|$ and $\langle \phi'_{(q;s)}(\hat{j}_1 \dots \hat{j}_{2m})|$. The inner product of these bases vectors for each of the vector space, $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ are normalized so that

$$\langle \phi_{(p;r)}(\hat{j}_1^* \hat{j}_2^* \dots \hat{j}_{2m}^*)|\phi_{(p';r')}(\hat{j}_1 \hat{j}_2 \dots \hat{j}_{2m})\rangle = \delta_{(p)(p')}\delta_{(r)(r')} ,$$

$$\langle \phi'_{(q;s)}(\hat{j}_1^* \hat{j}_2^* \dots \hat{j}_{2m}^*)|\phi'_{(q';s')}(\hat{j}_1 \hat{j}_2 \dots \hat{j}_{2m})\rangle = \delta_{(q)(q')}\delta_{(r)(r')} . \tag{4.1}$$

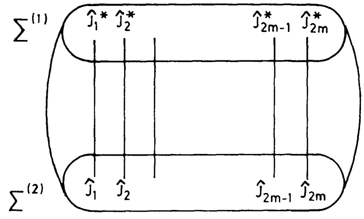


**Fig. 11.** An identity braid in a three manifold with boundaries $\Sigma^{(1)}$ and $\Sigma^{(2)}$, each an $S^2$

The two primed and unprimed bases are related by duality of the conformal blocks given by Theorem 1:

$$|\phi'_{(q;s)}(\hat{j}_1\hat{j}_2\ldots\hat{j}_{2m})\rangle=\sum_{(p;r)} a_{(p;r)(q;s)}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \\ & \vdots \\ j_{2m-1} & j_{2m}\end{bmatrix}|\phi_{(p;r)}(\hat{j}_1\hat{j}_2\ldots\hat{j}_{2m})\rangle \tag{4.2}$$

with duality matrices as in Eq. (2.3).

The Chern–Simons functional integral over the three-manifold of Fig. 11 may now be written in terms of any one of the above bases:

$$v_I\begin{pmatrix}\hat{j}_1^* & \hat{j}_2^* & \cdot\cdot\,\hat{j}_{2m}^* \\ \hat{j}_1 & \hat{j}_2 & \cdot\cdot\,\hat{j}_{2m}\end{pmatrix}=\sum_{(p;r)(p';r')} M_{(p;r)(p';r')}|\phi^{(1)}_{(p;r)}(\hat{j}_1^*\,\cdot\cdot\,\hat{j}_{2m}^*)\rangle|\phi^{(2)}_{(p';r')}(\hat{j}_1\,\cdot\cdot\,\hat{j}_{2m})\rangle\,.$$

Here we have put superscripts (1) and (2) on the bases vectors to indicate explicitly that they belong to the vector spaces $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ respectively. Now notice glueing two copies of this manifold along two oppositely oriented boundaries, each an $S^2$, yields the same manifold. Hence

$$\sum_{(p';r')} M_{(p;r)(p';r')}M_{(p';r')(p'';r'')}=M_{(p;r)(p'';r'')}\,.$$

This immediately leads to $M_{(p;r)(p';r')}=\delta_{(p)(p')}\delta_{(r)(r')}$, so that the functional integral over the three-manifold of Fig. 11 can be written as

$$v_I\begin{pmatrix}\hat{j}_1^* & \hat{j}_2^* & \ldots\,\hat{j}_{2m}^* \\ \hat{j}_1 & \hat{j}_2 & \ldots\,\hat{j}_{2m}\end{pmatrix}=\sum_{(p;r)}|\phi^{(1)}_{(p;r)}(\hat{j}_1^*\ldots\hat{j}_{2m}^*)\rangle|\phi^{(2)}_{(p;r)}(\hat{j}_1\ldots\hat{j}_{2m})\rangle\,. \tag{4.3a}$$

Equivalently we could write this functional integral in terms of the primed basis using (4.2) and orthogonality property of duality matrices (2.4) as

$$v_I\begin{pmatrix}\hat{j}_1^* & \ldots\,\hat{j}_{2m}^* \\ \hat{j}_1 & \ldots\,\hat{j}_{2m}\end{pmatrix}=\sum_{(q;s)}|\phi'^{(1)}_{(q;s)}(\hat{j}_1^*\ldots\hat{j}_{2m}^*)\rangle|\phi'^{(2)}_{(q;s)}(\hat{j}_1\ldots\hat{j}_{2m})\rangle\,. \tag{4.3b}$$

The conformal blocks $\phi_{(p;r)}(j_1\ldots j_{2m})$ as shown in Fig. 2(a) of the conformal field theory and the corresponding basis vectors $|\phi_{(p;r)}(\hat{j}_1\hat{j}_2\ldots\hat{j}_{2m})\rangle$ are eigenfunctions of the odd indexed braiding generators. $B_{2l+1}$, $l=0,1\ldots(m-1)$ of Fig. 7. On the other hand the conformal blocks $\phi'_{(q;s)}(j_1j_2\ldots j_{2m})$ (Fig. 2b) and the associated basis vectors $|\phi'_{(q;s)}(\hat{j}_1\hat{j}_2\ldots\hat{j}_{2m})\rangle$ are eigenfunctions of the even indexed braid generators, $B_{2l}$, $l=1,2,\ldots(m-1)$:

$$\begin{aligned}B_{2l+1}|\phi_{(p;r)}(\ldots\hat{j}_{2l+1}\hat{j}_{2l+2}\ldots)\rangle&=\lambda_{p_l}(\hat{j}_{2l+1},\hat{j}_{2l+2})|\phi_{(p;r)}(\ldots\hat{j}_{2l+2}\hat{j}_{2l+1}\ldots)\rangle\,,\\ B_{2l}|\phi'_{(q;r)}(\hat{j}_1\,\cdot\cdot\,\hat{j}_{2l}\hat{j}_{2l+1}\,\cdot\cdot\,\hat{j}_{2m})\rangle&=\lambda_{q_l}(\hat{j}_{2l},\hat{j}_{2l+1})|\phi_{(q;s)}(\hat{j}_1\,\cdot\cdot\,\hat{j}_{2l+1}\hat{j}_{2l}\,\cdot\cdot\,\hat{j}_{2m})\rangle\,.\end{aligned} \tag{4.4}$$

The eigenvalues of the half-twist matrices depend on the relative orientation of the twisted strands:

$$\begin{aligned}\lambda_t(\hat{j},\hat{j}')=\lambda_t^{(+)}(j,j')\quad&\equiv(-)^{j+j'-t}q^{(c_j+c_{j'})/2+c_{\min(j,j')}-c_t/2}\quad\text{if }\varepsilon\varepsilon'=+1\\ =(\lambda_t^{(-)}(j,j'))^{-1}&\equiv(-)^{|j-j'|-t}q^{|c_j-c_{j'}|/2-c_t/2}\quad\text{if }\varepsilon\varepsilon'=-1\,,\end{aligned} \tag{4.5}$$

where $c_j = j(j+1)$ is the quadratic Casimir for spin $j$ representation. When $\varepsilon\varepsilon' = +1$ above, the two-strands have the same orientation and the braid generator introduces a right-handed half-twist as shown in Fig. 12a. On the other hand for $\varepsilon\varepsilon' = -1$, the two strands are anti-parallel and the braid generator introduces a left-handed half-twist as shown in Fig. 12b. Thus $\lambda_t^{(+)}(j,j')$ and $\lambda_t^{(-)}(j,j')$ above are the eigenvalues of the half-twist matrix which introduce right-handed half-twists in parallely and anti-parallely oriented strands respectively. These eigenvalues are obtained from the monodromy properties of the conformal blocks of Fig. 2 of the corresponding conformal theory [19] and further compensated for the change of framing introduced due to the twisting of the strands [3, 6]. There is some ambiguity with regard to the $q$-independent phases in these expressions for the eigenvalues. However, this ambiguity along with that in the phase of the duality matrix $a_{jl}$ of Eq. (2.1) are relatively fixed by consistency requirements as will be discussed in the Appendix I below.

Equations (4.4), (4.5) and (4.2) define representations of braids. This we express in the form of a theorem:

**Theorem 4.** *A class of representations for generators of the groupoid of coloured-oriented braids of Fig. 7 are given (in the basis* $|\phi_{(p;r)}\rangle$*) by*

$$\left[B_{2l+1}\begin{pmatrix} \hat{j}_1^* & \cdot\cdot & \hat{j}_{2l+1}^* & \hat{j}_{2l+2}^* & \cdot\cdot & \hat{j}_{2m}^* \\ \cdot\cdot & \cdot\cdot & \hat{j}_{2l+2} & \hat{j}_{2l+1} & \cdot\cdot & \cdot\cdot \end{pmatrix}\right]_{(p;r)(p';r')}$$

$$= \lambda_{p_l}(\hat{j}_{2l+1}, \hat{j}_{2l+2})\,\delta_{(p)(p')}\,\delta_{(r)(r')} \qquad l = 0, 1, \ldots (m-1)\,,$$

and

$$\left[B_{2l}\begin{pmatrix} \hat{j}_1^* & \cdot\cdot & \hat{j}_{2l}^* & \hat{j}_{2l+1}^* & \cdot\cdot & \hat{j}_{2m}^* \\ \hat{j}_1 & \cdot\cdot & \hat{j}_{2l+1} & \hat{j}_{2l} & \cdot\cdot & \hat{j}_{2m} \end{pmatrix}\right]_{(p;r)(p';r')}$$

$$= \sum_{(q;s)} \lambda_{q_l}(\hat{j}_{2l}, \hat{j}_{2l+1})\, a_{(p';r')(q;s)}\begin{bmatrix} j_1 & j_2 \\ \vdots & \vdots \\ j_{2l-1} & j_{2l+1} \\ j_{2l} & j_{2l+2} \\ \vdots & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix} a_{(p;r)(q;s)}\begin{bmatrix} j_1 & j_2 \\ \vdots & \vdots \\ j_{2l-1} & j_{2l} \\ j_{2l+1} & j_{2l+2} \\ \vdots & \vdots \\ j_{2m-1} & j_{2m} \end{bmatrix}$$

$$l = 1, 2 \ldots (m-1)\,. \tag{4.6}$$

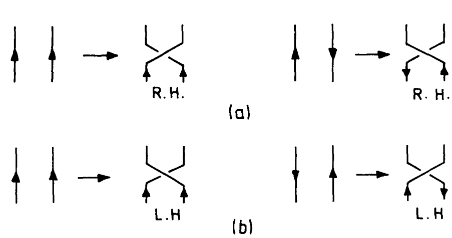


**Fig. 12.** Braid generators introduce (a) right-handed half-twists in parallely oriented strands and (b) left-handed half-twists in anti-parallel strands

Using the identities given in Appendix I, these can readily be verified to satisfy the defining relations (3.1) of the braid generators.

