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Chern–Simons Theory with Finite Gauge Group

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Abstract. We construct in detail a $2 + 1$ dimensional gauge field theory with finite gauge group. In this case the path integral reduces to a finite sum, so there are no analytic problems with the quantization. The theory was originally introduced by Dijkgraaf and Witten without details. The point of working it out carefully is to focus on the algebraic structure, and particularly the construction of quantum Hilbert spaces on closed surfaces by cutting and pasting. This includes the "Verlinde formula." The careful development may serve as a model for dealing with similar issues in more complicated cases.

A typical course in quantum field theory begins with a thorough examination of a "toy model," usually the ϕ^4 theory. Our purpose here is to provide a detailed description of a "toy model" for *topological* quantum field theory, suitable for use as a foundation for more sophisticated developments. We carry through all the steps of the path integral quantization: start with a lagrangian, construct the classical action, construct a measure, and do the integral. When the gauge group is finite the "path integral" reduces to a finite sum. This remark makes it clear that the analytical difficulties simplify enormously, and that there should be no essential problem in carrying out the process. Many interesting features remain, however. The algebraic and topological structure are essentially unchanged, and are much clearer when not overshadowed by the analysis. And even the analysis does not entirely disappear: the details of the construction of the state spaces requires a much more precise formulation of the classical theory than is usually given, and reveals some incompleteness in the understanding of the classical theory for continuous Lie groups [F1].

Chern–Simons theory with finite gauge group was introduced by Dijkgraaf and Witten [DW], who essentially cataloged the possible lagrangians and gave some

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sample calculations. Special cases were considered by Segal [S2] and Kontsevich [K]. More abstract and mathematically-oriented versions are considered in [Q1, Q2], and connections with the representation-theoretic approach of Reshetikhin and Turaev are described by Ferguson [Fg] and Yetter [Y].

Let Γ denote the gauge group, which we assume to be finite. In Sects. 1 and 2 we carry out the quantization: describe the lagrangians, classical actions, and path integrals. The fields in this version are regular covering spaces $P \to X$ whose group of deck transformations is Γ. The action is a (torsion) characteristic number associated to a class in $H^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. We represent this class in singular cohomology by a singular cocycle, and use this to write the action (1.6) as an integral over X. (The cochain plays the role of the lagrangian.) We caution that the resulting theory depends in a subtle way on this choice of cocycle: a different choice gives a theory which is isomorphic in an appropriate sense, but the isomorphism between the two is not canonical: it depends on further choices. We must also make careful sense of integration of singular cocycles over manifolds with boundary; this is explained in Appendix B. The classical theory is somewhat unusual in that the action on manifolds with boundary is not a number, but rather an element in a complex line determined by the restriction of the field to the boundary. These lines, which properly belong to the hamiltonian theory, are the source of much of the structure in the theory. We defer some of the details of the classical theory to [F1] and [F3], which explores the classical Chern–Simons theory for *arbitrary* compact gauge groups. The quantization in Sect. 2 is straightforward. Of general interest is Lemma 2.4, which explains the behavior of symmetry groups and measures under gluing, and (2.18), which proves the gluing law in the quantum theory.

One distinguishing feature of the Chern–Simons theory (with any gauge group) in $2 + 1$ dimensions is its relationship to conformal field theory in $1 + 1$ dimensions. As a result one constructs quantum Hilbert spaces not only for closed surfaces, but also for surfaces with boundary,[1] and these vector spaces obey a gluing law related to the gluing law of the path integral. We explore these ideas, which we term "modular structure," in Sect. 3. (The name derives from the term "modular functor," which was coined by Segal [S1].) The quantization of the cylinder produces a semisimple coalgebra A, and the quantum spaces attached to surfaces with boundary are A-comodules. The gluing law (3.26) is expressed in terms of a cotensor product. (We review the relevant algebra in Appendix A.) In rational conformal field theories (see [MS], for example) one has a set of "labels" which index the conformal blocks. They appear in the definition of a modular functor [S1] and also in most treatments of Chern–Simons theory [W]. Here they index the irreducible corepresentations of A, as we discuss in Sect. 4. The gluing law for surfaces with labeled boundary is (4.11). The numerical factors which describe the behavior of the inner products also appear in Kevin Walker's careful treatment [Wa] of the $SU(2)$ theory. The Verlinde algebra [V] is easily derived from this gluing law.

We remark that all of these theories have "level" zero.[2] The cohomology class in $H^3(B\Gamma; \mathbb{R}/\mathbb{Z}) \cong H^4(B\Gamma; \mathbb{Z})$, which in the $SU(2)$ theory gives the level, is pure

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[1] This requires fixing certain boundary data at the outset. The role of these choices is investigated in [F5]

[2] As a result there are no central extensions of diffeomorphism groups, as there are in theories with continuous gauge group

torsion. In the simplest theory (the "untwisted theory") this class vanishes. Then the path integral essentially counts representations of the fundamental group into Γ. We include some computations for the untwisted theory in Sect. 5. Here we also find an *algebra* structure on A, and so A becomes a Hopf algebra. It is the dual of the Hopf algebra considered by Dijkgraaf, Pasquier and Roche [DPR], and plays the role of the quantum group in these finite theories. In Sect. 5 we also calculate the action of $SL(2;\mathbb{Z})$ on the quantum space attached to the torus in arbitrary twisted theories.

In a subsequent paper [F5] we investigate more fully the relationship between the path integral and the Hopf algebras, or quantum groups, which arise in the field theory. There we show how to interpret the path integral over manifolds of lower dimension, and how the quantum group emerges from those considerations.

The axioms for topological quantum field theory were first formulated by Atiyah [A], who also gives a mathematical perspective on the subject. These ideas are developed much further in [Q1, Q2] where more examples are pursued. There is an expository account of gluing laws in topological field theory in [F4].

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1. Classical Theory

The basic data of a Lagrangian classical field theory is a space of fields $\mathscr{C}_X$ for each spacetime X and a functional $S_X\colon \mathscr{C}_X \to \mathbb{R}$ called the *action*. The fields and action must be *local*, i.e., computable by cutting and pasting. Usually the action is the integral of a locally defined differential form over spacetime, and so is local by standard properties of the integral. Also, the action usually depends on a metric on spacetime. Here the theory is topological so no metric is needed. The formal properties of a classical topological field theory are carefully stated in Theorem 1.7. (See [S1, A], and Theorem 2.13 for analogous axiomatizations of topological quantum field theory.) In the theory we consider here a field is a finite regular covering space whose group of deck transformations is a fixed finite group Γ. Unlike fields in ordinary theories, which are usually functions, these fields (covering spaces) have automorphisms. This complication leads to our use of categorical language, which is designed to handle mathematical objects with automorphisms. We must keep track of automorphisms to make proper sense of cutting and pasting. The action for spacetimes without boundary is a characteristic class which takes its values in $\mathbb{R}/\mathbb{Z}$. For spacetimes with boundary the characteristic class is not standardly defined. We introduce an integration theory for singular cocycles in Appendix B, and express the characteristic class as an integral. Hence as in standard field theories our action is an integral over spacetime. We remark that we can also define the action using a refinement of integration in de Rham theory [F2]. In $2+1$ dimensions the classical field theory in this section is a special case of the classical Chern–Simons theory which is defined for arbitrary compact Lie groups [F1, F3].

Fix a finite group Γ. A *classifying space* $B\Gamma$ for Γ is a connected space with homotopy groups $\pi_1(B\Gamma,*)\cong\Gamma$ and $\pi_j(B\Gamma,*)=0, j\geqq 2$, for any basepoint $*\in B\Gamma$. In the topology literature this is also called an Eilenberg–MacLane space

$K(\Gamma, 1)$. We construct a classifying space as follows. Denote $\Gamma = \{g_1, \ldots, g_N\}$. Suppose H is an infinite dimensional separable complex Hilbert space, and let $E\Gamma$ be the Stiefel manifold of ordered orthonormal sets $\{v_{g_1}, \ldots, v_{g_N}\} \subset H$ labeled by the elements of Γ. Then $g \in \Gamma$ acts on $\{v_{g_1}, \ldots, v_{g_N}\}$ by permutation:

$$\{v_{g_1}, \ldots, v_{g_N}\} \cdot g = \{v_{g_1}, \ldots, v_{g_N g}\} .$$

This defines a free right action of Γ on $E\Gamma$, and we set $B\Gamma = E\Gamma/\Gamma$. Since $E\Gamma$ is contractible, the quotient $B\Gamma$ has the requisite properties.

