Commun. Math. Phys. 161, 289–309 (1994)

Communications in Mathematical Physics
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Level Spacing Distributions and the Bessel Kernel

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Received: 19 April 1993

Abstract: Scaling models of random $N \times N$ hermitian matrices and passing to the limit $N \to \infty$ leads to integral operators whose Fredholm determinants describe the statistics of the spacing of the eigenvalues of hermitian matrices of large order. For the Gaussian Unitary Ensemble, and for many others as well, the kernel one obtains by scaling in the "bulk" of the spectrum is the "sine kernel" $\dfrac{\sin \pi(x-y)}{\pi(x-y)}$. Rescaling the GUE at the "edge" of the spectrum leads to the kernel $\dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$, where Ai is the Airy function. In previous work we found several analogies between properties of this "Airy kernel" and known properties of the sine kernel: a system of partial differential equations associated with the logarithmic differential of the Fredholm determinant when the underlying domain is a union of intervals; a representation of the Fredholm determinant in terms of a Painlevé transcendent in the case of a single interval; and, also in this case, asymptotic expansions for these determinants and related quantities, achieved with the help of a differential operator which commutes with the integral operator. In this paper we show that there are completely analogous properties for a class of kernels which arise when one rescales the Laguerre or Jacobi ensembles at the edge of the spectrum, namely

$$\frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)},$$

where $J_\alpha(z)$ is the Bessel function of order α. In the cases $\alpha = \mp\frac{1}{2}$ these become, after a variable change, the kernels which arise when taking scaling limits in the bulk of the spectrum for the Gaussian orthogonal and symplectic ensembles. In particular, an asymptotic expansion we derive will generalize ones found by Dyson for the Fredholm determinants of these kernels.

1. Introduction and Statement of Results

A. Introduction. Scaling models of random $N \times N$ hermitian matrices and passing to the limit $N \to \infty$ leads to integral operators whose Fredholm determinants describe the statistics of the spacing of the eigenvalues of hermitian matrices of large order [18, 25]. Which integral operators (or, more precisely, which kernels of integral operators) result depends on the matrix model one starts with and at which location in the spectrum the scaling takes place.

For the simplest model, the Gaussian Unitary Ensemble (GUE), and for many others as well (see, e.g., [16, 17, 23, 24]), the kernel one obtains by scaling in the "bulk" of the spectrum is the "sine kernel"

$$\frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Precisely, this comes about as follows. If $\{\phi_k(x)\}_{k=0}^{\infty}$ is the sequence obtained by orthonormalizing the sequence $\{x^k e^{-x^2/2}\}$ over $(-\infty, \infty)$ and if

$$K_N(x,y) = \sum_{k=0}^{N-1} \phi_k(x)\,\phi_k(y)\,, \tag{1.1}$$

then in the GUE the probability density that n of the eigenvalues (irrespective of order) lie in infinitesimal intervals about $x_1, \ldots, x_n$ is equal to

$$R_n(x_1, \ldots, x_n) = \det(K_N(x_i, x_j))_{i,j=1,\ldots,n}\,.$$

The density of eigenvalues at a fixed point z is $R_1(z)$, and this is $\sim \sqrt{2N}/\pi$ as $N \to \infty$. Rescaling at z leads to the sine kernel because of the relation.

$$\lim_{N\to\infty} \frac{\pi}{\sqrt{2N}} K_N\left(z + \frac{\pi x}{\sqrt{2N}}, z + \frac{\pi y}{\sqrt{2N}}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Rescaling the GUE at the "edge" of the spectrum, however, leads to a different kernel. The edge corresponds to $z \sim \sqrt{2N}$, at which point the density is $\sim 2^{\frac{1}{2}} N^{\frac{1}{6}}$, and we have there the scaling limit [3, 11, 22],

$$\lim_{N\to\infty} \frac{1}{2^{\frac{1}{2}} N^{\frac{1}{6}}} K_N\left(\sqrt{2N} + \frac{x}{2^{\frac{1}{2}} N^{\frac{1}{6}}}, \sqrt{2N} + \frac{y}{2^{\frac{1}{2}} N^{\frac{1}{6}}}\right)$$
$$= \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y},$$

where Ai is the Airy function. In previous work [28] we found several analogies between properties of this "Airy kernel" and known properties of the sine kernel: a system of partial differential equations associated with the logarithmic differential of the Fredholm determinant when the underlying domain is a union of intervals [15]; a representation of the Fredholm determinant in terms of a Painlevé transcendent in the case of a single interval [15]; and, also in this case, asymptotic expansions for these determinants and related quantities [5, 2, 29, 6, 20], achieved with the help of a differential operator which commutes with the integral operator. (See [27] for further discussion of these properties of the sine kernel.)

In this paper we show that there are completely analogous properties for a class of kernels which arise when one rescales the Laguerre or Jacobi ensembles at the edge of the spectrum. For the Laguerre ensemble the analogue of the sequence of functions $\{\phi_k(x)\}$ in (1.1) is obtained by orthonormalizing the sequence

$$\{x^k x^{\alpha/2} e^{-x/2}\}$$

over $(0, \infty)$ (here $\alpha > -1$), whereas for Jacobi one orthonormalizes

$$\{x^k(1-x)^{\alpha/2}(1+x)^{\beta/2}\}$$

over $(-1, 1)$. (Here $\alpha, \beta > -1$.) In the Laguerre ensemble of (positive) hermitian $N \times N$ matrices the eigenvalue density satisfies [4, 23], for a fixed $x < 1$,

$$R_1(4Nx) \sim \frac{1}{2\pi}\sqrt{\frac{1-x}{x}}\,.$$

This limiting law is to be contrasted with the well-known Wigner semi-circle law in the GUE. The new feature here is the "hard edge" for $x \sim 0$. At this edge we have the scaling limit [11]:

$$\lim_{N\to\infty} \frac{1}{4N} K_N\left(\frac{x}{4N}, \frac{y}{4N}\right) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}\,,$$

where $J_\alpha(z)$ is the Bessel function of order α. Both limits follow from the asymptotic formulas for the generalized Laguerre polynomials. (Scaling in the bulk will just lead to the sine kernel and scaling at the "soft edge," $x \sim 1$, will lead to the Airy kernel.) The same kernel arises when scaling the Jacobi ensemble at -1 or 1. (Recall that in the Jacobi ensemble both ± 1 are hard edges; see e.g. [23].)

For later convenience we introduce now a parameter λ and define our "Bessel kernel" by

$$K(x,y) := \lambda \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}\,, \quad (x \neq y)\,, \tag{1.2a}$$

$$= \frac{\lambda}{4}(J_\alpha(\sqrt{x})^2 - J_{\alpha+1}(\sqrt{x})J_{\alpha-1}(\sqrt{x})) \quad (x = y)\,. \tag{1.2b}$$

Before stating our results, we mention that in the cases $\alpha = \mp\frac{1}{2}$ we have, when $\lambda = 1$,

$$2\sqrt{xy}\,K(x^2, y^2) = \frac{\sin(x-y)}{\pi(x-y)} \pm \frac{\sin(x+y)}{\pi(x+y)}\,, \tag{1.3}$$

which are kernels which arise when taking scaling limits in the bulk of the spectrum for the Gaussian orthogonal and symplectic ensembles [18]. In particular, an asymptotic expansion we derive will generalize ones found by Dyson [5] for the Fredholm determinants of these kernels.