Now let us place an arbitrary weaving pattern instead of an identity braid inside the three-manifold with two boundaries (each an $S^2$) discussed above with specific spin-orientation assignments as shown in Fig. 13. The spin-orientation assignment $(\hat{l}_1, \hat{l}_2 \ldots \hat{l}_m)$ on the lower boundary are just a permutations of $(\hat{j}_1^*, \hat{j}_2^* \ldots \hat{j}_m^*)$. The braid inside indicated as a shaded box can be represented in terms of a word $\mathscr{B}$ in the braid generators $B_i$ above. The Chern–Simons functional integral over this three-manifold can thus be obtained by $\mathscr{B}$ (written in terms generators $B_i$) acting on the state (4.3) from below:

$$v_{\mathscr{B}}\begin{pmatrix} \hat{j}_1 & \hat{j}_1^* & \ldots & \hat{j}_m & \hat{j}_m^* \\ \hat{l}_1 & \hat{l}_1^* & \ldots & \hat{l}_m & \hat{l}_m^* \end{pmatrix} = \sum_{(p;r)} |\phi^{(1)}_{(p;r)}\rangle \mathscr{B} |\phi^{(2)}_{(p;r)}\rangle \,. \tag{4.7}$$

We wish to plat this braid. This can be done by glueing one copy each of the three-ball shown in Fig. 14(a) from below and above with spin-orientation assignments matching at the punctures. The functional integral over this three-ball (Fig. 14(a)) can again be thought of to be a vector in the Hilbert space associated with the boundary. Thus we write the functional integral (normalized by multiplying by $Z^{-1/2}$, where $Z$ is the partition function on $S^3$) in terms of a basis of this Hilbert space as

$$v(\hat{j}_1\hat{j}_1^* \ldots \hat{j}_m\hat{j}_m^*) = \sum_{(p;r)} N_{(p;r)} |\phi_{(p;r)}(\hat{j}_1\hat{j}_1^* \ldots \hat{j}_m\hat{j}_m^*)\rangle \,,$$

where the coefficients $N_{(p;r)}$ are to be fixed. Notice applying an arbitrary combination of odd indexed braid generators $B_{2l+1}$ on Fig. 14(a) does not change this manifold; the half-twists so introduced can simply be undone. That shows that the vector $v(\hat{j}_1\hat{j}_1^* \ldots \hat{j}_m\hat{j}_m^*)$ is proportional to $|\phi_{(0;0)}(\hat{j}_1\hat{j}_1^* \ldots \hat{j}_m\hat{j}_m^*)\rangle$ which is the eigen-function of the generators $B_{2l+1}$ with eigenvalue one. Thus the only non-zero coefficient is $N_{(0;0)}$. Further if we glue two copies of the three-ball of Fig. 14(a) onto each other along their oppositely oriented boundaries, we obtain an $S^3$ containing $m$ unlinked unknots carrying spins $j_1, j_2, \ldots j_m$ respectively. The invariant for this link is given simply by the product of invariants for individual unknots. Now for cabled knots such as two unknots, the invariants satisfy the fusion rules of the associated conformal field theory. Thus for unknots $V_{j_1}[U]\, V_{j_2}[U] = \Sigma_j V_j[U]$, where the spins $(j_1, j_2, j)$ are related by the fusion rules of the conformal field theory. For spin 1/2 representation using skein relations we can obtain $V_{1/2}[U] = [2]$, where square brackets define $q$-numbers as $[x] = (q^{x/2} - e^{-x/2})/(q^{1/2} - q^{-1/2})$. Using this along with $V_0[U] = 1$, the invariant

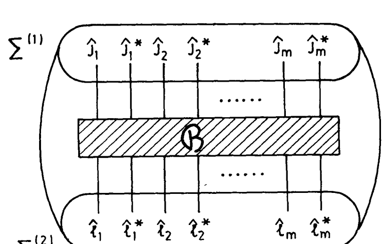


**Fig. 13.** A three-manifold containing an arbitrary coloured oriented $2m$-braid $\mathscr{B}$

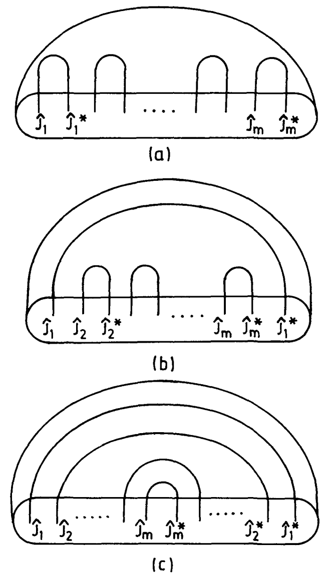


**Fig. 14.** Three-balls containing $m$ Wilson lines

for unknot $U$ can be seen to be given by the $q$-dimension of the representation living on the knot $V_j[U]=[2j+1]$. This discussion leads to $N_{(0;0)}=\prod[2j_i+1]^{1/2}$ above. Thus

$$v(\hat{j}_1\hat{j}_1^* \dots \hat{j}_m\hat{j}_m^*)=\left(\prod_{i=1}^{m}[2j_i+1]^{1/2}\right)|\phi_{(0;0)}(\hat{j}_1\hat{j}_1^* \dots \hat{j}_m\hat{j}_m^*)\rangle\,. \tag{4.8}$$

Now we are ready to plat the braid in the manifold of Fig. 13 by glueing to it manifolds of the type shown in Fig. 14(a) from below and above. This, invoking Theorem 2, leads us to our main theorem:

**Theorem 5.** *The expectation value* (1.4) *of a Wilson operator for an arbitrary link $L$ with a plat representation in terms of a coloured-oriented braid* $\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_1^* & \dots & \hat{j}_m & \hat{j}_m^*\\ \hat{l}_1 & \hat{l}_1^* & \dots & \hat{l}_m & \hat{l}_m^*\end{pmatrix}$ *generated by a word written in terms of the braid generators $B_i$, $i=1,2,\dots(2m-1)$, is given by*

$$V[L]=\left(\prod_{i=1}^{m}[2j_i+1]\right)$$
$$\times\left\langle\phi_{(0;0)}(\hat{l}_1^*\hat{l}_1 \dots \hat{l}_m^*\hat{l}_m)|\mathscr{B}\begin{pmatrix}\hat{j}_1 & \hat{j}_1^* & \dots & \hat{j}_m & \hat{j}_m^*\\ \hat{l}_1 & \hat{l}_1^* & \dots & \hat{l}_m & \hat{l}_m^*\end{pmatrix}|\phi_{(0;0)}(\hat{j}_1^*\hat{j}_1 \dots \hat{j}_m^*\hat{j}_m)\right\rangle\,. \tag{4.9}$$

This main theorem along with Theorem 4 allows us to calculate the link invariant for any arbitrary link. Before illustrating this with an explicit example in the next section, we shall extend our discussion developed above to write down the Chern–Simons functional integral over a three-ball with Wilson lines as shown in Fig. 14(b). One way to obtain this functional integral is by applying a weaving pattern generated by $B_{2m-1}B_{2m}\ldots B_3B_2$ on the functional integral (4.8) for the three-ball shown in Fig. 14(a). Alternatively, since this functional integral is unchanged by applying even-indexed braid generators $B_{2l}$, it is proportional to $|\phi'_{(0;0)}(\hat{j}_1\hat{j}_2\hat{j}_2^*\ldots\hat{j}_m\hat{j}_m^*\hat{j}_1^*)\rangle$ which is the eigenfunction of these generators with eigenvalue one. This functional integral over the ball of Fig. 14(b) (normalized by multiplying by $Z^{-1/2}$) is given by the vector

$$\nu'(\hat{j}_1\hat{j}_2\hat{j}_2^*\ldots\hat{j}_m\hat{j}_m^*\hat{j}_1^*)$$
$$=(-)^{2j_1}\left(\prod_{i=1}^{m}(-)^{2\min(j_1,j_i)}[2j_i+1]^{1/2}\right)|\phi'_{(0;0)}(\hat{j}_1\hat{j}_2\hat{j}_2^*\ldots\hat{j}_m\hat{j}_m^*\hat{j}_1^*)\rangle\,. \tag{4.10}$$