Now a *principal Γ bundle* over a manifold M is a manifold P with a free right Γ action such that $P/\Gamma = M$. Notice that $P \to M$ is a finite regular covering space. A map of Γ bundles $\varphi\colon P' \to P$ is a smooth map which commutes with the Γ action. There is an induced map $\bar{\varphi}\colon M' \to M$ on the quotients. If $M' = M$ and $\bar{\varphi} = \mathrm{id}$, then we term φ a *morphism*. Notice that any morphism has an inverse. Hence if there exists a morphism $\varphi\colon P' \to P$, we say that P' is equivalent to P. A morphism $\varphi\colon P \to P$ is an *automorphism* (or *gauge transformation* or *deck transformation*). Let $\mathscr{C}_M$ be the category of principal Γ bundles over M and bundle morphisms. Since every morphism is invertible, this category is a groupoid. Finally, denote by $\overline{\mathscr{C}_M}$ the set of equivalence classes of Γ bundles over M. It is a finite set if M is compact. In fact, if M is connected then there is a natural identification

$$\overline{\mathscr{C}_M} \cong \mathrm{Hom}(\pi_1(M, m), \Gamma)/\Gamma \tag{1.1}$$

for any basepoint $m \in M$, where Γ acts by conjugation. The identification is via monodromy.

If $P \to M$ is a Γ bundle, then there exists a bundle map $F\colon P \to E\Gamma$. Such an F is called a *classifying map* for P. (This explains the term "classifying space.") To construct F note that the Γ bundle $(P \times E\Gamma)/\Gamma \to M$ has contractible fibers. Hence there exist sections $M \to (P \times E\Gamma)/\Gamma$, or equivalently Γ maps $P \to E\Gamma$. All sections are homotopic, whence all classifying maps are homotopic through Γ maps.

After these preliminaries we are ready to define the classical theory. Fix an integer $d \geqq 0$. The "spacetimes" in our theory have dimension $d + 1$ and the "spaces" have dimension d.[3] The ingredients of such a theory are a space of fields $\mathscr{C}_X$ attached to every compact oriented spacetime X and an action functional S_X defined on $\mathscr{C}_X$. In our theory we take $\mathscr{C}_X$ to be the discrete space of principal Γ bundles defined above.

Now the action. Fix a class[4] $\alpha \in H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ and a singular cocycle $\hat{\alpha} \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ which represents α. The simplest theory has $\alpha = \hat{\alpha} = 0$, in which case the action (and so the whole classical theory) is completely trivial. Suppose X is a *closed* oriented $(d+1)$-manifold. Fix $P \in \mathscr{C}_X$ and let $F\colon P \to E\Gamma$ be a classifying map. There is a quotient map $\bar{F}\colon X \to B\Gamma$. Set

$$S_X(P) = \alpha(\bar{F}_*[X]) \in \mathbb{R}/\mathbb{Z} , \tag{1.2}$$

where $[X] \in H_{d+1}(X)$ is the fundamental class. Since all classifying maps for P are homotopic through Γ maps, the action (1.2) does not depend on the choice of F.

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[3] One can consider CW complexes (of arbitrary dimension) instead of manifolds of a fixed dimension [Q1, Q2, K]. We restrict to manifolds to make contact with standard physical theories

[4] Notice that $H^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z}) \cong H^{d+2}(B\Gamma)$ since the real cohomology of $B\Gamma$ is trivial. We could consider complex cohomology classes in $H^{d+1}(B\Gamma; \mathbb{C}/\mathbb{Z})$, and then the resulting theories are *unitary* only if the class is real

Also, the action on closed manifolds only depends on the cohomology class α, not on the particular cocycle $\hat{\alpha}$. We often write the action as $e^{2\pi i S_X(P)} \in \mathbb{T}$; it takes values in the circle group $\mathbb{T}$ of complex numbers with unit norm.

The action on manifolds with boundary is not a number, but rather is an element in a complex line associated to the boundary. We first abstract the construction of vector spaces in situations where one must make choices. We like to call this the *invariant section construction.*[5] Suppose that the set of possible choices and isomorphisms of these choices forms a groupoid $\mathscr{C}$. Let $\mathscr{L}$ be the category whose objects are metrized complex lines (one dimensional inner product spaces) and whose morphisms are unitary isomorphisms. Suppose we have a functor $\mathscr{F}: \mathscr{C} \to \mathscr{L}$. Define $V_{\mathscr{F}}$ to be the inner product space of *invariant* sections of the functor $\mathscr{F}$: An element $v \in V_{\mathscr{F}}$ is a collection $\{v(C) \in \mathscr{F}(C)\}_{C \in \mathrm{Obj}(\mathscr{C})}$ such that if $C_1 \xrightarrow{\psi} C_2$ is a morphism, then $\mathscr{F}(\psi)v(C_1) = v(C_2)$. Supose $\mathscr{C}$ is *connected*, that is, there is a morphism between any two objects. Then $\dim V_{\mathscr{F}} = 0$ or $\dim V_{\mathscr{F}} = 1$, the latter occurring if and only if $\mathscr{F}$ *has no holonomy*, i.e., $\mathscr{F}(\psi) = \mathrm{id}$ for every automorphism $C_1 \xrightarrow{\psi} C$. We will apply this construction many times in this paper, both in the classical theory and in its quantization.

Return to the finite group Γ and cocycle $\hat{\alpha} \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$. Let $Q \to Y$ be a principal Γ bundle over a closed oriented d-manifold Y. We now define a metrized line L_Q. Consider the category $\mathscr{C}_Q$ whose objects are classifying maps $f: Q \to E\Gamma$ for Q and whose morphisms $f \xrightarrow{h} f'$ are homotopy classes rel boundary of Γ-homotopies $h: [0,1] \times Q \to E\Gamma$ from f to f'. The category $\mathscr{C}_Q$ is connected since any two classifying maps are Γ-homotopic. Define a functor $\mathscr{F}_Q: \mathscr{C}_Q \to \mathscr{L}$ as follows. Let

$$\mathscr{F}_Q(f) = I_{Y, \bar{f}^*\hat{\alpha}} \tag{1.3}$$

be the metrized integration line of Proposition B.1, and $\mathscr{F}_Q(f \xrightarrow{h} f')$ the map

$$\exp\left(2\pi i \int_{[0,1]\times Y} \bar{h}^*\hat{\alpha}\right): I_{Y,\bar{f}'^*\hat{\alpha}} \to I_{Y,\bar{f}'^*\hat{\alpha}}\,. \tag{1.4}$$

The integral in (1.4) is also defined in Proposition B.1. That $\mathscr{F}_Q$ is a functor follows from (B.2). That (1.4) only depends on the homotopy class of h follows from (B.3). Furthermore, $\mathscr{F}_Q$ has no holonomy.[6] For if $f \xrightarrow{h} f$ is an automorphism, then h determines a bundle map $h: S^1 \times Q \to E\Gamma$ by gluing. Since the bundle $S^1 \times Q \to S^1 \times Y$ extends over $D^2 \times Y$, so too does the map $\bar{h}: S^1 \times Y \to B\Gamma$ extend to a map $\bar{H}: D^2 \times Y \to B\Gamma$, and so

$$\exp\left(2\pi i \int_{S^1\times Y} \bar{h}^*\hat{\alpha}\right) = \exp\left(2\pi i \int_{D^2\times Y} \bar{H}^*\hat{\alpha}\right) = 1 \tag{1.5}$$

by Proposition B.1(e), which proves that $\mathscr{F}_Q$ has no holonomy. Let L_Q be the metrized line of invariant sections of $\mathscr{F}_Q$.