We now state the results we have obtained.

B. The System of Partial Differential Equations. We set

$$J := \bigcup_{j=1}^{m} (a_{2j-1}, a_{2j}) \quad (a_j \geqq 0) \tag{1.4}$$

and write $D(J;\lambda)$ for the Fredholm determinant of K (the operator with kernel $K(x,y)$) acting on J. If we think of this as a function of $a=(a_1,\ldots,a_{2m})$, then

$$d\log D(J;\lambda) = -\sum_{j=1}^{2m}(-1)^j R(a_j,a_j)\,da_j\,, \tag{1.5}$$

where $R(x,y)$ is the kernel of $K(I-K)^{-1}$. We introduce the notations

$$\phi(x) := \sqrt{\lambda}J_\alpha(\sqrt{x}), \quad \psi(x) := x\phi'(x) \tag{1.6}$$

and the quantities

$$q_j := (I-K)^{-1}\phi(a_j), \quad p_j := (I-K)^{-1}\psi(a_j), \quad (j=1,\ldots,2m)\,, \tag{1.7}$$

$$u := (\phi,(I-K)^{-1}\phi), \quad v := (\phi,(I-K)^{-1}\psi)\,, \tag{1.8}$$

where the inner products refer to the domain J. The differential equations are

$$\frac{\partial q_j}{\partial a_k} = (-1)^k\frac{q_jp_k-p_jq_k}{a_j-a_k}q_k \quad (j\neq k)\,, \tag{1.9}$$

$$\frac{\partial p_j}{\partial a_k} = (-1)^k\frac{q_jp_k-p_jq_k}{a_j-a_k}p_k \quad (j\neq k)\,, \tag{1.10}$$

$$a_j\frac{\partial q_j}{\partial a_j} = p_j+\frac{1}{4}q_ju-\sum_{k\neq j}(-1)^k a_k\frac{q_jp_k-p_jq_k}{a_j-a_k}q_k\,, \tag{1.11}$$

$$a_j\frac{\partial p_j}{\partial a_j} = \frac{1}{4}(\alpha^2-a_j+2v)q_j-\frac{1}{4}p_ju-\sum_{k\neq j}(-1)^k a_k\frac{q_jp_k-p_jq_k}{a_j-a_k}p_k\,, \tag{1.12}$$

$$\frac{\partial u}{\partial a_j} = (-1)^j q_j^2\,, \tag{1.13}$$

$$\frac{\partial v}{\partial a_j} = (-1)^j p_jq_j\,. \tag{1.14}$$

Moreover the quantities $R(a_j,a_j)$ appearing in (1.5) are given by

$$a_jR(a_j,a_j) = \sum_{k\neq j}(-1)^k a_k\frac{(q_jp_k-p_jq_k)^2}{a_j-a_k}+p_j^2-\frac{1}{4}(\alpha^2-a_j+2v)q_j^2+\frac{1}{2}p_jq_ju\,. \tag{1.15}$$

These equations are quite similar to Eqs. (1.4)–(1.9) of [28], as is their derivation.

C. The Ordinary Differential Equation. For the special case $J=(0,s)$ the above equations can be used to show that $q(s;\lambda)$, the quantity q of the last section corresponding to the endpoint s, satisfies

$$s(q^2-1)(sq')' = q(sq')^2+\frac{1}{4}(s-\alpha^2)q+\frac{1}{4}sq^3(q^2-2) \quad \left(' = \frac{d}{ds}\right) \tag{1.16}$$

with boundary condition

$$q(s;\lambda) \sim \frac{\sqrt{\lambda}}{2^\alpha\Gamma(1+\alpha)}s^{\alpha/2}, \quad s\to 0\,. \tag{1.17}$$

This equation is reducible to a special case of the P_V differential equation;[1] explicitly, if $q(s) = (1 + y(x))/(1 - y(x))$ with $s = x^2$, then $y(x)$ satisfies P_V with $\alpha' = -\beta' = \alpha^2/8, \gamma' = 0$ and $\delta' = -2$. (We have primed the usual P_V parameters to avoid confusion with the α in our kernel. We mention that this special P_V can be expressed algebraically in terms of a third Painlevé transcendent and its first derivative [13]. We mention also that an argument can be given that (1.16) must be reducible to one of the 50 canonical types of differential equations found by Painlevé, without an explicit verification being necessary. This will be discussed at the end of Sect. II B.) It is sometimes convenient to transform (1.16) by making the substitution

$$q(s) = \cos\psi(s)\,,$$

so that ψ satisfies

$$\psi'' + \frac{1}{s}\psi' = \frac{1}{8s}\sin(2\psi) - \frac{\alpha^2}{4s^2}\frac{\cos\psi}{\sin^3\psi}\,. \tag{1.18}$$

The Fredholm determinant is expressible in terms of q by the formula

$$D(J;\lambda) = \exp\left(-\frac{1}{4}\int_0^s \log\frac{s}{t}\, q(t)^2\, dt\right). \tag{1.19}$$

Denoting by $R(s)$ minus the logarithmic derivative of $D(J;\lambda)$ with respect to s, we have also the representation

$$R(s) = \frac{1}{4}\cos^2\psi(s) + s\left(\frac{d\psi}{ds}\right)^2 - \frac{\alpha^2}{4s}\cot^2\psi(s)\,. \tag{1.20}$$

Furthermore, $R(s)$ itself satisfies a differential equation which in the Jimbo–Miwa–Okamoto σ notation for Painlevé III (see, in particular, (3.13) in [14]) is

$$(s\sigma'')^2 + \sigma'(\sigma - s\sigma')(4\sigma' - 1) - \alpha^2(\sigma')^2 = 0\,, \tag{1.21}$$

where $\sigma(s) = sR(s)$; it has small s expansion

$$\begin{aligned}\sigma(s;\lambda) = {} & c_\alpha s^{1+\alpha}\left[1 - \frac{1}{2(2+\alpha)}s + \frac{3+2\alpha}{16(1+\alpha)(2+\alpha)(3+\alpha)}s^2 + \cdots\right] \\ & + \frac{1}{1+\alpha}c_\alpha^2 s^{2+2\alpha}\left[1 - \frac{3+2\alpha}{2(2+\alpha)^2}s + \cdots\right] \\ & + \frac{1}{(1+\alpha)^2}c_\alpha^3 s^{3+3\alpha}[1 + \cdots] + \cdots,\end{aligned} \tag{1.22}$$

where

$$c_\alpha = \frac{\lambda}{2^{2\alpha+2}}\frac{1}{\Gamma(1+\alpha)\Gamma(2+\alpha)}\,.$$

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[1] The Painlevé V differential equation is

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\left(\alpha' y + \frac{\beta'}{y}\right) + \frac{\gamma' y}{x} + \frac{\delta' y(y+1)}{y-1}\,,$$

where α', β', γ', and δ' are constants.

We mention that in the special case $\alpha = 0$ and $\lambda = 1$ we have $D(J;1) = e^{-s/4}$, $q(s;1) = 1$, $\psi(s,1) = 0$, and $\sigma(s,1) = s/4$ exactly [7,8,11].

