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The Classification of Affine $SU(3)$ Modular Invariant Partition Functions

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Abstract: A complete classification of the *physical* modular invariant partition functions for the WZNW models is known for very few affine algebras and levels, the most significant being all levels of $SU(2)$, and level 1 of all simple algebras. In this paper we solve the classification problem for $SU(3)$ modular invariant partition functions, all levels. Our approach will also be applicable to other affine Lie algebras, and we include some preliminary work in that direction, including a sketch of a new proof for $SU(2)$.

1. Introduction

The classification of all rational conformal field theories (RCFTs) is clearly a desirable pursuit. In spite of tremendous progress in our understanding of RCFTs, we still find ourselves far from our ultimate goal. The problem can be somewhat simplified by focusing on the building blocks, the Wess–Zumino–Novikov–Witten (WZNW) models [41, 28, 16] associated with simple Lie algebras. Unfortunately, a full classification of even these models is still lacking. Only in the special cases of $SU(2)_k$ [6, 23, 15] and level 1 for all simple affine algebras [19, 9, 11] has a list of *physical* modular invariant partition functions been proven to be complete. The generalization of these proofs to higher ranks and levels has been plagued with difficulties due to the explosively increasing numbers of *non-physical* modular invariants. In this article we attempt to develop the tools necessary for this generalization, and successfully apply the new technique to $SU(3)_k$.

The partition function of a WZNW conformal field theory associated with affine Lie algebra (= current Lie algebra on S^1) [20, 25, 3] $\hat{g}$ and level k can be written as

$$Z = \sum N_{\lambda_L \lambda_R} \chi^k_{\lambda_L} \chi^{k*}_{\lambda_R} . \tag{1.1}$$

χ^k_λ is the *normalized character* [21] of the representation of $\hat{g}$ with (horizontal) highest weight λ and level k; it is a function of a complex vector z and a complex number τ. The algebra $\hat{g}$ is the untwisted affine extension $g^{(1)}$ of a simple Lie algebra g (this extends in the obvious way to semi-simple algebras). The (finite) sum in Eq. (1.1) is over the horizontal highest weights λ_L, λ_R of level k.

There are three properties the sum in Eq. (1.1) must satisfy in order to be interpreted as the partition function of a physically sensible conformal field theory:

(P1) *modular invariance.* This is equivalent to the two conditions:

$$Z(z_L z_R | \tau + 1) = Z(z_L z_R | \tau) , \tag{1.2a}$$

$$\exp[-k\pi i(z_L^2/\tau - z_R^{*2}/\tau^*)] Z(z_L/\tau, z_R/\tau | -1/\tau) = Z(z_L z_R | \tau); \tag{1.2b}$$

(P2) *positivity* and *integrality.* The coefficients $N_{\lambda_L \lambda_R}$ in Eq. (1.1) must be non-negative integers; and

(P3) *uniqueness of vacuum.* $\lambda = 0$ is a possible highest weight vector, for any g and k. We must have $N_{00} = 1$ (in the following sections we will change notations slightly, and this will become $N_{\rho\rho} = 1$).

We will call any modular invariant function Z of the form (1.1), an *invariant.* Z will be called *positive* if in addition each $N_{\lambda_L \lambda_R} \geqq 0$, and *physical* if it satisfies (P1), (P2), and (P3). Our task is to find all physical invariants corresponding to each algebra g and level k.

An invariant satisfying (P1), (P2) and (P3) is still not necessarily the partition function of a conformal field theory obeying duality and CPT-invariance. If it is, we will call it *strongly physical.* These are the invariants of interest to physics. We will discuss the additional properties satisfied by strongly physical invariants (most importantly, that they become automorphism invariants when written in terms of the characters of their maximal chiral algebras) at the beginning of Sect. 5.

Much work has been done over the past few years on finding these physical invariants. But there has been comparatively little progress in the task of determining all physical invariants belonging to certain choices of g and k: all physical invariants for $g = A_1$ are known, for any level k [6, 23, 15]; all level 1 physical invariants have been found for simple g – namely, $g = A_n$ [19, 9], and $g = B_n$, C_n, D_n, $E_{6,7,8}$, F_4, G_2 [11]; and all A_2 level k ones are known when $k + 3$ is prime [33] (this work has recently been extended – see the Note at the end of this paper). Some work in classifying the *heterotic* physical invariants has also been done [12].

Unfortunately, enough simplifications apply to the level 1 cases, and to the A_1 case, to make it unclear how to extend those arguments to more general cases. In this paper we will focus on the case $g = A_2$, although our primary interest lies in developing tools applicable to other algebras (see Sect. 6). There are several known physical invariants for A_2 [7, 26]. These will be given in Eqs. (2.7). The question this paper addresses is the completeness of this list. Two results in this direction are already known: the list is complete for $k + 3$ prime [33]; the list is complete for $k \leqq 32$ [18].

In Sect. 2 we will introduce the notation and terminology used in the later sections, and sketch the strategy taken. Section 3 will find all *permutation invariants* (see Eq. (3.1)) of A_2, for each level. In Sect. 4 we find, for each k, a list of weights λ for which $N_{0\lambda}$ can be non-zero for some level k physical invariant N; this list shows, among other things, that the only A_2 physical invariants for $k \equiv 2$, 4, 7, 8, 10, 11 (mod 12) are permutation invariants. Thus Sects. 3 and 4 succeed in finding all A_2 physical invariants for those levels. In Sect. 5 we complete the classification for the remaining levels (except for the levels 3, 5, 6, 9, 12, 15 and 21, which we avoided because of extra complications arising at those k), but to do this we need to impose further physical conditions (namely, the duality and unitarity of the underlying conformal field theory) so that the powerful analysis of [26] can be applied – we find all A_2 strongly physical invariants for those levels. Together with [18] this concludes the A_2 classification problem. In the final section we investigate how well this approach extends to other algebras. The appendix includes a detailed sketch of how this approach applies to A_1.

The key advantage the approach developed in this paper has over previous approaches is that explicit construction of the commutant is avoided, and positivity is imposed from the beginning. This significantly simplifies the analysis required.

The only remaining question for the A_2 classification problem is to see if our proof, which found all *physical* invariants for half the levels and all *strongly physical* ones for the other half, can be strengthened so as to find all *physical* ones for all levels – although all assumptions we have imposed are physically valid, it would be nice to reduce these to the smallest number possible. A more interesting and important question is to find other algebras which can be handled by analogous methods.

2. Terminology and Sketch of Proof

Before we begin the main body of this paper, it is necessary to introduce some notation and terminology. For a much more complete description of the rich theory of Kač–Moody algebras, see e.g. [21, 17, 22]. We will restrict attention here to the algebra $g = A_2$, but similar comments hold for the other algebras. The few facts about lattices which we need are included in e.g. [8].

The root = coroot lattice of $g = A_2$ is also called A_2. Let β_1, β_2 denote the fundamental weights of A_2, and write $\rho = \beta_1 + \beta_2$; β_1 and β_2 span the *dual lattice* A_2^* of A_2. Throughout this paper we will identify the weight $\lambda = m\beta_1 + n\beta_2$ with its Dynkin labels (m, n).

An integrable irreducible representation of the affine Lie algebra $\hat{g} = A_2^{(1)}$ is given by a positive integer k (called the *level*) and a *highest weight* $\lambda \in A_2^*$. The set of all possible highest weights corresponding to level k representations is

$$P_+^k \stackrel{\text{def}}{=} \{m\beta_1 + n\beta_2 \,|\, m, n \in \mathbf{Z},\ 0 \leqq m, n, m + n \leqq k\}\ . \tag{2.1a}$$

We will find it more convenient to use instead the related set

$$P^k = P_{++}^{k+3} \stackrel{\text{def}}{=} \{m\beta_1 + n\beta_2 \,|\, m, n \in \mathbf{Z},\ 0 < m, n, m + n < k + 3\}\ . \tag{2.1b}$$

Clearly, $P^k = P_+^k + \rho$, and $\rho \in P^k$. *For the remainder of this paper*, the character corresponding to the level k representation with highest weight $\lambda = m\beta_1 + n\beta_2 \in P_+^k$ will be denoted

$$\chi_{\lambda+\rho}^k = \chi_{m+1,n+1}^k\ .$$

The trivial representation of level k, which is given by highest weight $\lambda = 0$, corresponds then to the character $\chi_\rho^k = \chi_{11}^k$, and (P3) becomes $N_{11,11} = 1$.

Let $\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2$ be the simple roots of $\hat{A}_2$. The 6 outer automorphisms of $\hat{A}_2$ are generated by h (order 2) and ω (order 3), where $h(\hat{\alpha}_0) = \hat{\alpha}_0, h(\hat{\alpha}_1) = \hat{\alpha}_2, h(\hat{\alpha}_2) = \hat{\alpha}_1$, and $\omega(\hat{\alpha}_0) = \hat{\alpha}_1, \omega(\hat{\alpha}_1) = \hat{\alpha}_2, \omega(\hat{\alpha}_2) = \hat{\alpha}_0$. On the weights $(m, n) \in P^k$ these become

$$h(m, n) = (n, m) , \tag{2.2a}$$

$$\omega(m, n) = (k + 3 - m - n, m) . \tag{2.2b}$$

Note that $\omega^2(m, n) = (n, k + 3 - m - n)$. We will be encountering h and ω throughout the paper.

The *Weyl–Kač character formula* gives us a convenient expression for the character χ_λ^k:

$$\chi_\lambda^k(z, \tau) = \frac{\sum_{w \in W} \varepsilon(w) \Theta\left(\frac{\lambda}{\sqrt{k+3}} + \sqrt{k+3} A_2\right)(\sqrt{k+3}\, w(z) | \tau)}{D(z|\tau)} , \tag{2.3a}$$

where $$D(z|\tau) \stackrel{\text{def}}{=} \sum_{w \in W} \varepsilon(w) \Theta\left(\frac{\rho}{\sqrt{3}} + \sqrt{3} A_2\right)(\sqrt{3}\, w(z) | \tau) , \tag{2.3b}$$

and $$\Theta(v + \Lambda)(z|\tau) \stackrel{\text{def}}{=} \sum_{x \in \Lambda} \exp[\pi i \tau (x + v)^2 + 2\pi i z \cdot (x + v)] . \tag{2.3c}$$

Here, W is the 6 element Weyl group of A_2 and $\varepsilon(w) = \det w \in \{\pm 1\}$. The variable $\tau \in \mathbf{C}$ satisfies $\operatorname{Im} \tau > 0$, and $z = z_1 \beta_1 + z_2 \beta_2$ is a complex vector. Unlike much of the literature, we will retain $z \neq 0$, so an invariant here will usually be different from its *charge conjugate* (2.7h).

By the *commutant* Ω^k we mean the (complex) space of all functions

$$Z(z_L z_R | \tau) = \sum_{\lambda, \lambda' \in P^k} N_{\lambda\lambda'} \chi_\lambda^k(z_L, \tau) \chi_{\lambda'}^k(z_R, \tau)^* \tag{2.4}$$

invariant under the modular group, i.e. those Z in (2.4) satisfying (P1). It is not hard to show that two functions Z and Z' are equal iff their *coefficient* (or *mass*) *matrices* N and N' are equal; we will use the invariant Z interchangeably with its matrix N.

The functions χ_λ^k behave quite nicely under the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$:

$$\chi_\lambda^k(z, \tau + 1) = \sum_{\lambda' \in P^k} (T^{(k)})_{\lambda\lambda'} \chi_{\lambda'}^k(z, \tau), \quad \text{where} \tag{2.5a}$$

$$(T^{(k)})_{\lambda\lambda'} = \exp\left[\pi i \frac{\lambda^2}{k+3} - \pi i \frac{2}{3}\right] \delta_{\lambda\lambda'} \tag{2.5b}$$

$$= e_k(-m^2 - mn - n^2 + k + 3) \delta_{m,m'} \delta_{n,n'} ; \tag{2.5c}$$

$$\chi_\lambda^k(z/\tau, -1/\tau) = \exp[k\pi i z^2/\tau] \sum_{\lambda' \in P^k} (S^{(k)})_{\lambda\lambda'} \chi_{\lambda'}^k(z, \tau), \quad \text{where} \tag{2.5d}$$

$$(S^{(k)})_{\lambda\lambda'} = \frac{i^3}{(k+3)\sqrt{3}} \sum_{w \in W} \varepsilon(w) \exp\left[-2\pi i \frac{w(\lambda') \cdot \lambda}{k+3}\right] \tag{2.5e}$$

$$= \frac{-i}{\sqrt{3}(k+3)} \{e_k(2mm' + mn' + nm' + 2nn')$$
$$+ e_k(-mm' - 2mn' - nn' + nm')$$
$$+ e_k(-mm' + mn' - 2nm' - nn')$$
$$- e_k(-2mn' - mm' - nn' - 2nm')$$
$$- e_k(2mm' + mn' + nm' - nn')$$
$$- e_k(-mm' + mn' + nm' + 2nn')\} , \tag{2.5f}$$

where in (2.5c, f) we have $\lambda = m\beta_1 + n\beta_2$, $\lambda' = m'\beta_1 + n'\beta_2$ and the function e_k is defined by $e_k(x) \stackrel{\text{def}}{=} \exp\left[\frac{-2\pi i x}{3(k+3)}\right]$. The matrices $T^{(k)}$ and $S^{(k)}$ are unitary and symmetric.

