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General Formulation of Griffiths' Inequalities

J. GINIBRE

Laboratoire de Physique Théorique et Hautes Energies, 91-Orsay, France

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Abstract. We present a general framework in which Griffiths inequalities on the correlations of ferromagnetic spin systems appear as natural consequences of general assumptions. We give a method for the construction of a large class of models satisfying the basic assumptions. Special cases include the Ising model with arbitrary spins, and the plane rotator model. The general theory extends in a straightforward way to the non-commutative (quantum) case, but non-commutative examples satisfying all the assumptions are lacking at the moment.

Introduction

Recently, Griffiths [1] obtained remarkable inequalities for the correlation functions of Ising ferromagnets with two-body interactions. These inequalities were subsequently generalized by Kelly and Sherman [2] to systems with interactions involving an arbitrary number of spins, and by Griffiths to systems with arbitrary spins [3]. These inequalities have received several applications of physical interest. They have been used to prove the existence of the infinite volume limit for the correlation functions of Ising ferromagnets [1], to settle the question of the existence of phase transitions in one dimensional systems with moderately long range interactions [4], to obtain upper and lower bounds on critical temperatures [5], and to establish rigorous inequalities on critical point exponents [6]. It is therefore of interest to extend the inequalities to the largest possible class of models.

In this paper we make a first step in this direction by giving a general formulation which seems appropriate for this problem, both for classical and quantum systems. We obtain sufficient conditions for the inequalities to hold, analyze these conditions, and construct examples which satisfy them. These include as special cases the Ising model with arbitrary spins and the plane rotator model.

In Section 1, we develop the general theory for the classical case. In Section 2, we describe a number of models which fit into the general scheme, and for which the inequalities hold. We also give some counter examples indicative of the limits of the theory. Section 3 contains a more general formulation of the theory, which contains that of Section 1 as a special case, and seems adequate to study quantum systems. The purpose of this exposition in two overlapping stages is to give a self-contained description of the classical case, which is slightly simpler, and for which a fairly large class of non trivial examples exist.

Basically, the problem is an algebraic one. Nevertheless, series and therefore convergence problems arise here and there. Therefore some assumptions of a topological nature are required, in order to make definite statements. Their choice is mainly a matter of convenience. In what follows, we shall make assumptions which are natural, sufficient to treat the applications, and sufficient for the convergence problems to become trivial. The arbitrariness of this choice will not be mentioned any more, but should be kept in mind.

1. General Theory in the Classical (≡ Commutative) Case

Let K be a compact space, $\mathfrak{A} \equiv \mathscr{C}(K)$ the algebra of complex continuous functions on K. Elements of K will be denoted by x, y, etc., elements of $\mathfrak{A}$ by f, g, h, etc. $\mathfrak{A}$ is a commutative C^*-algebra [7] with the norm $\|f\| = \operatorname{Sup}_x |f(x)|$. It has a unit element, namely the function which is constant and equal to one, and which we denote by $\mathbb{1}$. Let σ be a state on $\mathfrak{A}$, or equivalently a positive measure on K with total mass equal to one. We shall be interested in subsets Q of $\mathfrak{A}$ satisfying some of the following conditions:

(Q1) Q is a convex cone, closed in the topology of $\mathfrak{A}$, containing $\mathbb{1}$ and closed under multiplication and complex conjugation. In particular, for any $f \in Q$, $g \in Q$ and any $\alpha \geqq 0$, $\beta \geqq 0$, the functions $\alpha f + \beta g$, fg and $\bar{f}$ belong to Q.

(Q2) For any finite family $f_1, \ldots, f_n$ of elements of Q, the following quantity is real positive[1]:

$$\sigma(f_1 \ldots f_n) = \int d\sigma(x) f_1(x) \ldots f_n(x) \geqq 0 . \tag{1.1}$$

(Q3) For any finite family $f_1, \ldots, f_n$ of elements of Q and for any sequence of plus or minus signs, the following quantity is real positive:

$$\int d\sigma(x)\, d\sigma(y) \prod_{i=1}^{n} (f_i(x) \pm f_i(y)) \geqq 0 . \tag{1.2}$$

This choice calls for the following remarks.

[1] In all this paper, a positive means $a \geqq 0$, a strictly positive means $a > 0$.

Remark 1. If Q satisfies (Q1), then (Q2) simply means that σ belongs to the polar cone [8] of the convex cone that consists of the real elements in Q.

Remark 2. The conditions (Q2) and (Q3) are not independent. (Q2) is a consequence of (Q3). In fact:

$$\int d\sigma(x) \prod_{i=1}^{n} f_i(x) = \tfrac{1}{2} \int d\sigma(x)\, d\sigma(y) \left[\prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y) \right]. \tag{1.3}$$

Now:

$$\begin{aligned} \prod_{i=1}^{n} f_i(x) + \prod_{i=1}^{n} f_i(y) = {} & \tfrac{1}{2}(f_1(x) + f_1(y)) \left(\prod_{i=2}^{n} f_i(x) + \prod_{i=2}^{n} f_i(y) \right) \\ & + \tfrac{1}{2}(f_1(x) - f_1(y)) \left(\prod_{i=2}^{n} f_i(x) - \prod_{i=2}^{n} f_i(y) \right) \end{aligned} \tag{1.4}$$

By iteration, the integrand in (1.3) decomposes as a linear combination with positive coefficients of integrands of the type that occur in (1.2). Therefore (Q3) implies (Q2).

On the other hand, if all signs are plus signs, (1.2) reduces to a consequence of (Q2). The inequality (1.2) is obviously satisfied if the number of minus signs is odd, in which case the *LHS* of (1.2) is zero. Other cases are not trivial.

Remark 3. In all the applications of Section 2, Q will be a real cone. Complex cones will rather provide us with counterexamples. One could also include in (Q1) the condition that Q is real. The present form is chosen in analogy with the more general case described in Section 3.

The relevant conditions for our purposes are (Q2) and (Q3). We now show that the natural subsets of $\mathfrak{A}$ to consider are those satisfying (Q1). Let S be a self conjugate subset of $\mathfrak{A}$ (that is, with f, S also contains $\bar{f}$). Let $Q(S)$ be the smallest subset of $\mathfrak{A}$ that contains S and satisfies (Q1), or equivalently the intersection of all subsets of $\mathfrak{A}$ that contain S and satisfy (Q1), or equivalently the norm closure of the set of polynomials of elements of S and $\mathbb{1}$ with positive coefficients. $Q(S)$ will be called the multiplicative convex cone generated by S.

Proposition 1. *Let S be a self conjugate subset of $\mathfrak{A}$. If S satisfies* (Q2) *(resp.* (Q3)*), then $Q(S)$ also satisfies* (Q2) *(resp.* (Q3)*).*

Proof. The statement for (Q2) follows from the previous description of $Q(S)$ and the stability of (Q2) under convex combinations, multiplication and norm closure.

Suppose now that S satisfies (Q3). (Q3) is obviously stable under convex combinations and norm closure. In order to prove that $Q(S)$ satisfies (Q3) it is sufficient to prove that (1.2) holds when each f_i is a product of elements in S. The argument is the same as in Remark 2.