[5] This is a special case of a general construction (the "limit" or "inverse limit" or "projective limit") in category theory [Mac]

[6] Here it is crucial that our classifying maps are defined on the bundles and not just on the base spaces

Now suppose $P \to X$ is a Γ bundle over a compact oriented $(d+1)$-manifold X. For each classifying map $F\colon P \to E\Gamma$ consider the quantity

$$e^{2\pi i S_X(P,F)} = \exp\left(2\pi i \int_X \bar{F}^*\hat{\alpha}\right) \in L_{\partial X, \overline{\partial F}^*\hat{\alpha}} . \tag{1.6}$$

Here $\partial F = F|_{\partial P}$ is the restriction of F to the boundary. Suppose F' is another classifying map for P. Choose a homotopy $K\colon [0,1] \times P \to E\Gamma$ with $K|_{\{0\}\times P} = F$ and $K|_{\{1\}\times P} = F'$. Let $k = K|_{[0,1]\times\partial P}$ be the induced homotopy from ∂F to $\partial F'$. Denote $I = [0,1]$. There is a product class $[I]\times[X] \in H_{d+2}(I\times X, \partial(I\times X))$ with

$$\begin{aligned}\partial([I]\times[X]) &= \partial[I]\times[X] \cup (-1)^{\dim I}[I]\times\partial[X] \\ &= \{1\}\times[X] \cup -\{0\}\times[X] \cup -[I]\times[\partial X] .\end{aligned}$$

Choose representative chains i, x for $[I]$ and $[X]$. Then since $\hat{\alpha}$ is closed,

$$0 = \bar{F}'^*\hat{\alpha}(x) - \bar{F}^*\hat{\alpha}(x) - \bar{k}^*\hat{\alpha}(i\times\partial x) .$$

Applying $\exp(2\pi i\,\cdot)$ we see by (1.4) that

$$\mathscr{F}_{\partial P}(\partial F \xrightarrow{k} \partial F')\, e^{2\pi i S_X(P,F)} = e^{2\pi i S_X(P,F')} .$$

Thus (1.6) determines an invariant section of $\mathscr{F}_{\partial P}$, i.e., an element (of unit norm)

$$e^{2\pi i S_X(P)} \in L_{\partial P} .$$

This defines the action on manifolds with boundary.

The following theorem expresses what we mean by the statement "S_X is the action of a local Lagrangian field theory."

Theorem 1.7. *Let Γ be a finite group and $\hat{\alpha} \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ a cocycle. Then the assignments*

$$\begin{aligned} Q &\mapsto L_Q , \quad Q \in \mathscr{C}_Y , \\ P &\mapsto e^{2\pi i S_X(P)} , \quad P \in \mathscr{C}_X \end{aligned} \tag{1.8}$$

defined above for closed oriented d-manifolds Y and compact oriented $(d+1)$-manifolds X satisfy:

(a) (*Functoriality*) *If $\psi\colon Q' \to Q$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\psi}\colon Y' \to Y$, then there is an induced isometry*

$$\psi_*\colon L_{Q'} \to L_Q \tag{1.9}$$

and these compose properly. If $\varphi\colon P' \to P$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\varphi}\colon X' \to X$, then

$$(\partial\varphi)_*(e^{2\pi i S_{X'}(P')}) = e^{2\pi i S_X(P)} , \tag{1.10}$$

where $\partial\varphi\colon \partial P' \to \partial P$ is the induced map over the boundary.

(b) (*Orientation*) *There is a natural isometry*

$$L_{Q,-Y} \cong \overline{L_{Q,Y}} , \tag{1.11}$$

and

$$e^{2\pi i S_{-X}(P)} = \overline{e^{2\pi i S_X(P)}} \,. \tag{1.12}$$

(c) (*Additivity*[7]) *If $Y = Y_1 \sqcup Y_2$ is a disjoint union, and Q_i are bundles over Y_i, then there is a natural isometry*

$$L_{Q_1 \sqcup Q_2} \cong L_{Q_1} \otimes L_{Q_2} \,. \tag{1.13}$$

If $X = X_1 \sqcup X_2$ is a disjoint union, and P_i are bundles over X_i, then

$$e^{2\pi i S_{X_1 \sqcup X_2}(P_1 \sqcup P_2)} = e^{2\pi i S_{X_1}(P_1)} \otimes e^{2\pi i S_{X_2}(P_2)} \,. \tag{1.14}$$

(d) (*Gluing*) *Suppose $Y \subsetneq X$ is a closed oriented codimension one submanifold and X^{cut} is the manifold obtained by cutting X along Y. Then $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. Suppose P is a bundle over X, P^{cut} the induced bundle over X^{cut}, and Q the restriction of P to Y. Then*

$$e^{2\pi i S_X(P)} = \operatorname{Tr}_Q\left(e^{2\pi i S_{X^{\text{cut}}}(P^{\text{cut}})}\right) , \tag{1.15}$$

where Tr_Q is the contraction

$$\operatorname{Tr}_Q \colon L_{\partial P^{\text{cut}}} \cong L_{\partial P} \otimes L_Q \otimes \overline{L_Q} \to L_{\partial P} \tag{1.16}$$

using the hermitian metric on L_Q.

Several comments are in order. We allow the empty set $\emptyset$ as a manifold: $L_\emptyset = \mathbb{C}$ and $S_\emptyset = 0$. From a functorial point of view, (a) implies that $Q \mapsto L_Q$ defines a functor

$$\mathscr{C}_Y \to \mathscr{L}$$

and that each X determines an invariant section $e^{2\pi i S_X(\cdot)}$ of the composite functor $\mathscr{C}_X \to \mathscr{C}_{\partial X} \to \mathscr{L}$, where the first arrow is restriction to the boundary. The invariance of the action (1.10) on closed manifolds X means that if $P' \cong P$, then $S_X(P') = S_X(P)$. Hence the action passes to a function

$$S_X \colon \overline{\mathscr{C}_X} \to \mathbb{R}/\mathbb{Z} \,. \tag{1.17}$$

Bundle morphisms over compact oriented d-manifolds Y act on the corresponding lines, via (1.9). So there is a line bundle

$$\mathscr{L}_Y \to \mathscr{C}_Y \tag{1.18}$$

with a lift of the morphisms in $\mathscr{C}_Y$. (Since $\mathscr{C}_Y$ is a discrete set of points, $\mathscr{L}_Y$ is a discrete union of lines.) If X is a compact oriented $(d+1)$-manifold, there is an induced line bundle $\mathscr{L}_X \to \mathscr{C}_X$, obtained by pulling back $\mathscr{L}_{\partial X}$ via the restriction map $\mathscr{C}_X \to \mathscr{C}_{\partial X}$.The action $e^{2\pi i S_X(\cdot)}$ is an invariant section of $\mathscr{L}_X \to \mathscr{C}_X$. In particular, the group of automorphisms Aut P of $P \to X$ acts on the line over $P \in \mathscr{C}_X$. If this action is nontrivial, then $e^{2\pi i S_X(P)} = 0$. Theorem 1.7 expresses in a (necessarily) complicated way the fact that "S_X" is a local functional of local fields defined as the integral of a local expression (c), (d); is invariant under symmetries of the fields (a); and changes sign under orientation reversal (b).

[7] Although (1.14) looks like a multiplicative property, it expresses the additivity of the classical action S_X. However, S_X is not defined if $\partial X \neq \emptyset$, which is why we use the exponential notation $e^{2\pi i S_X(\cdot)}$

Proof. The proof of Theorem 1.7 is straightforward but tedious. For example, to construct the map (1.9), choose a classifying map $f': Q' \to E\Gamma$ for Q', and let $f \circ \psi^{-1}: Q \to E\Gamma$ be the induced classifying map for Q. Fix a representative $y' \in C_d(Y')$ of $[Y']$, and let $y = \bar{\psi}_*(y') \in C_d(Y)$ be the corresponding representative of $[Y]$. These choices determine trivializations $L_{Q'} \cong \mathbb{C}$ and $L_Q \cong \mathbb{C}$. Relative to these trivializations we define ψ_* in (1.9) to be the identity. A routine check shows that this is independent of the choices and composes properly. The constructions of the isometries (1.11) and (1.13) are similar. Equations (1.10), (1.12), (1.14), and (1.16) follow from the corresponding properties of the integral (Proposition B.1).