Note that $Z = \sum N_{\lambda\lambda'}\chi_\lambda^k \chi_{\lambda'}^{k*} \in \Omega^k$ iff both

$$(T^{(k)})^\dagger N(T^{(k)}) = N , \tag{2.6a}$$

$$(S^{(k)})^\dagger N(S^{(k)}) = N . \tag{2.6b}$$

Recall the outer automorphisms h and ω given in (2.2). The known physical invariants of A_2 are:

$$\mathscr{A}_k \stackrel{\text{def}}{=} \sum_{\lambda \in P^k} |\chi_\lambda^k|^2 , \tag{2.7a}$$

$$\mathscr{D}_k \stackrel{\text{def}}{=} \sum_{(m,n)\in P^k} \chi_{m,n}^k \chi_{\omega^{k(m-n)}(m,n)}^{k*}, \quad \text{for } k \not\equiv 0 \pmod 3 \text{ and } k \geqq 4 ; \tag{2.7b}$$

$$\mathscr{D}_k \stackrel{\text{def}}{=} \frac{1}{3} \sum_{\substack{(m,n)\in P^k \\ m \equiv n \pmod 3}} |\chi_{m,n}^k + \chi_{\omega(m,n)}^k + \chi_{\omega^2(m,n)}^k|^2, \quad \text{for } k \equiv 0 \pmod 3 ; \tag{2.7c}$$

$$\mathscr{E}_5 \stackrel{\text{def}}{=} |\chi_{1,1}^5 + \chi_{3,3}^5|^2 + |\chi_{1,3}^5 + \chi_{4,3}^5|^2 + |\chi_{3,1}^5 + \chi_{3,4}^5|^2$$
$$+ |\chi_{3,2}^5 + \chi_{1,6}^5|^2 + |\chi_{4,1}^5 + \chi_{1,4}^5|^2 + |\chi_{2,3}^5 + \chi_{6,1}^5|^2 ; \tag{2.7d}$$

$$\mathscr{E}_9^{(1)} \stackrel{\text{def}}{=} |\chi_{1,1}^9 + \chi_{1,10}^9 + \chi_{10,1}^9 + \chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2$$
$$+ 2|\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2 ; \tag{2.7e}$$

$$\mathscr{E}_9^{(2)} \stackrel{\text{def}}{=} |\chi_{1,1}^9 + \chi_{10,1}^9 + \chi_{1,10}^9|^2 + |\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2 + 2|\chi_{4,4}^9|^2$$
$$+ |\chi_{1,4}^9 + \chi_{7,1}^9 + \chi_{4,7}^9|^2 + |\chi_{4,1}^9 + \chi_{1,7}^9 + \chi_{7,4}^9|^2$$
$$+ |\chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2$$
$$+ (\chi_{2,2}^9 + \chi_{2,8}^9 + \chi_{8,2}^9)\chi_{4,4}^{9*} + \chi_{4,4}^9(\chi_{2,2}^{9*} + \chi_{2,8}^{9*} + \chi_{8,2}^{9*}) ; \tag{2.7f}$$

$$\mathscr{E}_{21} \stackrel{\text{def}}{=} |\chi^{21}_{1,1} + \chi^{21}_{5,5} + \chi^{21}_{7,7} + \chi^{21}_{11,11} + \chi^{21}_{22,1} + \chi^{21}_{1,22}$$
$$+ \chi^{21}_{14,5} + \chi^{21}_{5,14} + \chi^{21}_{11,2} + \chi^{21}_{2,11} + \chi^{21}_{10,7} + \chi^{21}_{7,10}|^2$$
$$+ |\chi^{21}_{16,7} + \chi^{21}_{7,16} + \chi^{21}_{16,1} + \chi^{21}_{1,16} + \chi^{21}_{11,8} + \chi^{21}_{8,11}$$
$$+ \chi^{21}_{11,5} + \chi^{21}_{5,11} + \chi^{21}_{8,5} + \chi^{21}_{5,8} + \chi^{21}_{7,1} + \chi^{21}_{1,7}|^2 ; \tag{2.7g}$$

together with their *conjugations* Z^c under h, defined by:

$$Z^c = \sum_{\lambda,\lambda' \in P^k} N_{\lambda,h(\lambda')} \chi^k_\lambda \chi^{k*}_{\lambda'} = \sum_{m,n,m',n'} N_{mn,n'm'} \chi^k_{mn} \chi^{k*}_{m'n'}, \tag{2.7h}$$

where Z is given by (2.4). Note that $\mathscr{D}_3 = \mathscr{D}_3^c$, $\mathscr{D}_6 = \mathscr{D}_6^c$, $\mathscr{E}_9^{(1)} = \mathscr{E}_9^{(1)c}$, and $\mathscr{E}_{21} = \mathscr{E}_{21}^c$. In the case of restricted characters $\chi(0, \tau)$, $Z = Z^c$.

Our goal is to prove that this list is complete: in particular we will prove

Theorem 1(a). *For* $k \equiv 2, 4, 7, 8, 10, 11 \pmod{12}$, *and* $k = 1$, *the set of all physical invariants for* A_2 *is given by Eqs.* (2.7);

1(b). *For* $k \equiv 0, 1, 3, 5, 6, 9 \pmod{12}$, $k \neq 3, 5, 6, 9, 12, 15, 21$, *the set of all strongly physical invariants for* A_2 *is given by Eqs.* (2.7).

(The terms *physical invariant* and *strongly physical invariant* are defined in Sect. 1.)

Two partial results are already known. In [33] this theorem is proven for $k + 3$ prime. They accomplish this by very explicitly computing a basis for the commutant, then finding all the positive invariants, and lastly imposing the uniqueness condition $N_{11,11} = 1$. Unfortunately this explicitness makes it very difficult to apply their approach to more general k. A second partial result is the computer search in [18]. Using the Roberts–Terao–Warner lattice method [31, 40], it finds a basis for the commutant for a given k, and then imposes positivity and uniqueness of the vacuum. The proof given in [11] that lattice partition functions span the commutant guarantees the completeness of this search. In this way it has verified that the list in Eqs. (2.7) is complete for all $k \leqq 32$ (it also applies this technique to the three other rank 2 algebras). This program thus fills in all of the holes of Theorem 1(b).

The approach taken here is somewhat different.

Call an invariant *ρ-decoupled* if $N_{\rho,\lambda} = N_{\lambda,\rho} = 0$ for all $\lambda \neq \rho$. Hence such an invariant can be written in the form

$$Z = a|\chi^k_\rho|^2 + \sum_{\lambda,\lambda' \neq \rho} N_{\lambda,\lambda'} \chi^k_\lambda \chi^{k*}_{\lambda'} .$$

For example, the only ρ-decoupled invariants in Eqs. (2.7) are (2.7a, b) and their conjugates. A valuable observation was made in [11] (see also [14, 26]):

Lemma 1. *A ρ-decoupled physical invariant is a permutation invariant (defined in Eq.* (3.1) *below).*

All permutation invariants are found in the following section. In Sect. 4 we proceed to show that for some levels, any physical invariant must be a permutation invariant, thus proving Theorem 1 for those levels.

A second observation made in [11] (see also the Note at the end of this paper) connects more directly with the lattice method of [31, 40]. First note the following:

It is proven in [21] that for any $\lambda \in A_2^*$, either

$$\sum_{w \in W} \varepsilon(w)\Theta\left(\frac{w(\lambda)}{\sqrt{k+3}} + \sqrt{k+3}A_2\right)\Big/ D = 0 \tag{2.8a}$$

holds identically (where Θ and D are defined in Eqs. (2.3c, b), respectively), or there exists a $w' \in W$ and a $\lambda' \in P^k$ such that $\lambda \equiv w'(\lambda') \pmod{(k+3)A_2}$, and hence

$$\sum_{w} \varepsilon(w)\Theta\left(\frac{w(\lambda)}{\sqrt{k+3}} + \sqrt{k+3}A_2\right)\Big/ D = \varepsilon(w')\chi_{\lambda'}^k . \tag{2.8b}$$

By the *parity* $\varepsilon(\lambda)$ of λ we mean $\varepsilon(\lambda) = 0$ if (2.8a) holds, and $\varepsilon(\lambda) = \varepsilon(w')$ if (2.8b) does. When $\varepsilon(\lambda) \neq 0$, let $[\lambda]_k$ denote the (unique) weight λ' in (2.8b).

Now choose any $\lambda_L, \lambda_R \in P^k$. We showed in [11], *for each* ℓ *relatively prime to* $3(k+3)$, *that* $\varepsilon(\ell\lambda_L)\varepsilon(\ell\lambda_R) \neq 0$, and that for any level k invariant Z in (2.4),

$$N_{\lambda_L\lambda_R} = \varepsilon(\ell\lambda_L)\varepsilon(\ell\lambda_R)N_{[\ell\lambda_L]_k[\ell\lambda_R]_k} . \tag{2.9}$$

We did this by first showing it for lattice partition functions, where it is obvious, and then referring to the result that lattice partition functions span the commutant. A similar derivation of (2.9) can be made using the construction in [4] (and generalized in [11]) of the *Weyl-unfolded commutant.*

"ℓ relatively prime to $3(k+3)$" is equivalent here (and in Lemma 2 below) to the statement "ℓ relatively prime to the order L of the vector $(\lambda_L; \lambda_R)$ with respect to the lattice $((k+3)A_2; (k+3)A_2)$" – indeed that is how (2.9) is expressed in [11]. Examples of (2.9) for $k = 5$ and $k = 9$ are given in [11]. Of course, it also holds for all other algebras. Equation (2.9) (as well as Lemma 2 below) is used in [18] to eliminate "redundant" coefficients $N_{\lambda\lambda'}$, and hence moderate memory problems. Its main value for our purpose lies in its trivial consequence:

Lemma 2. *Let* $\lambda, \lambda' \in P^k$. *If some* ℓ *relatively prime to* $3(k+3)$ *satisfies* $\varepsilon(\ell\lambda)\varepsilon(\ell\lambda') = -1$, *then* $N_{\lambda,\lambda'} = 0$ *for any positive invariant* N.

The analogue of Lemma 2 holds for all algebras. Lemma 2 constitutes an extremely strong constraint on which $\lambda, \lambda' \in P^k$ may *couple* – i.e. have $N_{\lambda,\lambda'} \neq 0$ – in some positive invariant N. It hints that the space Ω_+^k spanned by the positive invariants of level k may have much smaller dimension than the full commutant Ω^k and so may be a much more convenient space to work with. Indeed, although the dimension of the commutant Ω^k goes to infinity with k, $\dim \Omega_+^k = 4$ for many k [18]. Our approach involves using Lemma 2 to keep our analysis restricted as much as possible to the space Ω_+^k, instead of Ω^k.

A final tool that we will mention here also holds for any positive invariant of any algebra and level, and exploits the fact that the product NN' of two invariants is also an invariant (this can be read off from Eqs. (2.6)). It is proved using the Perron–Frobenius theory of non-negative matrices [29, 10, 24, 35], can be thought of as a generalization of Theorem 4 in [11]. It will be used in Sect. 5 to significantly restrict the possibilities for the coefficient matrix N of physical invariants.

Any matrix M can be written as a direct sum $\bigoplus_i M_i$ of *indecomposable* blocks M_i. By a *non-negative matrix* we mean a square matrix M with non-negative real entries. Any such matrix has a non-negative real eigenvalue $r = r(M)$ with the property that $r \geqq |s|$ for all other (possibly complex) eigenvalues s of M. The number $r(M)$ has many nice properties, for example:

$$\min_i \sum_j M_{ij} \leqq r(M) \leqq \max_i \sum_j M_{ij} , \tag{2.10a}$$

and if M is indecomposable, either equality holds iff each row sum $\sum_j M_{ij}$ is equal; and

$$\max_i M_{ii} \leqq r(M) , \tag{2.10b}$$

and if M is indecomposable and symmetric, equality happens in (2.10b) iff M is a 1×1 matrix $M = (M_{11})$. Also, there is an eigenvector v with eigenvalue r whose components v_i are all non-negative reals.

Lemma 3. *Let $Z = \sum N_{\lambda\lambda'}\chi_\lambda\chi_{\lambda'}$ be a positive invariant, for any algebra and any level. Write N as a direct sum of indecomposable blocks*

$$N = \bigoplus_{\ell=0}^{L} N_\ell = \begin{pmatrix} N_0 & 0 & \cdots & 0 \\ 0 & N_1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & N_L \end{pmatrix} , \tag{2.10c}$$

where N_0 is the block "containing" $N_{\rho\rho}$. Then $r(N_\ell) \leqq r(N_0)$ for all ℓ. If in addition N is a symmetric matrix, and if for all ℓ with $r(N_\ell) = r(N_0)$ we have $(N_\ell)^2 = c_\ell N_\ell$ for some constant c_ℓ, then for each m, either $r(N_m) = r(N_0)$ or $N_m = (0)$.

Proof. Suppose $r(N_{\ell_0}) > r(N_0)$ for ℓ_0, and choose any r satisfying $r(N_0) < r < r(N_{\ell_0})$. Consider the limits as $n \to \infty$ of each $\left(\frac{1}{r} N_\ell\right)^n$. It is easy to show (e.g. using Jordan blocks) that if all eigenvalues λ of a matrix M have norm $|\lambda| < 1$, then the limit of M^n is the 0-matrix. In particular, the limit of $\left(\frac{1}{r} N_0\right)^n$ will be 0. What happens to $\left(\frac{1}{r} N_{\ell_0}\right)^n$?

Let v be an eigenvector of N_{ℓ_0} with eigenvalue $r_{\ell_0} = r(N_{\ell_0})$, whose components are non-negative reals. Then $\left(\frac{1}{r} N_{\ell_0}\right)^n v = (r_{\ell_0}/r)^n v$. By positivity, this implies that $\left(\frac{1}{r} N_{\ell_0}\right)^n$ will have some arbitrarily large components as n increases.

The matrix $\left(\frac{1}{r} N\right)^n$ will correspond to a positive invariant, for each n, and will be the direct sum of the blocks $\left(\frac{1}{r} N_\ell\right)^n$. Taking n sufficiently large, Eq. (5.2) of [11] can now be used to give us a contradiction.

Thus $r(N_\ell) \leqq r(N_0)$.

If $N_\ell^2 = m_\ell N_\ell$, then by the above argument $r(N_\ell) = m_\ell$. The remainder of the proof is as in Theorem 4 of [11]. QED

The conditions in the last sentence of the lemma can be weakened somewhat, but this is all that we will need in this paper. A commonly occurring example of a matrix M with the property $M^2 = mM$ is the $n \times n$ matrix

$$M_{n,\ell} = \begin{pmatrix} \ell & \ell & \cdots & \ell \\ \vdots & \vdots & & \vdots \\ \ell & \ell & \cdots & \ell \end{pmatrix}. \tag{2.10d}$$

Here, $r(M_{n,\ell}) = m = \ell n$.

The strategy adopted in this paper is three-fold.

Sect. 3 Find all permutation invariants for each level k. We accomplish this by repeatedly exploiting the facts that this permutation must be a symmetry of both $S^{(k)}_{\lambda\mu}$ and the fusion rules $N^{(k)}_{\lambda\mu\nu}$.

Sect. 4 For each k, use Lemma 2 to find all weights $\lambda \in P^k$ which can couple to ρ in some positive invariant N. The argument is elementary but tedious and involves investigating several cases. There are surprisingly few such λ; the results are compiled in Lemma 4. There will always be at least one such weight, namely ρ itself. When this is the only one, then Lemma 1 tells us that any physical invariant of that level must necessarily be a permutation invariant, and so must be on the list found in Sect. 3.

Sect. 5 The remaining levels, which have nontrivial ρ-couplings, must now be considered. To do them, we use [26], together with Lemma 3, to write down the characters of all possible maximal extensions of $\hat{A}_2$ consistent with Lemma 4; if there are any such extensions, we then find their symmetries by mimicking the argument of Sect. 3.

In this paper we only make use of Lemma 2 for $\lambda' = \rho$. It is quite possible that apply it to other weights will permit us to avoid using [26] in Sect. 5, and so could yield a classification proof for those levels which assumes only (P1), (P2), (P3), instead of exploiting in addition the existence and properties of the maximally extended chiral algebras of the theory. A more interesting possibility is to exploit more of the rich algebraic structure of Ω^k.