D. Asymptotics. Again we take $J = (0,s)$ and consider asymptotics as $s \to \infty$. From the random matrix point of view the interesting quantities are

$$E(n;s) := \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda^n} D(J;\lambda)|_{\lambda=1}\,. \tag{1.23}$$

This is the probability that exactly n eigenvalues lie in J. The asymptotics of $E(0,s) = D(J;1)$ are obtained from (1.19) using the asymptotics of $q(s;1)$ or equivalently $\sigma(s;1)$ obtained from (1.16) or (1.21), respectively. (Our derivation is heuristic since as far as we are aware the corresponding Painlevé connection problem has not been rigorously solved.) We find that as $s \to \infty$,

$$\begin{aligned} E(0;s) = \tau_\alpha \frac{e^{-s/4+\alpha\sqrt{s}}}{s^{\alpha^2/4}} \\ \times\left(1 + \frac{\alpha}{8}s^{-\frac{1}{2}} + \frac{9\alpha^2}{128}s^{-1} + \left(\frac{3\alpha}{128} + \frac{51\alpha^3}{1024}\right)s^{-\frac{3}{2}}\right. \\ \left. + \left(\frac{75\alpha^2}{1024} + \frac{1275\alpha^4}{32768}\right)s^{-2} + \cdots\right), \end{aligned} \tag{1.24}$$

where τ_α is a constant which cannot be determined from the asymptotics of q (or σ) alone. However, as we mentioned above, when $\alpha = \mp\frac{1}{2}$ this expansion must agree with those obtained from formulas (12.2.6) of [18] (see also (12.6.17)–(12.6.19) in [18]) after replacing s by $\pi^2 t^2$. This leads to the conjecture

$$\tau_\alpha = \frac{G(1+\alpha)}{(2\pi)^{\alpha/2}}\,,$$

where G is the Barnes G-function [1]. This conjecture is further supported by numerical work similar to that described for the analogous conjecture in [28].

As in [28], there are two approaches to the asymptotics of $E(n;s)$ for general n. We use the notation

$$r(n;s) := \frac{E(n;s)}{E(0;s)}\,. \tag{1.25}$$

In the first approach (see also [2,27]) one differentiates (1.21) successively with respect to λ. Using the known asymptotics of $\sigma(s;1)$ and the differential equation (1.21) satisfied by $\sigma(s;\lambda)$ for all λ, one can find asymptotic expansions for the quantities

$$\sigma_n(s) := \left.\frac{\partial^n\sigma}{\partial\lambda^n}\right|_{\lambda=1}\,,$$

and these in turn can be used to find expansions for the $r(n;s)$. This approach is inherently incomplete since yet another undetermined constant enters the picture. And there are also computational problems since when one expresses the $r(n,s)$ in terms of the $\sigma_n(s)$ a large amount of cancellation occurs, with the result that even the first-order asymptotics of $r(n;s)$ are out of reach by this method when n is large.

The second approach uses the easily-established identity

$$r(n;s) = \sum_{i_1 < \cdots < i_n} \frac{\lambda_{i_1} \cdots \lambda_{i_n}}{(1-\lambda_{i_1}) \cdots (1-\lambda_{i_n})}, \tag{1.26}$$

where $\lambda_0 > \lambda_1 > \cdots$ are the eigenvalues of the integral operator K with $\lambda = 1$ acting on $(0,s)$. It turns out that this operator, rescaled so that it acts on $(0,1)$, commutes with the differential operator $\mathscr{L}$ defined by

$$\mathscr{L}f(x) = (x(1-x)f'(x))' - \left(\frac{\alpha^2}{4x} + \frac{sx}{4}\right) f(x),$$

with appropriate boundary conditions on f. Applying the WKB method to the equation, and a simple relationship between the eigenvalues of K (as functions of s) and its eigenfunctions, we are able to derive the following asymptotic formula for the eigenvalues as $s \to \infty$:

$$1 - \lambda_i \sim \frac{2\pi}{\Gamma(\alpha+i+1)i!} s^{i+\frac{\alpha+1}{2}} e^{-2\sqrt{s}} 2^{4i+2\alpha+2}. \tag{1.27}$$

From this and (1.26) we deduce

$$r(n;s) \sim \left\{ \prod_{k=0}^{n-1} \Gamma(\alpha+k+1)k! \right\} \pi^{-n} 2^{-n(2n+2\alpha+1)} s^{-\frac{n^2}{2}-\frac{\alpha}{2}n} e^{2n\sqrt{s}}. \tag{1.28}$$

For the special case $\alpha = 0$, the quantity $r(1;s)$ can be expressed exactly in terms of Bessel functions (see (2.30) below).

II. Differential Equations

A. Derivation of the System of Equations. We shall use two representations for our kernel. The first is just our definition (1.2a) using the notation (1.6),

$$K(x,y) = \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x-y}. \tag{2.1}$$

The second is the integral representation

$$K(x,y) = \frac{1}{4}\int_0^1 \phi(xt)\phi(yt)dt. \tag{2.2}$$

This follows from the differentiation formula

$$zJ_\alpha'(z) = \alpha J_\alpha(z) - zJ_{\alpha+1}(z),$$

which gives the alternative representation

$$\lambda \frac{\sqrt{x}J_{\alpha+1}(\sqrt{x})J_\alpha(\sqrt{y}) - J_\alpha(\sqrt{x})\sqrt{y}J_{\alpha+1}(\sqrt{y})}{2(x-y)}$$

for $K(x,y)$, and the Christoffel–Darboux type formula (7.14.1(9)) of [9].

Our derivation will use, several times, the commutator identity

$$[L,(I-K)^{-1}] = (I-K)^{-1}[L,K](I-K)^{-1}, \tag{2.3}$$

which holds for arbitrary operators K and L, and the differentiation formula

$$\frac{d}{da}(I-K)^{-1} = (I-K)^{-1}\frac{dK}{da}(I-K)^{-1}, \tag{2.4}$$

which holds for an arbitrary operator depending smoothly on a parameter a. We shall also use the notations

$$M = \text{multiplication by the independent variable}, \quad D = \text{differentiation},$$

and a subscript on an operator indicates the variable on which it acts.

It will be convenient to think of our operator K as acting, not on J, but on $(0, \infty)$ and to have kernel

$$K(x,y)\chi_J(y),$$

where χ_J is the characteristic function of J. We continue to denote the resolvent kernel of K by $R(x,y)$ and note that it is smooth in x but discontinuous at $y = a_j$. The quantities $R(a_j, a_j)$ appearing in (1.5) are interpreted to mean

$$\lim_{\substack{y \to a_j \\ y \in J}} R(a_j, y),$$

and similarly for p_j and q_j in formulas (1.7). The definitions (1.8) of u and v must be modified to read

$$u = (\phi\chi_J, (I-K)^{-1}\phi), \quad v = (\phi\chi_J, (I-K)^{-1}\psi), \tag{2.5}$$

where now the inner products are taken over $(0, \infty)$. Notice that since

$$(I-K)^{-1}\xi = (I-K)^{-1}\xi\chi_J \quad \text{in } J$$

for any function ξ, this agrees with the original definitions (1.8) of u and v.