The metrized line bundle (1.18) passes to a (possibly degenerate) metrized line bundle

$$\overline{\mathscr{L}_Y} \to \overline{\mathscr{C}_Y} \tag{1.19}$$

over the finite set of equivalence classes. The fiber $L_{[Q]}$ is the space of invariant sections of the functor $Q \mapsto L_Q$ as Q ranges over the equivalence class $[Q]$. If Aut Q acts nontrivially on L_Q then $L_{[Q]} = 0$; if Aut Q acts trivially then $\dim L_{[Q]} = 1$.

2. Quantum Theory

There is one crucial piece of data which must be added to a classical field theory to define path integral quantization: a measure on the space of fields. In many quantum field theories this is only done formally. In our theory this is easy to do precisely since the space of fields is discrete. We simply count each bundle according to the number of its automorphisms (2.1). The path integral for closed spacetimes is then defined directly (2.9) as the integral of the action over the space of (equivalence classes of) fields, whereas for manifolds with boundary the path integral is a function of the field on the boundary (2.12). In usual field theories the fields on space (and on spacetime) have continuous parameters; then extra geometry[8] is introduced to carry out "canonical quantization," and again this is only a formal procedure in many cases of interest. Here no extra geometry is needed since the space of fields is discrete. The quantum Hilbert space is the space of all functions of fields (2.10) (which respect the symmetry group); here it is finite dimensional. The path integral and quantum Hilbert spaces satisfy a set of axioms we spell out in Theorem 2.13 (cf. [Se, A]). They mostly follow from the corresponding properties of the classical theory (Theorem 1.7) and the gluing properties of the measure (Lemma 2.4). We emphasize that since the configuration space $\overline{\mathscr{C}_X}$ of equivalence classes of fields is finite, the path integral reduces to a finite sum.

We introduce a measure μ on the collection $\mathscr{C}_M$ of principal Γ bundles over any manifold M. Namely, set

$$\mu_P = \frac{1}{\# \operatorname{Aut} P}, \tag{2.1}$$

where Aut P is the group of automorphisms of P. If $P' \cong P$, then Aut $P' \cong$ Aut P, so that $\mu_{P'} = \mu_P$. Hence there is an induced measure on the set of equivalence classes $\overline{\mathscr{C}_M}$.

[8] In geometric quantization this is called a "polarization"

Suppose M is a manifold with boundary, and $Q \in \mathscr{C}_{\partial M}$ is a Γ bundle over the boundary. Define the category

$$\mathscr{C}_M(Q) = \{\langle P, \theta\rangle : P \in \mathscr{C}_M, \theta : \partial P \to Q \text{ is an isomorphism}\}$$

of bundles over M whose boundary has a specified isomorphism to Q. A morphism $\varphi: \langle P', \theta'\rangle \to \langle P, \theta\rangle$ is an isomorphism $\varphi: P' \to P$ such that $\theta' = \theta \circ \partial\varphi$. Two elements $\langle P', \theta'\rangle$, $\langle P, \theta\rangle \in \mathscr{C}_M(Q)$ are equivalent if there exists a morphism $\langle P', \theta'\rangle \to \langle P, \theta\rangle$. We denote the set of equivalence classes by $\overline{\mathscr{C}_M}(Q)$. Equation (2.1) defines a measure on $\mathscr{C}_M(Q)$, where we now interpret "Aut$\langle P, \theta\rangle$" in the sense just described. Notice that such automorphisms are trivial over components of M with nonempty boundary. Again the measure passes to the quotient $\overline{\mathscr{C}_M}(Q)$. Finally, if $\psi: Q' \to Q$ is a morphism, there is an induced measure preserving map

$$\psi_* : \overline{\mathscr{C}_M}(Q') \to \overline{\mathscr{C}_M}(Q) . \tag{2.2}$$

Next, we investigate the behavior of these measures under cutting and pasting. Suppose $N \subsetneq M$ is an oriented codimension one submanifold and M^{cut} the manifold obtained by cutting M along N. Then $\partial M^{\mathrm{cut}} = \partial M \sqcup N \sqcup -N$. Fix a bundle $Q \to N$. Then $\mathscr{C}_{M^{\mathrm{cut}}}(Q \sqcup Q)$ is the category of triples $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle$, where $P^{\mathrm{cut}} \to M^{\mathrm{cut}}$ is a Γ bundle and $\theta_i: P^{\mathrm{cut}}|_N \to Q$ are isomorphisms over the two copies of N in M^{cut}. Consider the gluing map

$$\begin{aligned} g_Q &: \overline{\mathscr{C}_{M^{\mathrm{cut}}}}(Q \sqcup Q) \to \overline{\mathscr{C}_M} , \\ &\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle \mapsto P^{\mathrm{cut}}/(\theta_1 = \theta_2) \end{aligned} \tag{2.3}$$

or equivalence classes.

Lemma 2.4. *The gluing map g_Q satisfies*

(a) *g_Q maps onto the set of bundles over M whose restriction to N is isomorphic to Q.*
(b) *Let $\phi \in \mathrm{Aut}\, Q$ act on $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle \in \mathscr{C}_{M^{\mathrm{cut}}}(Q \sqcup Q)$ by*

$$\phi \cdot \langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle = \langle P^{\mathrm{cut}}, \phi \circ \theta_1, \phi \circ \theta_2\rangle .$$

Then the stabilizer of this action at $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle$ is the image $\mathrm{Aut}\, P \to \mathrm{Aut}\, Q$ *determined by the θ_i, where $P = g_Q(\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle)$.*
(c) *There is an induced action on equivalence classes $\overline{\mathscr{C}_{M^{\mathrm{cut}}}}(Q \sqcup Q)$, and* Aut Q *acts transitively on $g_Q^{-1}([P])$ for any $[P] \in \overline{\mathscr{C}_M}$.*
(d) *For all $[P] \in \overline{\mathscr{C}_M}$ we have*

$$\mu_{[P]} = \mathrm{vol}\,(g_Q^{-1}([P])) \cdot \mu_Q . \tag{2.5}$$

Proof. If $P \to M$ is a bundle and $\theta: P|_N \to Q$ an isomorphism, then $g_Q(\langle P^{\mathrm{cut}}, \theta, \theta\rangle) \cong P$, where P^{cut} is the pullback of P under the gluing map $M^{\mathrm{cut}} \to M$. This proves (a). If $g_Q(\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle) \cong g_Q(\langle \widetilde{P^{\mathrm{cut}}}, \tilde\theta_1, \tilde\theta_2\rangle)$, then there exists an isomorphism $\varphi: P^{\mathrm{cut}} \to \widetilde{P^{\mathrm{cut}}}$ such that $\tilde\theta_1 \varphi \theta_1^{-1} = \tilde\theta_2 \varphi \theta_2^{-1} \in \mathrm{Aut}\, Q$. Call this element ϕ. Then φ determines an isomorphism $\phi \cdot \langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle \cong \langle \widetilde{P^{\mathrm{cut}}}, \tilde\theta_1, \tilde\theta_2\rangle$. This proves (c). If $\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle = \langle \widetilde{P^{\mathrm{cut}}}, \tilde\theta_1, \tilde\theta_2\rangle$, then φ determines an element of Aut P, and ϕ is the restriction of this element to Q. This proves (b). Now the exact sequence

$$1 \to \mathrm{Aut}\langle P^{\mathrm{cut}}, \theta_1, \theta_2\rangle \to \mathrm{Aut}\, P \to \mathrm{Aut}\, Q$$

together with (b) and (c) imply

$$\# \operatorname{Aut} P = \# \operatorname{Aut} \langle P^{\text{cut}}, \theta_1, \theta_2 \rangle \frac{\# \operatorname{Aut} Q}{\#(g_Q^{-1}(P))}.$$

This is equivalent to (2.5).