Let $f = g_1 \dots g_r$ where $g_j \in S$ for $1 \leqq j \leqq r$. Then

$$f(x) \pm f(y) = \frac{1}{2}(g_1(x) + g_1(y))(g_2 \dots g_r(x) \pm g_2 \dots g_r(y)) + \frac{1}{2}(g_1(x) - g_1(y))(g_2 \dots g_r(x) \mp g_2 \dots g_r(y)). \tag{1.5}$$

By iteration, we obtain a decomposition of $f(x) \pm f(y)$ as a polynomial with positive coefficients of expressions of the type $g(x) \pm g(y)$, where $g \in S$. The result follows immediately.

We now introduce the following notation. Let h and f belong to $\mathfrak{A}$. We define:

$$Z_h = \int d\sigma(x) \exp[-h(x)] \tag{1.6}$$

$$\langle f \rangle_h = Z_h^{-1} \int d\sigma(x) f(x) \exp[-h(x)] \tag{1.7}$$

provided $Z_h \neq 0$.

In the applications, K will be the phase space of a physical system, h the hamiltonian, f any observable, Z_h the partition function, and $\langle f \rangle_h$ the thermal average of f.

We are now prepared to state and prove the basic inequalities. The first one reduces to the following statements.

Proposition 2. *(Griffiths' first inequality). Let S be a self conjugate subset of $\mathfrak{A}$, satisfying $(Q\,2)$. Let $h \in \mathfrak{A}$.*

(1) If $e^{-h} \in Q(S)$ and $Z_h \neq 0$, then $\langle f \rangle_h$ is real positive for all $f \in Q(S)$.
(2) If $-h \in Q(S)$, then $e^{-h} \in Q(S)$ and $Z_h \geqq 1$.

Proof. $Q(S)$ satisfies $(Q\,1)$ by definition and $(Q\,2)$ by Proposition 1. (1) is obvious. (2) follows from $(Q\,1)$ and $(Q\,2)$ by expanding e^{-h} as a power series in $-h$. The series is norm convergent in $\mathfrak{A}$. Term by term integration gives $Z_h \geqq 1$.

We turn to the second inequality.

Proposition 3. *(Griffiths' second inequality). Let S be a self conjugate subset of $\mathfrak{A}$, satisfying $(Q\,3)$. Then for any f, g and $-h$ in $Q(S)$, the following quantity is real positive:*

$$\langle fg \rangle_h - \langle f \rangle_h \langle g \rangle_h \geqq 0 . \tag{1.8}$$

Proof. $Q(S)$ satisfies $(Q\,1)$ by definition and $(Q\,3)$ by Proposition 1. From the definition (1.7), we obtain

$$2Z_h^2(\langle fg \rangle_h - \langle f \rangle_h \langle g \rangle_h) = \int d\sigma(x)\, d\sigma(y)\, (f(x) - f(y)) \times (g(x) - g(y)) \exp(-h(x) - h(y)) \tag{1.9}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d\sigma(x)\, d\sigma(y)\, (f(x) - f(y))\, (g(x) - g(y))\, (-h(x) - h(y))^n . \tag{1.10}$$

Term by term integration is allowed by the uniform convergence of the series, and each integral is real positive by $(Q3)$.

These results call for the following remarks.

Remark 4. Proposition 2 is a rather weak result, and holds under much more general circumstances. For instance, the exponential can be replaced by any entire function with positive Taylor coefficients at the origin. We have kept unnecessarily restrictive assumptions because they seem to be useful for Proposition 3, which is the really interesting result.

Remark 5. In the proof of Proposition 3, we do not need the full strength of the condition $(Q3)$, but only a special case of it. The main reason for imposing $(Q3)$ in the present form is its factorization property, already used in the proof of Proposition 1, and to be used again for Proposition 5 below.

We now face the problem of finding sufficiently general self conjugate subsets S of $\mathfrak{A}$ that satisfy $(Q2)$ or $(Q3)$. It will prove useful to consider subsets S of $\mathfrak{A}$ which satisfy the following condition:

(S) The product of any two elements $f \in S$, $g \in S$, has an expansion:

$$fg = \sum_n a_n f_n \tag{1.11}$$

where $a_n \geqq 0$ and $f_n \in S$ for all n, and where the series converges in the norm topology of $\mathfrak{A}$.

Let now $C(S)$ be the convex cone generated by S and $\mathbb{1}$, i.e. the norm closure of the set of linear combinations of elements of S and $\mathbb{1}$ with positive coefficients. The search for S is simplified by the following result.

Proposition 4. *Let S be a self conjugate subset of $\mathfrak{A}$. Then:*

(1) *if S satisfies (S), $Q(S) = C(S)$,*

(2) *if S satisfies (S) and if $\sigma(f)$ is real positive for all $f \in S$, then S satisfies $(Q2)$.*

Proof. (2) is obvious. We prove (1). Certainly $C(S) \subset Q(S)$. Now (S) implies that $C(S)$ satisfies $(Q1)$. Therefore $C(S) = Q(S)$.

An essential tool in the construction of models is the following factorization property.

Proposition 5. *Let $K = K_1 \times K_2$ be the product of two compact spaces, let $\sigma = \sigma_1 \times \sigma_2$ be the product of two probability measures on K_1 and K_2 respectively, let S_1 and S_2 be self conjugate subsets of $\mathscr{C}(K_1)$ and $\mathscr{C}(K_2)$ respectively, and let $S = S_1 S_2 \subset \mathscr{C}(K_1 \times K_2)$ be the set of functions of the type $f(x_1, x_2) = f_1(x_1) f_2(x_2)$, where $f_1 \in S_1$ and $f_2 \in S_2$.*

Then, if S_1 and S_2 both satisfy (S) (resp. $(Q2)$, resp. $(Q3)$), S also satisfies (S) (resp. $(Q2)$, resp. $(Q3)$).

Proof. The property is obvious for (*S*) and (*Q* 2). For (*Q* 3), it follows from repeated use of the identity:

$$\begin{aligned} f_1(x_1)f_2(x_2) \pm f_1(y_1)f_2(y_2) &= \frac{1}{2}(f_1(x_1)+f_1(y_1))(f_2(x_2) \pm f_2(y_2)) \\ &\quad + \frac{1}{2}(f_1(x_1)-f_1(y_1))(f_2(x_2) \mp (f_2(y_2)). \end{aligned} \tag{1.12}$$

In the following section, we shall use Propositions 4 and 5 to construct examples of subsets $S \subset \mathfrak{A}$ satisfying (*Q* 2) or (*Q* 3), and for which therefore Propositions 2 and 3 hold.

2. Examples

In this section, we give explicit examples of systems (K, σ, S) such that S satisfies (*S*) and (*Q* 2) or (*Q* 3). We describe successively:

two basic examples where S satisfies (*S*) and (*Q* 2) (Examples 1 and 2),
two basic examples where S satisfies (*S*) and (*Q* 3) (Examples 2 and 4).