We have, by (2.2),

$$((MD)_x + (MD)_y)K(x,y) = \frac{1}{4}\int_0^1 t\frac{\partial}{\partial t}(\phi(xt)\phi(yt))\,dt = \frac{1}{4}\phi(x)\phi(y) - K(x,y).$$

But it is easy to see that

$$[MD, L] \doteq ((MD)_x + (MD)_y + I)L(x,y) \tag{2.6}$$

for any operator L with kernel $L(x,y)$, where "$\doteq$" means "has kernel." Taking $L(x,y) = K(x,y)\chi_J(y)$ gives

$$[MD, K] \doteq \frac{1}{4}\phi(x)\phi(y)\chi_J(y) - \sum(-1)^k a_k K(x,a_k)\delta(y-a_k).$$

(Recall the form (1.4) of J.) It follows from this and (2.3) that

$$[MD, (I-K)^{-1}] \doteq \frac{1}{4}Q(x)(I-K^t)^{-1}\chi_J\phi(y) - \sum(-1)^k a_k R(x,a_k)\rho(a_k,y), \tag{2.7}$$

where $Q(x)$, and an analogous function $P(x)$, are defined by

$$Q(x) := (I-K)^{-1}\phi, \quad P(x) := (I-K)^{-1}\psi, \tag{2.8}$$

where $\rho(x,y) = R(x,y) + \delta(x-y)$ is the distributional kernel of $(I-K)^{-1}$, and where K^t is the transpose of the operator K. (Note that K takes smooth functions to smooth functions while its transpose takes distributions to distributions.) Observe that

$$q_j = Q(a_j), \quad p_j = P(a_j) .$$

Next we consider commutators with M and use the first representation (2.1) of $K(x,y)$. We have immediately

$$[M,K] = (\phi(x)\psi(y) - \psi(x)\phi(y))\chi_J(y) ,$$

and so, by (2.3) again,

$$[M,(I-K)^{-1}] \doteq Q(x)(I-K^t)^{-1}\psi\chi_J(y) - P(x)(I-K^t)^{-1}\phi\chi_J(y) . \tag{2.9}$$

Notice that since

$$(I-K^t)^{-1}\psi\chi_J = (I-K)^{-1}\psi = P \quad \text{on } J ,$$

and similarly for ϕ, Q, we deduce

$$R(x,y) = \frac{Q(x)P(y) - P(x)Q(y)}{x-y} \quad (x,y \in J) .$$

In particular we have

$$R(a_j,a_k) = \frac{q_j p_k - p_j q_k}{a_j - a_k} \quad (j \neq k) , \tag{2.10}$$

$$R(x,x) = Q'(x)P(x) - P'(x)Q(x) \quad (x \in J) . \tag{2.11}$$

In order to compute $R(a_j,a_j)$, and also the derivatives in (1.11) and (1.12), we must find $Q'(x)$ and $P'(x)$. We begin with the obvious

$$xQ'(x) = MD(I-K)^{-1}\phi(x) = (I-K)^{-1}MD\phi(x) + [MD,(I-K)^{-1}]\phi(x) .$$

Using (2.7), and recalling (1.6) and (2.5), we find that

$$xQ'(x) = P(x) + \frac{1}{4}Q(x)u - \sum(-1)^k a_k R(x,a_k)q_k . \tag{2.12}$$

Similarly, replacing ϕ by ψ in this derivation gives

$$xP'(x) = (I-K)^{-1}MD\psi(x) + \frac{1}{4}Q(x)v - \sum(-1)^k a_k R(x,a_k)p_k . \tag{2.13}$$

To evaluate the first term on the right side we use the fact that ϕ satisfies the differential equation

$$x^2\phi''(x) + x\phi'(x) + \frac{1}{4}(x-\alpha^2)\phi(x) = 0 , \tag{2.14}$$

which may be rewritten $MD\psi(x) = \frac{1}{4}(\alpha^2 - x)\phi$. Hence

$$\begin{aligned}(I-K)^{-1}MD\psi(x) &= \frac{\alpha^2}{4}Q(x) - \frac{1}{4}(I-K)^{-1}M\phi(x) \\ &= \frac{\alpha^2}{4}Q(x) - \frac{x}{4}Q(x) + \frac{1}{4}[M,(I-K)^{-1}]\phi(x) .\end{aligned} \tag{2.15}$$

But we find, using (2.9), that

$$[M,(I-K)^{-1}]\phi(x) = Q(x)v - P(x)u\,,$$

and combining this with (2.13) and (2.15) gives

$$xP'(x) = \frac{1}{4}(\alpha^2 - x)Q(x) + \frac{1}{2}Q(x)v - \frac{1}{4}P(x)u - \sum(-1)^k a_k R(x,a_k)p_k\,. \quad (2.16)$$

It follows from (2.11), (2.12) and (2.16) that for $x \in J$,

$$a_j R(a_j,a_j) = p_j^2 - \frac{1}{4}(\alpha^2 - a_j + 2v)q_j^2 + \frac{1}{2}p_j q_j u + \sum_{k \neq j} a_k R(a_j,a_k)(q_j p_k - p_j q_k)\,.$$

In view of (2.10) this is Eq. (1.15).

We now derive the differential equations (1.9)–(1.14). First, we have the easy fact that

$$\frac{\partial}{\partial a_k} K \doteq (-1)^k K(x,a_k)\delta(y-a_k)\,,$$

and so by (2.4),

$$\frac{\partial}{\partial a_k}(I-K)^{-1} \doteq (-1)^k R(x,a_k)\rho(y,a_k)\,. \quad (2.17)$$

At this point we use the notations $Q(x,a)$, $P(x,a)$ for $P(x)$ and $Q(x)$ to remind ourselves that they are functions of a as well as x. We deduce immediately from (2.17) and (2.8) that

$$\frac{\partial}{\partial a_k}Q(x,a) = (-1)^k R(x,a_k)q_k, \quad \frac{\partial}{\partial a_k}P(x,a) = (-1)^k R(x,a_k)p_k\,. \quad (2.18)$$

Since $q_j = Q(a_j,a)$ and $p_j = P(a_j,a)$ this gives

$$\frac{\partial q_j}{\partial a_k} = (-1)^k R(a_j,a_k)q_k, \quad \frac{\partial p_j}{\partial a_k} = (-1)^k R(a_j,a_k)p_k, \quad (j \neq k)\,.$$

In view of (2.10) again, these are Eqs. (1.9) and (1.10). Moreover

$$\frac{\partial q_j}{\partial a_j} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial a_j}\right)Q(x,a)\bigg|_{x=a_j}, \quad \frac{\partial p_j}{\partial a_j} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial a_j}\right)P(x,a_j)\bigg|_{x=a_j}.$$

Equations (1.11) and (1.12) follow from this, (2.18), (2.12), (2.16) and (2.10).

Finally, using the definition of u in (2.5), the fact

$$\frac{\partial}{\partial a_j}\chi_J(y) = (-1)^j\delta(y-a_j)\,,$$

and (2.17) we find that

$$\frac{\partial u}{\partial a_j} = (-1)^j\phi(a_j)q_j + (-1)^j(\phi\chi_J, R(\cdot\,,a_j))q_j\,.$$

But

$$(\phi\chi_J, R(\cdot\,,a_j)) = \int_J R(x,a_j)\phi(x)dx = \int_J R(a_j,x)\phi(x)dx\,,$$

since $R(x, y) = R(y, x)$ for $x, y \in J$. Since $R(y, x) = 0$ for $x \notin J$ the last integral equals

$$\int_0^\infty R(a_j, x)\phi(x)dx = q_j - \phi(a_j) .$$

This gives (1.13), and (1.14) is completely analogous.