We remark that this computation is valid both for automorphisms which are the identity over ∂M and automorphisms which are unrestricted on ∂M.

Now we are ready to carry out the quantization. The quantum theory assigns a complex inner product space $E(Y)$ to every closed oriented d-manifold Y and a vector $Z_X \in E(\partial X)$ to every compact oriented $(d+1)$-manifold X. If X is closed, then Z_X is a complex number. We begin with the $\alpha = \hat{\alpha} = 0$ theory, which is surely the simplest quantum field theory that one could imagine. For X closed the path integral is

$$Z_X = \int_{\overline{\mathscr{C}_X}} d\mu([P]) = \operatorname{vol}(\overline{\mathscr{C}_X}). \tag{2.6}$$

The vector space attached to Y is

$$E(Y) = L^2(\overline{\mathscr{C}_Y}). \tag{2.7}$$

If X is a manifold with boundary, then the path integral is

$$Z_X(Q) = \int_{\overline{\mathscr{C}_X(Q)}} d\mu([P]) = \operatorname{vol}(\overline{\mathscr{C}_X(Q)}), \quad Q \in \mathscr{C}_{\partial X}. \tag{2.8}$$

Since (2.2) is a measure preserving map, $Z_X(Q)$ only depends on the equivalence class of Q, so defines an element of $E(\partial X)$ as desired.

The formulae for the twisted case ($\hat{\alpha} \neq 0$) are obtained by substituting the nontrivial action $e^{2\pi i S_X(\cdot)}$ for the trivial action 1 in (2.6)–(2.8). Thus for X closed we define the partition function

$$Z_X = \int_{\overline{\mathscr{C}_X}} d\mu([P])\, e^{2\pi i S_X([P])}, \tag{2.9}$$

where $S_X([P])$ is the action (1.17) on the quotient. We emphasize that the (path) integral in (2.9) is a finite sum. For Y a closed oriented d-manifold we have the possibly degenerate metrized line bundle (1.19). Set

$$E(Y) = L^2(\overline{\mathscr{C}_Y}, \overline{\mathscr{L}_Y}). \tag{2.10}$$

In other words, $E(Y)$ is the space of invariant sections of the functor

$$\mathscr{F}_Y \colon \mathscr{C}_Y \to \mathscr{L},$$
$$Q \mapsto L_Q.$$

If v, v' are invariant sections, then

$$(v, v')_{E(Y)} = \sum_{[Q] \in \overline{\mathscr{C}_Y}} \mu_Q(v(Q), v'(Q))_{L_Q}, \tag{2.11}$$

where Q is a bundle in the equivalence class $[Q]$. Finally, if X is a manifold with boundary, set

$$Z_X(Q) = \int_{\overline{\mathscr{C}_X(Q)}} d\mu([P])\, e^{2\pi i S_X([P])} \in L_Q, \quad Q \in \mathscr{C}_{\partial X}\,. \tag{2.12}$$

A simple application of (2.2) and (1.10) proves that (2.12) defines an invariant section of the functor $\mathscr{F}_{\partial X}$, so an element in $E(\partial X)$.

The following theorem expresses what we mean by the statement "$E(Y)$ and Z_X define a (unitary) topological quantum field theory." It is essentially the list of axioms in [A].

Theorem 2.13. *Let Γ be a finite group and $\hat{\alpha} \in C^{d+1}(B\Gamma; \mathbb{R}/\mathbb{Z})$ a cocycle. Then the assignments*

$$Y \mapsto E(Y)\,,$$
$$X \mapsto Z_X\,, \tag{2.14}$$

defined above for closed oriented d-manifolds Y and compact oriented $(d+1)$-manifolds X satisfy:

(a) (*Functoriality*) *Suppose $f\colon Y' \to Y$ is an orientation preserving diffeomorphism. Then there is an induced isometry*

$$f_*\colon E(Y') \to E(Y) \tag{2.15}$$

and these compose properly. If $F\colon X' \to X$ is an orientation preserving diffeomorphism, then

$$(\partial F)_*(Z_{X'}) = Z_X\,, \tag{2.16}$$

where $\partial F\colon \partial X' \to \partial X$ is the induced map over the boundary.
(b) (*Orientation*) *There is a natural isometry*

$$E(-Y) \cong \overline{E(Y)}\,,$$

and

$$Z_{-X} = \overline{Z_X}\,.$$

(c) (*Multiplicativity*) *If $Y = Y_1 \sqcup Y_2$ is a disjoint union, then there is a natural isometry*

$$E(Y_1 \sqcup Y_2) \cong E(Y_1) \otimes E(Y_2)\,.$$

If $X = X_1 \sqcup X_2$ is a disjoint union, then

$$Z_{X_1 \sqcup X_2} = Z_{X_1} \otimes Z_{X_2}\,.$$

(d) (*Gluing*) *Suppose $Y \subsetneq X$ is a closed oriented codimension one submanifold and X^{cut} is the manifold obtained by cutting X along Y. Write $\partial X^{\mathrm{cut}} = \partial X \sqcup Y \sqcup -Y$. Then*

$$Z_X = \mathrm{Tr}_Y(Z_{X^{\mathrm{cut}}})\,, \tag{2.17}$$

where Tr_Y is the contraction

$$\mathrm{Tr}_Y\colon E(\partial X^{\mathrm{cut}}) \cong E(\partial X) \otimes E(Y) \otimes \overline{E(Y)} \to E(\partial X)$$

using the hermitian metric on $E(Y)$.

Proof. The map f induces a measure preserving functor $f^*: \mathscr{C}_Y \to \mathscr{C}_{Y'}$, which lifts to $f^*: L_Y \to L_{Y'}$ in view of (1.9). Then (2.15) is the induced pullback on invariant sections. The diffeomorphism invariance (2.16) follows from (1.10). The assertions in (b) are direct consequences of Theorem 1.7(b). For (c) we use Theorem 1.7(c) and the fact that for any disjoint union

$$\mathscr{C}_{M_1 \sqcup M_2} = \mathscr{C}_{M_1} \times \mathscr{C}_{M_2} .$$

It remains to prove the gluing law (d). Fix a bundle $Q' \to \partial X$. Then for each $Q \to Y$ and each $P^{\mathrm{cut}} \in \mathscr{C}_{X^{\mathrm{cut}}}(Q' \sqcup Q \sqcup Q)$ we have

$$e^{2\pi i S_X(g_Q(P^{\mathrm{cut}}))} = \mathrm{Tr}_Q(e^{2\pi i S_{X^{\mathrm{cut}}}(P^{\mathrm{cut}})})$$

by (1.15). Fix a set of representatives $\{Q\}$ for $\overline{\mathscr{C}_Y}$. Let $\overline{\mathscr{C}_X}(Q')_Q$ denote the equivalence classes of bundles over X whose restriction to ∂X is Q' and to Y is Q. Recall the gluing map (2.3) and Eq. (2.5) relating the measures. Then

$$\begin{aligned} Z_X(Q') &= \int_{\overline{\mathscr{C}_X}(Q')} d\mu([P])\, e^{2\pi i S_X([P])} \\ &= \sum_{Q \in \{Q\}} \int_{\overline{\mathscr{C}_X}(Q')_Q} d\mu([P])\, e^{2\pi i S_X([P])} \\ &= \sum_{Q \in \{Q\}} \int_{\overline{\mathscr{C}_{X^{\mathrm{cut}}}}(Q' \sqcup Q \sqcup Q)} d\mu([P^{\mathrm{cut}}])\, \mu_Q \mathrm{Tr}_Q(e^{2\pi i S_{X^{\mathrm{cut}}}([P^{\mathrm{cut}}])}) \\ &= \sum_{Q \in \{Q\}} \mu_Q \mathrm{Tr}_Q(Z_{X^{\mathrm{cut}}}(Q' \sqcup Q \sqcup Q)). \end{aligned} \tag{2.18}$$

The definition (2.11) of the inner product shows that this is equivalent to (2.17).