They are restrictions of Examples 1 and 2 respectively.

two counter examples to optimistic guesses concerning the possibility of generalizing example 4 (Counter examples 1 and 2).

An example where (*S*) and (*Q* 3) are satisfied, but which falls outside of the class considered in the previous section, since the corresponding K is not compact.

Finally, we turn to physical applications constructed from Examples 3 and 4 and describing generalized spin systems (Models 1 to 4).

Example 1. (*S*) and (*Q* 2) satisfied. K is an arbitrary compact set, σ an arbitrary probability measure on K, $S \equiv Q(S)$ is the set of real positive continuous functions on K.

Example 2. (*S*) and (*Q* 2) satisfied. K is a compact group G, σ is a positive definite measure on G, or equivalently a measure of the form $\sigma = \mu^* * \mu$, where μ is a bounded measure on G, and μ^* is defined by $d\mu^*(x) = \overline{d\mu(x^{-1})}$. $S \equiv Q(S)$ is the set of positive definite functions on G [9]. (*S*) follows from the fact that the product of two positive definite functions on G is again positive definite, and (*Q* 2) from the fact that f positive definite implies

$$\int d\mu(x)\, \overline{d\mu(y)}\, f(xy^{-1}) \geqq 0\,.$$

Example 3. (*S*) and (*Q* 3) satisfied. K is an arbitrary compact set, σ an arbitrary probability measure on K. Let g be a real function in $\mathscr{C}(K)$. Let T be the set of real positive continuous non decreasing functions φ on $g(K)$, the image of K under g. We define:

$$S \equiv Q(S) = \{ f \in \mathscr{C}(K) : \exists \varphi \in T \quad \text{such that} \quad f = \varphi \circ g \} .$$

(S) is obviously satisfied. We prove that S also satisfies (Q3). Let $f_i = \varphi_i \circ g \in S$ for $i = 1, \ldots, n$. The integrand in (1.2) is a product of factors which are either of the type $(f_i(x) + f_i(y))$ and therefore positive, or of the type $f_i(x) - f_i(y)$, in which case they have the sign of $g(x) - g(y)$. Since it suffices to consider the case of an even number of minus signs, the integrand in (1.2) is positive, and (Q3) is proved.

Example 4. (S) and (Q3) satisfied. This example is a special case of Example 2. We take G commutative, σ is the Haar measure on G and $Q(S)$ is the set of all *real* positive definite functions on G. We moreover restrict our attention to the case where G is the direct product of a finite number of circles T_1 and a finite number of finite cyclic groups Z_{p_i}, where $Z_p = Z/pZ$ is the additive group of integers modulo p. Because of Bochner's theorem [10], we can take S to be the set of the real parts of the characters of G.

An element of G is a family $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$ of r angles $\theta_i \in [0, 2\pi)$. For each i, θ_i can take either all values in $[0, 2\pi)$, or values of the form $2\pi k/p_i$ with k integer, depending on whether this specific θ_i belongs to a circle T_1 or to a cyclic group Z_{p_i}.

The characters of G are the functions $\chi_{\boldsymbol{m}}(\boldsymbol{\theta}) = \exp(i\boldsymbol{m} . \boldsymbol{\theta})$ where $\boldsymbol{m} = (m_1, \ldots, m_r)$ is a set of r integers (some of them taken modulo some p_i). The functions in S are the functions $f_{\boldsymbol{m}}(\boldsymbol{\theta}) = \cos(\boldsymbol{m} . \boldsymbol{\theta})$. Condition ($S$) follows from the identity

$$\cos(\boldsymbol{m} . \boldsymbol{\theta}) \cos(\boldsymbol{m}' . \boldsymbol{\theta}) = \frac{1}{2} [\cos((\boldsymbol{m} + \boldsymbol{m}') . \boldsymbol{\theta}) + \cos((\boldsymbol{m} - \boldsymbol{m}') . \boldsymbol{\theta})] . \tag{2.1}$$

We now show that S satisfies (Q3). Let $f_{\boldsymbol{m}_i}$, $i = 1, \ldots, n$, be a finite family of functions in S. We want to prove that the following quantity is positive:

$$J = \int d\boldsymbol{\theta}\, d\boldsymbol{\theta}' \prod_{i=1}^{n} (f_{\boldsymbol{m}_i}(\boldsymbol{\theta}) \pm f_{\boldsymbol{m}_i}(\boldsymbol{\theta}')) . \tag{2.2}$$

Here $\int d\boldsymbol{\theta} = \prod_{j=1}^{r} \int d\theta_j$, and for each j, $\int d\theta_j$ means either integration in $(0, 2\pi)$ or summation over the discrete values $2\pi k/p$ with $0 \leqq k < p - 1$, for some positive integer p.

Now for all i,

$$\begin{cases} \cos \boldsymbol{m}_i . \boldsymbol{\theta} + \cos \boldsymbol{m}_i . \boldsymbol{\theta}' = 2 \cos \left(\boldsymbol{m}_i \cdot \dfrac{\boldsymbol{\theta} + \boldsymbol{\theta}'}{2} \right) \cos \left(\boldsymbol{m}_i . \dfrac{\boldsymbol{\theta}' - \boldsymbol{\theta}}{2} \right) \\[2ex] \cos \boldsymbol{m}_i . \boldsymbol{\theta} - \cos \boldsymbol{m}_i . \boldsymbol{\theta}' = 2 \sin \left(\boldsymbol{m}_i \cdot \dfrac{\boldsymbol{\theta} + \boldsymbol{\theta}'}{2} \right) \sin \left(\boldsymbol{m}_i . \dfrac{\boldsymbol{\theta}' - \boldsymbol{\theta}}{2} \right) . \end{cases} \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain for J an expression of the type:

$$J = \int d\boldsymbol{\theta}\, d\boldsymbol{\theta}' F\left(\frac{\boldsymbol{\theta}+\boldsymbol{\theta}'}{2}\right) F\left(\frac{\boldsymbol{\theta}'-\boldsymbol{\theta}}{2}\right) \tag{2.4}$$

where $F(\boldsymbol{\omega})$ is a periodic function of each component ω_i of $\boldsymbol{\omega}$ with period 2π. Each factor in the integrand in (2.4) is a periodic function of each θ_j and θ'_j with period 4π, in such a way that the product has period 2π in θ_j and θ'_j.

For each component (θ_j, θ'_j) of $(\boldsymbol{\theta}, \boldsymbol{\theta}')$, we now change the integration variables from (θ_j, θ'_j) to $\alpha_j = (\theta'_j + \theta_j)/2$ and $\beta_j = (\theta'_j - \theta_j)/2$. This is straightforward if θ_j, θ'_j are continuous variables in $(0, 2\pi)$, but requires some care if θ_j and θ'_j are discrete variables. We consider a specific component j and drop the subscript j. We proceed as follows (see Fig. 1).