3. The Permutation Invariants

By a *permutation invariant* (sometimes called an *automorphism invariant*) we mean an invariant of the form

$$Z = \sum_{\lambda \in P^k} \chi_\lambda \chi^*_{\sigma\lambda}, \tag{3.1a}$$

$$\text{i.e.} \quad N_{\lambda\lambda'} = N^\sigma_{\lambda\lambda'} \stackrel{\text{def}}{=} \delta_{\lambda',\sigma\lambda} \tag{3.1b}$$

for some permutation σ of P^k. In this section we will find all A_2 permutation invariants, for each k. In particular, we will prove the following theorem:

Theorem 2. The only level k permutation invariants for A_2 are $\mathscr{A}_k$, $\mathscr{A}^c_k$ for $k \equiv 0 \pmod 3$, and $\mathscr{A}_k$, $\mathscr{A}^c_k$, $\mathscr{D}_k$, $\mathscr{D}^c_k$ for $k \not\equiv 0 \pmod 3$.

Many permutation invariants, for each algebra, have been constructed (see e.g. [2]), but their methods cannot claim to find them all. For example, the $k = 4$ G_2 and $k = 3$ F_4 exceptional permutation invariants found in a computer search in [37] were missed by [2], and also cannot be obtained using simple currents [37]. Until now, only for A_1 [6, 23, 15] recently [13] all those for $g = A_1 \oplus \cdots \oplus A_1$ have also been found.

Throughout this section let $k' = k + 3$, and assume N^σ is a permutation invariant.

That the matrix N^σ in (3.1b) must commute with $S^{(k)}$ and $T^{(k)}$ (see (2.6)) is equivalent to

$$S^{(k)}_{\lambda\lambda'} = S^{(k)}_{\sigma\lambda,\sigma\lambda'}\,, \tag{3.2a}$$

$$T^{(k)}_{\lambda\lambda'} = T^{(k)}_{\sigma\lambda,\sigma\lambda'}\,, \tag{3.2b}$$

for all $\lambda, \lambda' \in P^k$. Note that (2.5c) tells us that (3.2b) is equivalent to the condition that

$$m^2 + mn + n^2 \equiv m'^2 + m'n' + n'^2 \pmod{3k'} \tag{3.2c}$$

for all $(m, n) \in P^k$, where $\sigma(m, n) = (m', n')$.

It can be shown (Theorem 3 in [11]) that any permutation invariant must be physical, so

$$\sigma(1, 1) = (1, 1)\,. \tag{3.3a}$$

Also, we know $(S^{(k)})^2 = C^{(k)}$, the *charge conjugation matrix* defined by $C^{(k)}_{mn,m'n'} = \delta_{m,n'}\delta_{n,m'}$, so N^σ must commute with $C^{(k)}$. This means

$$\sigma(m, n) = (m', n') \quad \text{iff} \quad \sigma(n, m) = (n', m')\,. \tag{3.3b}$$

Verlinde's formula [36, 27] gives us a relation between the fusion coefficients $N^{(k)}_{\lambda\mu\nu}$ and the $S^{(k)}$ matrix:

$$N^{(k)}_{\lambda\mu\nu} = \sum_{\lambda' \in P^k} \frac{S^{(k)}_{\lambda\lambda'} S^{(k)}_{\mu\lambda'} S^{(k)}_{\nu\lambda'}}{S^{(k)}_{\rho\lambda'}}\,. \tag{3.4a}$$

Therefore (3.2a) tells us that

$$N^{(k)}_{\lambda\mu\nu} = N^{(k)}_{\sigma\lambda,\sigma\mu,\sigma\nu}\,. \tag{3.4b}$$

Equation (3.4b) is useful to us, because these fusion coefficients have been computed for A_2 [5]. The formula will be given in the following paragraph.

Write $\lambda = \lambda_1\beta_1 + \lambda_2\beta_2$, $\mu = \mu_1\beta_1 + \mu_2\beta_2$, $\nu = \nu_1\beta_1 + \nu_2\beta_2$. Define

$$A = \frac{1}{3}[2(\lambda_1 + \mu_1 + \nu_1) + \lambda_2 + \mu_2 + \nu_2]\,,$$

$$B = \frac{1}{3}[\lambda_1 + \mu_1 + \nu_1 + 2(\lambda_2 + \mu_2 + \nu_2)]\,,$$

$$k_{\min} = \max\{\lambda_1 + \lambda_2, \mu_1 + \mu_2, \nu_1 + \nu_2, A - \min\{\lambda_1, \mu_1, \nu_1\}, B - \min\{\lambda_2, \mu_2, \nu_2\}\}\,,$$

$$k_{\max} = \min\{A, B\}\,,$$

$$\delta = \begin{cases} 1 & \text{if } k_{\max} > k_{\min} \text{ and } A, B \in \mathbf{Z} \\ 0 & \text{otherwise} \end{cases}$$

Then [5] says (changing their notation slightly and recalling that $k' = k + 3$)

$$N^{(k)}_{\lambda\mu\nu} = \begin{cases} 0 & \text{if } k' \leqq k_{\min} \text{ or } \delta = 0 \\ k' - k_{\min} & \text{if } k_{\min} \leqq k' \leqq k_{\max} \text{ and } \delta = 1 \\ k_{\max} - k_{\min} & \text{if } k' \geqq k_{\max} \text{ and } \delta = 1 \end{cases} . \tag{3.5}$$

The first step in the proof of Theorem 2 is to show that "point-wise" σ acts like an outer automorphism:

Claim. $\sigma(m, n) \in \{(m, n), (n, m), (m, k' - m - n), (n, k' - m - n), (k' - m - n, m), (k' - m - n, n)\}$.

Proof. Take $\lambda = \mu = \nu = m\beta_1 + n\beta_2$. Then (3.5) becomes

$$N^{(k)}_{\lambda\lambda\lambda} = \begin{cases} \min\{m, n\} & \text{if } k' \geqq m + n + \min\{m, n\} \\ k' - m - n & \text{otherwise} \end{cases} . \tag{3.6a}$$

Define $\mathscr{S}^k_a = \{(m, n) \in P^k \mid m = a \text{ or } n = a\}$, $\tilde{\mathscr{S}}^k_b = \{(m, n) \in P^k \mid m + n = b\}$. Equations (3.4b) and (3.6a) now imply

$$\sigma(m, n) \in \mathscr{S}^k_m \cup \mathscr{S}^k_n \cup \mathscr{S}^k_{k'-m-n} \cup \tilde{\mathscr{S}}^k_{k'-m} \cup \tilde{\mathscr{S}}^k_{k'-n} \cup \tilde{\mathscr{S}}^k_{m+n} . \tag{3.6b}$$

Now let us ask the question: when can $\sigma(m, n) = (m, n')$? Equations (3.2a) and (3.3a) would then imply $S^{(k)}_{mn,11} = S^{(k)}_{mn',11}$. Equation (2.5f) reduces this to

$$\sin\left(\frac{2\pi n}{k'}\right) - \sin\left(\frac{2\pi(n + m)}{k'}\right) = \sin\left(\frac{2\pi n'}{k'}\right) - \sin\left(\frac{2\pi(m + n')}{k'}\right) . \tag{3.7a}$$

Define $f_\alpha(x) = \sin(x) - \sin(x + \alpha)$. We are interested in finding all solutions $f_\alpha(x) = f_\alpha(y)$, where $x, y, \alpha > 0$ and $x + \alpha, y + \alpha < 2\pi$. Note that the derivative $f'_\alpha(x)$ is positive for $x \in (-\alpha/2, \pi - \alpha/2)$, and negative for $x \in (\pi - \alpha/2, 2\pi - \alpha/2)$. Also, f_α is symmetric about its local maxima: $f_\alpha(x + \pi - \alpha/2) = f_\alpha(-x + \pi - \alpha/2)$. What these facts mean is that, in the interval $x, y \in (0, 2\pi - \alpha)$, $f_\alpha(x) = f_\alpha(y)$ has the two solutions $x = y$ and $y = -x + 2\pi - \alpha$. Hence the only possible solutions to (3.7a) are

$$n' = n \quad \text{and} \quad n' = k' - m - n . \tag{3.7b}$$

The identical calculation and conclusion holds for $\sigma(m, n) = (n', m)$. Thus

$$\sigma(m, n) \in \mathscr{S}^k_m \Rightarrow \sigma(m, n) \in \{(m, n), (m, k' - m - n), (n, m), (k' - m - n, m)\} . \tag{3.7c}$$

The remaining five possibilities in (3.6b) reduce to identical arguments. QED to claim

The claim, together with (3.2c), tells us that the only possibilities for $\sigma(1, 2)$ are:

$$\sigma(1, 2) \in \{(1, 2), (2, 1)\} \qquad \text{if } k \equiv 0 \pmod 3 , \tag{3.8a}$$

$$\sigma(1, 2) \in \{(1, 2), (2, 1), (2, k), (k, 2)\} \qquad \text{if } k \equiv 1 \pmod 3 , \tag{3.8b}$$

$$\sigma(1, 2) \in \{(1, 2), (2, 1), (k, 1), (1, k)\} \qquad \text{if } k \equiv 2 \pmod 3 . \tag{3.8c}$$

Note that the possibilities for $k \equiv 0 \pmod 3$ are realized by $\mathscr{A}_k$ and $\mathscr{A}^c_k$, respectively, and for $k \equiv \pm 1 \pmod 3$ by $\mathscr{A}_k$, $\mathscr{A}^c_k$, $\mathscr{D}_k$ and $\mathscr{D}^c_k$, respectively. Since the (matrix) product of two permutation invariants is another permutation invariant, to prove Theorem 2 for each k it suffices to show that the only permutation invariant satisfying $\sigma(1,2)=(1,2)$ is $\mathscr{A}_k$.

Suppose for contradiction that $\sigma(1,2)=(1,2)$, but $\sigma(1,a)=(a,1)$ for some $2\leqq a\leqq k+1$. Then $S^{(k)}_{12,1a}=S^{(k)}_{12,a1}$, i.e.

$$c_k(5a+4)+c_k(a+5)+c_k(4a-1)=c_k(4a+5)+c_k(5a+1)+c_k(a-4)\,, \tag{3.9a}$$

where $c_k(x)=\cos\left(2\pi\dfrac{x}{3k'}\right)$. Making the substitution $b=a+\frac{1}{2}$, we would like to show that

$$p(b,k)\stackrel{\text{def}}{=}c_k\left(5b+\frac{3}{2}\right)+c_k\left(b+\frac{9}{2}\right)+c_k(4b-3)-c_k(4b+3)$$
$$-c_k\left(5b-\frac{3}{2}\right)-c_k\left(b-\frac{9}{2}\right) \tag{3.9b}$$

does not vanish at $b=\frac{5}{2},\frac{7}{2},\ldots,k+\frac{3}{2}$.

Using the obvious trigonometric identities, we can rewrite $p(b,k)$ as a polynomial in $c_k(b)$ and $s_k(b)=\sin\left(2\pi\dfrac{b}{3k'}\right)$ – in particular, $p(b,k)=p^{(k)}_5(c_k(b))+s_k(b)\cdot p^{(k)}_4(c_k(b))$, where $p^{(k)}_5$ and $p^{(k)}_5$ are, respectively degress 5 and 4 polynomials. Note from (3.9b) that $p(-b,k)=-b,k)$, so $p^{(k)}_5$ must be identically zero, and

$$p(b,k)=s_k(b)\cdot p^{(k)}_4(c_k(b))\,. \tag{3.9c}$$

We are interested in the roots of this function, in the range $b\in(0,\frac{3}{2}k')$. Since $s_k(b)$ does not vanish, and c_k is one-to-one, for those b, for fixed k there can be at most 4 zeros for $p^{(k)}_4$ and hence $p(b,k)$ in that range. But $b=\frac{1}{2},\frac{3}{2},k+\frac{5}{2},k+\frac{7}{2}$ are 4 distinct zeros for $p(b,k)$. Therefore they are the *only zeros* in the range $b\in(0,\frac{3}{2}k')$, and so $p(b,k)$ cannot vanish at $b=\frac{5}{2},\frac{7}{2},\ldots,k+\frac{3}{2}$. This means that we cannot have both $\sigma(1,2)=(1,2)$ and $\sigma(1,a)=(a,1)$, for any $a=2,3,\ldots,k+1$.

The other four possibilities $\sigma(1,a)=(1,k'-1-a)$, $(a,k'-1-a)$, $(k'-1-a,1)$, and $(k'-1-a,a)$ all succumb to similar reasoning. Thus we have shown:

$$\sigma(1,2)=(1,2)\Rightarrow\sigma(1,a)=(1,a)\quad\forall(1,a)\in P^k\,. \tag{3.10a}$$

Remember, to prove Theorem 2 it suffices to show $\sigma(1,2)=(1,2)$ implies $\sigma(a,b)=(a,b)$ $\forall(a,b)\in P^k$. Suppose instead $\sigma(1,2)=(1,2)$ but $\sigma(a,b)=(b,a)$. Take $\lambda=(a,b)$, $\mu=(b,1)$, $\nu=(1,a)$ and $\lambda'=(b,a)$. Then $N^{(k)}_{\lambda\mu\nu}=N^{(k)}_{\lambda'\mu\nu}$, by (3.10a), (3.3b) and (3.4b). But Eq. (3.5) tells us $N^{(k)}_{\lambda\mu\nu}=1$, while $N^{(k)}_{\lambda'\mu\nu}=0$ unless $a=b$.

Similar calculations show $\sigma(a,b)=(a,k'-a-b)$ only when $b=k'-a-b$, and $\sigma(a,b)=(k'-a-b,b)$ only when $a=k'-a-b$. The remaining two anomalous possibilities are slightly more difficult: $\sigma(a,b)=(b,k'-a-b)$ only when $3b=k'$, $b\leqq a$; and $\sigma(a,b)=(k'-a-b,a)$ only when $3a=k'$, $a\leqq b$.

Now, if $a>b=k'/3$ and $\sigma(a,b)=(b,k'-a-b)$, then σ being a permutation implies $\sigma(b,k'-a-b)\neq(b,k'-a-b)$, so from the above paragraph we must have either $3(k'-a-b)=k'$ or $b\leqq k'-a-b$, i.e. either $a=b=k'/3$ or $a\leqq b$ – a contradiction. Therefore the only way for either remaining anomalous possibility to be realized as if $a=b=k'/3$.

Thus, we have shown

$$\sigma(1,2) = (1,2) \Rightarrow \sigma(a,b) = (a,b) \quad \forall (a,b) \in P^k \,, \tag{3.10b}$$

i.e. that the only permutation invariant with $\sigma(1,2) = (1,2)$ is the identity, which concludes the proof of Theorem 2.