3. Surfaces with Boundary

Now we specialize to $d = 2$ – that is, to the $2 + 1$ dimensional theory – and examine the more detailed structure associated to surfaces with boundary. The classical theory of Sect. 1 assigns a metrized complex line L_Q to each Γ bundle $Q \to Y$ over a closed oriented 2-manifold. In this section we construct lines when Y has a boundary, but only after fixing certain choices on the boundary.[9] These lines obey a gluing law, which we state in Theorem 3.2. The basepoints and boundary parametrizations which appear in that theorem are part of the process of fixing choices on the boundary. By gluing cylinders these lines lead to certain central extensions of subgroups of Γ (3.13), which fit together into a central extension of a certain groupoid (3.9). The quantization then extends the definition of $E(Y)$ to surfaces with boundary and provides a method for computing this vector space by cutting and gluing. There is a rich algebraic structure: The vector space attached to the cylinder is a *coalgebra*, each boundary component of a surface determines a *comodule* structure on its quantum space, and the gluing law appears naturally in

[9] The dependence on these choices leads to a consideration of *gerbes* [B, BM, F5]

terms of *cotensor products*.[10] An important point is the behavior of the inner product under gluing (3.26), which comes quite naturally in our approach. The properties of these quantum Hilbert spaces are stated in Theorem 3.21, which is the main result in this section. We analyze the algebraic structures more closely in Sect. 4. Throughout this section we work with a fixed cocycle $\hat{\alpha} \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. We remark that there are several simplifications if $\hat{\alpha} = 0$, some of which we discuss in Sect. 5.

We begin with the classical theory. The important point is to rigidify the data over the boundary of a surface. Hence fix the standard circle $S^1 = [0, 1]/0 \sim 1$.[11] Consider first Γ bundles $R \to S^1$ with a basepoint in R chosen over the basepoint in S^1. Morphisms are required to preserve the basepoints. We denote the category of these pointed bundles and morphisms by $\mathscr{C}'_{S^1}$. Notice that there are no nontrivial automorphisms, since a deck transformation which is the identity at one point is the identity everywhere (on any connected space). Further, the basepoint determines a holonomy map $\mathscr{C}'_{S^1} \to \Gamma$, and the induced map $\overline{\mathscr{C}'_{S^1}} \to \Gamma$ on equivalence classes is a bijection. Summarizing, if R_1, R_2 are Γ bundles over S^1 with the same holonomy, then there is a *unique* isomorphism $R_1 \to R_2$ which preserves basepoints.

For each $g \in \Gamma$ fix once and for all a pointed bundle $\mathbf{R}_g \to S^1$ with holonomy g and a classifying map

$$\phi_g : \mathbf{R}_g \to E\Gamma . \tag{3.1}$$

Let Y be a compact oriented 2-manifold. Fix a diffeomorphism $S^1 \to (\partial Y)_i$ for each component $(\partial Y)_i$ of ∂Y. A boundary component is labeled " + " if the parametrization preserves orientation and " − " otherwise. The images of the basepoint in S^1 give a basepoint on each component of ∂Y. Then the Γ bundles $Q \to Y$ with basepoints chosen over the basepoints of ∂Y form a category $\mathscr{C}'_Y$; morphisms in this category are required to preserve the basepoints.

Theorem 3.2. *Let Y be a compact oriented 2-manifold with parametrized boundary. Then there is a functor*

$$Q \mapsto L_Q, \quad Q \in \mathscr{C}'_Y ,$$

which attaches to each Γ bundle $Q \to Y$ with basepoints a metrized line L_Q. It generalizes the corresponding functor (1.8) *for closed surfaces, and satisfies the functoriality*[12] (1.9), *orientation* (1.11), *and additivity* (1.13) *properties. In addition it satisfies:*

(*Gluing*) *Suppose $S \hookrightarrow Y$ is a closed oriented codimension one submanifold and Y^{cut} the manifold obtained by cutting along S. Then $\partial Y^{\text{cut}} = \partial Y \sqcup S \sqcup -S$ and we use parametrizations which agree on S and $-S$. Suppose $Q \in \mathscr{C}'_Y$ is a bundle over Y and $Q^{\text{cut}} \in \mathscr{C}'_{Y^{\text{cut}}}$ the induced bundle over Y^{cut}.* (*We choose basepoints over S and $-S$ which agree.*) *Then there is a natural isometry*

$$L_Q \cong L_{Q^{\text{cut}}} . \tag{3.3}$$

---

[10] A more elaborate development of the formal properties gives a tensor product description, see [Q2, ¶¶8, 9] for a treatment of finite gauge groups and an axiomatization

[11] For the integration theory of Appendix B we also need to fix a standard cycle $s \in C_1(S^1)$ which represents the fundamental class

[12] The functoriality holds for maps which preserve the basepoints and the boundary parametrizations

Proof. Let $Q \to Y$ be a pointed bundle. Using the boundary parametrizations we identify ∂Y as a disjoint union of circles. Since pointed bundles over the circle have no automorphisms, there is a unique based isomorphism of ∂Q with a disjoint union of our standard pointed bundles $\mathbf{R}_g$. Then the ϕ_g chosen above (3.1) determine a classifying map $\phi\colon \partial Q \to E\Gamma$. Let $\mathscr{C}_Q$ denote the category of classifying maps $f\colon Q \to E\Gamma$ which extend ϕ. A morphism $f \xrightarrow{h} f'$ is a homotopy h which is constant on ∂Q, or better a homotopy class rel boundary of such homotopies. As in (1.4) we define a functor $\mathscr{F}_Q\colon \mathscr{C}_Q \to \mathscr{L}$ by

$$\mathscr{F}_Q(f) = I_{Y,\bar{f}^*\hat{\alpha}}$$

and

$$\mathscr{F}_Q(f \xrightarrow{h} f') = e^{-2\pi i S_{S^1\times Y}(S^1\times Q)} \exp\left\{2\pi i \int_{[0,1]\times Y} \bar{h}^*\hat{\alpha}\right\}\colon I_{Y,\bar{f}^*\hat{\alpha}} \to I_{Y,\bar{f}'^*\hat{\alpha}}\,, \qquad (3.4)$$

but now we must reinterpret the formulae. First, the integration lines are defined for manifolds with boundary in Proposition B.5 of Appendix B. If $\partial Y = \emptyset$, then $S_{S^1\times Y}(S^1 \times Q) = 0$ by the argument in (1.5). If $\partial Y \neq \emptyset$, then the prefactor in (3.4) is an element in the line $L_{S^1\times\partial Q}$, which is trivialized by the classifying map ϕ and the trivialization (B.9). (This follows from the construction of $L_{S^1\times\partial Q}$ in Sect. 1. The integration line (1.3) in that construction is trivialized using the parametrization of ∂Y and (B.9). So this prefactor is a complex number. Since

$$\partial([0,1]\times Y) = \{1\}\times Y \cup -\{0\}\times Y \cup -[0,1]\times\partial Y\,,$$

the second factor in (3.4) is an element of

$$I_{Y,\bar{f}'^*\hat{\alpha}} \otimes I^*_{Y,\bar{f}^*\hat{\alpha}} \otimes I^*_{[0,1]\times\partial Y,\,\phi^*\hat{\alpha}}\;. \qquad (3.5)$$

Here we use Proposition B.5, in particular the gluing law (B.6). But the boundary parametrization and the trivialization (B.8) trivialize the last factor in (3.5). Hence (3.4) is well-defined. The fact that $\mathscr{F}_Q$ defines a functor and that this functor has no holonomy are routine checks. Both use the compatibility of (B.8) and (B.9) under gluing. Define L_Q to be the line of invariant sections of $\mathscr{F}_Q$.