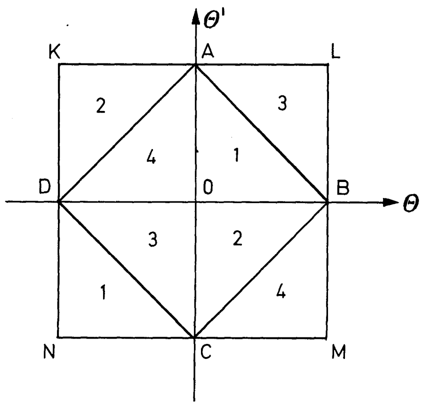

Fig. 1. Change of integration variables in Eq. (2.4). Regions with the same number give equal contributions

Using the periodicity of the integrand in (2.4) we first replace the integration in the square $OALB(0 \leqq \theta < 2\pi,\ 0 \leqq \theta' < 2\pi)$ by one fourth of the integration in the square $KLMN(-2\pi \leqq \theta < 2\pi,\ -2\pi \leqq \theta' < 2\pi)$. The latter integration is twice the integration in the square $ABCD$ $(-\pi \leqq \alpha < \pi,\ -\pi \leqq \beta < \pi)$. In the continuous case, the new integration domain is obviously the product of the intervals $-\pi \leqq \alpha < \pi$ and $-\pi \leqq \beta < \pi$, the measure being $d\alpha\, d\beta$. In the discrete case where θ and θ' represent elements in a finite cyclic group Z_p, one has to consider separately the two subdomains that consist of points with both α and β even multiples or odd multiples of π/p. Each subdomain is the product of identical domains for the variables α and β. The case $p = 2$ is shown on Fig. 2 as an example.

Finally, J reduces to one term, or to a sum of terms, of the type $(\int d\boldsymbol{\alpha} F(\boldsymbol{\alpha}))^2$. This is positive, and $(Q\,3)$ is proved.

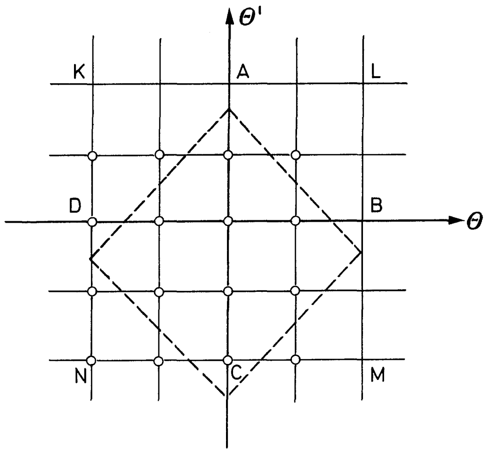

Fig. 2. Change of variables in Eq. (2.4) in the discrete case ($G=Z_2$). The final integration range consists of the points interior to the dotted square

We next show that the restriction to *real* positive definite functions cannot be dropped in general in Example 4 if (Q3) is to be fulfilled.

Counter Example 1. (Q3) not satisfied. K is the unit circle T_1, σ is the Haar measure $d\sigma(\theta)=d\theta/2\pi$, S is the set of characters of T_1, $Q(S)$ is the set of positive definite functions on T_1. Then (Q3) does not hold, as shown by the following counter example:

$$\int \frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}(e^{i\theta}-e^{i\theta'})^2(e^{-i\theta}+e^{-i\theta'})^2=\int\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}(-4\sin^2(\theta-\theta'))=-2\,. \tag{2.5}$$

In a previous paper [11], we obtained Griffiths' second inequality (1.8) as a consequence of the positive definiteness of a suitable function. The group G corresponding to this case is a direct product $(Z_2)^N$. In this case, every element of G is of order two, so that all the characters and positive definite functions on G are real. We denote by x, y, etc., the elements of G and by ϱ, μ, λ, etc. the characters of G, i.e. the elements of the dual group $\hat{G}$. Then, for any (real) positive definite functions f and $-h$ on G, the function $\varphi_f(\varrho)=\langle\varrho\rangle_h\langle\varrho f\rangle_h$ is a real positive definite function on $\hat{G}$. In particular, $\varphi_f(\varrho)$ is bounded everywhere by its value at the origin $\mathbb{1}$ of $\hat{G}$. Now if f is real positive definite, and $\mu\in\hat{G}$, $f\mu$ is also real positive definite. Therefore:

$$\varphi_{f_\mu}(\mathbb{1})-\varphi_{f_\mu}(\mu)=\langle f\mu\rangle_h-\langle f\rangle_h\langle\mu\rangle_h\geqq 0\,. \tag{2.6}$$

We now show that the positive definiteness of $\varphi_f(\varrho)$ for (real) positive definite f does not hold in general if the characters of G are complex.

Counter Example 2. The function $\varphi_f(\varrho)=\langle\varrho\rangle_h\langle\varrho f\rangle_h$ is not positive definite in general. We start with an arbitrary compact commutative group G, to be chosen more precisely later. We take $-h$ to be a real positive definite function on G

$$-h=\sum_{\lambda\in\hat{G}} J(\lambda)\,\lambda \tag{2.7}$$

with $J(\lambda)=J(\bar{\lambda})=\overline{J(\lambda)}\geqq 0$. We first consider the case where $f=\mu\in\hat{G}$. By Bochner's theorem [10], $\varphi_\mu(\varrho)$ is positive definite iff its Fourier transform $\hat{\varphi}_\mu(x)$ is real positive. We consider $\hat{\varphi}_\mu(x)$:

$$\begin{aligned} Z_h^2\hat{\varphi}_\mu(x) &= \sum_\varrho \bar{\varrho}(x)\int dy\,dz\,\varrho(y)\,\varrho(z)\,\mu(y)\exp\Big[\sum_\lambda J(\lambda)\,(\lambda(y)+\lambda(z))\Big] \\ &= \int dy\,\mu(y)\exp\Big[\sum_\lambda J(\lambda)\,(1+\lambda(x))\,\overline{\lambda(y)}\Big] \\ &= \sum_{n\geqq 0}\frac{1}{n!}\sum_{\lambda_1,\dots\lambda_n} J(\lambda_1)\dots J(\lambda_n)\prod_{i=1}^n(1+\lambda_i(x)) \\ &\quad\times\int dy\,\mu(y)\prod_{i=1}^n\bar{\lambda}_i(y)\,. \end{aligned} \tag{2.8}$$

This is not real in general, as shown by the following example. We take $\mu\neq\mathbb{1}$, $J(\mu)=J(\bar{\mu})=J>0$, $J(\lambda)=0$ for $\lambda\neq\mu$ and $\lambda\neq\bar{\mu}$. We take furthermore J small enough for the first order term in (2.8) to be dominant. Then:

$$\hat{\varphi}_\mu(x)=Z_h^{-2}(J(1+\mu(x))+0(J^2))\,. \tag{2.9}$$

This is not real in general if μ is not real.