4. The ρ-Coupling Lemma

Again write $k' = k+3$. Let $\mathscr{R}^k$ be the set of all $\lambda \in P^k$ such that there exists a positive invariant with $N_{\rho,\lambda} \neq 0$. For example, the known A_2 physical invariants (2.7) tell us that $\mathscr{R}^5 \supseteq \{(1,1),(3,3)\}$, $\mathscr{R}^6 \supseteq \{(1,1),(7,1),(1,7)\}$ and $\mathscr{R}^7 \supseteq \{(1,1)\}$. If $\mathscr{R}^k = \{\rho\}$ then by Lemma 1 any level k physical invariant will be a permutation invariant, and will be listed in Theorem 2.

Let $\lambda = a\beta_1 + b\beta_2 \in \mathscr{R}^k$. Then it must satisfy $\lambda^2 \equiv \rho^2 \pmod{2k'}$, i.e.

$$a^2 + ab + b^2 \equiv 3 \pmod{3k'} \,. \tag{4.1a}$$

It is easy to see from that equation that any $\lambda \in \mathscr{R}^k$ must have order k' with respect to $k'A_2$, and hence the vector $(\rho;\lambda)$ has order $L = k'$ with respect to $(k'A_2; k'A_2)$. Now, investigating the behavior of the Weyl group W on A_2^* allows a simple formula for the parity $\varepsilon(\mu)$ (see (2.8a)) of an arbitrary vector $\mu = c\beta_1 + d\beta_2$: for any real number x define by $\{x\}$ the unique number congruent to $x \pmod{k'}$ satisfying $0 \leqq \{x\} < k'$, then

$$\varepsilon(\mu) = \begin{cases} 0 & \text{if } \{c\}, \{d\} \text{ or } \{c+d\} = 0 \\ +1 & \text{if } \{c\} + \{d\} < k' \text{ and } \{c\}, \{d\}, \{c+d\} > 0 \,. \\ -1 & \text{if } \{c\} + \{d\} > k' \text{ and } \{c\}, \{d\}, \{c+d\} > 0 \end{cases} \tag{4.1b}$$

Then Lemma 2 implies that:

$$\begin{aligned} &0 < \{\ell\} < k'/2,\ \ell \text{ relatively prime to } k', \Rightarrow \{\ell a\} + \{\ell b\} < k', \\ &k'/2 < \{\ell\} < k',\ \ell \text{ relatively prime to } k', \Rightarrow \{\ell a\} + \{\ell b\} > k' \,. \end{aligned} \tag{4.1c}$$

Lemma 4 (ρ-coupling). *The only solutions to Eqs.* (4.1a, c) *are:*

(i) *for* $k \equiv 2, 4, 7, 8, 10, 11 \pmod{12}$:

$$(a,b) \in \{(1,1)\} \,; \tag{4.2a}$$

(ii) *for* $k \equiv 1, 5 \pmod{12}$:

$$(a,b) \in \left\{(1,1), \left(\frac{k+1}{2}, \frac{k+1}{2}\right)\right\} ; \tag{4.2b}$$

(iii) *for* $k \equiv 0, 3, 6 \pmod{12}$:

$$(a,b) \in \{(1,1), (1,k+1), (k+1,1)\} \,; \tag{4.2c}$$

(iv) *for* $k \equiv 9 \pmod{12}$, $k \neq 21, 57$:

$$(a,b) \in \left\{(1,1), (1,k+1), \left(2, \frac{k+1}{2}\right), (k+1,1), \left(\frac{k+1}{2}, 2\right), \left(\frac{k+1}{2}, \frac{k+1}{2}\right)\right\} . \tag{4.2d}$$

We will say k is in class (i) if $k \equiv 2, 4, 7, 8, 10, 11 \pmod{12}$, in class (ii) if $k \equiv 1, 5 \pmod{12}$, etc. The only k's missing from this list are $k = 21, 57$, each of which have 12 possibilities for (a, b). For $k = 21$, these are precisely the 12 weights in (2.7g) lying in the block containing $\chi^{21}_{1,1}$ – namely $(1, 1)$, $(5, 5)$, $(7, 7), \ldots, (7, 10)$. For $k = 57$ these are given in (5.11). The reason $k = 21$ and $k = 57$ are singled out here turns out to be the same (see Claim 1, and the proof of Claim 3) as the reason $k = 10$ and $k = 28$ are singled out in the ρ-coupling Lemma for A_1 (see Sect. 6 and the Appendix). Indeed, $21 + 3 = 2(10 + 2)$ and $57 + 3 = 2(28 + 2)$.

We will prove Lemma 4 later in the section. For now let us consider what would happen if it were true. For example, for $k \equiv 2, 4, 7, 8, 10, 11 \pmod{12}$, or $k = 1$, $\mathscr{R}^k = \{\rho\}$; and for $k \equiv 1, 5 \pmod{12}$, $\mathscr{R}^k \subseteq \left\{\rho, \frac{k+1}{2}\rho\right\}$. Then for half of the possible levels, we will have reduced the completeness proof to the classification of the permutation invariants, and considerable information about the remaining levels will have been deduced. Lemma 4 turns out to the sufficient to complete the proof of Theorem 1 for all k (this is done in Sect. 5).

Incidently, taking $\ell = -1$ in (2.9) shows that, e.g., $N_{1,1;1,k+1} = N_{1,1;k+1,1}$, for any invariant N and level k. We will need this and many other consequences of (2.9) in Sect. 5.

Lemma 4 can be thought of as related to the A_1 completeness proofs in [6, 23, 15] and [30], and the $A_2 k'$ prime proof in [33], though it was obtained independently. However it captures the big advantage the approach developed in this paper has over those older approaches: through it we impose positivity from the start; because of it we avoid explicit construction of the commutant.

Before trying to understand the somewhat lengthy proof of Lemma 4 given below, it may be wise for the reader to consult the related, but considerably simpler, proof given at the end of Sect. 5 of [11] for ρ-coupling for level 1 of C_n, odd n, or the proof for ρ-coupling of A_1, all levels, given in the appendix of this paper. We will find a strong relationship between the A_1 proof, and that of A_2. In particular we will need the following result, proven in the appendix:

Claim 1. *Let $K > a$ be positive integers, and a be odd. Suppose that for $0 < \ell < K$, ℓ relatively prime to $2K$, we have $\{\ell a\}_{2K} < K$, where $\{x\}_y$ is the unique number congruent* $(\bmod\, y)$ *to x satisfying $0 \leqq \{x\}_y < y$. Then*

(a) *for K odd, $a = 1$;*
(b) *for K even and $K \neq 6, 10, 12, 30$, $a = 1$ or $K - 1$;*
(c) *for $K = 6$, $a = 1, 3, 5$; for $K = 10$, $a = 1, 3, 7, 9$; for $K = 12$, $a = 1, 5, 7, 11$; and for $K = 30$, $a = 1, 11, 19, 29$.*

Claim 2. *For any k and any $(a, b) \in P^k$, (a, b) satisfies the parity condition* (4.1c) iff *$\omega(a, b)$ does. Moreover, if (a, b) satisfies the condition*

$$a^2 + ab + b^2 \equiv 3 \pmod{k'}\,, \tag{4.3}$$

then so will $\omega(a, b)$, and if 3 divides k', then (a, b) will satisfy the norm condition (4.1a) iff *$\omega(a, b)$ will.*

ω is the outer automorphism defined in (2.2b). The proof of Claim 2 is a straightforward calculation. For example, if $\{\ell a\} + \{\ell b\} < k'$, then $\{\ell k' - \ell a - \ell b\} + \{\ell a\} = k' - \{\ell a\} - \{\ell b\} + \{\ell a\} = k' - \{\ell b\} < k'$.

Because of Claim 2, we will restrict our attention of the remainder of this section to any weight $(a, b) \in P^k$ satisfying the parity condition (4.1c) and the norm condition (4.3) (and if $k' \equiv 0 \pmod 3$ the stronger norm condition (4.1a)). By claim 2 this set of possible (a, b) is invariant under the 6 outer automorphisms. At the conclusion of our arguments we will have a finite set of solutions (a, b) to (4.1c) and (4.3); it suffices then to find those weights among them which satisfy (4.1a).

Proof of Lemma 4 when 4 divides k'. We learn from the norm condition (4.3) that two of a, b and $k' - a - b$ will be odd and one will be even; from Claim 2 we may assume for now that a and b are odd. Let $0 < \ell < k'/2$, ℓ relatively prime to k'. Then $\ell' = \ell + k'/2$ will also be relatively prime to k' but will lie in the range $k'/2$ to k'. Then (4.1c) tells us

$$\{\ell a\} + \{\ell b\} < k' < \{\ell' a\} + \{\ell' b\} . \tag{4.4}$$

But a is odd, so $\{\ell' a\} = (k'/2 + \ell a\}$ equals $k'/2 + \{\ell a\}$ if $\{\ell a\} < k'/2$, or $-k'/2 + \{\ell a\}$ if $\{\ell a\} > k'/2$. A similar comment applies to b. From (4.4) we now immediately get that both $\{\ell a\}, \{\ell b\} < k'/2$. Thus, putting $K = k'/2$ we read off from Claim 1 that the only possibilities for a and b are 1 and $(k+1)/2$, unless $k' = 12, 20, 24, 60$. From these we can also compute the possibilities for $k' - a - b$. Equation (4.1a) now suffices to reduce this list of possibilities to those given in Lemma 4. QED to classes (ii) and (iv)

Thus it suffices now to consider $k \equiv 0, 2, 3 \pmod 4$. As before let $(a, b) \in P^k$ be any weight satisfying (4.1c) and (4.3) (and (4.1a) if 3 divides k'). First we will prove two useful results.

Claim 3. *For $k \equiv 0, 2, 3 \pmod 4$, if $a = b$ then $a = b = 1$.*

Proof. Clearly $a < k'/2$. First consider k' odd. Let $M > 0$ be the smallest integer for which $2^M < k'/2 < 2^{M+1}$. Similarly, let $N \geqq 0$ be the smallest integer for which $2^N a < k'/2 < 2^{N+1} a$. Assume for contradiction that $a > 1$. Then $0 \leqq N < M$. Take $\ell = 2^{N+1} < k'/2$. Then we get $\{\ell a\} + \{\ell a\} = 2\{2^{N+1} a\} > k'$, contradicting (4.1c).

For k' even, (4.3) says a must be odd. We can now directly apply Claim 1(a) with $K = k'/2$, to again get $a = 1$. QED to Claim 3

Claim 4. *The greatest common divisors of a and k', of b and k', and of $k' - a - b$ and k', equal either 1 or 2.*

Proof. Suppose a prime $p \neq 2$ divides both a and k'. Then (4.1a) implies $p \neq 3$, and (4.3) that $b^2 \equiv 3 \pmod p$ – i.e. 3 is a quadratic residue of p, so $p \geqq 11$.

Let $\ell_m = 1 + mk'/p$, $m = 0, 1, \ldots, p-1$. Except possibly for one value of m, call it m_0, each ℓ_m will be relatively prime to k'. Assume $k' \neq p$; if $k' = p$ the ranges given below for m will be slightly different but otherwise the same argument holds. Therefore, for $m = 0, \ldots, \frac{p-1}{2}$ (except possibly for $m = m_0$), (4.1c) says $\{b + mbk'/p\} < k' - a$, and for $m = \frac{p+1}{2}, \ldots, p-1$ (except possibly $m = m_0$), $\{b + mbk'/p\} > k' - a$. Because $p \geqq 11$, it can be shown that these two inequalities can only be satisfied if $bk'/p \equiv \pm k'/p \pmod{k'}$, i.e. $b \equiv \pm 1 \pmod p$, in which case $b^2 \equiv 1 \not\equiv 3 \pmod p$.

Therefore, $p = 2$ is the only prime that can divide both a and k'. Since (4.3) shows 4 cannot divide both, the only possibilities for the gcd are 1 or 2.

The same calculation applies to $\gcd(b,k')$ and, using Claim 2, to $\gcd(k'-a-b,k')$.

QED to Claim 4

Proof of Lemma 4 for k′ odd. From Claim 2 we may assume $1 \leqq a, b \leqq k'/2$. It suffices to show $a = b = 1$.

First take $\ell = (k'-1)/2$; it is relatively prime to k' and less than $k'/2$. If a is even, $\{\ell a\} = k'a/2$, and if a is odd, $\{\ell a\} = k'/2 - a/2$. Hence $\{\ell a\} + \{\ell b\} = ik' + (k'-a-b)/2$, where $i = 1/2, 1, 3/2$ depending on whether 0, 1 or both of a, b are even. But $i \geqq 1$ contradicts (4.1c). Therefore both a and b must be odd.

Equation (4.3) tells us $\{a^2\} + \{ab\} + \{b^2\} = 3 + mk'$, for some integer m. Since by definition $0 \leqq \{\ldots\} < k'$, we have $m = 0, 1$, or 2. But $m = 2$ would imply $\{a^2\} + \{ab\} = 3 + 2k' - \{b^2\} > k'$, which contradicts (4.1c) with $\ell = a < k'/2$, by Claim 4.

Next suppose $m = 1$, i.e.

$$\{a^2\} + \{ab\} + \{b^2\} = k' + 3\,. \tag{4.5}$$

Choose $\ell = (k'+a)/2$, $\ell' = (k'+b)/2$ – again Claim 4 tells us these are relatively prime to k'. Then $\ell a \equiv k'/2 + a^2/2 \pmod{k'}$, so $\{\ell a\} = \{a^2\}/2 + k'/2$ if $\{a^2\}$ is odd, and $\{a^2\}/2$ if $\{a^2\}$ is even. Similarly, $\{\ell b\} = \{\ell' a\} = \{ab\}/2 + k'/2$ or $\{ab\}/2$, depending on whether $\{ab\}$ is odd or even, resp., and $\{\ell' b\} = \{b^2\}/2 + k'/2$ if $\{b^2\}$ is odd, and $\{b^2\}/2$ if $\{b^2\}$ is even. But (4.5) tells us that $\{a^2\} + \{ab\} + \{b^2\}$ is even, so either all three are even, or 2 are odd and 1 is even. If $\{a^2\}$ or $\{ab\}$ are even, then using ℓ in (4.1c) gives $k' < \{a^2\} + \{ab\}$, contradicting $a < k'/2$; otherwise using ℓ' contradicts $b < k'/2$.

Thus $m \neq 1$, so $m = 0$ is forced. This gives us $a^2 \equiv ab \equiv b^2 \equiv 1 \pmod{k'}$; Claim 4 then implies $a \equiv b \pmod{k'}$, which Claim 3 tells us forces $a = b = 1$. QED to Lemma 4 for k' odd.

Proof for Lemma 4 for $k' \equiv 2 \pmod 4$. This is the final, and messiest, possibility; its proof uses tools resembling those in the Appendix. From (4.3) we get that both a and b cannot be even, so by Claim 2 we may assume a, b are both odd. Define M by $2^M < k'/2 < 2^{M+1}$, so $k'/2^M < 4$. Let us begin with a useful fact.