We leave the verification of (3.3) to the reader.

As a corollary we obtain a metrized line bundle

$$\mathscr{L}_Y \to \mathscr{C}'_Y$$

which generalizes (1.19). Furthermore, we remark that (2.1) defines a measure on $\mathscr{C}'_Y$ which is invariant under morphisms, so passes to a measure on the set of equivalence classes $\overline{\mathscr{C}'_Y}$. If each component of Y has nonempty boundary, then this measure has unit mass on each bundle.

The fact that pointed bundles over S^1 have no automorphisms makes gluing well-defined on *equivalence classes* of bundles. More precisely, consider a compact oriented 2-manifold Y with parametrized boundary, an oriented codimension one submanifold $S \subsetneq Y$, and the resulting cut manifold Y^{cut}. Suppose $Q^{\text{cut}} \in \mathscr{C}'_{Y^{\text{cut}}}$ restricts to isomorphic bundles over the two copies of S in Y^{cut}. Then there is a well-determined bundle $g(Q^{\text{cut}}) \in \mathscr{C}'_Y$ obtained by gluing. (Compare (2.3).) In other words, there is a gluing map

$$g\colon \mathscr{B}_{Y^{\text{cut}}} \subset \mathscr{C}'_{Y^{\text{cut}}} \to \mathscr{C}'_Y$$

defined on the subset $\mathscr{B}_{Y^{\text{cut}}}$ of bundles which are isomorphic on the two copies of S. Since the isomorphism used to perform the gluing is unique, there is an induced gluing

$$g\colon \overline{\mathscr{B}_{Y^{\text{cut}}}} \subset \overline{\mathscr{C}'_{Y^{\text{cut}}}} \to \overline{\mathscr{C}'_Y} \tag{3.6}$$

on equivalence classes of bundles. Suppose $Y^{\text{cut}} = Y_1 \sqcup Y_2$ and $Q^{\text{cut}} = Q_1 \sqcup Q_2$. Further, suppose the copy of S in Y_1 has a $+$ parametrization and the copy in Y_2 has a $-$ parametrization. Then we denote the glued bundle over Y as

$$Q_1 \circ_S Q_2 = Q_1 \circ Q_2 \,.$$

Theorem 3.2(d) implies that the gluing lifts to the associated lines. In other words, there is an isometry

$$L_{Q_1} \otimes L_{Q_2} \to L_{Q_1 \circ Q_2} \,. \tag{3.7}$$

As gluing makes sense on the equivalence classes, we write $[Q]_1 \circ [Q]_2$ for the glued element in $\overline{\mathscr{C}'_Y}$. Then (3.7) induces an isometry

$$L_{[Q]_1} \otimes L_{[Q]_2} \to L_{[Q]_1 \circ [Q]_2} \,. \tag{3.8}$$

(Recall, however, that these "lines" may be the zero vector space.)

We apply this first to the cylinder $Y = [0, 1] \times S^1$ which we cut along $\{1/2\} \times S^1$. Then $\overline{\mathscr{C}'_{Y^{\text{cut}}}} = \overline{\mathscr{C}'_Y} \times \overline{\mathscr{C}'_Y}$, and the gluing provides a *groupoid* structure on

$$\mathscr{G} = \overline{\mathscr{C}'_{[0,1]\times S^1}} \,. \tag{3.9}$$

Let $\{*\} \in S^1$ be the basepoint. An element[13] $[T]_{\langle x, g\rangle} \in \mathscr{G}$ is given by a pair $\langle x, g\rangle \in \Gamma \times \Gamma$, where x is the holonomy around $\{0\} \times S^1$ and g is the parallel transport along $[0, 1] \times \{*\}$. This parallel transport is well-defined since the bundle has a basepoint over $\{0\} \times \{*\}$ and one over $\{1\} \times \{*\}$. The groupoid composition is then

$$[T]_{\langle x_1, g_1\rangle} \circ [T]_{\langle x_2, g_2\rangle} = [T]_{\langle x_1, g_2 g_1\rangle}, \quad \text{if } x_2 = g_1 x_1 g_1^{-1} \,, \tag{3.10}$$

and is undefined if $x_2 \neq g_1 x_1 g_1^{-1}$. Let $L_{\langle x, g\rangle} = L_{[T]_{\langle x, g\rangle}}$ denote the line attached to the equivalence class $[T]_{\langle x,g\rangle}$. Note that $\dim L_{\langle x,g\rangle} = 1$ since pointed bundles over the cylinder have trivial automorphism groups. Then (3.7) in this context is an isometry

$$L_{\langle x_1, g_1\rangle} \otimes L_{\langle x_2, g_2\rangle} L_{\langle x_1, g_2 g_1\rangle}, \quad \text{if } x_2 = g_1 x_1 g_1^{-1} \,. \tag{3.11}$$

In particular, for $g_1 = g_2 = e$ and any $x \in \Gamma$ this gives a trivialization

$$L_{\langle x, e\rangle} \cong \mathbb{C} \,. \tag{3.12}$$

Restricting to unit vectors the isometries (3.11) define a central extension $\hat{\mathscr{G}}$ of the groupoid $\mathscr{G}$ by $\mathbb{T}$. For each $x \in \Gamma$ the set

$$\{[T]_{\langle x, g\rangle} \colon gx = xg\}$$

[13] We use "T" to denote bundles over the cylinder and "Q" to denote bundles over arbitrary surfaces

is closed under the composition (3.10) and is isomorphic[14] to the centralizer C_x of x in Γ. The lines (3.11) then give a central extension

$$1 \to \mathbb{T} \to \hat{C}_x \to C_x \to 1 . \tag{3.13}$$

The equivalence class of this extension can be expressed in terms of the cohomology class $[\hat{\alpha}] \in H^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. Namely, there is a homotopy equivalence

$$\operatorname{Map}(S^1, B\Gamma) \sim \coprod_{[x]} BC_x ,$$

where x runs over representatives of the conjugacy classes in Γ. Let

$$e\colon S^1 \times \operatorname{Map}(S^1, B\Gamma) \to B\Gamma$$

be the evaluation map and

$$\pi\colon S^1 \times \operatorname{Map}(S^1, B\Gamma) \to \operatorname{Map}(S^1, B\Gamma)$$

the projection.

Proposition 3.14. *The cohomology class of the extension* (3.13) *is* (*a component of*) *the transgression*

$$\pi_* e^* [\hat{\alpha}] \in \bigoplus_{[x]} H^2(BC_x; \mathbb{R}/\mathbb{Z}) .$$

Proof. First, we recall that the central extension (3.13) determines a class in $H^2(C_x; \mathbb{R}/\mathbb{Z})$, the second group cohomology. To construct a cocycle, for each $g \in C_x$ choose an element $\hat{g} \in \hat{C}_x$ covering g. Then set

$$c(g_1, g_2) = \widehat{g_1 g_2}\, \hat{g}_2^{-1} \hat{g}_1^{-1} \subset \mathbb{R}/\mathbb{Z}, \quad g_1, g_2 \in C_x . \tag{3.15}$$

This is a cocycle for the group cohomology.