We next consider the case of a *real* positive definite f. We take $f=\mu+\bar{\mu}$, so that $\varphi_f(\varrho)=\langle\varrho\rangle_h\langle\varrho(\mu+\bar{\mu})\rangle_h$. Its Fourier transform is then real:

$$\hat{\varphi}_f(x)=\hat{\varphi}_\mu(x)+\hat{\varphi}_{\bar{\mu}}(x)=2\,\mathrm{Re}\,\hat{\varphi}_\mu(x)\,. \tag{2.10}$$

One may then ask whether $\varphi_f(\varrho)$ is positive definite, or equivalently whether its Fourier transform $\hat{\varphi}_f(x)$ is positive. The following example shows that this is false in general. We take now $G=T_1$, we call the variable θ instead of x, we take $\mu(\theta)=e^{2i\theta}$, $J(\lambda)=J(\bar{\lambda})=J>0$ for $\lambda(\theta)=e^{i\theta}$, $J(v)=0$ otherwise, and J small enough for the lowest order term in $\hat{\varphi}_f(x)$ to be dominant. Then:

$$\hat{\varphi}_f(\theta)=Z_h^{-2}\,\mathrm{Re}(J^2(1+e^{i\theta})^2+0(J^3)) \tag{2.11}$$

$$\hat{\varphi}_f(\theta)=2Z_h^{-2}(J^2\cos\theta(1+\cos\theta)+0(J^3))\,. \tag{2.12}$$

This has the sign of $\cos\theta$, and φ_f is therefore not positive definite.

Counter example 2 shows that the proof and generalization of Griffiths' second inequality given in Ref. [11] are accidental, since they depend in an essential way on the reality of the characters of G, which is not required in the general situation of Example 4.

In a recent paper, Sherman [12] obtained a generalization of Griffiths' second inequality (1.8) for the group $G=(Z_2)^N$, which is weaker than the positive definiteness of $\varphi_\mu(\varrho)$, and states only that for any subgroup H of $\hat{G}$ and for any real positive definite f, g and $-h$ on G, the following quantity is real positive:

$$\sum_{\varrho\in H} (\langle\varrho\rangle_h\langle\varrho f g\rangle_h - \langle\varrho f\rangle_h\langle\varrho g\rangle_h) \geqq 0 . \tag{2.13}$$

It would be interesting to determine whether (2.13) still holds for the groups considered in Example 4. We now present an example where (Q3) holds, but where K is not compact.

Example 5. (S) and (Q3) satisfied. K is the real line. The measure σ is defined by $d\sigma(x)=\exp(-x^2/2)\,dx$, S consists of the Hermite polynomials $H_n(x)$ with the usual sign conventions. These polynomials are orthogonal with respect to σ. (S) follows from the orthogonality relations and the fact that for all l, m, n, the following quantity is positive:

$$\int d\sigma(x)\, H_l(x)\, H_m(x)\, H_n(x) \geqq 0 . \tag{2.16}$$

We now prove (Q3). We consider the quantity:

$$J=\int d\sigma(x)\, d\sigma(y) \prod_{i=1}^{n} (H_{m_i}(x) \pm H_{m_i}(y)) . \tag{2.17}$$

We make a change of variables from (x, y) to $u=(x+y)/\sqrt{2}$, $v=(x-y)/\sqrt{2}$:

$$J=\int d\sigma(u)\, d\sigma(v) \prod_{i=1}^{n} \left(H_{m_i}\left(\frac{u+v}{\sqrt{2}}\right) \pm H_{m_i}\left(\frac{u-v}{\sqrt{2}}\right)\right) . \tag{2.18}$$

The Hermite polynomials satisfy the identity [13]:

$$H_m\left(\frac{u+v}{\sqrt{2}}\right) = 2^{-m/2} \sum_{l=0}^{m} \binom{m}{l} H_{m-l}(u)\, H_l(v) . \tag{2.19}$$

Furthermore, H_m has the parity of m. Therefore:

$$H_m\left(\frac{u+v}{\sqrt{2}}\right) \pm H_m\left(\frac{u-v}{\sqrt{2}}\right) = 2^{1-m/2} \sum_{l\left\{\substack{\text{even}\\ \text{odd}}\right\}} \binom{m}{l} H_{m-l}(u)\, H_l(v) \tag{2.20}$$

where even corresponds to plus, and odd to minus. Using (2.20) repeatedly, we decompose the integrand in (2.18) as a polynomial with positive coefficients of Hermite polynomials of u and v. The integrations over u

and v factorize in each monomial, and it follows immediately from (S), the orthogonality relations and the positivity of H_0, that J is positive. This proves $(Q\,3)$.

This example is of restricted interest, since convergence requirements allow only for quadratic Hamiltonians in this case.

Using Examples 3 and 4 as building blocks and proposition 5 as a glue, we can now construct a large variety of models where $(Q\,3)$ is satisfied, and which therefore satisfy Propositions 2 and 3. We select a few of them on the basis of their physical interest.

Model 1. Ising model with spin $\frac{1}{2}$. This is a special case of Example 4 with $G=(Z_2)^N$. We have therefore obtained one more proof of a well-known result [1, 2, 11].

Model 2. Generalized Ising model with arbitrary spins. Let Λ be a finite set of N sites. We shall construct a system (K, σ, S) as the product of individual subsystems (K_r, σ_r, S_r) associated with each site $r \in \Lambda$. By Proposition 5, S will satisfy (S) and $(Q\,3)$ if each S_r does. Let r be a site in Λ. The system (K_r, σ_r, S_r) is itself constructed as the product of two systems:

– a system of the type of Example 4, with $G=Z_2$,

– a system of the type of Example 3, with $K=[0, 1]$, and $g(x)=x$.

This can be described equivalently as follows. K_r is the closed interval $[-1, +1]$. σ_r is any positive normalized even measure on K_r. In particular, $d\sigma_r(x)=d\sigma_r(-x)$. If σ has a finite mass $a\delta(x)$ at the origin, it should be understood as $\lim_{\varepsilon\to 0} \frac{a}{2}(\delta(x+\varepsilon)+\delta(x-\varepsilon))$. S_r is the set of functions on K_r that are of the form $f(x)=\varphi(|x|)$ or $f(x)=\varepsilon(x)\,\varphi(|x|)$, where $\varphi(t)$ is any positive continuous non decreasing function of $t\in[0, 1]$, and $\varepsilon(x)=+1$ (resp. -1) for $x>0$ (resp. $x<0$). The set $Q(S)$ which occurs in Propositions 2 and 3 is then the set of all norm limits of polynomials of all such functions for all sites in Λ, with positive coefficients.

Remark 6. The Ising model with arbitrary spins considered in Ref. [3] is the special case obtained by restricting S_r to be the set of functions $\{x_r^n\}$ where n is an arbitrary positive integer, and σ_r to be the measure defined by

$$d\sigma_r(x)=\frac{1}{2s_r+1}\sum_{m=-s_r}^{s_r}\delta\left(x-\frac{m}{s_r}\right)dx \tag{2.21}$$

where s_r is integer or half integer, and m takes all integer or half integer values from $-s_r$ to $+s_r$.