Claim 5. *$a = 1$ implies $b = 1$.*

Proof. The norm condition (4.3) becomes

$$b^2 + b \equiv 2 \pmod{k'}\,. \tag{4.6}$$

Take first $\ell = b$ in (4.1c); from (4.6) we get either $b = 1$ or $b > k'/2$. Suppose $b > k'/2$, and write $b = k'/2 + b'$, so b' is even and $0 < b' < k'/2$. Define N so that $k'/2 < 2^N b' < k'$. Then $0 < N \leqq M$. Taking $\ell = k'/2 + 2^N$, we get $k' < \{k'/2 + 2^N\} + \{(k'/2 + 2^N)b\} = k'/2 + 2^N + 2^N b' - k'/2 = 2^N + 2^N b'$.

Now take $\ell = (k'/2 + 2^N)b$. We get $k' > \{(k'/2 + 2^N)b\} + \{(k'/2 + 2^N)b^2\} = 2^N b' - k'/2 + \{k'/2 + 2^{N+1} - 2^N b'\}$, using (4.6). This forces $k'/2 + 2^{N+1} - 2^N b' > 0$. Hence $k' < 2^N + 2^N b' < k'/2 + 2^N + 2^{N+1}$, i.e. $k' < 3 \cdot 2^{N+1}$, so $b' < k'/2^N < 6$, which tells us either $b' = 2$ or $b' = 4$.

It is easy to verify that $b = k'/2 + 2$ cannot satisfy (4.6), and $b = k'/2 + 4$ can only if $20 \equiv 2 \pmod{k'}$, i.e. $k' = 18$ or 6. These values can be individually checked. QED to Claim 5

Thus by Claims 3 and 5 it suffices to show there can be no solutions $(a, b) \in P^k$ to (4.1c) and (4.3) for a, b odd, $1 < a < b$. Write out the binary expansions $a/k' = \sum_{i=1}^{\infty} a_i 2^{-i}$, $b/k' = \sum_{i=1}^{\infty} b_i 2^{-i}$, where each $a_i, b_i \in \{0, 1\}$.

Consider $\ell_i = k'/2 + 2^i$, $i = 1, \ldots, M$. Then

$$k' < \{\ell_i a\} + \{\ell_i b\} = \left\{\frac{k'}{2} + 2^i a\right\} + \left\{\frac{k'}{2} + 2^i b\right\}$$

$$= \{2^i a\} + \{2^i b\} + \begin{cases} k' & \text{if } a_{i+1} = b_{i+1} = 0 \\ 0 & \text{if } a_{i+1} + b_{i+1} = 1 \\ -k' & \text{if } a_{i+1} = b_{i+1} = 1 \end{cases}. \tag{4.7a}$$

But $\{\ldots\} < k'$, so (4.7a) forbids $a_{i+1} = b_{i+1} = 1$, for all $i = 1, 2, \ldots, M$ (the relation $a + b < k'$ forbids it for $i = 0$).

Define I by $k'/2^I < b < k'/2^{I-1}$, i.e. $b_i = 0$ for $i < I$ and $b_I = 1$. Consider first the case $I > 1$. Then (4.7a) tells us $k' < \{2^{I-1}a\} + \{2^{I-1}b\} = 2^{I-1}a + 2^{I-1}b$, i.e. $k'/2^{I-1} < a + b$. This strong inequality now forces $a_i + b_i = 1$ for $I \leqq i < M + 1$, i.e.

$$I > 1 \Rightarrow a + b = \frac{k'}{2^{I-1}} + \varepsilon, \quad \text{where } 0 < \varepsilon < 2\,. \tag{4.7b}$$

The case $I = 1$ is similar. Define $I' > 1$ to be the smallest index (other than $I = 1$) with $a_{I'} = 1$ or $b_{I'} = 1$. Then the identical argument gives

$$I = 1 \Rightarrow a + b = \frac{k'}{2} + \frac{k'}{2^{I'-1}} + \varepsilon, \quad \text{where } 0 < \varepsilon < 2\,. \tag{4.7c}$$

In both (4.7b, c), ε is fixed by the constraint that $a + b$ must be even. Thus we have essentially removed one degree of freedom. First we will eliminate I, $I' = 2, 3$.

Claim 6. *Either $I > 3$, or $I = 1$ and $I' > 3$.*

Proof. Suppose first that $I = 2$. Then $a + b = k'/2 + 1$, so $\{ab\} = k'/2 - 2$, $\{a^2\} = a + 2$, $\{b^2\} = b + 2$. Therefore either $\frac{a(a-1)}{2} \equiv 1 \pmod{k'}$ (if $a \equiv -1 \pmod 4$), or $\frac{a(a-1)}{2} + \frac{a}{2}k' \equiv 1 \pmod{k'}$ (if $a \equiv +1 \pmod 4$). Then $a \equiv +1 \pmod 4$ would violate (4.1c) with $\ell = \frac{a-1}{2} + \frac{k'}{2}$, so $a \equiv -1 \pmod 4$. Similarly, we must have $b \equiv -1 \pmod 4$, so $\frac{k'}{2} + 1 \equiv 2 \pmod 4$, i.e. $k' \equiv 2 \pmod 8$. Now take $\ell = \frac{k'+2}{4}$; we get $\left\{\frac{-k'}{2} + \frac{a}{2}\right\} + \left\{\frac{-k'}{4} + \frac{b}{2}\right\} = \frac{3k'}{2} + \frac{k'+2}{4} > k'$, contradicting (4.1c).

Now suppose $I = 3$, i.e.

$$a < \frac{k'}{8} < b < \frac{k'}{4}, \quad a + b = \frac{k'}{4} + \begin{cases} \frac{1}{2} & \text{if } k' \equiv -2 \pmod 8 \\ \frac{3}{2} & \text{if } k' \equiv +2 \pmod 8 \end{cases}.$$

Taking $\ell = k'/2 - a - b$ eliminates $a \equiv b \equiv 3 \pmod 4$. Perhaps the easiest way to handle $I = 3$ is to address the 4 possibilities for $k' \pmod{16}$ individually. For $k' \equiv 2, 6 \pmod{16}$ it turns out $\{a^2\} + \{ab\} > k'$, violating (4.1c) with $\ell = a$. For $k' \equiv 10 \pmod{16}$, $a \not\equiv b \pmod 4$ so take $\ell = \frac{k'}{4} + \frac{1}{2}$.

The harder possibility is $k' \equiv 14 \pmod{16}$. Then $a + b = \frac{k'}{4} + \frac{1}{2} \equiv 0 \pmod 4$, so $a \not\equiv b \pmod 4$. Here we have $\left(\frac{k'}{4}\right)^2 \equiv \frac{7}{8} k' \pmod{k'}$, so for $k' > 14$ $\{ab\} = \frac{k'}{8} - \frac{11}{4}$ and $\{b^2\}$ equals either $\frac{5}{8}k' + \frac{b}{2} + \frac{11}{4}$ or $\frac{1}{8}k' + \frac{b}{2} + \frac{11}{4}$. Taking $\ell = \frac{3}{2}k' - 4b$ gives the contradiction $11 + k' - 2b - 11 > k'$. For the remaining case, $k' = 14$, choosing $\ell = a$ gives a contradiction.

Now suppose $I = 1$. Then by (4.7c), $I' = 2$ would violate $a + b < k'$. $I' = 3$ can be handled similarly to $I = 3$. QED to Claim 6

Write $k' = 2 \cdot 3^L \cdot k''$, where $k'' \equiv \pm 1 \pmod 6$. Consider first the case $L = 0$. Define J by $\frac{k'}{3 \cdot 2^J} < b < \frac{k'}{3 \cdot 2^{J-1}}$ if $I > 1$, and if $I = 1$ define J to be the smallest number such that either $\frac{k'}{3 \cdot 2^J} < a < \frac{k'}{3 \cdot 2^{J-1}}$ or $\frac{k'}{3 \cdot 2^J} < b' < \frac{k'}{3 \cdot 2^{J-1}}$. Note that $J = I - 1$ or $I - 2$, and $J = I' - 1$ or $I' - 2$, in the 2 cases. By Claim 6 we know $J > 1$. Putting $\ell'_i = k'/2 - 3 \cdot 2^i$ into (4.1c) presents us with a familiar calculation:

$$I > 1 \Rightarrow a + b = \frac{k'}{3 \cdot 2^{J-1}} + \varepsilon' ; \tag{4.8a}$$

$$I = 1 \Rightarrow a + b = \frac{k'}{2} + \frac{k'}{3 \cdot 2^{J-1}} + \varepsilon' ; \tag{4.8b}$$

where $0 < \varepsilon' < 2$. Equating Eqs. (4.7b) with (4.8a) gives us $k'/(3 \cdot 2^{I-1}) = \varepsilon' - \varepsilon$ if $J = I - 1$, and $k'/(3 \cdot 2^{I-1}) = \varepsilon - \varepsilon'$ if $J = I - 2$. In either case we get $a < k'/2^I < 3$, i.e. $a = 1$. The $I = 1$ calculation is identical.

Now consider $L > 1$. We get from (4.3) that $a \equiv b \pmod 3$. $a \equiv -1 \pmod 3$ is dealt with using $\ell = k'/6 - 2$, so a must be $\equiv +1 \pmod 3$. We will do $I > 1$ (the proof for $I = 1$ is similar).

An easy calculation from $a > 1$ shows $2^{I-1} < k'/3$. Therefore both $\ell' = k'/6 + 2^{I-2}$ and $\ell'' = k'/6 + 2^{I-1}$ are less than $k'/2$, and $\ell''' = k'/6 - 2^{I-3}$ is positive. ℓ' gives $a + b < 2k'/(3 \cdot 2^{I-2})$, while ℓ'' implies $2^{I-1}b + k'/6 > k'$. Now ℓ''' yields $k' > k'/6 - 2^{I-3}a + 7k'/6 - 2^{I-3}b$, contradicting the ℓ' inequality.

Finally consider $L = 1$. As before we will only give the proof for $I > 1$ – it is similar for $I = 1$. As before, we can force $a \equiv b \equiv +1 \pmod 3$.

Define J by $k'/2^J < 3b < k'/2^{J-1}$. Then $J = I - 1$ or $I - 2$, so $J > 1$ by Claim 6. Assume for now that $J > 2$; then $\ell' = k'/6 + 3 \cdot 2^{J-2}$ and $\ell'' = k'/6 + 3 \cdot 2^{J-1}$ give us $a + b < 2k'/(9 \cdot 2^{J-2})$ and $b > 5k'/(18 \cdot 2^{J-1})$. The former tells us $J = I - 2$, while the latter demands $J = I - 1$.

The remaining possibility, namely $J = 2$ and $I = 4$, is eliminated by taking $\ell = \ell'' = k'/6 + 6$. QED to Lemma 4 for $k' \equiv 2 \pmod 4$

5. The Remaining Levels

In Sect. 4 we concluded the proof that Eqs. (2.7) exhaust all physical invariants, for half the levels. In this section we will conclude that A_2 classification problem for the remaining levels.

Until now the only properties of the partition functions we have exploited are (P1), (P2) and (P3): the invariants are *physical* invariants. However there are other conditions known to be satisfied by the partition functions of all (unitary, CPT-invariant) conformal field theories. We will make these extra properties explicit in the following two paragraphs; any physical invariant satisfying them will be called a *strongly physical invariant.*

[26] tells us that to any level k strongly physical invariant Z there are associated two maximally extended chiral algebras, $\mathscr{A}$ and $\bar{\mathscr{A}}$, which may or may not be isomorphic. These algebras are extensions of the affine algebra $\hat{A}_2$; they both equal $\hat{A}_2$ iff Z is a permutation invariant. Let ch_i and $\bar{ch}_j$ be their characters. These can be written as finite linear combinations

$$ch_i = \sum_{\lambda \in P^k} m_{i\lambda} \chi_\lambda^k, \quad \bar{ch}_j = \sum_{\lambda \in P^k} \bar{m}_{j\lambda} \chi_\lambda^k \tag{5.1a}$$

of characters χ_λ^k of $\hat{A}_2$, where the coefficients $m_{i\lambda}$, $\bar{m}_{j\lambda}$ are non-negative integers. Let ch_0 and $\bar{ch}_0$ be the unique ones with $m_{0\rho}, \bar{m}_{0\rho} \neq 0$. $\mathscr{A}$ and $\bar{\mathscr{A}}$ must have an equal number n_c of characters. Then

$$Z = \sum_{i=0}^{n_c - 1} ch_i \bar{ch}_{\pi i}^*, \tag{5.1b}$$

for some permutation π of the indices $\{0, \ldots, n_c - 1\}$. In other words, every strongly physical invariant is a sort of permutation invariant when the chiral algebras are maximally extended.

Consider the matrices S_{ij}^e and $\bar{S}_{ij}^e$ which describe the behaviour of the extended characters ch_i and $\bar{ch}_i$, respectively, under the transformation $\tau \to -1/\tau$, as in (2.5d). Then we know from [26] that S^e and $\bar{S}^e$ are both unitary and symmetric, and

$$S_{0j}^e \geqq S_{00}^e > 0, \qquad \bar{S}_{0j}^e \geqq \bar{S}_{00}^e > 0. \tag{5.2}$$

Now consider any level k strongly physical invariant Z, given by (5.1b). One immediate consequence of the above comments is that the function

$$Z' = \sum_{i=0}^{n_c - 1} |ch_i|^2 \tag{5.3a}$$

also is a physical invariant. We will call an invariant of this form (i.e. diagonal in the extension) a *block-diagonal*. In a sense to be made clear later, this observation will allow us to simplify our argument by permitting us to consider the existence (or non-existence) solely of physical invariants of the form (5.3a). Note that the coefficient matrix N' of Z' in (5.3a) is symmetric and must satisfy

$$(N'_{\lambda\lambda'})^2 \leqq N'_{\lambda\lambda} N'_{\lambda'\lambda'}, \qquad \forall \lambda, \lambda' \in P^k. \tag{5.3b}$$

Consider any $\lambda \in P^k$, and compute the sum

$$s(\lambda, \mathscr{A}) \stackrel{\text{def}}{=} \sum_{\lambda' \in P^k} m_{0\lambda'} S_{\lambda'\lambda}^{(k)} = \sum_{i=0}^{n_c - 1} m_{i\lambda} S_{0i}^e. \tag{5.4}$$

The second equality here follows because S^e is symmetric. But $m_{i\lambda} \geqq 0$ and by (5.2) $S^e_{0i} > 0$, so the RHS of (5.4) is non-negative and will be zero only if all $m_{i\lambda} = 0$. This gives us a simple but powerful test for the character ch_0. In particular, we can read off from Lemma 4 the possibilities for ch_0; most of these will have $s(\lambda) < 0$ or $s(\lambda)$ non-real for some $\lambda \in P^k$ and so can be dismissed.