Similarly the Chern–Simons functional integral for the three-ball Fig. 14(c) can be constructed by applying the braid $g_{2m}\equiv(B_{m+1}B_m)(B_{m+2}B_{m+1}B_mB_{m-1})(B_{m+3}B_{m+2}\ldots B_{m-2})\ldots(B_{2m-1}\ldots B_3B_2)$ on the vector (4.8) representing functional integral over the manifold of Fig. 14(a). We write

$$|\tilde{\phi}(\hat{j}_1\hat{j}_2\ldots\hat{j}_m\hat{j}_m^*\ldots\hat{j}_2^*\hat{j}_1^*)\rangle=g_{2m}|\phi_{(0;0)}(\hat{j}_1\hat{j}_1^*\hat{j}_2\hat{j}_2^*\ldots\hat{j}_m\hat{j}_m^*)\rangle\,. \tag{4.11}$$

Then the Chern–Simons normalized functional integral over this three-ball is

$$\tilde{\nu}(\hat{j}_1\hat{j}_2\ldots\hat{j}_m\hat{j}_m^*\ldots\hat{j}_2^*\hat{j}_1^*)=\left(\prod_{i=1}^{m}[2j_i+1]^{1/2}\right)|\tilde{\phi}(\hat{j}_1\hat{j}_2\ldots\hat{j}_m\hat{j}_m^*\ldots\hat{j}_2^*\hat{j}_1^*)\rangle\,. \tag{4.12}$$

This functional integral allows us to obtain a result equivalent to Theorem 5 for the links as represented by closure of braids. To do so, for a braid $\mathscr{B}_m\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & \ldots & \hat{j}_m\\ \hat{j}_1^* & \hat{j}_2^* & \ldots & \hat{j}_m^*\end{pmatrix}$ with $m$ strands, construct another braid by adding $m$ untangled strands to obtain a $2m$-strand braid $\hat{\mathscr{B}}_m\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & \ldots & \hat{j}_m & \hat{j}_m^* & \ldots & \hat{j}_2^* & \hat{j}_1^*\\ \hat{j}_1^* & \hat{j}_2^* & \ldots & \hat{j}_m^* & \hat{j}_m & \ldots & \hat{j}_2 & \hat{j}_1\end{pmatrix}$ as shown in Fig. 15 with the spin-orientation assignments as indicated. Then the closure of the original $m$-strand braid $\mathscr{B}_m$ in $S^3$ is obtained by glueing two copies, one each from above and below, of the three-ball of Fig. 14(c) onto the manifold of Fig. 15 with proper matching of spin-orientations on the punctures on the boundaries. Thus, we may state the result for links represented as closure of braids as:

**Theorem 6.** *For a link represented by the closure of an m-strand coloured-oriented braid* $\mathscr{B}_m\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & \ldots & \hat{j}_m\\ \hat{j}_1^* & \hat{j}_2^* & \ldots & \hat{j}_m^*\end{pmatrix}$, *the link invariant is given in terms of the extended*

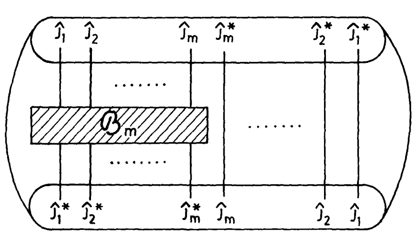


**Fig. 15.** $m$-braid $\mathscr{B}_m$ extended to $2m$-braid $\hat{\mathscr{B}}_m$

*2m-braid* $\hat{\mathscr{B}}_m\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & \dots & \hat{j}_m & \hat{j}_m^* & \dots & \hat{j}_2^* & \hat{j}_1^* \\ \hat{j}_1^* & \hat{j}_2^* & \dots & \hat{j}_m^* & \hat{j}_m & \dots & \hat{j}_2 & \hat{j}_1\end{pmatrix}$ *constructed by adding m untangled strands as in Fig.* 15, *by*

$$V[L]=\left(\prod_{i=1}^{m}[2j_i+1]\right)\langle\tilde{\phi}(\hat{j}_1\hat{j}_2\,..\,\hat{j}_m\hat{j}_m^*\,..\,\hat{j}_2^*\hat{j}_1^*)|$$

$$\times\hat{\mathscr{B}}_m\begin{pmatrix}\hat{j}_1 & \hat{j}_2 & .. & \hat{j}_m & \hat{j}_m^* & .. & \hat{j}_2^* & \hat{j}_1^* \\ \hat{j}_1^* & \hat{j}_2^* & .. & \hat{j}_m^* & \hat{j}_m & .. & \hat{j}_2 & \hat{j}_1\end{pmatrix}|\tilde{\phi}(\hat{j}_1^*\hat{j}_2^*\,..\,\hat{j}_m^*\hat{j}_m\,..\,\hat{j}_2\hat{j}_1)\rangle\,. \quad (4.13)$$

Here the $2m$-strand braid is written as a word in terms of the braid generators $B_1, B_2, \dots B_{m-1}$ introducing a weaving pattern in the first $m$ strands only and the vector $|\tilde{\phi}\rangle$ is given by Eq. (4.11) above.

Theorem 5 or equivalently Theorem 6 provides a complete and explicit solution of $SU(2)$ Chern–Simons gauge theory on $S^3$.

## 5. Applications of the Main Theorem

To illustrate the use of the main Theorem 5, let us calculate the invariant for Borromean rings. This link is made from three knots. We shall place spin $j_1, j_2$ and $j_3$ on these knots. Figure 16 shows this links with orientation and spin assignments as indicated. A plat representation for this link has also been drawn. The link is given as a plat of a six strand braid $B_2B_4^{-1}B_3B_1B_4^{-1}B_3B_2^{-1}B_4^{-1}$. To apply Theorem 5, first we evaluate $B_2^{-1}B_4^{-1}|\phi_{(0)}\rangle$. This we do by first converting the basis vector $|\phi_{(0)}\rangle$ to $|\phi'_{(l)}\rangle$ through duality matrix, and since $B_2^{-1}B_4^{-1}$ introduces right-handed half twists in anti-parallel strands:

$$B_2^{-1}B_4^{-1}|\phi_{(0)}(\hat{j}_2\hat{j}_2^*\hat{j}_1\hat{j}_1^*\hat{j}_3\hat{j}_3^*)\rangle$$

$$=\sum_{(l_i)}\lambda_{l_1}^{(-)}(j_1j_2)\lambda_{l_2}^{(-)}(j_1j_3)a_{(0)(l)}\begin{bmatrix}j_2 & j_2\\ j_1 & j_1\\ j_3 & j_3\end{bmatrix}|\phi'_{(l)}(\hat{j}_2\hat{j}_1\hat{j}_2^*\hat{j}_3\hat{j}_1^*\hat{j}_3^*)\rangle\,.$$

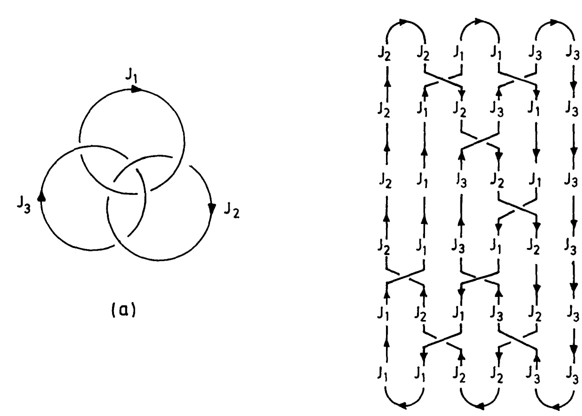


**Fig. 16.** (a) Borromean rings and (b) a plat representation for it

Next we apply $B_3$ (which introduces a left-handed half-twist in the anti-parallel strands) on this vector. For this we change the basis back to $|\phi_{(m)}\rangle$ through duality transformation. Repeating such steps, we finally have using Theorem 5, the invariant for the Borromean rings of Fig. 16 as

$$\begin{aligned} V_{j_1 j_2 j_3} &= [2j_j+1][2j_2+1][2j_3+1] \\ &\quad \times \langle \phi_{(0)}(\hat{j}_1^* \hat{j}_1 \hat{j}_2^* \hat{j}_2 \hat{j}_3^* \hat{j}_3) | B_2 B_4^{-1} B_3 B_1 B_4^{-1} B_3 B_2^{-1} B_4^{-1} | \phi_{(0)}(\hat{j}_2 \hat{j}_2^* \hat{j}_1 \hat{j}_1^* \hat{j}_3 \hat{j}_3^*) \rangle \\ &= [2j_1+1][2j_2+1][2j_3+1] \sum (\lambda_{q_1}^{(-)}(j_1 j_2))^{-1} \lambda_{q_2}^{(-)}(j_2 j_3) \lambda_{p_0}^{(+)}(j_1 j_2) \\ &\quad \times (\lambda_{p_1}^{(-)}(j_1 j_3))^{-1} (\lambda_{n_2}^{(+)}(j_1 j_2))^{-1} (\lambda_{m_1}^{(-)}(j_2 j_3))^{-1} \lambda_{l_1}^{(-)}(j_1 j_2) \lambda_{l_2}^{(-)}(j_1 j_3) \\ &\quad \times a_{(0)(q)} \begin{bmatrix} j_1 & j_1 \\ j_2 & j_2 \\ j_3 & j_3 \end{bmatrix} a_{(p)(q)} \begin{bmatrix} j_1 & j_2 \\ j_1 & j_3 \\ j_2 & j_3 \end{bmatrix} a_{(p)(n)} \begin{bmatrix} j_2 & j_1 \\ j_3 & j_1 \\ j_2 & j_3 \end{bmatrix} \\ &\quad \times a_{(m)(n)} \begin{bmatrix} j_2 & j_1 \\ j_3 & j_2 \\ j_1 & j_3 \end{bmatrix} a_{(m)(l)} \begin{bmatrix} j_2 & j_1 \\ j_2 & j_3 \\ j_1 & j_3 \end{bmatrix} a_{(0)(l)} \begin{bmatrix} j_2 & j_2 \\ j_1 & j_1 \\ j_3 & j_3 \end{bmatrix} . \end{aligned} \tag{5.1}$$

Similarly the knot invariants for example, for all the knots and links listed in the tables given in Rolfsen's book [18] may be calculated. We shall present the result of such calculations for knots up to eight crossings and two-component links up to seven crossings in Appendix II. Some of these invariants were calculated earlier in refs. 6.