We end this section with two relationships (analogues of (2.18) and (2.19) of [28]) which would allow us to express u and v in terms of the q_j and p_j if we wished to do so. (They will also be needed in the next section.) These are

$$2v + \frac{1}{4}u^2 + u = \sum_j (-1)^j a_j q_j^2 , \tag{2.19}$$

$$u = \sum_j (-1)^j (4p_j^2 - (\alpha^2 - a_j + 2v)q_j^2 + 2p_j q_j u) . \tag{2.20}$$

To obtain the first of these observe that (1.9) and (1.11) imply

$$\left(\sum_k a_k \frac{\partial}{\partial a_k}\right) q_j = p_j + \frac{1}{4} q_j u ,$$

while from (1.13) and (1.14),

$$\frac{\partial}{\partial a_j}\left(2v + \frac{1}{4}u^2\right) = 2(-1)^j q_j \left(p_j + \frac{1}{4} q_j u\right) .$$

If we multiply both sides of the previous formula by $(-1)^j a_j q_j$ and sum over j what we obtain may be written

$$\left(\sum_k a_k \frac{\partial}{\partial a_k}\right)\left(\sum_j (-1)^j a_j q_j^2\right) - \sum_k (-1)^k a_k q_k^2 = \left(\sum_k a_k \frac{\partial}{\partial a_k}\right)\left(2v + \frac{1}{4}u^2\right),$$

or equivalently

$$\left(\sum_k a_k \frac{\partial}{\partial a_k}\right)\left(\sum_j (-1)^j a_j q_j^2\right) = \left(\sum_k a_k \frac{\partial}{\partial a_k}\right)\left(2v + \frac{1}{4}u^2 + u\right).$$

It follows that the two sides of (2.19) differ by a function of $(a_1, \ldots, a_{2m})$ which is invariant under scalar multiplication. Since, as is easily seen, both sides vanish when all $a_j = 0$ their difference must vanish identically.

To deduce (2.20) we multiply (2.17) by a_k and sum over k and then add the result to (2.7), recalling (2.6), to obtain

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + I + \sum_k a_k \frac{\partial}{\partial a_k}\right) R(x, y) = \frac{1}{4} Q(x) Q(y)$$

for $x, y \in J$. This gives

$$\left(\sum a_k \frac{\partial}{\partial a_k}\right) a_j R(a_j, a_j) = \frac{1}{4} a_j q_j^2 = \frac{1}{4} a_j (-1)^j \frac{\partial u}{\partial a_j} .$$

If we multiply both sides of this by $(-1)^j$ and sum over j we deduce, by an argument similar to one just used, that

$$\sum_j (-1)^j a_j R(a_j, a_j) = \frac{1}{4} u . \tag{2.21}$$

Substituting for $a_j R(a_j, a_j)$ here the right side of (1.15) we see that the resulting double sum vanishes, and (2.20) results.

B. The Ordinary Differential Equation. In this section we specialize to the case $J = (0, s)$ and derive (among other things) the differential equation (1.16) and the representation (1.19). In the notation of the last section $m = 1, a_1 = 0, a_2 = s$. We shall write $q(s)$, $p(s)$, $R(s)$ for q_2, p_2, $R(s, s)$, respectively. Equations (1.11)–(1.14) become

$$sq' = p + \frac{1}{4}qu\,, \tag{2.22}$$

$$sp' = \frac{1}{4}(\alpha^2 - s)q + \frac{1}{2}qv - \frac{1}{4}pu\,, \tag{2.23}$$

$$u' = q^2\,, \tag{2.24}$$

$$v' = pq\,. \tag{2.25}$$

It is immediate from (2.21) and (2.24) that

$$(sR(s))' = \frac{1}{4}q(s)^2\,, \tag{2.26}$$

and since

$$\frac{d}{ds}\log D(J;\lambda) = -R(s)$$

(see (1.5)), we obtain the representation (1.19).

To obtain the differential equation (1.16) we apply $s\frac{d}{ds}$ to both sides of (2.22) and use (2.22), (2.23) and (2.24). What results is

$$s(sq')' = \frac{1}{4}(\alpha^2 - s)q + \frac{1}{16}(u^2 + 8v)q + \frac{1}{4}sq^3\,. \tag{2.27}$$

But (2.19) in this case is

$$u^2 + 8v = 4sq^2 - 4u\,,$$

and so the above can be written

$$s(sq')' = \frac{1}{4}(\alpha^2 - s)q - \frac{1}{4}uq + \frac{1}{2}sq^3\,. \tag{2.28}$$

Next, we square both sides of (2.22) and use (2.20), which now says

$$u = 4p^2 - (\alpha^2 - s + 2v)q^2 + 2pqu\,,$$

and find that

$$(sq')^2 = \frac{1}{4}u + \frac{1}{4}(\alpha^2 - s)q^2 + \frac{1}{16}q^2(u^2 + 8v)\,.$$

Combining this with (2.27) gives

$$sq(sq')' = (sq')^2 - \frac{1}{4}u + \frac{1}{4}sq^4\,,$$

and combining *this* with (2.28) gives the desired Eq. (1.16). The boundary condition (1.17) follows from the Neumann expansion of the defining expression (1.7) for q.

Using (1.18) one easily verifies that

$$R(s) = \frac{1}{4}\cos^2\psi + s\left(\frac{d\psi}{ds}\right)^2 - \frac{\alpha^2}{4s}\csc^2\psi + \frac{c}{s}$$

$$= \frac{1}{4}\cos^2\psi + s\left(\frac{d\psi}{ds}\right)^2 - \frac{\alpha^2}{4s}\cot^2\psi + \left(c - \frac{\alpha^2}{4}\right)\frac{1}{s}$$

satisfies (2.26), where c is a constant of integration. That this constant is equal to $\alpha^2/4$ follows from the small s expansion of $R(s)$. (Use the fact that for $s \to 0$, $R(s) \sim K(s, s)$ and that, as follows from (1.2b), there is no simple pole in s.) Equation (1.21) follows from (1.20) and (1.18).

Here is the argument why (1.16) must be reducible to some Painlevé equation (or one of the other simpler differential equations on Painlevé's list). The derivation of (1.16) used only the facts that the Bessel kernel had both forms (2.1) and (2.2) and that the function ϕ satisfied the differential equation (2.14). (Of course ψ in (2.1) must be defined as $MD\phi$.) This equation has a 2-complex-parameter family solution and this gives a 2-complex-parameter family of kernels defined by (2.1). They can be shown to satisfy (2.2). We replace the kernels $K(x, y)$ by $sK(sx, sy)$ and have them act on $(0, 1)$ rather than $(0, s)$. These operators on $(0, 1)$ depend analytically on the *complex* variable s (except for a branch point at $s = 0$) and the corresponding $q(s)$ can have, aside from a branch point at $s = 0$, only poles which occur at the values of s for which $\lambda = 1$ is an eigenvalue of the operator. (The resolvent of an analytic family of compact operators has a pole whenever $\lambda = 1$ is an eigenvalue.) Thus the general solution (i.e., 2-complex-parameter family of solutions) of (1.16) has only poles as moveable singularities. Since the equation is of the form $q'' =$ rational function of q' and q, it must be reducible to one of the Painlevé types.

We mention that this argument requires $|\alpha| < 1$ since it is only then that all solutions of (2.14) give compact, or even bounded, operators on L_2. For other α it may be that we just have to replace L_2 by an appropriate space of distributions.