Remark 7. Model 2 satisfies the following property, already obtained in Ref. [3] for the special case described in Remark 6. Choose $Q(S)$ as above. Suppose that for all r, σ_r is non-vanishing in any neighborhood of $+1$. This can always be achieved by a rescaling of the corresponding spin variable, and entails therefore no loss in generality. Let σ be the product measure $\sigma = \prod_r \sigma_r$ on K and let $\langle f \rangle_{h,\sigma}$ be the thermal average of $f \in Q(S)$ with hamiltonian h and measure σ(cf. (1.7)). Let $\varphi(t)$ be a positive continuous strictly increasing function of $t \in [0, 1]$, for instance $\varphi(t) = t^\alpha$ with $\alpha > 0$. Consider the new hamiltonian h_λ defined for $\lambda > 0$ by:

$$-h_\lambda = -h + \lambda \sum_{r \in \Lambda} \varphi(|x_r|) . \tag{2.22}$$

Then, by Proposition 3, $\langle f \rangle_{h_\lambda,\sigma}$ is an increasing function of λ. On the other hand, $\langle f \rangle_{h_\lambda,\sigma} = \langle f \rangle_{h,\sigma_\lambda}$ where σ_λ is the measure defined by $\sigma_\lambda = \prod_r \sigma_{\lambda r}$, where:

$$d\sigma_{\lambda r}(x) = d\sigma_r(x) \exp(\lambda \varphi(|x|)) \left[\int d\sigma_r(x) \exp[\lambda \varphi(|x|)]\right]^{-1} . \tag{2.23}$$

It is easily seen that when λ tends to infinity, σ_λ tends in a suitable sense (technically: in the sense of the $W*$ topology induced by $\mathfrak{A}$ on its dual space $\mathfrak{A}'$) to the measure $\tilde{\sigma} = \Pi \tilde{\sigma}_r$, where $d\tilde{\sigma}_r(x) = \frac{1}{2}(\delta(x+1) + \delta(x-1))\, dx$. Therefore, for any f and $-h \in Q(S)$, the following inequality holds:

$$0 \leqq \langle f \rangle_{h,\sigma} \leqq \langle f \rangle_{h,\tilde{\sigma}} . \tag{2.24}$$

In words, this means that the thermal averages of elements of $Q(S)$ for arbitrary spins are bounded by the averages of the same quantities for spin $\frac{1}{2}$, with the same hamiltonian.

Model 3. Plane rotators. Let Λ be a finite set of N sites. With each site is associated a classical spin, which is a unit vector in the two dimensional euclidean plane, or equivalently a point θ of the unit circle T_1. The phase space is therefore $K = (T_1)^N$. This is a special case of example 4. σ and S are chosen as in this example: σ is the Haar measure $d\sigma = \Pi\, d\theta_r/2\pi$, S is the set of functions $\cos(\boldsymbol{m} . \boldsymbol{\theta})$ where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$, $\boldsymbol{m} = (m_1, \ldots, m_r)$, and the m_j are arbitrary integers. $Q(S) \equiv C(S)$ is the convex cone of norm limits of polynomials (or equivalently linear combinations) of such functions, with positive coefficients. In the applications, one often considers special cases, for instance functions of the type $\cos(\theta_r - \theta_s)$, $r \in \Lambda$, $s \in \Lambda$.

Model 4. Classical Heisenberg model. Let Λ be a finite set of N sites. With each site r is associated a classical spin, which is a unit vector u_r in the three dimensional euclidean space, or equivalently a point of the two dimensional unit sphere. Such a point is described by two angles $\theta \in [0, \pi]$, and $\varphi \in T_1$, or equivalently, by two variables $x = \cos\theta \in [-1, +1]$ and $\varphi \in T_1$. The phase space is therefore the product of those of models 2 and 3: $K=[-1,+1]^N \times (T_1)^N$. Suitable σ and S can therefore be obtained as products of those considered in Models 2 and 3. Note however, that this is not sufficient to accomodate the usual interaction terms, which are linear in $\cos(\boldsymbol{u}_r, \boldsymbol{u}_s) = \cos\theta_r \cos\theta_s + \sin\theta_r \sin\theta_s \cos(\varphi_r - \varphi_s)$, and for which further generalizations of example 4 are needed.

Similar games can be played with higher dimensional classical spins.

3. General Theory in the Quantum (≡ Non Commutative) Case

In this section, we extend the theory of Section 1 to the non commutative case. The non commutativity complicates the algebraic structure, and several extensions are possible; we first present the most natural one, which is completely straightforward, and then a symmetrized theory which however has the severe drawback that the factorization properties are destroyed by the symmetrization.

Let $\mathfrak{A}$ be a Banach $*$-algebra[2] [14] with unit element $\mathbb{1}$. Other elements of $\mathfrak{A}$ will be denoted by A, B, H, etc. Let σ be a state on $\mathfrak{A}$, namely a positive linear functional normalized by $\sigma(\mathbb{1})=1$. We are interested in subsets Q of $\mathfrak{A}$ which satisfy some of the following conditions.

(Q1) Q is a norm closed, self adjoint, convex cone, closed under multiplication in $\mathfrak{A}$, and containing $\mathbb{1}$.

(Q2) For any finite family $A_1, \ldots, A_n$ of elements of Q, the quantity $\sigma(A_1 \ldots A_n)$ is real positive.

(Q3) For any finite family $A_1, \ldots, A_n$ of elements of Q and for any sequence of plus or minus signs, the following quantity is real positive:

$$(\sigma\otimes\sigma)\left(\prod_{i=1}^{n}(A_i\otimes\mathbb{1}\pm\mathbb{1}\otimes A_i)\right)\geqq 0. \tag{3.1}$$

The product in (3.1) is an element of $\mathfrak{A}\otimes\mathfrak{A}$, and $\sigma\otimes\sigma$ is a state on the latter algebra in an obvious way.

Remark 2 of Section 1 applies to the present case without modification. For any self-adjoint subset S of $\mathfrak{A}$, we define $Q(S)$ as in Section 1. Proposition 1 still holds (with self adjoint replacing self conjugate), the proof being identical.

The natural extension of Proposition 2 is the following statement.

Proposition 6. *Let S be a self-adjoint subset of $\mathfrak{A}$, satisfying (Q2), and let $H\in\mathfrak{A}$.*

(1) *If $e^{-tH}\in Q(S)$ for all $t\in[0,\varepsilon]$ for some $\varepsilon>0$, then the following quantity is real positive for all $A\in Q(S)$ and all $t\in[0,1]$*

$$\sigma(e^{-tH}Ae^{-(1-t)H})\geqq 0. \tag{3.2}$$

[2] It might appear natural to use C^*-algebras. The extra condition thereby imposed does not seem to be useful for the elementary considerations that follow.