## 6. Concluding Remarks

We have here presented an explicit method for obtaining the functional average of an arbitrary Wilson link operator (1.4) in an $SU(2)$ Chern–Simons theory. Either of the main Theorems 5 or 6 provides this complete solution. To develop this method, we have made use of theory of coloured-oriented braids. In addition, following Witten [3], we have used the equivalence of the Hilbert space of Chern–Simons functional integrals over a three-manifold with boundary with the vector space of the conformal blocks for the correlators of the associated Wess–Zumino conformal field theory on that boundary based on the same group and same level. This has helped us to find a whole class of new representations of generators of coloured-oriented braids. These in turn have finally led to the explicit solution of the Chern–Simons gauge theory. Of the new link invariants so obtained, the Jones polynomial is the simplest. It corresponds to a spin 1/2 representation living on all the components of the link. The new invariants appear to be more powerful than the Jones polynomial as these do distinguish knots which are known to have the same Jones polynomials.

Tables of the new invariants for knots and links of low crossing numbers have been presented in Appendix II. We could read off the invariants for any other links as well by the rules defined by Theorem 5 or 6. In particular, invariants for toral knots can be obtained in this way.

Theorems 5 and 6 can also be used for an efficient calculation of the invariants on a computer.

The method developed has obvious generalizations to other compact semi-simple gauge groups. It can also be extended to study Chern–Simons gauge theory on three-manifolds other than $S^3$.

## Appendix I

Here we list the duality matrix $a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$ relating the two bases of four-point correlators of $SU(2)_k$ Wess–Zumino field theory as shown in Fig. 1. We shall also give some of their useful properties. These duality matrices are given in terms of $q$-Racah coefficients as [19, 6]:

$$a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = (-)^{(j_1+j_2+j_3+j_4)}\sqrt{[2j+1][2l+1]}\begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & l \end{pmatrix}. \tag{A.1}$$

Here the triplets $(jj_1j_2)$, $(jj_3j_4)$, $(lj_1j_4)$ and $(lj_2j_3)$ satisfy the fusion rules of the conformal theory:

$$\begin{gathered} \max(|j_1-j_2|, |j_3-j_4|) \geqq j \geqq \min(j_1+j_2, j_3+j_4) \\ \max(|j_2-j_3|, |j_1-j_4|) \geqq l \geqq \min(j_2+j_3, j_1+j_4) \\ j_1+j_2+j \leqq k, \quad j_3+j_4+j \leqq k, \quad j_2+j_3+l \leqq k, \quad j_1+j_4+l \leqq k \\ j_1+j_2+j, \quad j_3+j_4+j, \quad j_2+j_3+l \text{ and } j_1+j_4+l \in \mathbf{Z}\,. \end{gathered} \tag{A.2}$$

The phase in (A.1) is so chosen that it is real; $(j_1+j_2+j_3+j_4)$ is always an integer.

The $SU(2)_q$ Racah–Wigner coefficients [15] are:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{pmatrix} &= \Delta(j_1, j_2, j_{12})\Delta(j_3, j_4, j_{12})\Delta(j_1, j_4, j_{23})\Delta(j_2, j_3, j_{23}) \\ &\times \sum_{m\geqq 0} (-)^m [m+1]!\{[m-j_1-j_2-j_{12}]! \\ &\times [m-j_3-j_4-j_{12}]![m-j_1-j_4-j_{23}]! \\ &\times [m-j_2-j_3-j_{23}]![j_1+j_2+j_3+j_4-m]! \\ &\times [j_1+j_3+j_{12}+j_{23}-m]![j_2+j_4+j_{12}+j_{23}-m]!\}^{-1}, \end{aligned} \tag{A.3}$$

where

$$\Delta(a, b, c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}\,.$$

Here the square brackets represent the $q$-numbers ($q = \exp(-2\pi i/(k+2))$):

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

and $[n]! = [n][n-1]\ldots[3][2][1]$. The $SU(2)$ spins are related as $\vec{j}_1 + \vec{j}_2 + \vec{j}_3 = \vec{j}_4$, $\vec{j}_1 + \vec{j}_2 = \vec{j}_{12}$, $\vec{j}_2 + \vec{j}_3 = \vec{j}_{23}$.

The $q$-Racah coefficients above satisfy the following properties [15]: Interchange of any two columns of $\begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & l \end{pmatrix}$ leaves it unchanged. Further

$$\begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & l \end{pmatrix} = \begin{pmatrix} j_1 & j_4 & l \\ j_3 & j_2 & j \end{pmatrix} = \begin{pmatrix} j_3 & j_2 & l \\ j_1 & j_4 & j \end{pmatrix} = \begin{pmatrix} j_3 & j_4 & j \\ j_1 & j_2 & l \end{pmatrix}, \tag{A.4}$$

$$\begin{pmatrix} j_1 & j_2 & 0 \\ j_3 & j_4 & l \end{pmatrix} = \frac{(-)^{l+j_2+j_3}\delta_{j_1j_2}\delta_{j_3j_4}}{\sqrt{[2j_2+1][2j_3+1]}}, \tag{A.5}$$

$$\sum_j [2j+1][2l+1] \begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & l \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ j_3 & j_4 & l' \end{pmatrix} = \delta_{ll'}, \tag{A.6}$$

$$\sum_x (-)^{j+l+x}[2x+1] q^{-C_{x/2}} \begin{pmatrix} j_1 & j_2 & x \\ j_3 & j_4 & j \end{pmatrix} \begin{pmatrix} j_1 & j_2 & x \\ j_4 & j_3 & l \end{pmatrix}$$
$$= \begin{pmatrix} j_3 & j_2 & j \\ j_4 & j_1 & l \end{pmatrix} q^{C_{j/2}+C_{l/2}} q^{-C_{j_1/2}-C_{j_2/2}-C_{j_3/2}-C_{j_4/2}}, \tag{A.7}$$

$$\sum_{l_1} (-)^{l_1+l_2+l_3+r_1+r_2}[2l_1+1] \begin{pmatrix} r_1 & j_3 & r_2 \\ j_4 & j_5 & l_1 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & r_1 \\ l_1 & j_5 & l_2 \end{pmatrix} \begin{pmatrix} l_2 & j_2 & l_1 \\ j_3 & j_4 & l_3 \end{pmatrix}$$
$$= (-)^{j_1+j_2+j_3+j_4+j_5} \begin{pmatrix} j_1 & l_3 & r_2 \\ j_4 & j_5 & l_2 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & r_1 \\ j_3 & r_2 & l_3 \end{pmatrix}, \tag{A.8}$$

where $C_j = j(j+1)$.

Using these we see that the duality matrices satisfy the orthogonality and symmetry properties as:

$$\sum_j a_{jl} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} a_{jl'} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = \delta_{ll'}, \tag{A.9}$$

$$a_{jl} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = a_{lj} \begin{bmatrix} j_1 & j_4 \\ j_3 & j_2 \end{bmatrix} = a_{lj} \begin{bmatrix} j_3 & j_2 \\ j_1 & j_4 \end{bmatrix} = a_{jl} \begin{bmatrix} j_3 & j_4 \\ j_1 & j_2 \end{bmatrix}. \tag{A.10}$$