C. $r(1; s)$ for $\alpha = 0$. If we set $\alpha = 0$ and make the change of variables $s = x^2$, the differential equation for ψ (recall (1.18)) becomes

$$\psi'' + \frac{1}{x}\psi' = \frac{1}{2}\sin(2\psi)\,, \tag{2.29}$$

and we want the solutions holomorphic at the origin. The linearization of this differential equation is the modified Bessel equation and all solutions of the linear equation are linear combinations of $I_0(x)$ and $K_0(x)$. Flaschka and Newell [10] have shown, using methods of monodromy preserving deformations and singular integral equations, that the general 2-parameter solution to (2.29) can be viewed as a "perturbation" of this linear solution. (Precisely, they derive a singular integral equation whose Neumann expansion in a particular limit gives $\psi(x)$ – the first term in this expansion is a linear combination of Bessel functions.) The one-parameter

family of solutions to (2.29) that are holomorphic at the origin has the representation [10]

$$\psi(x;\mu) = \mu\psi_1(x) + \frac{\mu^3}{3!}\psi_3(x) + \frac{\mu^5}{5!}\psi_5(x) + \cdots,$$

where $\psi_1(x) = I_0(x)$, and $\mu^2 = 1 - \lambda$. Note that we are using the slightly confusing notation $\psi(x;\mu)$ to denote the function ψ of (1.18) after the change of variables $s = x^2$ and $\mu^2 = 1 - \lambda$. Multiple integral representations (obtained from a Neumann expansion) for the higher ψ_j's can be easily derived from [10].

The resolvent kernel $R(s)$, $s = x^2$, is given in this special case by

$$\begin{aligned} 4R(s) &= \cos^2\psi(x) + \left(\frac{d\psi}{dx}\right)^2 \\ &= 1 + (1-\lambda)((\psi_1')^2 - \psi_1^2) + (1-\lambda)^2(\psi_1^4 - \psi_1\psi_3 + \psi_1'\psi_3') + \cdots. \end{aligned}$$

Thus (recall (1.23) and (1.25))

$$r(1;s) = -\int_0^s R_1(t)dt = -2\int_0^{\sqrt{s}} xR_1(x^2)dx\,,$$

where

$$R_1(t) = \frac{\partial}{\partial\lambda}R(t)\bigg|_{\lambda=1}.$$

Therefore

$$\begin{aligned} r(1;s) &= \frac{1}{2}\int_0^{\sqrt{s}} x(\psi_1^2(x) - (\psi_1'(x))^2)dx \\ &= \frac{1}{2}\int_0^{\sqrt{s}} x(I_0(x)^2 - I_1^2(x))dx \\ &= \frac{s}{2}\left(I_0^2(\sqrt{s}) - \frac{1}{\sqrt{s}}I_0(\sqrt{s})I_1(\sqrt{s}) - I_1^2(\sqrt{s})\right). \end{aligned} \tag{2.30}$$

The last equality follows from (5.542) of [12].

We point out the curious fact that (after letting $\psi \to i\psi$) the same differential equation (2.29) and closely related τ-function arise in the 2D Ising model [30, 21, 26] *except* that here the boundary condition is $\psi(x) \sim \mu K_0(x)$ as $x \to \infty$.

III. Asymptotics

A. Asymptotics of the σ-Equation. In the case of the finite N ensemble and $\alpha = 0$, Edelman [7, 8] and Forrester [11] (by a direct evaluation of the integrals defining the probability $E_N(0;s)$) have shown that $E(0;s)$ is exactly equal to $e^{-s/4}$. From this it follows, for $\alpha = 0$ and $\lambda = 1$, that $\sigma(s;1) = s/4$. For general α and $\lambda = 1$ it is therefore reasonable to assume an asymptotic expansion of the form:

$$\sigma(s;1) = c_1 s + c_2 s^{1/2} + c_3 + c_4 s^{-1/2} + \cdots, \quad s \to \infty\,.$$

Substituting this into the differential equation (1.21) results in equations that uniquely determine the coefficients c_j *once* a choice in the square root $\sqrt{\alpha^2}$ is made.

Since for $\alpha = \mp\frac{1}{2}$ our asymptotic expansion of $E(0;s)$ must agree with those of Dyson (recall 1.3)), we see that we must choose the square root $-\alpha$. A calculation then gives

$$\begin{aligned}\sigma(s;1) = \frac{s}{4} - \frac{\alpha}{2}s^{1/2} + \frac{\alpha^2}{4} + \frac{\alpha}{16}s^{-1/2} + \frac{\alpha^2}{16}s^{-1} \\ + \frac{\alpha}{256}(16\alpha^2 + 9)s^{-3/2} + \frac{\alpha^2}{64}(4\alpha^2 + 9)s^{-2} \\ + \frac{\alpha}{2048}(128\alpha^4 + 720\alpha^2 + 225)s^{-5/2} + \cdots s \to \infty\ ,\end{aligned} \tag{3.1}$$

from which (1.24) follows.

B. Asymptotics Via the Commuting Differential Operator. Throughout this section we take $\lambda = 1$. The operator K, when rescaled to act on $(0,1)$ instead of $(0,s)$, has kernel $sK(sx, sy)$. By (2.2) this is equal to

$$\frac{s}{4}\int_0^1 \phi(sxt)\phi(syt)dt\ ,$$

and so K (rescaled, as it will be throughout this section), is the square of the operator on $(0,1)$ with kernel

$$J(x,y) = \frac{\sqrt{s}}{2}\phi(sxy) = \frac{\sqrt{s}}{2}J_\alpha(\sqrt{sxy})\ .$$

This function satisfies the differential equation

$$x^2 J_{xx} + xJ_x + \left(\frac{sxy}{4} - \frac{\alpha^2}{4}\right)J = 0\ . \tag{3.2}$$

The operator J will commute with a differential operator

$$\mathscr{L} = \frac{d}{dx}\alpha(x)\frac{d}{dx} + \beta(x)$$

if $\alpha(0) = \alpha(1) = 0$ and if

$$\alpha(y)J_{yy} + \alpha'(y)J_y + \beta(y)J = \alpha(x)J_{xx} + \alpha'(x)J_x + \beta(x)J\ .$$

If we use (3.2) we see that this will be satisfied if

$$\begin{aligned}-\alpha(x)\left[x^{-1}J_x + \left(\frac{sy}{4x} - \frac{\alpha^2}{4x^2}\right)\right] + \alpha'(x)J_x + \beta(x)J \\ = \text{the same expression with } x \text{ and } y \text{ interchanged.}\end{aligned}$$

Equating the terms involving the first derivatives of J gives

$$(\alpha'(x) - x^{-1}\alpha(x))J_x = (\alpha'(y) - y^{-1}\alpha(y))J_y\ .$$

But $xJ_x = yJ_y$, so the above will hold if

$$x^{-1}(\alpha'(x) - x^{-1}\alpha(x)) = y^{-1}(\alpha'(y) - y^{-1}\alpha(y))\ .$$

This is satisfied if $\alpha(x)$ is a quadratic without constant term and, of course, we choose

$$\alpha(x) = x(1-x)\,.$$

What is required of β, then, is seen to be

$$\beta(x) - x(1-x)\left(\frac{sy}{4x} - \frac{\alpha^2}{4x^2}\right) = \beta(y) - y(1-y)\left(\frac{sx}{4y} - \frac{\alpha^2}{4y^2}\right),$$

which is satisfied by

$$\beta(x) = -\frac{\alpha^2}{4x} - \frac{sx}{4}\,.$$

We write the differential equation, for which the eigenfunctions $f(x)$ are the eigenfunctions of J, as

$$(x(1-x)f'(x))' + \left(\mu\sqrt{s} - \frac{\alpha^2}{4x} - \frac{sx}{4}\right)f(x) = 0\,. \tag{3.3}$$