(2) *It is sufficient for* e^{-tH} *to belong to* $Q(S)$ *for all* $t\in[0,\varepsilon]$ *that* $H=B+C$, *where* $-B\in Q(S)$ *and* $e^{-tC}\in Q(S)$ *for all* $t\in[0,\varepsilon]$. *(In particular, it is sufficient that* $-H\in Q(S)$.*)*

Proof. $Q(S)$ satisfies $(Q1)$ by definition and $(Q2)$ by Proposition 1. (1) is obvious. (2) follows from the perturbation expansion:

$$e^{-t(B+C)}=\sum_{n\geqq 0}\int_{0\leqq t_1\leqq\ldots\leqq t_n\leqq t} dt_1\ldots dt_n e^{-t_1C}(-B)\ldots(-B)e^{-(t-t_n)C}. \tag{3.3}$$

The natural extension of Proposition 3 is the following statement.

Proposition 7. *Let* S *be a self-adjoint subset of* $\mathfrak{A}$, *satisfying* $(Q3)$. *Then, for any* A, B *and* $-H$ *in* $Q(S)$ *and for any* s, $t(0\leqq s\leqq t\leqq 1)$, *the following quantity is real positive:*

$$\begin{aligned}&\sigma(e^{-sH}Ae^{-(t-s)H}Be^{-(1-t)H})\,\sigma(e^{-H})\\&\qquad-\sigma(e^{-sH}Ae^{-(1-s)H})\,\sigma(e^{-tH}Be^{-(1-t)H})\geqq 0.\end{aligned} \tag{3.4}$$

Proof. $Q(S)$ satisfies $(Q1)$ by definition and $(Q3)$ by Proposition 1. For any $A\in\mathfrak{A}$, we define $A_\pm\in\mathfrak{A}\otimes\mathfrak{A}$ by:

$$A_\pm=A\otimes\mathbb{1}\pm\mathbb{1}\otimes A. \tag{3.5}$$

Let Ω be the LHS of (3.4). Then

$$2\Omega=(\sigma\otimes\sigma)\,(e^{-sH_+}A_-e^{-(t-s)H_+}B_-e^{-(1-t)H_+}) \tag{3.6}$$

$$=\sum_{p,q,r\geqq 0}\frac{s^p(t-s)^q(1-t)^r}{p!q!r!}(\sigma\otimes\sigma)\left((-H_+)^pA_-(-H_+)^qB_-(-H_+)^r\right). \tag{3.7}$$

This is real positive by $(Q3)$.

Integrating (3.2) or (3.4) over s and t with positive measures, we obtain various inequalities which are consequences of Propositions 6 and 7 respectively. Special cases will be obtained below from weaker assumptions in Propositions 9 and 10.

The extension to the present case of the end of Section 1 is immediate and will only be sketched briefly. Condition (S) is formulated and Proposition 4 holds with self adjoint replacing self conjugate. The proof is identical with the previous one. Proposition 5 is reformulated in an obvious way as Proposition 8 below, the proof being unchanged.

Proposition 8. *Let* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *be two Banach* $*$*-algebras with unit element. Let* $\sigma_i(i=1,2)$ *be a state on* $\mathfrak{A}_i$. *Let* $\sigma=\sigma_1\otimes\sigma_2$ *be the product state on* $\mathfrak{A}_1\otimes\mathfrak{A}_2$. *Let* S_1 *and* S_2 *be self adjoint subsets of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *respectively, and let* $S\subset\mathfrak{A}$ *be the set of elements of the form* $A=A_1\otimes A_2$ *where* $A_1\in S_1$ *and* $A_2\in S_2$.

Then, if S_1 *and* S_2 *both satisfy* (S) *(resp.* $(Q2)$, *resp.* $(Q3)$*),* S *also satisfies* (S) *(resp.* $(Q2)$, *resp.* $(Q3)$*).*

We now present a symmetrized version of the theory which however is much less complete, because it lacks the factorization property expressed by proposition 8.

Let again $\mathfrak{A}$ be a Banach $*$-algebra with unit element $\mathbb{1}$, and σ a state on $\mathfrak{A}$. We are interested in subsets Q of $\mathfrak{A}$ which satisfy one of the following conditions.

$(Qs2)$. For any finite family $A_1, \ldots, A_n$ of elements of Q, the quantity $\sigma(\mathscr{S}(A_1 \ldots A_n))$ is real positive, where $\mathscr{S}$ means complete symmetrization with respect to the order of the factors:

$$\mathscr{S}(A_1 \ldots A_n) = \frac{1}{n!} \sum_{\pi} A_{\pi(1)} \ldots A_{\pi(n)}. \tag{3.8}$$

The sum in (3.8) runs over all permutations of $(1, \ldots, n)$.

$(Qs3)$. For any finite family $A_1, \ldots, A_n$ of elements of Q and for any sequence of plus or minus signs, the following quantity is real positive.

$$(\sigma \otimes \sigma)\left(\mathscr{S}\left(\prod_{i=1}^{n} (A_i \otimes \mathbb{1} \pm \mathbb{1} \otimes A_i)\right)\right) \geqq 0. \tag{3.9}$$

Remark 2 of Section 1 still applies to this case. We first show that $(Qs3)$ implies $(Qs2)$. By the same argument as in Section 1, we obtain the following identity in $\mathfrak{A} \otimes \mathfrak{A}$:

$$\prod_{i=1}^{n} (A_i \otimes \mathbb{1}) + \prod_{i=1}^{n} (\mathbb{1} \otimes A_i) = \sum_{\{\pm\}} c_{\{\pm\}} \prod_{i=1}^{n} (A_i \otimes 1 \pm 1 \otimes A_i) \tag{3.10}$$

where the sum runs over appropriate sequences of plus or minus signs, and the coefficients $c_{\{\pm\}}$ are positive. We apply $\mathscr{S}$ and then $(\sigma \otimes \sigma)$ to both sides of (3.10). The *LHS* becomes $2\sigma(\mathscr{S}(A_1 \ldots A_n))$, while the *RHS* is positive if $(Qs3)$ holds. Therefore $(Qs3)$ implies $(Qs2)$. We next show that (3.9) with plus signs everywhere is a consequence of $(Qs2)$. This follows immediately from the identity:

$$\mathscr{S}\left(\prod_{i=1}^{n} (A_i \otimes \mathbb{1} + \mathbb{1} \otimes A_i)\right) = \sum_{I} \left[\mathscr{S}\left(\prod_{i \in I} A_i\right) \otimes \mathbb{1}\right] \left[\mathbb{1} \otimes \mathscr{S}\left(\prod_{i \notin I} A_i\right)\right] \tag{3.11}$$

where the sum in the *RHS* runs over all possible subsets I of $(1, \ldots, n)$.

There is no natural analogue of $(Q1)$ and Proposition 1 in this case. Nevertheless, $(Qs2)$ and $(Qs3)$ extend in an obvious way from any given self-adjoint subset S of $\mathfrak{A}$ to the closed convex cone $C(S)$ generated by S and $\mathbb{1}$, as described in section 1.

$(Qs2)$ and $(Qs3)$ are weaker than $(Q2)$ and $(Q3)$ respectively, and therefore yield weaker results than those contained in Propositions 6 and 7.