Hence our argument will depend crucially on Lemma 4, so will be broken down into 4 cases: class (ii); class (iii); class (iv); and the exceptional value $k = 57$.

5.1. Class (ii). Consider first class (ii), i.e. all $k \equiv 1, 5 \pmod{12}$. We may consider $k > 1$, because by Lemmas 4 and 1, any physical invariant at $k = 1$ is a permutation invariant and thus is enumerated in Theorem 2. Suppose there exists a level k strongly physical invariant which is not a permutation invariant. We would like to show that, except for $k = 5$, this cannot happen. Write $\rho' = \left(\frac{k+1}{2}, \frac{k+1}{2}\right)$. Lemma 4 tells us that $N_{\rho,\lambda} = N_{\lambda,\rho} = 0$ except for $N_{\rho,\rho} = 1$, and $N_{\rho,\rho'}, N_{\rho',\rho}$. Write $ch_0 = \chi^k_\rho + a\chi^k_{\rho'}$, $\bar{ch}_0 = \chi^k_\rho + b\chi^k_{\rho'}$, for non-negative integers $a = N_{\rho',\rho}$, $b = N_{\rho,\rho'}$. At least one of a, b must be non-zero (otherwise by Lemma 1 N would be a permutation invariant) – without loss of generality say $a > 0$. Then the corresponding block-diagonal (5.3a) will also be a physical non-permutation invariant of level k. Let us then assume our invariant is in block-diagonal form. If we can show there is no block-diagonal invariant corresponding to these ch_i, we will have succeeded in showing no strongly physical invariant can exist at these levels unless it is a permutation invariant, and we will have completed the proof of Theorem 1(b) for these levels.

From (5.3b) we get $N_{\rho',\rho'} \geqq a^2$, where the inequality will hold iff $m_{i\rho'} \neq 0$ for some $i > 0$. However, taking $\lambda_L = \lambda_R = \rho$ and $\ell = \frac{k+1}{2}$ in Eq. (2.9) tells us $1 = N_{\rho,\rho} = N_{\rho'\rho'}$, so $a = 1$ and $m_{i\rho'} = 0$ for all $i > 0$.

Taking $\lambda = (1, 2)$, note that Eq. (2.5f) gives us

$$s(\lambda) = S^{(k)}_{\rho\lambda} + S^{(k)}_{\rho'\lambda} = \frac{4}{k'\sqrt{3}}\{\sin[2\pi/k'] - \sin[6\pi/k']\} .$$

$s(\lambda)$ equals 0 for $k = 5$, but is negative for all larger $k \equiv 1 \pmod 4$. By the discussion after (5.4), this means no strongly physical invariant (except possibly for $k = 5$) can have $ch_0 = \chi^k_{\rho'} + \chi^k_\rho$, which concludes the proof of Theorem 1(b) for class (ii).

5.2. Class (iii). Class (iii), i.e. $k \equiv 0, 3, 6 \pmod{12}$, is more difficult. As in class (ii), it suffices to consider block-diagonal invariants. From Lemma 4 we read off $ch_0 = \chi^k_\rho + a\chi^k_{\rho'} + b\chi^k_{\rho''}$, where now we take $\rho' = (k+1, 1)$ and $\rho'' = (1, k+1)$, and where at least one of a, b is positive. Taking $\ell = -1$ and $(\lambda; \lambda') = (\rho; \rho')$ in (2.9) tells us $a = b \geqq 1$. We would first like to show $a = 1$.

$\omega(\rho) = \rho'$ and $\omega^2(\rho) = \rho''$, where ω is defined in (2.2). Put $\lambda' = (m, n)$; then for any λ we get from (2.5f)

$$S^{(k)}_{\omega(\lambda),\lambda'} = \exp[2\pi i(m-n)/3]S^{(k)}_{\lambda\lambda'},\ S^{(k)}_{\omega^2(\lambda),\lambda'} = \exp[2\pi i(-m+n)/3]S^{(k)}_{\lambda\lambda'} . \quad (5.5a)$$

Substituting in $\lambda' = (1, 2)$, we find that

$$S^{(k)}_{\rho,(1,2)} + aS^{(k)}_{\rho',(1,2)} + aS^{(k)}_{\rho'',(1,2)} = (1-a)S^{(k)}_{\rho,(1,2)} , \quad (5.5b)$$

where $S^{(k)}_{\rho,(1,2)} > 0$. For $a > 1$ $s(\lambda)$ will be negative, so we must have $a = 1$.

Similar calculations show that $m_{i,(m,n)}$ will be zero for all $i = 0, 1, \ldots, n_c - 1$ iff $m \not\equiv n \pmod 3$, and that $m_{i,\rho'} = m_{i,\rho''} = 0$ for all $i = 1, \ldots, n_c - 1$.

We can partition the indices $\{0, \ldots, n_c - 1\}$ into disjoint sets I_ℓ, where i, j lie in the same set I_ℓ iff there exists a $\lambda \in P^k$ such that $m_{i\lambda}, m_{j\lambda} > 0$. For example we have just shown that one set, call it I_0, equals $\{0\}$. Rewrite (5.3a) as

$$Z = \sum_\ell \sum_{i \in I_\ell} |ch_i|^2 . \tag{5.6a}$$

This is equivalent to writing N as a direct sum of indecomposable matrices N_l as in (2.10c), where

$$N_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \overset{\text{def}}{=} M_{3,1} \tag{5.6b}$$

is the block "containing" χ^k_ρ, $\chi^k_{\rho'}$ and $\chi^k_{\rho''}$. What are the possibilities for the other N_l? The following result is the heart of the class (iii) proof.

Claim. *N can be written as a direct sum of matrices, N_ℓ, where either*

$$N_\ell = (0) \overset{\text{def}}{=} M_{1,0}, \quad N_\ell = (3) \overset{\text{def}}{=} M_{1,3} \quad \text{or} \quad N_\ell = M_{3,1} . \tag{5.6c}$$

In other words, each ch_i either equals χ^k_λ for some λ (in which case there also are i', $i'' \neq i$ for which $ch_{i'} = ch_{i''} = ch_i$), or $ch_i = \chi^k_{\lambda_1} + \chi^k_{\lambda_2} + \chi^k_{\lambda_3}$ for some distinct weights $\lambda_1, \lambda_2, \lambda_3$. Moreover, $ch_i = \chi^k_\lambda$ can only happen for $\lambda = (k'/3,\ k'/3)$, and $ch_i = \chi^k_{\lambda_1} + \chi^k_{\lambda_2} + \chi^k_{\lambda_3}$ can only happen for $\lambda_2 = \omega(\lambda_1)$ and $\lambda_3 = \omega^2(\lambda_1)$, up to a possible reordering of the λ_i.

Proof. From (5.6b) and (2.10a) we find $r(N_0) = 3$. It is an easy combinatorial exercise to find all possibilities for N_ℓ with $r(N_\ell) = 3$: N_ℓ equals either

$$M_{1,3},\ M_{3,1},\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \overset{\text{def}}{=} M', \text{ or } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{\text{def}}{=} M'' . \tag{5.6d}$$

The main things to keep in mind when showing (5.6d) is complete are Eqs. (2.10), and the fact that N_ℓ is the coefficient matrix for $\sum_{i \in I_\ell} |ch_i|^2$. For example, if any entry of N_ℓ is at least 3, then by (5.3b) a diagonal entry of N_ℓ is at least 3, so by (2.10b) N_ℓ must equal $(3) = M_{1,3}$.

We wish to show that no N_ℓ can equal either M' or M''. Since $M_{1,3}^2 = 3M_{1,3}$ and $M_{3,1}^2 = 3M_{3,1}$, Lemma 3 would then conclude the proof of the first statement of the claim.

First let us make some general remarks. Because of (5.6d), we know all of the possible extended characters ch look like $ch = \sum_{j=1}^h \chi^k_{\lambda_j}$ for $h = 1, 2$ or 3, where each $\lambda_j = (m_j, n_j) \in P^k$ is distinct. Then we know

$$\sum_{j=1}^h S^{(k)}_{m_j n_j; m'n'} = 0 \tag{5.7a}$$

must hold for each $(m', n') \in P^k$ with $m' \not\equiv n' \pmod 3$. In fact, because the *triality* $m' - n' \pmod 3$ is preserved both by Weyl reflections and adding vectors in $k'A_2$, we may drop the assumption in (5.7a) that $(m',n')\in P^k$ and demand only that $m'\not\equiv n' \pmod 3$ (for $(m',n')\notin P^k$ define $S^{(k)}_{mn,m'n'}$ by (2.5f)).

Writing $(m',n')=(\ell+\ell',\ell)$, $x_\ell=\exp[-2\pi i\ell/k']$ and $y_{\ell'}=\exp[-2\pi i\ell'/3k']$, (5.7a) becomes

$$0=y_{\ell'}^{2k'}x_\ell^{k'}\sum_{j=1}^{h}\{y_{\ell'}^{2m_j+n_j}x_{\ell'}^{m_j+n_j}+y_{\ell'}^{-m_j+n_j}x_\ell^{-m_j}+y_{\ell'}^{-m_j-2n_j}x_\ell^{-n_j}$$

$$-y_{\ell'}^{-m_j-2n_j}x_\ell^{-m_j-n_j}-y_{\ell'}^{2m_j+n_j}x_{\ell'}^{m_j}-y_\ell^{-m_j+n_j}x_\ell^{n_j}\}\,, \tag{5.7b}$$

where the extra irrelevant factor in front is added for future convenience (to make the exponents all positive). This must hold for all $\ell,\ell'\in\mathbf{Z}$, with $\ell'\equiv 0 \pmod 3$. Consider the polynomial $p(x,y)$ obtained by replacing x_ℓ and $y_{\ell'}$ with the variables x and y, respectively, in the RHS of (5.7b). Divide $x^{k'}-1$ into $(y^{k'}-1)p(x,y)$; the remainder is

$$\sum_{j=1}^{h}\{y^{3k'-m_j+n_j}x^{k'-m_j}-y^{2k'-m_j+n_j}x^{k'-m_j}+y^{3k'-m_j-2n_j}x^{k'-n_j}$$

$$-y^{2k'-m_j-2n_j}x^{k'-n_j}-y^{3k'-m_j-2n_j}x^{k'-m_j-n_j}+y^{2k'-m_j-2n_j}x^{k'-m_j-n_j}$$

$$+y^{3k'+2m_j+n_j}x^{m_j+n_j}-y^{2k'+2m_j+n_j}x^{m_j+n_j}-y^{3k'+2m_j+n_j}x^{m_j}$$

$$+y^{2k'+2m_j+n_j}x^{m_j}-y^{3k'-m_j+n_j}x^{n_j}+y^{2k'-m_j+n_j}x^{n_j}\}\,. \tag{5.7c}$$

Then (5.7b) is equivalent to the statement that $y^{3k'}-1$ must divide (5.7c).

Let us look at one of these terms, say $y^{2k'+2m_j+n_j}x^{m_j}$. Because $y^{3k'}-1$ must divide the polynomial in (5.7c), there are only two possibilities: either that one of the $6\cdot h$ terms in (5.7c) with a coefficient of -1 will cancel $y^{2k'+2m_j+n_j}x^{m_j}$; or that one of the other $6\cdot h-1$ terms in (5.7c) with a coefficient of $+1$ will equal $y^{2m_j+n_j-k'}x^{m_j}$. Let us first show that the second possibility cannot be realized.

Suppose e.g. that $y^{2k'-m_i-2n_i}x^{k'-m_i-n_i}=y^{2m_j+n_j-k'}x^{m_j}$, for some $1\leqq i\leqq h$. That means $2k'-m_i-2n_i=2m_j+n_j-k'$ and $k'-m_i-n_i=m_j$, i.e. $2k'-n_i=m_j+n_j$. But this contradicts $n_i<k'$ and $m_j+n_j<k'$. The other possibilities all fail for similar reasons.

Thus the first possibility must be realized. Suppose e.g. that $y^{2k'-m_i+n_i}x^{k'-m_i}=y^{2k'+2m_j+n_j}x^{m_j}$, i.e. $2k'-m_i+n_i=2k'+2m_j+n_j$ and $k'-m_i=m_j$. This gives us $-k'+n_i=m_j+n_j$, which is likewise impossible. Indeed, the only positive terms in (5.7c) which can cancel $y^{2k'+2m_j+n_j}x^{m_j}$ are $y^{3k'-m_i+n_i}x^{n_i}$ for any i, which give us the equations $n_i=m_j$ and $m_i=k'-m_j-n_j$. In other words, for each j there must be an i such that $\omega(m_j,n_j)=(m_i,n_i)$.

Suppose $h=1$, i.e. $ch=\chi^k_{m,n}$. Then $\omega(m,n)=(m,n)$, which can only happen for $(m,n)=(k'/3,k'/3)$.

Suppose $h=2$. Then either $\omega(m_1,n_1)=(m_1,n_1)$ and $\omega(m_2,n_2)=(m_2,n_2)$, or $\omega(m_1,n_1)=(m_2,n_2)$ and $\omega(m_2,n_2)=(m_1,n_1)$. In either case, the only way this can happen is if $(m_1,n_1)=(m_2,n_2)=(k'/3,k'/3)$, contradicting $\lambda_1\neq\lambda_2$. Therefore h cannot equal 2. Hence $N_\ell=M'$ and $N_\ell=M''$ are both impossible, because both require an extended character with $h=2$.

Finally, suppose $h=3$. It is easy to verify that the only possibility here is $\omega(m_1,n_1)=(m_2,n_2)$ and $\omega(m_2,n_2)=(m_3,n_3)$, relabelling the indices if necessary. QED to claim

From the claim, and the earlier observation that $m_{i,(m,n)} \neq 0$ for some i iff $m \equiv n \pmod 3$, we know already what our block-diagonal invariant must look like: it is $\mathscr{D}_k$. Note that this extended algebra, which we will call $A^e_{2,k}$, has far fewer characters than $A_{2,k}$ does, so there are no "hybrid" invariants with e.g. the chiral algebras $\mathscr{A} = A^e_{2,k}$ and $\bar{\mathscr{A}} = A_{2,k}$: the only possibilities for a strongly physical invariant are $\mathscr{A} = \bar{\mathscr{A}} = A_{2,k}$ and $\mathscr{A} = \bar{\mathscr{A}} = A^e_{2,k}$. The first possibility corresponds to permutation invariants. Our task here is to enumerate all physical invariants corresponding to the second possibility, in other words to find all permutations π of the extended characters which obey $T^e_{\pi i,\pi j} = T^e_{ij}$ and

$$S^e_{\pi i,\pi j} = S^e_{ij}\,. \tag{5.8a}$$

Also, $N^e_{\pi h,\pi i,\pi j} = N^e_{hij}$, where

$$N^e_{hij} \stackrel{\text{def}}{=} \sum_{g=0}^{n_c-1} \frac{S^e_{hg} S^e_{ig} S^e_{jg}}{S^e_{0g}}\,. \tag{5.8b}$$

Write $ch_{m,n}$ for $\chi^k_{m,n} + \chi^k_{\omega(m,n)} + \chi^k_{\omega^2(m,n)}$ and $ch_{(i)}$, $i = 1, 2, 3$, for the extended characters equal to $\chi^k_{k'/3,k'/3}$. To avoid redundancy, we may restrict (m, n) here to lie in the set $P^e = \{(m,n) \in P^k \,|\, m < k' - m - n \text{ and } n \leqq k' - m - n\}$.