The boundary conditions are that $f(x)$ be bounded as $x \to 1$ and that $f(x)$ be asymptotic to a constant times $x^{\alpha/2}$ as $x \to 0$. The reason we wrote the eigenvalues as we did is that for each i the μ corresponding to the ith largest eigenvalue is bounded as $s \to \infty$. This is easily seen by an oscillation argument. So we assume i is fixed and proceed to find the asymptotics of the corresponding eigenfunction $f(x)$ as $s \to \infty$. We assume it normalized so that

$$f(x) \sim x^{\alpha/2} \text{ as } x \to 0\,. \tag{3.4}$$

1. The Region $x \ll 1$. The approximating equation is

$$f''(x) + \frac{1}{x}f'(x) + \left(\frac{\mu\sqrt{s}}{x} - \frac{\alpha^2}{4x^2} - \frac{s}{4}\right)f(x) = 0\,.$$

The solution of this equation which satisfies (3.4) is

$$x^{\alpha/2}e^{-\sqrt{s}x/2}\,\Phi\left(\frac{1+\alpha}{2} - \mu, 1+\alpha, \sqrt{s}x\right),$$

where Φ is the confluent hypergeometric function [9]. We deduce that when $x \ll 1$,

$$f(x) \sim x^{\alpha/2}e^{-\sqrt{s}x/2}\,\Phi\left(\frac{1+\alpha}{2} - \mu, 1+\alpha, \sqrt{s}x\right). \tag{3.5}$$

2. The Region $x \ll 1$, $\sqrt{s}x \gg 1$.

Case 1. $\mu \neq i + \dfrac{1+\alpha}{2}$ $(i = 0, 1, \ldots)$. Then from the known asymptotics of Φ as its argument tends to ∞ (6.13(3) of [9]) we deduce that

$$f(x) \sim \frac{\Gamma(1+\alpha)}{\Gamma\left(\dfrac{1+\alpha}{2} - \mu\right)}\, s^{-\frac{1}{2}\left(\frac{1+\alpha}{2}+\mu\right)} x^{-\frac{1}{2}-\mu} e^{\sqrt{s}x/2}\,. \tag{3.6}$$

Case 2. $\mu = i + \dfrac{1+\alpha}{2}$. Then

$$\Phi(-i, 1+\alpha, \sqrt{s}x) = \frac{\Gamma(\alpha+1)i!}{\Gamma(\alpha+i+1)} L_i^\alpha(\sqrt{s}x)\,, \tag{3.7}$$

where L_i^α is the generalized Laguerre polynomial (6.9(36) of [9]). So we find that in this case

$$f(x) \sim (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+i+1)} s^{\frac{i}{2}} x^{i+\frac{\alpha}{2}} e^{-\sqrt{s}x/2}\,. \tag{3.8}$$

3. The Region $\sqrt{s}x \gg 1, (1-x)s \gg 1$. Here we use the standard WKB approximation for the solutions of a differential equation $(pf')' + qf = 0$ given by $e^{\pm y}/(pq)^{\frac{1}{4}}$, where $y = \int\sqrt{q/p}$. In the case of our Eq. (3.3), the range of validity of the approximation is as indicated in the heading of this section. To be definite, we take

$$y = -\int_x^1 \sqrt{q(z)/p(z)}\,dz\,.$$

It is easy to compute that for the range in question we have

$$y = -\sqrt{s}\sqrt{1-x} - \mu\log x + 2\mu\log(1+\sqrt{1-x}) + o(1)\,.$$

Also, in this case pq is asymptotically $-s/4$ times $x^2(1-x)$. Hence

$$\begin{aligned} f(x) \sim\ & (a(s)x^{-\mu}(1+\sqrt{1-x})^{2\mu} e^{-\sqrt{s}\sqrt{1-x}} \\ & + b(s)x^{\mu}(1+\sqrt{1-x})^{-2\mu} e^{\sqrt{s}\sqrt{1-x}})\,, \end{aligned} \tag{3.9}$$

where $a(s)$ and $b(s)$ are constants depending on s.

4. The Region $x \to 1$. Letting $y = 1-x$, $f(x) = g(y)$ gives the approximating equation

$$(yg(y)')' - \frac{s}{4}g(y) = 0\,.$$

The general solution of this is a constant times $I_0(\sqrt{y})$, where I_0 is the modified Bessel function. Thus we deduce that as $x \to 1$,

$$f(x) \sim c(s)\, I_0(\sqrt{s(1-x)}) \tag{3.10}$$

for some $c(s)$.

5. Determination of $a(s)$, $b(s)$ *and* $c(s)$. From the asymptotics of I_0 at infinity and (3.10) we deduce that

$$f(x) \sim \frac{c(s)}{\sqrt{2\pi}\, s^{\frac{1}{4}}} \frac{e^{\sqrt{s}\sqrt{1-x}}}{(1-x)^{\frac{1}{4}}}$$

when $x \to 1$ and $s(1-x) \to \infty$. Comparing this with (3.9) shows that $a(s) = 0$ and that

$$c(s) \sim \sqrt{2\pi} s^{\frac{1}{4}} b(s)\,. \tag{3.11}$$

And now comparing (3.9) with (3.6) and (3.8) in their overlapping range of validity we see that we must be in Case 2 and that

$$b(s) \sim (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+i+1)} 2^{2n+\alpha+1} e^{-\sqrt{s}} . \tag{3.12}$$

What we shall need from all this is, first, the asymptotics of $f(1)$. This follows immediately from (3.10), (3.11), (3.12), and the fact that $I_0(0) = 1$:

$$f(1) \sim (-1)^n \sqrt{2\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+i+1)} s^{\frac{i}{2}+\frac{1}{4}} e^{-\sqrt{s}} 2^{2n+\alpha+1} . \tag{3.13}$$

We shall also need the asymptotics of $\int f(x)^2 dx$. It follows from the asymptotics we have derived that the main contribution to this integral comes from an arbitrarily small neighborhood of $x = 0$. It follows from (3.5), (3.7) and the fact

$$\int_0^\infty x^\alpha e^{-x} L_i^\alpha(x)^2 dx = \frac{\Gamma(\alpha+i+1)}{i!}$$

that

$$\int_0^1 f(x)^2 dx \sim \frac{\Gamma(\alpha+1)^2 i!}{\Gamma(\alpha+i+1)} s^{-\frac{\alpha}{2}-\frac{1}{2}} .$$

We put these two relations together to get what we *really* want, which is

$$\frac{f(1)^2}{\int_0^1 f(x)^2 dx} \sim \frac{2\pi}{\Gamma(\alpha+i+1) i!} s^{i+\frac{\alpha}{2}+1} e^{-2\sqrt{s}} 2^{4i+2\alpha+2} . \tag{3.14}$$

6. *The asymptotics of* λ_i. Since $\mathscr{L}$ commutes with our integral operator K, rescaled to act on $(0,1)$, the set $\{f_i\}$ of eigenfunctions corresponding to the eigenvalues $\mu_0 < \mu_1 < \ldots$ of $\mathscr{L}$ is the set of eigenfunctions of K corresponding to its eigenvalues, in some order.