Further

$$a_{jl} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = a_{jl} \begin{bmatrix} j_2 & j_1 \\ j_4 & j_3 \end{bmatrix} = (-)^{j_1+j_3-j-l} \sqrt{\frac{[2j+1][2l+1]}{[2j_1+1][2j_3+1]}}\, a_{j_1j_3} \begin{bmatrix} j & j_2 \\ l & j_4 \end{bmatrix}, \tag{A.11}$$

$$a_{0l} \begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix} = (-)^{j_1+j_3-l} \sqrt{\frac{[2l+1]}{[2j_2+1][2j_3+1]}}\, \delta_{j_1j_2}\delta_{j_3j_4};$$
$$a_{jl} \begin{bmatrix} 0 & j_2 \\ j_3 & j_4 \end{bmatrix} = \delta_{j_2j}\delta_{j_4l}, \tag{A.12}$$

and

$$(-)^{2\min(j_1,j_2)}(\lambda_m(\hat{j}_1\hat{j}_2))^{\pm 1} a_{m0} \begin{bmatrix} j_2 & j_1 \\ j_1 & j_2 \end{bmatrix} = \sum_l a_{0l} \begin{bmatrix} j_1 & j_1 \\ j_2 & j_2 \end{bmatrix} (\lambda_l(\hat{j}_1\hat{j}_2^*))^{\mp 1} a_{ml} \begin{bmatrix} j_1 & j_2 \\ j_1 & j_2 \end{bmatrix}, \tag{A.13}$$

$$\sum_{ml} a_{ms}\begin{bmatrix} j_3 & j_2 \\ j_1 & j_4 \end{bmatrix} \lambda_m(\hat{j}_2\hat{j}_3) a_{ml}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix} \lambda_l(\hat{j}_1\hat{j}_3) a_{pl}\begin{bmatrix} j_2 & j_1 \\ j_3 & j_4 \end{bmatrix} \lambda_p(\hat{j}_1\hat{j}_2)$$

$$= \sum_{ml} \lambda_s(\hat{j}_1\hat{j}_2) a_{ms}\begin{bmatrix} j_3 & j_1 \\ j_2 & j_4 \end{bmatrix} \lambda_m(\hat{j}_1\hat{j}_3) a_{ml}\begin{bmatrix} j_1 & j_3 \\ j_2 & j_4 \end{bmatrix} \lambda_l(\hat{j}_2\hat{j}_3) a_{pl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}, \quad \text{(A.14)}$$

$$\sum_{l_1} a_{r_2 l_1}\begin{bmatrix} r_1 & j_3 \\ j_4 & j_5 \end{bmatrix} a_{r_1 l_2}\begin{bmatrix} j_1 & j_2 \\ l_1 & j_5 \end{bmatrix} a_{l_1 l_2}\begin{bmatrix} l_2 & j_2 \\ j_3 & j_4 \end{bmatrix}$$

$$= a_{r_2 l_2}\begin{bmatrix} j_1 & l_3 \\ j_4 & j_5 \end{bmatrix} a_{r_1 l_3}\begin{bmatrix} j_1 & j_2 \\ j_3 & r_2 \end{bmatrix}. \quad \text{(A.15)}$$

Equation (A.14) reflects the generating relation of the braiding generators $B_iB_{i+1}B_i = B_{i+1}B_iB_{i+1}$. Both (A.13) and (A.14) follow immediately from the applications of the identity (A.7).

The $q$-independent phases in the eigenvalues $\lambda_l^{(\pm)}(j_1j_2)$ of the braiding matrices given in Eq. (4.5) and also that in the duality matrix (A.1) are somewhat ambiguous. The choice we make here differs from that in refs. 6. We have chosen these phases in such a way that $\lambda_0^{(-)}(j,j)=1$, $\lambda_l^{(\pm)}(0,j)=\delta_{lj}$ and $a_{jl}\begin{bmatrix} 0 & j_2 \\ j_3 & j_4 \end{bmatrix} = a_{jl}\begin{bmatrix} j_2 & 0 \\ j_4 & j_3 \end{bmatrix} = a_{jl}\begin{bmatrix} j_3 & j_4 \\ 0 & j_2 \end{bmatrix} = a_{jl}\begin{bmatrix} j_4 & j_3 \\ j_2 & 0 \end{bmatrix} = \delta_{jj_2}\delta_{lj_4}$. The braiding relation (A.14) is not sensitive to this ambiguity of phases. However, these phases are relatively fixed by requiring some consistency conditions. One such condition is obtained by gluing a copy each of the manifolds shown in Figs. 14(a) and (b) for $m=2$ (and with $j_1=j_2=j$) along their oppositely oriented boundaries. This yields an unknot $U$ carrying spin $j$ in an $S^3$. Thus we have, using Eqs. (4.8, 10) for $m=2$ and $j_1=j_2$, the consistency condition:

$$[2j+1]^2(-)^{2j}\langle\phi_0(\hat{j}\hat{j}^*\hat{j}\hat{j}^*)|\phi_0'(\hat{j}^*\hat{j}\hat{j}^*\hat{j})\rangle = [2j+1]^2(-)^{2j}a_{00}\begin{bmatrix} j & j \\ j & j \end{bmatrix} = V_j[U]\,. \quad \text{(A.16)}$$

Another consistency condition is Eq. (A.13). This reflects the equality of Chern–Simons functional integrals over two three-balls as shown in Fig. 17(a). A weaker condition on $\lambda_l^{(\pm)}(j_1,j_2)$ is

$$\sum_l [2l+1](\lambda_l^{(+)}(j_1j_2))^{\pm 2} = \sum_l [2l+1](\lambda_l^{(-)}(j_1j_2))^{\pm 2}\,. \quad \text{(A.17)}$$

This condition is obtained by gluing two copies each of the diagrams of Fig. 17(a) to represent the same Hopf link in two different ways. Yet another consistency condition is

$$[2j+1]^2\sum_l a_{0l}\begin{bmatrix} j & j \\ j & j \end{bmatrix} a_{0l}\begin{bmatrix} j & j \\ j & j \end{bmatrix}(\lambda_l^{(+)}(jj))^{\pm 1} = [2j+1]\,. \quad \text{(A.18)}$$

This equation represents the fact that each of the knots in Fig. 17(b) is an unknot $U$.

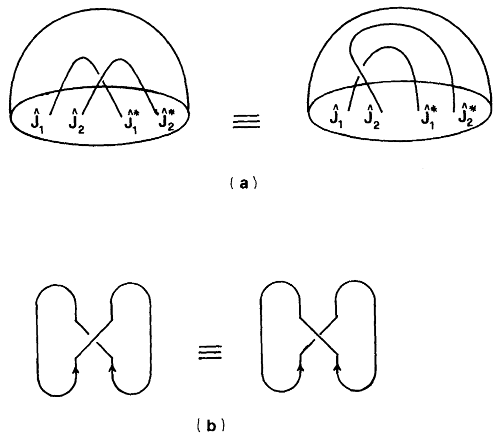


**Fig. 17.** Consistency conditions on $\lambda_j^{(\pm)}(j_1 j_2)$ and $a_{jl}\begin{bmatrix} j_1 & j_2 \\ j_3 & j_4 \end{bmatrix}$

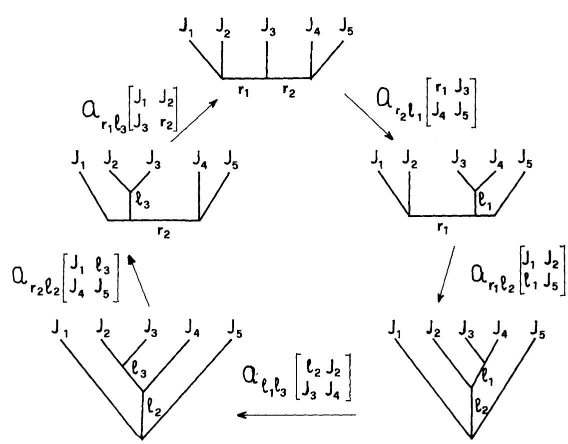

**Fig. 18.** A cycle of five duality transformations

Next, Eq. (A.15) follows directly from Eq. (A.8). This equation reflects that five duality transformations on conformal blocks of the conformal field theory as shown in Fig. 18 bring us back to the same block. However a phase may be picked up in the process. With the phase of duality matrices as fixed above, there is no such phase picked up by this cycle of five duality transformations.

The duality matrices for some low values of $j$ can easily be computed explicitly. For example,

$$a_{jl}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{[2]}\begin{pmatrix} -1 & \sqrt{[3]} \\ \sqrt{[3]} & 1 \end{pmatrix} \qquad \text{(A.19)}$$

and

$$a_{jl}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\frac{1}{[3]}\begin{pmatrix}1 & -\sqrt{[3]} & \sqrt{[5]}\\ -\sqrt{[3]} & \frac{[3]([5]-1)}{[4][2]} & \frac{[2]\sqrt{[5][3]}}{[4]}\\ \sqrt{[5]} & \frac{[2]\sqrt{[5][3]}}{[4]} & \frac{[2]}{[4]}\end{pmatrix}. \quad \text{(A.20)}$$

The general duality matrix $a_{(p;r)(q;s)}$ for $2m$-point correlators of Wess–Zumino conformal theory as depicted in Figs. 2 are given by Eq. (2.3) of Theorem 1. Using the above four-point duality matrices, these can be shown to satisfy the orthogonality and symmetry properties expressed in Eqs. (2.4). Further some special values of these $2m$-point duality matrices are

$$a_{(0;0)(q;s)}\begin{bmatrix}jj\\ jj\\ \vdots\\ jj\end{bmatrix}=\prod_{l=0}^{m-2}\left((-)^{2j-q_{l+1}}\frac{\sqrt{[2q_{l+1}+1]}}{[2j+1]}\right)\prod_{i=1}^{m-2}a_{js_{i-1}}\begin{bmatrix}j & q_i\\ s_i & j\end{bmatrix},$$

$$a_{(p;r)(0;0)}\begin{bmatrix}jj\\ jj\\ \vdots\\ jj\end{bmatrix}=\prod_{l=0}^{m-2}\left((-)^{2j-r_l}\frac{\sqrt{[2r_l+1]}}{[2j+1]}\right)\prod_{i=1}^{m-2}a_{jp_i}\begin{bmatrix}r_{i-1} & j\\ j & r_i\end{bmatrix}, \quad \text{(A.21)}$$

where $s_0\equiv q_0$, $s_{m-2}\equiv q_{m-1}$, $r_0=p_0$, $r_{m-2}=p_{m-1}$. Further a useful identity is:

$$\sum_{(q_i;\,i\neq l),\,(s_j;\,j\neq l,\,l-1)} a_{(p;r)(q;s)}\begin{bmatrix}j_1 & j_2\\ \vdots & \vdots\\ j_{2l-3} & j_{2l-2}\\ j_{2l-1} & j_{2l}\\ j_{2l+1} & j_{2l+2}\\ \vdots & \vdots\\ j_{2m-1} & j_{2m}\end{bmatrix} a_{(p';r')(q;s)}\begin{bmatrix}j_1 & j_2\\ \vdots & \vdots\\ j_{2l-3} & j_{2l-2}\\ j_{2l-1} & j_{2l+1}\\ j_{2l} & j_{2l+2}\\ \vdots & \vdots\\ j_{2m-1} & j_{2m}\end{bmatrix}$$

$$=\left(\prod_{i\neq l-1,\,l}\delta_{p_ip_i'}\right)\left(\prod_{j\neq l-1,\,l}\delta_{r_jr_j'}\right)a_{s_{l-1}p_{l-1}'}\begin{bmatrix}r_{l-2} & j_{2l-1}\\ j_{2l+1} & r_{l-1}'\end{bmatrix}$$

$$\times a_{s_{l-1}p_{l-1}}\begin{bmatrix}r_{l-2} & j_{2l-1}\\ j_{2l} & r_{l-1}\end{bmatrix}a_{s_lp_l'}\begin{bmatrix}r_{l-1}' & j_{2l}\\ j_{2l+2} & r_l\end{bmatrix}$$

$$\times a_{s_lp_{l-1}}\begin{bmatrix}r_{l-1} & j_{2l+1}\\ j_{2l+2} & r_l\end{bmatrix}a_{r_{l-1}'q_l}\begin{bmatrix}s_{l-1} & j_{2l+1}\\ j_{2l} & s_l\end{bmatrix}a_{r_{l-1}q_l}\begin{bmatrix}s_{l-1} & j_{2l}\\ j_{2l+1} & s_l\end{bmatrix}$$

$$l=0,1,2,\ldots(m-1). \quad \text{(A.18)}$$

Here $r_{-1}\equiv 0$.

## Appendix II

It may be worthwhile to present a tabulation of the new invariants for knots and links. This we present now for knots and links with low crossing numbers as listed by tables of Rolfsen [19]. The naming of knots and links will be given as in this book which reads clearly the crossing number (as the minimal number of double points in the link diagram). We shall not present the link diagrams as shown in these tables but instead give their plat representations so that Theorem 5 can readily be used to write down the invariants.

*IIA. Knots.* In this subsection invariants for knots up to crossing number eight will be given. All knots will carry spin $j$ representations and we shall shorten the notation for eigenvalues of the braid matrix introducing right-handed half-twists in parallely and antiparallely oriented strands and also the duality matrices as:

$$\lambda_l^{(\pm)} \equiv \lambda_l^{(\pm)}(j,j), \quad a_{ml} \equiv a_{ml}\begin{bmatrix} jj \\ jj \end{bmatrix},$$

$$a_{(m)\,(l)} \equiv a_{(m_0 m_1 m_2)\,(l_0 l_1 l_2)}\begin{bmatrix} jj \\ jj \\ jj \end{bmatrix}$$

The plat representation of knots studied here are given in Fig. 19 and their knot invariants $V_j$ using Theorem 5 are as follows:

$0_1$: $[2j+1]$

$3_1$: $\sum [2l+1](\lambda_l^{(+)})^{-3}$

$4_1$: $\sum \sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-2}(\lambda_l^{(-)})^2 a_{ml}$

$5_1$: $\sum [2l+1](\lambda_l^{(+)})^{-5}$

$5_2$: $\sum \sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-3}(\lambda_l^{(+)})^{-2} a_{ml}$

$6_1$: $\sum \sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-4}(\lambda_l^{(-)})^2 a_{ml}$

$6_2$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m}(\lambda_p^{(+)})^{-3}(\lambda_n^{(-)})^{-1}(\lambda_m^{(-)})^2 a_{pn} a_{mn}$

$6_3$: $\sum \sqrt{[2p+1][2l+1]}(-)^{p+l-2j}(\lambda_p^{(+)})^2(\lambda_n^{(-)})(\lambda_m^{(-)})^{-1}(\lambda_l^{(+)})^{-2} a_{pn} a_{mn} a_{ml}$

$7_1$: $\sum [2l+1](\lambda_l^{(+)})^{-7}$

$7_2$: $\sum \sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-5}(\lambda_l^{(+)})^{-2} a_{ml}$

$7_3$: $\sum \sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(+)})^4(\lambda_l^{(-)})^3 a_{ml}$

$7_4$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m}(\lambda_p^{(-)})^3(\lambda_n^{(+)})(\lambda_m^{(-)})^3 a_{pn} a_{mn}$

$7_5$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m}(\lambda_p^{(+)})^{-3}(\lambda_n^{(-)})^{-2}(\lambda_m^{(+)})^{-2} a_{pn} a_{mn}$

$7_6$: $\sum \sqrt{[2p+1][2l+1]}(-)^{p+l-2j}(\lambda_p^{(+)})^{-2}(\lambda_n^{(-)})^{-1}(\lambda_m^{(-)})^2(\lambda_l^{(-)})^{-2} a_{pn} a_{mn} a_{ml}$

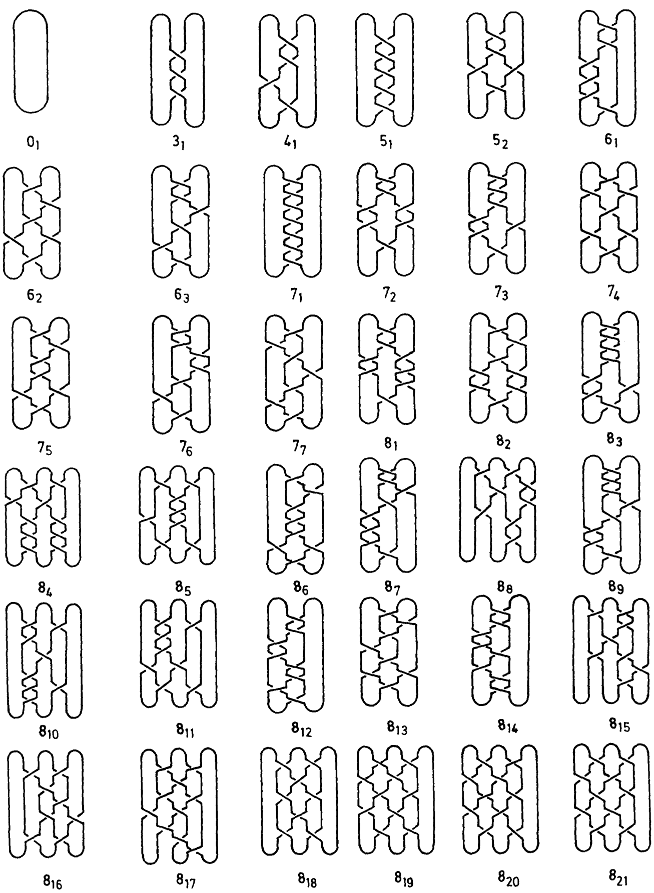


**Fig. 19.** Plat representations for knots up to eight crossing number

$$7_7:\quad \sum\sqrt{[2r+1][2m+1]}(-)^{r+m}(\lambda_r^{(-)})^2(\lambda_q^{(-)})^{-1}(\lambda_p^{(+)})^{-1}(\lambda_n^{(-)})^{-1}(\lambda_m^{(-)})^2 a_{rq}a_{pq}a_{pn}a_{mn}$$

$$8_1:\quad \sum\sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-6}(\lambda_l^{(-)})^2 a_{ml}$$

$$8_2:\quad \sum\sqrt{[2p+1][2m+1]}(-)^{p+m}(\lambda_p^{(+)})^{-5}(\lambda_n^{(-)})^{-1}(\lambda_m^{(-)})^2 a_{pn}a_{mn}$$

$$8_3:\quad \sum\sqrt{[2m+1][2l+1]}(-)^{m+l-2j}(\lambda_m^{(-)})^{-4}(\lambda_l^{(-)})^4 a_{ml}$$

$$8_4:\quad [2j+1]^3\sum(\lambda_{n_1}^{(-)})^3(\lambda_{n_2}^{(+)})^{-3}(\lambda_{m_0}^{(+)})(\lambda_{m_1}^{(+)})(\lambda_{l_1}^{(-)}\lambda_{l_2}^{(+)})^{-1}$$
$$\times\, a_{(0)(n)}a_{(m)(n)}a_{(m)(l)}a_{(0)(l)}$$

$$8_5:\quad [2j+1]^3\sum\lambda_{n_1}^{(-)}(\lambda_{n_2}^{(+)})^{-1}(\lambda_{m_0}^{(+)})(\lambda_{m_1}^{(+)})^3\lambda_{l_1}^{(-)}(\lambda_{l_2}^{(+)})^{-1}$$
$$\times\, a_{(0)(n)}a_{(m)(n)}a_{(m)(l)}a_{(0)(l)}$$

$$8_6:\quad \sum\sqrt{[2p+1][2m+1]}(-)^{p+m}(\lambda_p^{(+)})^{-3}(\lambda_n^{(-)})^{-3}(\lambda_m^{(-)})^2 a_{pn}a_{mn}a_{ml}$$

$$8_7:\quad \sum\sqrt{[2p+1][2l+1]}(-)^{p+l-2j}(\lambda_p^{(+)})^4(\lambda_n^{(-)})(\lambda_m^{(-)})^{-1}(\lambda_l^{(+)})^{-2}a_{pn}a_{mn}a_{ml}$$