Using (5.5a) we can easily find most of the entries of S^e_{ij}:

$$S^e_{mn,m'n'} = 3S^{(k)}_{mn,m'n'}, \tag{5.9a}$$

$$S^e_{mn;(i)} = S^{(k)}_{mn;k'/3,k'/3}, \quad i = 1, 2, 3, \tag{5.9b}$$

using obvious notation. The 9 remaining entries, $S^e_{(i)(j)} = S^e_{(j)(i)}$ for $i, j = 1, 2, 3$, satisfy several relations: e.g.

$$|S^e_{(i)(1)}|^2 + |S^e_{(i)(2)}|^2 + |S^e_{(i)(3)}|^2 = \frac{2}{3} + \frac{|\alpha|^2}{3}\,, \tag{5.9c}$$

where $\alpha = S^{(k)}_{k'/3,k'/3;k'/3,k'/3} = \dfrac{6}{\sqrt{3k'}}\sin(2\pi k'/9)$. Equation (5.9c) follows from (5.9b) and the unitarity of S^e and $S^{(k)}$.

It will not be necessary for us to explicitly compute the $S^e_{(i)(j)}$. It suffices to note from (5.9c) that for each i, there is a j for which $|S^e_{(i)(j)}| \geqq \sqrt{2}/3$. On the other hand, by (5.9a, b) the other entries of S^e are proportional to $1/k'$, and so will usually be much smaller.

Suppose π takes some $ch_{(i)}$ to some $ch_{m,n}$. A quick calculation shows $|S^e_{mn,m'n'}| \geqq \sqrt{2}/3$ can only happen for $k \leqq 19$, i.e. $k = 18, 15, 12, \ldots, 3$. An explicit computer calculation shows $|S^{(18)}| < \sqrt{2}/9$, which eliminates $k = 18$. Similarly $|S^e_{mn,(j)}| \geqq \sqrt{2}/3$ can only happen for $k \leqq 3$. Therefore, by (5.8a) for $k \geqq 18$ we must have π taking each $ch_{m,n}$ to some $ch_{m',n'}$. Let us restrict ourselves to these k. How π permutes the $ch_{(i)}$ is irrelevant to us, since those 3 characters are equal.

To find all such π for $k' > 18$ reduces to arguments familiar from earlier parts of this paper, so we will only give a 3-step sketch of the proof.

Let $\mathscr{S}^e_a = \{(m,n) \in P^e \,|\, m = a \text{ or } n = a\}$. Suppose we know $\pi(a,b) \in \mathscr{S}^e_a \cup \mathscr{S}^e_b$. Then using (5.9a) the argument surrounding Eqs. (3.7) applies here and tells us that $\pi(a, b) = (a, b)$ or (b, a).

Because of (5.9a, b) and (5.5a) we can see that

$$N^e_{mn,m'n',m''n''} = \sum_{i=0}^{2} N^{(k)}_{\omega^i(m,n),m'n',m''n''}, \text{ so} \tag{5.10a}$$

$$N^e_{mn,mn,mn} = \min\{m,n\} + \begin{cases} m + 2n - \frac{2k'}{3} & \text{if } \frac{k'}{3} - \frac{m}{2} < n \leqq \frac{k'}{3} \\ \frac{k'}{3} + m - n & \text{if } \frac{k'}{3} \leqq n < \frac{k'}{3} + m \\ 0 & \text{otherwise} \end{cases}$$

$$+ \begin{cases} n + 2m - \frac{2k'}{3} & \text{if } \frac{k'}{3} - \frac{n}{2} < m \leqq \frac{k'}{3} \\ \frac{k'}{3} + n - m & \text{if } \frac{k'}{3} \leqq m < \frac{k'}{3} + n\,. \\ 0 & \text{otherwise} \end{cases} \tag{5.10b}$$

Step 1. First prove that $\pi(1,a) = (1,a)$ or $(a,1)$ for all $(1,a) \in P^e$.
It suffices to show $\pi(1,a) \in \mathscr{S}^e_1$. This follows immediately from (5.10b), except for $a = k'/3$ when $k'/3 \equiv 1 \pmod 3$. If $\pi(1,k'/3) \notin \mathscr{S}^e_1$, then $\pi(1,a) = (2,m)$ or $(m,2)$ for some m. Now, we can show $\pi(3,3) = (3,3)$ (otherwise by (5.10b) it must equal $(2, k'/3+1)$ or $(k'/3+1, 2)$, which violates $S^e_{11,33} = S^e_{11,\pi(33)}$). Then $S^e_{33;1,k'/3} = S^e_{33;2m}$ implies $m = k'/3 - 2$, which violates $S^e_{11;1,k'/3} = S^e_{11;\pi(1,k'/3)}$.

Step 2. Next show that $\pi(1,4) = (1,4)$ implies $\pi(1,a) = (1,a)$ for all $(1,a) \in P^e$. The proof is similar to that used in proving (3.10a). In particular, (3.9a) becomes

$$c_k(6+9a) + c_k(9+3a) + c_k(3-6a) = c_k(9+6a) + c_k(6-3a) + c_k(3+9a)\,, \tag{5.10c}$$

which has the solutions $a = 0, 1, k' - \frac{1}{2}$ in the range $0 \leqq a < k'$.

Step 3. Finally, show that $\pi(1,4) = (1,4)$ implies $\pi(a,b) = (a,b)$ for all $(a,b) \in P^e$. This argument resembles the one surrounding (5.7c). Step 2 can be used to show that $x^{k'} - 1$ must divide the polynomial

$$\begin{aligned} &[e_k(3a+3b) - e_k(3b)]x^{a+2b} + [e_k(3b') - e_k(3a'+3b')]x^{a'+2b'} \\ &\quad + [e_k(-3a) - e_k(-3a-3b)]x^{-2a-b} + [e_k(-3a'-3b') \\ &\quad - e_k(-3a')]x^{-2a'-b'} \\ &\quad + [e_k(-3b) - e_k(3a)]x^{a-b} + [e_k(3a') - e_k(-3b')]x^{a'-b'}\,, \end{aligned} \tag{5.10d}$$

where $(a',b') = \pi(a,b)$. This can only happen for $(a',b') = (a,b)$.

The conclusion is that there are only two possible permutations π, except possibly for $k \leqq 15$. These give rise to the physical invariants $\mathscr{D}_k$ and $\mathscr{D}^c_k$. (The only level this argument breaks down at are $k = 3, 6, 9, 12, 15$.)

5.3. Class (iv). From Lemma 4, and using the previous arguments, the only possibilities for ch_0 are χ^k_ρ; $\chi^k_\rho + a\chi^k_{k+1,1} + a\chi^k_{1,k+1}$; $\chi^k_\rho + \chi^k_{(k+1)/2,(k+1)/2}$; and $\chi^k_\rho + a\chi^k_{k+1,1} + a\chi^k_{1,k+1} + \chi^k_{(k+1)/2,(k+1)/2} + a\chi^k_{(k+1)/2,2} + a\chi^k_{2,(k+1)/2}$, where

$a \geqq 1$. The first corresponds to a permutation invariant, and is classified in Theorem. 2: the permutation invariants are $\mathscr{A}_k$ and $\mathscr{A}_k^c$. The second is classified (for $k \neq 9$) by the class (iii) argument given above: the physical invariants are $\mathscr{D}_k$ and $\mathscr{D}_k^c$ (it turns out that the exceptionals $\mathscr{E}_9^{(2)}$ and $\mathscr{E}_9^{(2)c}$ also correspond to this ch_0). The third possibility is dealt with using the argument of class (ii) given earlier, and is realized by no physical invariants.

The fourth and final possibility can be dealt with using the Eq. (5.4) argument. In particular, the vector $\lambda = (1, 3)$ in (5.4) implies $a = 1$. Then the choice $\lambda = (1, 4)$ gives a contradiction, except for level $k = 9$ (where there is an exceptional, $\mathscr{E}_9^{(1)}$, which realizes this ch_0 possibility).

5.4. Level 57. Lemma 4 for $k = 57$ is: if $N_{\rho\lambda} \neq 0$ for a positive invariant of level 57, then

$$\lambda \in \{(1, 1), (1, 58), (2, 29), (11, 11), (11, 38), (19, 19), (19, 22), (29, 29), (22, 19), (29, 2), (38, 11), (58, 1)\} . \tag{5.11}$$

From here we can read off the possibilities for ch_0: there are 16 of them, half of which involve a parameter $a \geqq 1$. Of these, four were considered in the class (iv) argument given above. The remaining 12 all succumb to similar arguments: the weight (1, 3) used in (5.4) forces $a = 1$ in the 6 remaining possibilities involving the parameter a; the weight (2, 5) eliminates the 6 not involving a, and eliminates $a = 1$ in the other 6. The conclusion is that the only $k = 57$ strongly physical invariants are the permutation invariants $\mathscr{A}_{57}$, $\mathscr{A}_{57}^c$, $\mathscr{D}_{57}$ and $\mathscr{D}_{57}^c$.

6. Extensions to Other Algebras

The main motivation for pursuing a classification proof for A_2 is the hope that the methods developed there would also be of use for other algebras. And indeed that should be the case. The main issues are the simplicity of the form the ρ-coupling lemma for those algebras takes, and also how well we can manage finding all permutation invariants. For some examples we will now write down the ρ-couplings for A_1, and a little later on that of $A_1 \oplus A_1$ for relatively prime levels $k_1 + 2$, $k_2 + 2$, as well as conjectures for G_2 and C_2 which we have verified on a computer for the first hundred levels (the ρ-shifted weights are identified with their Dynkin labels):

ρ-coupling for A_1. (a) *For* $k \equiv 1, 2, 3 \pmod 4$, $k \neq 10$, *the only possible weight* λ *which can couple to* $\rho = 1$ *in a positive invariant is* $\lambda = 1$;

(b) *for* $k \equiv 0 \pmod 4$, $k \neq 28$, *the only possibilities are* $\lambda = 1$ *and* $k + 1$;

(c) *for* $k = 10$, $\lambda = 1, 7$; *and for* $k = 28$, $\lambda = 1, 11, 19, 29$.

ρ-coupling for G_2. (a*) *For* $k \neq 3, 4$, *the only possible weight* λ *which can couple to* $\rho = (1, 1)$ *in a positive invariant is* $\lambda = (1, 1)$;

(b) *for* $k = 3$, *the only possibilities are* $\lambda = (1, 1), (2, 2)$; *and for* $k = 4$, $\lambda = (1, 1)$, $(4, 1)$.

ρ-coupling for C_2. (a*) *For k odd,* $k \neq 3, 7$, *the only possible weight* λ *which can couple to* $\rho = (1, 1)$ *in a positive invariant is* $\lambda = (1, 1)$;

(b*) *for k even,* $k \neq 12$, *the only possibilities are* $\lambda = (1, 1), (1, k + 1)$;

(c) *for* $k = 3$, $\lambda = (1, 1), (3, 2)$; *for* $k = 7$, $\lambda = (1, 1), (3, 3), (1, 6), (7, 2)$; *and for* $k = 12$, $\lambda = (1, 1), (3, 4), (7, 1), (5, 5), (9, 2), (3, 8), (9, 4), (7, 7), (1, 13)$.

The results for A_1 are proven in the appendix. As yet unproven results are marked with an '$*$' (but because they hold for $k < 100$, they likely hold for all k). This shows that in many ways A_2 is the least tractible of the rank 2 algebras. According to these findings, for G_2 all physical invariants (except at levels 3 and 4) must be permutation invariants, and similarly for C_2 at odd levels (except 3 and 7). Note that for each of these algebras, at any level with irregular ρ-coupling behaviour (e.g. $k = 10$ and 28 for A_1) exceptional invariants can always be found. The only known exception to this rule is A_2 at $k = 57$.

The ρ-coupling possibilities for $A_1 \oplus A_1$ are more complicated, because there are now two independent levels k_1 and k_2. But it is easy to find these when e.g. $k_1 + 2$ and $k_2 + 2$ are relatively prime, using an argument very similar to those used in the Appendix (see [13] for a proof). The result is:

ρ-coupling for $A_1 \oplus A_1$. *When $k_1 + 2$ and $k_2 + 2$ are relatively prime, the only $\lambda = (m, n)$ which can couple to $\rho = (1, 1)$ in some positive invariant are*:

(i) *for k_1, k_2 odd, $k_1 \equiv k_2 \pmod 4$, then $\lambda = (1, 1)$;*
(ii) *for k_1, k_2 odd, $k_1 \not\equiv k_2 \pmod 4$, then $\lambda = (1, 1), (k_1 + 1, k_2 + 1)$;*
(iii) *for $k_1 \equiv 0 \pmod 4$, $k_1 \neq 28$, then $\lambda = (1, 1), (k_1 + 1, 1)$;*
(iv) *for $k_2 \equiv 2 \pmod 4$, $k_1 \neq 10$, then $\lambda = (1, 1)$;*
(v) *for $k_1 = 10$, $\lambda = (1, 1), (7, 1)$; and for $k_1 = 28$, $\lambda = (1, 1), (11, 1), (19, 1), (29, 1)$.*

Since we also found in [13] all permutation invariants for $A_1 \oplus A_1$, for all levels k_1, k_2, it is now an easy task, using the techniques of Sect. 5, to complete the $A_1 \oplus A_1$ classification when $k_1 + 2, k_2 + 2$ are relatively prime. These observations have also been generalized in [13] to all $A_1 \oplus \cdots \oplus A_1$, when $k_1 + 2, \ldots, k_L + 2$ are all relatively prime.

These findings suggest that Lemma 2 continues to be useful for algebras other than just A_2. Our proof in Sect. 3 to find all permutation invariants of A_2 made use of explicit formulas for the A_2 fusion rules [5], and those do not exist at present for the other algebras (except for A_1 [16] and hence all sums of A_1 and A_2). Our hope is that this will not constitute a serious stumbling block for future applications of these ideas. The only fusion rules which our proof crucially needed were $N_{\lambda\lambda\lambda}$, which may be simple enough to calculate explicitly. Moreover, since all the information obtainable from fusion rules is also encoded in the modular S-matrix [36, 27], though in not so accessible a form, it is possible that alternate proofs of Theorem 2 can be found which do not require explicit knowledge of any fusion rules of A_2.