Lemma 1. *The eigenvalues of K are simple.*

Proof. Since the eigenvalues of K are the squares of the eigenvalues of J what we have to show is that if $f_1(x)$ and $f_2(x)$ are eigenfunctions of $\mathscr{L}$ (they need not correspond to any particular μ here) and if for some ν we have either

$$\int_0^1 J_\alpha(\sqrt{sxy}) f_i(y) dy = \nu f_i(x) \quad (i = 1,2)$$

or

$$\int_0^1 J_\alpha(\sqrt{sxy}) f_1(y) dy = \nu f_1(x), \quad \int_0^1 J_\alpha(\sqrt{sxy}) f_2(y) dy = -\nu f_2(x) ,$$

then f_1 and f_2 are linearly dependent. We shall assume them normalized so that they both have the value 1 at $x = 1$.

Notice first that $\nu \neq 0$ for otherwise if we expand the f_1 equation near $x = 0$ we would find that

$$\int_0^1 x^{\frac{\alpha}{2}+k} f_1(x) dx = 0, \quad k = 0, 1, \ldots,$$

and f_1 would be identically 0.

Next we define

$$H(x) = e^{-x/2} J_\alpha(\sqrt{se^{-x}}), \quad g_i(x) = e^{-x/2} f_i(e^{-x}) ,$$

make the obvious variable changes, and find that our relations become

$$\int_0^\infty H(x+y) g_i(y) dy = v g_i(x), \quad (i = 1, 2) , \tag{3.15}$$

$$\int_0^\infty H(x+y) g_1(y) dy = v g_1(x), \quad \int_0^\infty H(x+y) g_2(y) dy = -v g_2(x) . \tag{3.16}$$

What comes now is almost identical to the proof of Lemma 1 of [28]. Assuming first that (3.15) holds, we differentiate twice this relation with $i = 1$ and then integrate by parts twice to obtain

$$v g_1(x) = -H'(x) + H(x) g_1'(0) + \int_0^\infty H(x+y) g_1(y) dy .$$

If we multiply both sides by $g_2(x)$ and integrate, using (3.15) and its differentiated version, we obtain (recall that $g_i(0) = 1$)

$$v \int_0^\infty g_1''(x) g_2(x) dx = -v g_2'(0) + v g_1'(0) + v \int_0^\infty g_2(y) g_1''(y) dy .$$

Thus, since $v \neq 0$, we have $g_1'(0) = g_2'(0)$. Equivalently, $f_1'(1) = f_2'(1)$. But since also $f_1(1) = f_2(1)$, it follows from Eq. (2) that the corresponding eigenvalues μ must be the same, and so the eigenfunctions are the same.

Next, assume (3.16) holds. Differentiating both sides of the first relation once and integrating by parts give

$$v g_1'(x) = -H(x) - \int_0^\infty H(x+y) g_1'(y) dy .$$

Multiplying both sides of this by $g_2(x)$ and integrating, using the second part of (3.16), we obtain

$$v \int_0^\infty g_1'(x) g_2(x) dx = -v + v \int_0^\infty g_2(y) g_1'(y) dy ,$$

contradicting $v \neq 0$. ■

Now that we know the eigenvalues λ_i of K are simple; we can order them so that $\lambda_0 > \lambda_1 > \ldots$. There is a permutation σ of $\mathbf{N}$ such that the eigenvalue corresponding to f_i is $\lambda_{\sigma(i)}$. This permutation is independent of s since the eigenvalues μ_i and λ_i both vary continuously with s. The next lemma allows us to compute $d\lambda_i/ds$ for each i.

Lemma 2. *Let λ be a simple eigenvalue of an integral operator on $(0,1)$ with symmetric kernel of the form $s K(sx, sy)$, where $K(x,y)$ is smooth and symmetric. Let f be a corresponding eigenfunction. Then*

$$\frac{d\lambda}{ds} = \frac{\lambda}{s} \frac{f(1)^2}{\int_0^1 f(x)^2 dx} .$$

Proof. We may assume f normalized so that $\int_0^1 f(x)^2 dx = 1$. Then according to Lemma 3 of [28] we have

$$\frac{d\lambda}{ds} = \int_0^1 \int_0^1 \frac{\partial}{\partial s}(sK(sx, sy)) f(y) f(x) dy\, dx\ .$$

(This holds no matter what the form of the kernel.) In the case at hand the first factor in the integrand equals

$$K(sx, sy) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)K(sx, sy)\ ,$$

and we easily deduce from this that the integral itself equals

$$\frac{\lambda}{s}\int_0^1 f(x)^2 dx + \int_0^1 x f'(x) f(x) dx + \int_0^1 y f'(y) f(y) dy$$

and integration by parts shows that the expression in parentheses equals $f(1)^2$. ■

Remark. It is easily seen that the conclusion of the lemma holds when the kernel has a mild singularity at 0 such as our Bessel kernel has when $\alpha < 0$.

Lemma 3. *For each i we have $\lambda_i \to 1$ as $s \to \infty$.*

Proof. The Hankel transform, when rescaled by the variable change $x \mapsto \sqrt{x}$, is the integral operator H on $(0, \infty)$ with kernel $\frac{1}{2}J_\alpha(\sqrt{xy})$ and so our operator K on $(0, s)$ may be thought of as $P_s H P_s H P_s$, where P_s denotes the projection from $L_2(0, \infty)$ to $L_2(0, s)$. Since, as is well-known, $H^2 = I$, the minimax characterization of the eigenvalues shows that for each i the eigenvalue of $P_s H P_s$ with i^{th} largest absolute value tends to ± 1 as $s \to \infty$. Since our operator is the square of this one, the statement of the lemma follows. ■

We can now deduce the asymptotic formula (15) for the eigenvalues λ_i. We apply Lemma 2 to the eigenvalue $\lambda_{\sigma(i)}$ associated with the eigenfunction $f_i(x)$ of (3.3) and use (3.14) to deduce

$$\frac{d \log \lambda_{\sigma(i)}}{ds} \sim \frac{2\pi}{\Gamma(\alpha + i + 1) i!} s^{i + \frac{\alpha}{2}} e^{-2\sqrt{s}} 2^{4i + 2\alpha + 2}\ .$$

Recalling Lemma 3 we see that we can integrate from s to ∞ and we obtain

$$1 - \lambda_{\sigma(i)} \sim \frac{2\pi}{\Gamma(\alpha + i + 1) i!} s^{i + \frac{\alpha+1}{2}} e^{-2\sqrt{s}} 2^{4i + 2\alpha + 2}\ .$$

It remains to show that $\sigma(i) = i$ for all i. But it is clear from the above formula that $i < j$ implies that $\lambda_{\sigma(i)} > \lambda_{\sigma(j)}$ for large s (and so for all s) and therefore $\sigma(i) < \sigma(j)$. Since σ: $\mathbf{N} \to \mathbf{N}$ is onto, we must have $\sigma(i) = i$ for all i.

Acknowledgments. We wish to thank Professor F.J. Dyson for bringing to our attention the notion of "hard edge" random matrix ensembles. This work was supported in part by the National Science Foundation, DMS-9001794 and DMS-9216203, and this support is gratefully acknowledged.

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Communicated by N. Yu. Reshetikhin