$8_8$: $[2j+1]^3 \sum (\lambda_{n_1}^{(-)})^{-1} (\lambda_{n_2}^{(+)})^2 (\lambda_{m_1}^{(+)})^{-1} (\lambda_{m_2}^{(-)})^2 (\lambda_{l_1}^{(-)})^{-1} \lambda_{l_2}^{(+)}$
$\times a_{(0)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_9$: $\sum \sqrt{[2p+1][2l+1]} (-)^{p+l-2j} (\lambda_p^{(+)})^3 \lambda_n^{(-)} (\lambda_m^{(-)})^{-1} (\lambda_l^{(+)})^{-3} a_{pn} a_{mn} a_{ml}$

$8_{10}$: $[2j+1]^3 \sum (\lambda_{n_1}^{(+)})^3 (\lambda_{n_2}^{(-)})^{-1} (\lambda_{m_1}^{(+)})^{-1} (\lambda_{l_1}^{(+)})^2 (\lambda_{l_2}^{(-)})^{-1}$
$\times a_{(0)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{11}$: $[2j+1]^3 \sum (\lambda_{n_1}^{(-)})^{-1} (\lambda_{n_2}^{(+)}) (\lambda_{m_0}^{(+)})^{-1} (\lambda_{m_1}^{(-)})^{-1} (\lambda_{l_1}^{(-)})^{-3} \lambda_{l_2}^{(+)}$
$\times a_{(0)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{12}$: $\sum \sqrt{[2p+1][2l+1]} (-)^{p+l-2j} (\lambda_p^{(-)})^{-2} (\lambda_n^{(-)})^2 (\lambda_m^{(-)})^{-2} (\lambda_l^{(-)})^2 a_{pn} a_{mn} a_{ml}$

$8_{13}$: $\sum \sqrt{[2r+1][2l+1]} (-)^{r+l} (\lambda_r^{(-)})^3 (\lambda_q^{(+)}) (\lambda_p^{(-)}) (\lambda_n^{(-)})^{-1} (\lambda_m^{(+)})^{-2} a_{rq} a_{pq} a_{pn} a_{mn}$

$8_{14}$: $\sum \sqrt{[2q+1][2l+1]} (-)^{q+l} (\lambda_q^{(-)})^2 (\lambda_p^{(-)})^{-1} (\lambda_n^{(+)})^{-1} (\lambda_m^{(-)})^{-2} (\lambda_l^{(+)})^{-2}$
$\times a_{pq} a_{pn} a_{mn} a_{ml}$

$8_{15}$: $[2j+1]^3 \sum (\lambda_{q_2}^{(-)})^{-1} (\lambda_{p_2}^{(+)})^{-1} (\lambda_{n_1}^{(-)} \lambda_{n_2}^{(-)})^{-1} (\lambda_{m_1}^{(+)})^{-1} (\lambda_{l_1}^{(-)})^{-1} (\lambda_{l_2}^{(-)})^{-2}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{16}$: $[2j+1]^3 \sum (\lambda_{q_1}^{(-)} \lambda_{q_2}^{(-)})^{-1} \lambda_{p_1}^{(-)} (\lambda_{p_2}^{(+)})^{-1} \lambda_{n_2}^{(+)} \lambda_{m_1}^{(-)} (\lambda_{l_1}^{(-)} \lambda_{l_2}^{(-)})^{-1}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{17}$: $[2j+1]^3 \sum (\lambda_{s_2}^{(-)})^{-1} (\lambda_{r_1}^{(+)})^{-1} \lambda_{q_1}^{(-)} (\lambda_{p_0}^{(+)})^{-1} \lambda_{p_1}^{(-)} \lambda_{n_1}^{(+)} \lambda_{n_2}^{(+)} \lambda_{m_1}^{(-)} (\lambda_{l_1}^{(-)} \lambda_{l_2}^{(-)})^{-1}$
$\times a_{(0)(s)} a_{(r)(s)} a_{(r)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{18}$: $[2j+1]^3 \sum \lambda_{q_1}^{(-)} \lambda_{q_2}^{(-)} (\lambda_{p_1}^{(-)})^{-1} (\lambda_{n_1}^{(+)} \lambda_{n_2}^{(+)})^{-1} (\lambda_{m_1}^{(-)})^{-1} \lambda_{l_1}^{(-)} \lambda_{l_2}^{(-)}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{19}$: $[2j+1]^3 \sum \lambda_{q_1}^{(-)} \lambda_{q_2}^{(-)} \lambda_{p_0}^{(+)} \lambda_{p_1}^{(-)} \lambda_{n_1}^{(+)} \lambda_{n_2}^{(+)} (\lambda_{m_0}^{(+)})^{-1} \lambda_{m_1}^{(-)} \lambda_{l_1}^{(-)} \lambda_{l_2}^{(-)}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{20}$: $[2j+1]^3 \sum (\lambda_{q_1}^{(-)})^{-1} \lambda_{q_2}^{(-)} (\lambda_{p_0}^{(+)} \lambda_{p_1}^{(-)})^{-1} \lambda_{n_1}^{(+)} (\lambda_{n_2}^{(+)})^{-1} (\lambda_{m_0}^{(+)})^{-1} \lambda_{m_1}^{(-)} \lambda_{l_1}^{(-)} (\lambda_{l_2}^{(-)})^{-1}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

$8_{21}$: $[2j+1]^3 \sum (\lambda_{q_1}^{(-)} \lambda_{q_2}^{(-)})^{-1} (\lambda_{p_0}^{(+)})^{-1} \lambda_{p_1}^{(-)} (\lambda_{n_1}^{(+)} \lambda_{n_2}^{(+)})^{-1} (\lambda_{m_0}^{(+)} \lambda_{m_1}^{(-)})^{-1} \lambda_{l_1}^{(-)} \lambda_{l_2}^{(-)}$
$\times a_{(0)(q)} a_{(p)(q)} a_{(p)(n)} a_{(m)(n)} a_{(m)(l)} a_{(0)(l)}$

In the expressions above all the indices are summed over positive integers from 0 to $\min(2j, k-2j)$. In these calculations we have made use of identities (A.12) and (A.13). The results for knots up to crossing number seven as presented in the first of ref. 6 are the same. Further these invariants for $j=1/2$ and $j=1$ respectively agree with Jones-one variable polynomials [2] and those obtained by Wadati *et al* from the three-state exactly solvable model [14]. To do this comparison we need to multiply these polynomials of refs. 2 and 14 by $[2j+1]$, $j=1/2$ and 1 respectively, to account for differences of normalization before comparing with our results above.

Notice for $q=1$ (which corresponds to $k\to\infty$), the invariant for any knot is simply the ordinary dimension of the representation living on it, $V_j(q=1)=2j+1$.

Change of orientation does not affect the knots and invariants do not depend on the orientation. Thus orientation for knots may not be specified. However, mirror reflected knots are not isotopically equivalent in general. For any chiral knot in the above list, the invariant for the obverse is obtained by conjugation which amounts to replacing various braid matrix eigenvalues $\lambda_l^{(\pm)}$ by their inverses in the expression.

*IIB. Links.* Now we shall list the invariants for two component links with crossing number up to seven as listed in Rolfsens book [18]. Unlike in the case of knots above, here we need to specify the orientations on the two components. There are four possible ways of putting arrows on these knots. Simultaneous reversing of orientations on all the components knots does not change the invariant. Hence there are only two independent ways of specifying the orientations on the knots of a two component link. We have made a specific choice of relative orientations of the component knots as indicated in Fig. 20 where we have given plat representations of these links. We have also placed spin $j_1$ and $j_2$ representations on the components as indicated. Then from Theorem 5, the invariants $V_{j_1 j_2}$ for these links are:

$0_1$: $[2j_1+1][2j_2+1]$

$2_1$: $\sum[2l+1](\lambda_l^{(+)}(j_1j_2))^{-2}$

$4_1$: $\sum[2l+1](\lambda_l^{(+)}(j_1j_2))^{-4}$

$5_1$: $[2j_1+1]^2[2j_2+1]\sum(\lambda_{p_1}^{(-)}(j_1j_2))^{-1}\lambda_{p_2}^{(+)}(j_1j_2)(\lambda_{m_1}^{(+)}(j_1j_1))^{-1}$

$$\times(\lambda_{l_1}^{(-)}(j_1j_2))^{-1}\lambda_{l_2}^{(+)}(j_1j_2)a_{(0)(p)}\begin{bmatrix} j_1j_1\\ j_2j_2\\ j_1j_1\end{bmatrix}$$

$$\times\, a_{(m)(p)}\begin{bmatrix} j_1j_2\\ j_1j_1\\ j_2j_1\end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_1j_2\\ j_1j_1\\ j_2j_1\end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_1j_1\\ j_2j_2\\ j_1j_1\end{bmatrix}$$