Perhaps a more serious problem facing generalizations of these techniques to higher ranks is the dependence of many steps on explicit knowledge of the modular S-matrix. The Weyl group of the algebra increases fantastically as the rank; so will the complexity of the explicit formula for the modular S-matrix.

It was proven in [38] that there is an exact rank-level duality between C_n level k and C_k level n; in particular there is a one-to-one correspondence between the physical invariants of one and those of the other. Thus finding all the physical invariants of C_2 would mean we have also found all the level 2 invariants of C_n. There also is an approximate rank-level duality between A_n level k and A_{k-1} level $n + 1$ [39, 1]. This suggests that the situation for levels 2 and 3 of A_n should be approximately as accessible as that for arbitrary levels of A_1 and A_2. This will be another direction for our future research.

7. Comments

In this paper we first find all *permutation invariants* of A_2, for each level k. We then prove that for $k \equiv 2, 4, 7, 8, 10, 11 \pmod{12}$, the only level k *physical invariants* of A_2 are permutation invariants. Together, these two statements allow us to write down all physical invariants for A_2 of those levels: $\mathscr{A}_k$, $\mathscr{A}_k^c$, $\mathscr{D}_k$ and $\mathscr{D}_k^c$ (see Eqs. (2.7a, b, h)).

To handle the remaining levels, we make use of additional results known to be satisfied by the partition functions [26]. These allow us to find all strongly physical invariants for $k \equiv 0, 1, 3, 5, 6, 9 \pmod{12}$, except for 7 levels which have been completely treated by the computer program of [18]. Thus the classification problem for A_2 modular invariant partition functions has now been completed.

Two questions suggest themselves: (i) At present our only proof for levels $k \equiv 0, 1, 3, 5, 6, 9 \pmod{12}$ requires results from [26]; although these must hold for the partition function of any physically reasonable conformal field theory, they do not necessarily hold for invariants satisfying only the three conditions (P1), (P2) and (P3). It would be desirable to reduce as much as possible the required assumptions, even though all assumptions used are physically well-motivated. Can our classification of strongly physical invariants for those levels be extended somehow into a classification of physical invariants (the terms "physical" and "strongly physical" are defined in Sect. 1)? (ii) Can the methods developed here give classification proofs for the other affine algebras?

A natural way to try to answer question (i) in the affirmative is to apply Lemma 2 to weights other than just ρ, in other words to generalize the proof of Lemma 4 to $\lambda' + \rho$. There is a good chance this approach would work, but it could result in a much lengthier argument.

The main thrust of our future research (see e.g. [13]) will be directed towards (ii), i.e. applying these arguments to other algebras, starting with the remaining rank 2 algebras and $A_1 \oplus \cdots \oplus A_1$, and levels 2 and 3 of A_n. Section 6 discusses our initial findings.

Appendix: ρ-Coupling for A_1

An important step in the A_2 classification proof given in this paper is the ρ-coupling Lemma proven in Sect. 4. Its proof (see Claim 1 there) assumes knowledge of the ρ-coupling lemma for A_1, given in Sect. 6. Because of this, and because the ρ-coupling proofs for A_1 is more transparent but similar in spirit to that of A_2, we have included here the A_1 proof. After giving it, a few brief comments on how to finish off a classification proof for A_1 are provided. Claim 1 in Sect. 4 is the A_1 ρ-coupling Lemma, if we were to ignore the A_1 norm condition; its proof will be completed at the end of this appendix.

The proof given below for A_1 ρ-coupling is certainly not intended to be the shortest such; because our primary interest is in proving Claim 1, we will exploit the A_1 norm condition (A.1a) as rarely as possible.

Write $k' = k + 2 = 2^L k''$, where k'' is odd. Define the integer M by $k'/2 \leqq 2^M < k'$. Identify a weight $\lambda = m\beta_1$ of $\hat{A}_1$ by its Dynkin label m. Suppose $N_{1,a} > 0$ for some A_1 positive invariant N of level k. The norm condition reads

$$a^2 \equiv 1 \pmod{4k'}\,. \tag{A.1a}$$

Hence a must be odd. The parity $\varepsilon(m)$ of some weight m is simply

$$\varepsilon(m) = \begin{cases} +1 & \text{if } 0 < \{m\}_{2k'} < k' \\ -1 & \text{if } k' < \{m\}_{2k'} < 2k' , \\ 0 & \text{if } \{m\}_{2k'} = 0 \text{ or } k' \end{cases} \tag{A.1b}$$

where throughout this Appendix we use the notation $\{x\}_y$ for the unique number satisfying both $\{x\}_y \equiv x \pmod y$ and $0 \leqq \{x\}_y < y$. Using this, Lemma 2 becomes

$$0 < \ell < k', \ell \text{ relatively prime to } 2k', \Rightarrow \{\ell a\}_{2k'} < k' ;$$
$$k' < \ell < 2k', \ell \text{ relatively prime to } 2k', \Rightarrow \{\ell a\}_{2k'} > k' ; \tag{A.1c}$$

We want to find all integers $1 \leqq a < k'$ satisfying both (A.1a, c). Equations (A.1) are the analogues of Eqs. (4.1).

Assume first that k' is odd (i.e. that $L = 0$), and define $N \geqq 0$ so that $a2^N < k' < a2^{N+1} < 2k'$. If $a > 1$, then $N < M$. Put $\ell = k' - 2^{N+1}$; it will lie between 0 and k', and will be relatively prime to $2k'$. Then (A.1c) implies $k' > \{ak' - a2^{N+1}\}_{2k'} = 3k' - a2^{N+1} > k'$, a contradiction. Therefore, k' odd implies a must equal 1.

Thus it suffices to consider k' with $L > 0$. Let $a_2 = \{a\}_{2^{L+1}}$. There are two different cases: either $a_2 \leqq 2^L$ (to be called case 1), or $a_2 > 2^L$ (to be called case 2). If $k' = 2^L$, there will only be case 1.

Consider case 1 first. Define $\ell_i = k'' + 2^i$, for $i = 1, \ldots, M-1$. Then these ℓ_i will necessarily be relatively prime to $2k'$, and they all will lie in the range $0 < \ell_i < k'$. Let $b = a/k'$ and $c = 1 - a_2/2^L$. Then (A.1c) tells us that no $1 \leqq i < M$ can have $1 \leqq \frac{a_2}{2^L} + \{2^i b\}_2 \leqq 2$. Write out the binary expansion $b = \sum_{i=1}^{\infty} b_i 2^{-i}$ of b (so each $b_i = 0$ or 1). Then we have, for each $i = 1, \ldots, M-1$, that $\{2^i b\}_2 > 1 + c$ if $b_i = 1$, and $\{2^i b\}_2 < c$ if $b_i = 0$.

Assume inductively that $b_1 = \cdots = b_n = 0$ for some $1 \leqq n < M-1$, but $b_{n+1} = 1$. Then $2^n b = \{2^n b\}_2 < c$, but $2^{n+1} b = \{2n^{+1} b\}_2 > 1 + c$. Hence, $1 + c < 2^{n+1} b < 2c$, i.e. $1 < c$, which is false.

A similar calculation holds if $b_1 = \cdots = b_n = 1$ but $b_{n+1} = 0$. Thus there are exactly two possibilities: either $b_i = 0$ for all $i = 1, \ldots, M-1$, or $b_i = 1$ for $i = 1, \ldots, M-1$ – i.e. either $a < k'/2^{M-1}$ or $a > k' - k'/2^{M-1}$ But $k'/2^{M-1} \leqq 4$, so a odd implies either $a = 1$ or 3, $a = k' - 1$ or $k' - 3$. Equation (A.1a) now forces $a = 1$ or (if $L = 1$) $a = k + 1$.

Case 2 is harder, and we will begin by proving it for $L = 1$. As before, take $\ell_i = 2^i + k'/2$ for $i = 1, \ldots, M-1$. Equation (A.1c) however now reads $\frac{1}{3} < \{2^i b\}_2 < \frac{3}{2}$, since $a_2 = 3$ here.

Consider first $b_1 = 0$. Then $\frac{1}{2} \leqq 2b$ implies $b_2 = 1$, and $4b < \frac{3}{2}$ implies $b_3 = 0$. In fact, b_i continues alternating between 0 and 1, for $i = 1, \ldots, M$. The same conclusion holds if $b_1 = 1$. Therefore, for M even, $a = k'/3 + \varepsilon$ or $a = 2k'/3 - \varepsilon$, where $-k'/(3 \cdot 2^M) \leqq \varepsilon < k'/(3 \cdot 2^{M-1})$, and for M odd, $a = k'/3 + \varepsilon'$ or $a = 2k'/3 - \varepsilon'$, where $-k'/(3 \cdot 2^{M-1}) \leqq \varepsilon' < k'/(3 \cdot 2^M)$. ε and ε' are fixed by the requirement that a be odd. There are 3 possibilities: if $k' \equiv 0 \pmod 3$, we have $a = k'/3 + 1$ or $a = 2k'/3 - 1$; if $k' \equiv \pm 1$, we have $a = k'/3 \mp 1/3$. Equation (A.1a) tells us that for

$k' \equiv 0 \pmod 3$, these a can only work for $k' \equiv 30 \pmod{36}$; for $k' \equiv \pm 1 \pmod 3$ they cannot satisfy (A.1c).

So consider $k' \equiv 30 \pmod{36}$. Then $\ell = 6 + k'/6$ will be relatively prime to $2k'$. We find that $\ell(k'/3 + 1) \equiv 6 - k'/6 \pmod{2k'}$. This then contradict (A.1c), unless $6 - k'/6 > 0$, i.e. $k' < 36$, i.e. $k' = 30$. This concludes the proof of case 2, for $L = 1$.

All that remains for us to prove is case 2 for $L > 1$. Choosing $\ell_i = 2^i + k''$ gives us $c' < \{2^i b\}_2 < 1 + c'$ for $i = 1, \ldots, M - 1$, where $c' = 2 - a_2/2^L$. Choosing $\ell'_i = k'' - 2^i$ gives us $1 - c' < \{2^i b\}_2 < 2 - c'$ for $i = 1, \ldots, M - L$. Adding these, we get $\frac{1}{2} < \{2^i b\}_2 < \frac{3}{2}$ for $i = 1, \ldots, M - L$. Therefore by the case $2L = 1$ argument we get that b_i alternates between 0 and 1 for $i = 1, \ldots, M - L + 1$. From this and the ℓ_i, ℓ'_i inequalities we see that, unless $M - L = 1$, the binary expansion of c' either looks like $c' = 0.10\ldots$ or $c' = 0.01\ldots$ (in which case by the ℓ_i inequalities we cannot have $b_j = b_{j+1} = b_{j+2}$ for any $j < M$), and if $M - L > 2$ we have $c' = 0.101\ldots$ or $c' = 0.010\ldots$ (in which case we cannot have $b_j = b_{j+1}$ for any $j < M$).

Now take $\ell''_i = 2^i - k''$ for $i = M - L + 1, \ldots, M$. This means either $\{2^i b\}_2 + c' < 1$ or $\{2^i b\}_2 + c' \geqq 2$ for these i. Then by the case 1 argument, we must have $b_{M-L+1} = \cdots = b_M$.

Thus, either $M - L = 1$, or both $M - L = 2$ and $L = 2$. If $M - L = 2$ and $L = 2$, then $k' = 4 \cdot 5$ or $k' = 4 \cdot 7$. Otherwise $M - L = 1$, i.e. $k' = 3 \cdot 2^L$. From the above calculations we can read off that $a = k'/2 \pm 1$ here. Equation (A.1a) reduces to $2^{L-2} \cdot 3 \equiv \mp 1 \pmod 4$. The only possible solution is $L = 2$ (i.e. $k' = 12$) and $a = k'/2 + 1 = 7$.

This completes the classification of the case 2 ρ-couplings a, and hence the proof of the ρ-coupling lemma for A_1, except for 4 levels where the argument broke down: $k = 10, 18, 26, 28$. These can be explicitly worked out on a computer.

Little work now remains to obtain a new classification proof for A_1. The A_1 permutation invariants can be easily enumerated using the expression $S^{(k)}_{mn} = \sqrt{2/k'} \sin(\pi mn/k')$. Apart from the exceptional level $k = 10$, this classifies all physical invariants of level $k \equiv 1, 2, 3 \pmod 4$. To find the strongly physical invariants of level $k \equiv 0 \pmod 4$, the methods of Sect. 5 suffice, and indeed reduce ultimately to an example in [26].

Proof of Claim 1 in Sect. 4 If we replace the K in Claim 1 with $k' = k + 2$ here, we see that it is simply the A_1 situation, ignoring the norm condition. The only places we used (A.1a) were in the case 1 proof when we eliminated $a = 3$ etc.; the case 2 proof for $L = 1$, when we eliminated all but $k' \equiv 30 \pmod{36}$; and the case 2 proof for $L > 1$, when we threw out $k' = 2^L \cdot 3$ for $L > 2$.

Note that a will satisfy (A.1c) iff $k' - a$ will, so it suffices to consider $a \leqq k'/2$.

Consider first the case 1 proof, and $a = 3$. Because $3 = a_2 < 2^L$, we must have $L > 1$. If $k' \equiv -1 \pmod 3$ use $\ell = (k' + 1)/3$, while if $k' \equiv +1 \pmod 3$ use $\ell = (k' + 2)/3 + k''$. If $k' \equiv 0, 3 \pmod 9$ take $\ell = k'/3 + 1$, while if $k' \equiv -3 \pmod 9$ use $\ell = k'/3 + 3$ (this fails for $k' < 9$, but there are no such k' divisible by 4 and 3).

Now consider the case 2 proof for $L = 1$. Taking $\ell = 3$ eliminates $k' \equiv -1 \pmod 3$, and for $k' > 12$ taking $\ell = k'/2 + 6$ eliminates $k' \equiv +1 \pmod 3$. The only $k' \leqq 12$ with $k' \equiv +1 \pmod 3$ and $L = 1$ is $k = 8$. For $k' \equiv 6 \pmod{36}$ use $\ell = 4 + k'/6$ (this fails for $k = 4$), and for $k' \equiv 18 \pmod{36}$ use $\ell = 2 + k'/6$.

Finally, consider the case 2 proof for $L > 1$, where $k' = 2^L \cdot 3$. For $k' > 14$, take $\ell = 7$ (this fails for $k = 10$).

The only $k = K - 2$ which escaped our arguments are $k = 4, 8, 10, 18, 26, 28$. These can be explicitly worked out. This concludes the proof of Claim 1.

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Note added in proof. After this paper was completed we recieved two related papers [32, 34]. Reference [32] also finds all permutation invariants of A_2, but without using the fusion rule calculations in [5]. This is valuable because similar arguments may work on e.g. A_n. Reference [34] also identifies the important relation (2.9), which we found in [11]. Using it they find all A_2 physical invariants for $k \equiv 2$, 4 (mod 6), and all strongly physical invariants when $k + 3$ is a power of 2 or a power of 3.

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