Proposition 9. *Let S be a self-adjoint subset of $\mathfrak{A}$ satisfying $(Qs2)$, and let $H \in \mathfrak{A}$.*

(1) *If* $e^{-tH} \in C(S)$ *for all* $t \in [0, \varepsilon]$ *for some* $\varepsilon > 0$, *then the following quantity is real positive for all* $A \in C(S)$.

$$\sigma\left(\int_0^1 dt\, e^{-tH} A e^{-(1-t)H}\right) \geqq 0\,. \tag{3.12}$$

(2) *It is sufficient for* e^{-tH} *to belong to* $C(S)$ *for all* $t \in [0, \varepsilon]$ *that* $H = B + C$, *where* $-B \in C(S)$ *and* $e^{-tC} \in C(S)$ *for all* $t \in [0, \varepsilon]$.

Proof. (1): The following limit holds in norm in $\mathfrak{A}$.

$$\int_0^1 dt\, e^{-tH} A e^{-(1-t)H} = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} e^{-jH/n} A e^{-(n-j)H/n}\,. \tag{3.13}$$

The quantity in the *RHS* is the symmetric product of A and n factors equal to $\exp(-H/n)$. (3.12) then follows from $(Qs\,2)$.

(2) is proved by applying the same argument to each term in the *RHS* of (3.3).

Similarly Proposition 7 is replaced by the following result.

Proposition 10. *Let* S *be a self-adjoint subset of* $\mathfrak{A}$ *satisfying* $(Qs\,3)$. *Then, for any* A, B *and* $-H$ *in* $C(S)$, *the following quantity is real positive.*

$$\begin{aligned}\sigma&\left(\int_{0\leqq s\leqq t\leqq 1} ds\, dt\, e^{-sH}(A e^{-(t-s)H} B + B e^{-(t-s)H} A)\, e^{-(1-t)H}\right) \sigma(e^{-tH})\\ &-\sigma\left(\int_0^1 ds\, e^{-sH} A e^{-(1-s)H}\right) \sigma\left(\int_0^1 dt\, e^{-tH} B e^{-(1-t)H}\right) \geqq 0\,.\end{aligned} \tag{3.14}$$

Proof. Let Ω be the *LHS* of (3.14). Using the notation (3.5), we can rewrite it as:

$$2\Omega = (\sigma\otimes\sigma)\left(\int_{0\leqq s\leqq t\leqq 1} ds\,dt\, e^{-sH_+}(A_- e^{-(t-s)H_+} B_- + B_- e^{-(t-s)H_+} A_-) e^{-(1-t)H_+}\right) \tag{3.15}$$

$$\begin{aligned}2\Omega = \sum_{p,q,r\geqq 0} \int_{0\leqq s\leqq t\leqq 1} ds\, dt\, \frac{s^p(t-s)^q(1-t)^r}{p!\,q!\,r!}\\ \cdot(\sigma\otimes\sigma)\left((-H_+)^p A_-(-H_+)^q B_-(-H_+)^r + (A\leftrightarrow B)\right)\end{aligned} \tag{3.16}$$

$$= \sum_{n\geqq 0} \frac{1}{(n+2)!} (\sigma\otimes\sigma)\left(\sum_{p+q+r=n} ((-H_+)^p A_-(-H_+)^q B_-(-H_+)^r + (A\leftrightarrow B))\right) \tag{3.17}$$

$$= \sum_{n\geqq 0} \frac{1}{n!} (\sigma\otimes\sigma)\left(\mathscr{S}(A_- B_-(-H_+)^n)\right) \tag{3.18}$$

where $\mathscr{S}$ is defined by (3.8). This is positive by $(Qs\,3)$.

Remark 8. Propositions 9 and 10 become simpler in the special case where σ is a central state on $\mathfrak{A}$, i.e. when $\sigma(AB) = \sigma(BA)$ for any $A \in \mathfrak{A}$,

$B \in \mathfrak{A}$. This occurs in practice when $\mathfrak{A}$ is an algebra of $N \times N$ matrices, and $\sigma(A) = N^{-1} \mathit{Tr} A$. In this case, the quantity

$$\sigma(e^{-tH} A e^{-(1-t)H})$$

no longer depends on t. The inequality (3.14) in Proposition 10 takes the simple form:

$$\sigma\left(\int_0^1 ds\, A e^{-sH} B e^{-(1-s)H}\right) \sigma(e^{-H}) - \sigma(A e^{-H})\, \sigma(B e^{-H}) \geqq 0\,. \quad (3.19)$$

If σ is the trace over a matrix algebra, the first factor in (3.19) is simply Bogoliubov's scalar product [15], and (3.19) expresses the fact that if the hamiltonian contains a term $-\lambda B$, then the thermal average of A is an increasing function of λ.

We have not found any natural analogue of the factorization property (Proposition 8) in the present case.

Other extensions of the general theory of Section 1 to the non commutative case are possible. We mention only the following one, which lies between the non symmetric and the totally symmetric versions.

We define in $\mathfrak{A}$ a symmetric product $(A, B) \to A \circ B = \frac{1}{2}(AB + BA)$. This product is commutative, but not associative. One can reformulate (Q1, 2, 3) and (S) with the symmetric product replacing the ordinary product. One then obtains, instead of (3.2, 4) or (3.12, 14), intermediate inequalities of no special interest. The analogue of Proposition 4 still holds, but the important factorization property (Proposition 8) again breaks down.

We conclude this section by giving a non commutative example where (S) and (Q2) hold.

Example 6. $\mathfrak{A}$ is the algebra of $N \times N$ complex matrices. $S \equiv Q(S)$ is the set of matrices with real positive coefficients, and σ is defined by $\sigma(A) = \mathit{Tr}\, \varrho A / \mathit{Tr}\, \varrho$, where $\varrho \in S$. (S) and (Q2) are obviously satisfied, and Proposition 6 holds. A special case of this example has been described elsewhere [16]. It implies a version of Griffiths' first inequality for the isotropic Heisenberg model, which has been announced by Hurst and Sherman [17].

4. Conclusion

In this paper, we have presented a theory from which Griffiths' inequalities emerge in a natural way as consequences of two general conditions (Q2) and (Q3). We have given several examples satisfying these conditions, and including as special cases the Ising model with arbitrary spins, and the plane rotator model, both with many-body interactions. The theory allows for a natural although not unique non-commutative extension, which should be appropriate for quantum systems. In this case, we have only an example which fulfills (Q 2).

The general theory is completely elementary. Nevertheless, it brings into the foreground condition (Q 3) which seems to be the heart of the matter, and suggests that further progress would follow from a deeper analysis of this condition. In particular, it would be interesting to obtain a larger class of systems for which it is fulfilled. Examples are lacking in the noncommutative case. In the commutative case, a good candidate could be the convex cone of real positive definite functions on a non-commutative compact group, for instance the group of rotations in the usual euclidean space. For such systems as the classical Heisenberg model and similar models with higher dimensional spins, this would furthermore provide a larger class of interactions and observables satisfying (Q 3) than can be obtained by the method of Section 2.

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J. Ginibre
Laboratoire de Physique
Théorique et Hautes Energies
Bâtiment 211
Faculté des Sciences
F 91 Orsay