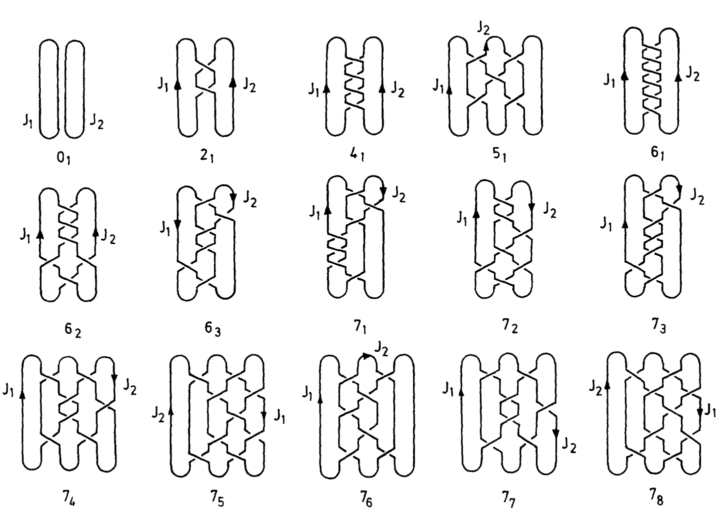


**Fig. 20.** Plat representation for two-component links up to seven crossing number

$6_1$: $\sum [2l+1](\lambda_l^{(+)}(j_1j_2))^{-6}$

$6_2$: $\sum \sqrt{[2l+1][2m+1]}(-)^{m+l-2\max(j_1,j_2)}(\lambda_m^{(-)}(j_1j_2)\lambda_l^{(+)}(j_1j_2))^{-3}a_{ml}\begin{bmatrix} j_1j_2 \\ j_1j_2 \end{bmatrix}$

$6_3$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m-2(j_1+j_2)}(\lambda_p^{(-)}(j_1j_2))^2(\lambda_n^{(-)}(j_1j_2))^{-2}(\lambda_m^{(-)}(j_1j_2))^2$

$$\times a_{pn}\begin{bmatrix} j_1j_2 \\ j_2j_1 \end{bmatrix} a_{mn}\begin{bmatrix} j_1j_2 \\ j_2j_1 \end{bmatrix}$$

$7_1$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m-2(j_1+j_2)}(\lambda_p^{(-)}(j_1j_2))^4(\lambda_n^{(-)}(j_2j_2))^{-1}(\lambda_m^{(+)}(j_1j_2))^{-2}$

$$\times a_{pn}\begin{bmatrix} j_1j_2 \\ j_2j_1 \end{bmatrix} a_{mn}\begin{bmatrix} j_1j_2 \\ j_1j_2 \end{bmatrix}$$

$7_2$: $\sum \sqrt{[2p+1][2l+1]}(-)^{p+l-2\max(j_1,j_2)}(\lambda_p^{(-)}(j_1j_2))^{-3}(\lambda_n^{(+)}(j_1j_2))^{-1}$

$(\lambda_m^{(-)}(j_2j_2))^{-1}(\lambda_l^{(-)}(j_1j_2))^2$

$$\times a_{pn}\begin{bmatrix} j_1j_2 \\ j_1j_2 \end{bmatrix} a_{mn}\begin{bmatrix} j_1j_1 \\ j_2j_2 \end{bmatrix} a_{ml}\begin{bmatrix} j_1j_1 \\ j_2j_2 \end{bmatrix}$$

$7_3$: $\sum \sqrt{[2p+1][2m+1]}(-)^{p+m-2(j_1+j_2)}(\lambda_p^{(-)}(j_1j_2))^2(\lambda_n^{(-)}(j_2j_2))^{-3}(\lambda_m^{(+)}(j_1j_2))^{-2}$

$$\times a_{pn}\begin{bmatrix} j_1j_2 \\ j_2j_1 \end{bmatrix} a_{mn}\begin{bmatrix} j_1j_2 \\ j_2j_1 \end{bmatrix}$$

$7_4$: $[2j_1+1][2j_2+1]^2 \sum \lambda_{n_1}^{(-)}(j_1j_2)\lambda_{n_2}^{(-)}(j_2j_2)(\lambda_{m_1}^{(-)}(j_1j_2))^{-2}$

$$\times \lambda_{m_2}^{(+)}(j_2j_2)\lambda_{l_1}^{(-)}(j_1j_2)\lambda_{l_2}^{(-)}(j_2j_2)$$

$$\times a_{(0)(m)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix} a_{(m)(n)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix}$$

$7_5$: $[2j_1+1][2j_2+1]^2 \sum (\lambda_{q_1}^{(-)}(j_1j_2)\lambda_{q_2}^{(-)}(j_1j_2))(\lambda_{p_1}^{(-)}(j_2j_2))^{-1}\lambda_{p_2}^{(+)}(j_1j_2)(\lambda_{n_2}^{(+)}(j_2j_2))^{-1}$

$$\times (\lambda_{m_1}^{(-)}(j_1j_2))^{-1}\lambda_{m_2}^{(+)}(j_1j_2)\lambda_{l_1}^{(-)}(j_1j_2)(\lambda_{l_2}^{(-)}(j_2j_2))^{-1}$$

$$\times a_{(0)(q)}\begin{bmatrix} j_2j_2 \\ j_1j_1 \\ j_2j_2 \end{bmatrix} a_{(p)(q)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_1j_2 \end{bmatrix} a_{(p)(n)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_2j_1 \end{bmatrix}$$

$$\times a_{(m)(n)}\begin{bmatrix} j_2j_2 \\ j_2j_1 \\ j_2j_1 \end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_1j_2 \end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_2j_2 \\ j_1j_1 \\ j_2j_2 \end{bmatrix}$$

$7_6$: $$[2j_1+1][2j_2+1]^2\sum(\lambda_{q_1}^{(-)}(j_1j_2)\lambda_{q_2}^{(-)}(j_2j_2))^{-1}\lambda_{p_2}^{(-)}(j_1j_2)$$
$$\times\lambda_{n_1}^{(+)}(j_2j_2)\lambda_{m_1}^{(-)}(j_1j_2)(\lambda_{l_1}^{(-)}(j_1j_2)\lambda_{l_2}^{(-)}(j_2j_2))^{-1}$$
$$\times a_{(0)(q)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix} a_{(p)(q)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(p)(n)}\begin{bmatrix} j_1j_2 \\ j_2j_1 \\ j_2j_2 \end{bmatrix}$$
$$\times a_{(m)(n)}\begin{bmatrix} j_1j_2 \\ j_2j_1 \\ j_2j_2 \end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix}$$

$7_7$: $$[2j_1+1][2j_2+1]^2\sum(\lambda_{n_1}^{(-)}(j_1j_2))^{-1}(\lambda_{n_2}^{(-)}(j_2j_2))(\lambda_{m_1}^{(-)}(j_1j_2))^{-2}$$
$$\times\lambda_{m_2}^{(+)}(j_2j_2)(\lambda_{l_1}^{(-)}(j_1j_2))^{-1}\lambda_{l_2}^{(-)}(j_2j_2)$$
$$\times a_{(0)(n)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix} a_{(m)(n)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_1j_2 \\ j_1j_2 \\ j_2j_2 \end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_2j_2 \end{bmatrix}$$

$7_8$: $$[2j_1+1][2j_2+1]^2\sum\lambda_{q_1}^{(-)}(j_1j_2)(\lambda_{q_2}^{(-)}(j_1j_2))^{-1}(\lambda_{p_1}^{(-)}(j_2j_2)\lambda_{p_2}^{(+)}(j_1j_2))^{-1}$$
$$\times(\lambda_{n_2}^{(+)}(j_2j_2))^{-1}(\lambda_{m_1}^{(-)}(j_2j_2))^{-1}\lambda_{m_2}^{(+)}(j_1j_2)\lambda_{l_1}^{(-)}(j_1j_2)$$
$$\times(\lambda_{l_2}^{(-)}(j_1j_2)^{-1}$$
$$\times a_{(0)(q)}\begin{bmatrix} j_2j_2 \\ j_1j_1 \\ j_2j_2 \end{bmatrix} a_{(p)(q)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_1j_2 \end{bmatrix} a_{(p)(n)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_2j_1 \end{bmatrix}$$
$$\times a_{(m)(n)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_2j_1 \end{bmatrix} a_{(m)(l)}\begin{bmatrix} j_2j_1 \\ j_2j_2 \\ j_1j_2 \end{bmatrix} a_{(0)(l)}\begin{bmatrix} j_2j_2 \\ j_1j_1 \\ j_2j_2 \end{bmatrix}.$$

Again identity (A.13) has been used in above calculations. A corresponding useful identity for six-point duality matrices is:

$$\sum_{(q)}(\lambda_{q_1}(\hat{j}_1^*\hat{j}_2))^r(\lambda_{q_2}(\hat{j}_2^*\hat{j}_3))^s a_{(0)(q)}\begin{bmatrix} j_1j_1 \\ j_2j_2 \\ j_3j_3 \end{bmatrix} a_{(p)(q)}\begin{bmatrix} j_1j_2 \\ j_1j_3 \\ j_2j_3 \end{bmatrix}$$
$$=(-)^{2\min(j_1,j_2)+2\min(j_2,j_3)}(\lambda_{p_0}(\hat{j}_1\hat{j}_2))^{-r}(\lambda_{p_2}(\hat{j}_2\hat{j}_3))^{-s}a_{(p)(0)}\begin{bmatrix} j_2j_1 \\ j_1j_3 \\ j_3j_2 \end{bmatrix},$$

where $r=\pm 1$, $s=\pm 1$.

For mirror reflected links, the invariants are obtained by conjugation.

Notice for $q=1$ (that is, $k \to \infty$), every one of these invariants reduces to the product of the ordinary dimensions of the representations placed on the two component knots, $V_{j_1 j_2}(q=1)=(2j_1+1)(2j_2+1)$.

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