Next, the group cohomology is isomorphic to the (singular) cohomology of the classifying space as follows. Let $\hat{\beta} \in C^2(BC_x; \mathbb{R}/\mathbb{Z})$ be a singular cocycle. Then for each $g \in BC_x$ fix a based loop $\gamma_g\colon S^1 \to BC_x$ whose homotopy class corresponds to g under the isomorphism $\pi_1(BC_x) \cong C_x$. Denote $I_g = I_{S^1, \gamma_g^* \hat{\beta}}$ for the integration line of Proposition B.1. Choose an element $\varepsilon_g \in I_g$ of unit norm. For $g_1, g_2 \in C_x$ the composite loop $\gamma_{g_1}\gamma_{g_2}$ is homotopic to $\gamma_{g_1 g_2}$. Let

$$k_{g_1, g_2}\colon [0, 1] \times S^1 \to BC_x$$

be a homotopy. By the integration theory of Appendix B this determines an isometry[15]

$$\theta_{g_1, g_2} = \int_{[0,1] \times S^1} k_{g_1, g_2}^* \hat{\beta}\colon I_{g_1} \otimes I_{g_2} \to I_{g_1 g_2} . \tag{3.16}$$

[14] In fact, it is anti-isomorphic to C_x because the composition law (3.10) reverses the order of multiplication

[15] We use the $d = 1$ case of (B.6) to identify the integration line of $\gamma_{g_1}\gamma_{g_2}$ with $I_{g_1} \otimes I_{g_2}$ (cf. the footnote preceding Proposition B.5). Also, by (B.3) the map θ_{g_1, g_2} only depends on the homotopy class of k_{g_1, g_2}

Define $c'(g_1, g_2) \in \mathbb{R}/\mathbb{Z}$ by

$$e^{2\pi i c'(g_1, g_2)}\, \theta_{g_1, g_2}(\varepsilon_{g_1} \otimes \varepsilon_{g_2}) = \varepsilon_{g_1 g_2} \,. \tag{3.17}$$

An easy check shows that c' is a cocycle for group cohomology.

With these preliminaries aside we proceed to the proof. For each $g \in C_x$ fix a pointed bundle $T_{\langle x, g\rangle} \to [0, 1] \times S^1$ with the correct holonomy. By gluing together the two ends of the cylinder, we obtain a bundle $\check{T}_{\langle x, g\rangle} \to S^1 \times S^1$. By (3.3) the lines corresponding $T_{\langle x, g\rangle}$ and $\check{T}_{\langle x, g\rangle}$ are isomorphic, so we pass freely between them. Choose a classifying map $f_{\langle x, g\rangle}\colon T_{\langle x, g\rangle} \to E\Gamma$ which restricts to ϕ_x on each boundary component. Let $\check{f}_{\langle x, g\rangle}\colon \check{T}_{\langle x, g\rangle} \to E\Gamma$ be the induced classifying map. By (1.3), $\check{f}_{\langle x, g\rangle}$ determines an isometry

$$L_{\langle x, g\rangle} = L_{\check{T}_{\langle x, g\rangle}} \cong I_{S^1 \times S^1, \bar{f}^*_{\langle x, g\rangle}\hat{\alpha}} \,.$$

Denote this line as I_g and fix an element $\varepsilon_g \in I_g$ of unit norm. From the point of view of the central extension (3.13), it is an element $\hat{g} \in \hat{C}_x$ which lifts $g \in C_x$. (Recall that $\hat{C}_x$ is defined as the set of elements of unit norm in the lines $L_{\langle x, g\rangle}$.) For each $g_1, g_2 \in C_x$ we have two trivializations of $L_{\langle x, g_1 g_2\rangle}$, via (3.11); their ratio is a cocycle $c(g_1, g_2)$ of the central extension (3.15). On the other hand, their ratio may be computed from (1.4). Namely, choose a homotopy

$$h_{g_1, g_2}\colon [0, 1] \times S^1 \times S^1 \mapsto E\Gamma$$

from the "composite" $f_{\langle g_1 x g_1^{-1}, g_2\rangle} f_{\langle x, g_1\rangle}$ (computed by gluing the first map over $\{1\} \times S^1$ to the second map over $\{0\} \times S^1$ and rescaling) to $f_{\langle x, g_1 g_2\rangle}$. Then

$$\theta_{g_1, g_2} = \int_{[0, 1] \times S^1 \times S^1} \bar{h}^*_{g_1, g_2}\hat{\alpha}\colon I_{g_1} \otimes I_{g_2} \to I_{g_1 g_2} \,, \tag{3.18}$$

and the desired cocycle $c(g_1, g_2) \in \mathbb{R}/\mathbb{Z}$ is determined by (3.17) as before. Let

$$\bar{k}_{g_1, g_2}\colon [0, 1] \times S^1 \to \mathrm{Map}\,(S^1, B\Gamma)$$

be the map $\bar{h}_{g_1, g_2}$ as a function of its first two variables. Then (3.18) implies

$$\theta_{g_1, g_2} = \int_{[0, 1] \times S^1} \bar{k}^*_{g_1, g_2}\, \pi_* e^*(\hat{\alpha}) \,. \tag{3.19}$$

A comparison of (3.19) and (3.16) shows that c, which is defined as a cocycle in the group cohomology for the central extension, corresponds to a cocycle for the cohomology class $\pi_* e^*[\hat{\alpha}]$, as claimed.

Finally, we remark that the measure (2.1) assigns unit mass of each $[T]_{\langle x, g\rangle} \in \mathscr{G}$, so is obviously left and right invariant under the groupoid composition law.

Now suppose Y is a compact oriented 2-manifold with parametrized boundary. Suppose $(\partial Y)_i$ is a $+$ boundary component. Then if $[Q] \in \overline{\mathscr{C}'_Y}$ and $[T] \in \mathscr{G}$ agree on $(\partial Y)_i \subset Y$ and $\{0\} \times S^1 \subset [0, 1] \times S^1$ there is a glued bundle $[Q] \circ [T] \in \overline{\mathscr{C}'_Y}$. Thus a $+$ boundary component determines a right $\mathscr{G}$ action on $\overline{\mathscr{C}'_Y}$. The isometry (3.8)

$$L_{[Q]} \otimes L_{[T]} \to L_{[Q] \circ [T]} \tag{3.20}$$

is a lift to a right $\hat{\mathscr{G}}$ action on an extension of $\overline{\mathscr{C}'_Y}$. Similarly, a $-$ boundary component determines a left action. It is easy to see that these actions preserve the measure on $\overline{\mathscr{C}'_Y}$.

The quantization generalizes (2.10). Namely, for each component oriented 2-manifold Y with parametrized boundary, set

$$E(Y) = L^2\,(\overline{\mathscr{C}'_Y}, \overline{\mathscr{L}_Y})\,.$$

The preceding classical structure has quantum implications. To state these we need the following algebraic notions. (See Appendix A for a more detailed review of coalgebras and comodules.) Suppose A is a coalgebra over $\mathbb{C}$, E_R is a right A-comodule and E_L is a left A-comodule. Then the *cotensor product* $E_R \boxtimes_A E_L$ is the vector subspace of $E_R \otimes_{\mathbb{C}} E_L$ annihilated by $\Delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_L$, where Δ_R, Δ_L are the coproducts. If E_R, E_L are unitary comodules, then $E_R \boxtimes_A E_L$ inherits the subspace inner product. More generally, if E has a left A-comodule structure Δ_L and a right A-comodule structure Δ_R, then we define

$$\mathrm{Inv}_A(E) \subset E$$

to be the subspace annihilated by $\Delta_L - P\Delta_R$, where $P\colon E \otimes_{\mathbb{C}} A \to A \otimes_{\mathbb{C}} E$ is the natural isomorphism.

The following generalizes Theorem 2.13.

Theorem 3.21. *The assignment*

$$Y \mapsto E(Y)$$

of a hermitian vector space to a compact oriented 2-manifold with parametrized boundary agrees with (2.14) *on closed manifolds and satisfies*:

(a) (*Functoriality*) *If $f\colon Y' \to Y$ is an orientation preserving diffeomorphism which preserves the boundary parametrizations, then there is an induced isometry*

$$f_*\colon E(Y') \to E(Y)$$

and these compose properly.

(b) (*Orientation*) *There is a natural isometry*

$$E(-Y) \cong \overline{E(Y)}\,. \tag{3.22}$$

(c) (*Multiplicativity*) *If $Y = Y_1 \sqcup Y_2$ is a disjoint union, then there is a natural isometry*

$$E(Y_1 \sqcup Y_2) \cong E(Y_1) \otimes E(Y_2)\,. \tag{3.23}$$

(d) (*Coalgebra*) *Let S be a parametrized closed oriented* 1-*manifold. Then*

$$A_S = E([0,1] \times S)$$

is a unitary[16] *coalgebra with antiinvolution. There are natural isomorphisms*

$$A_{-S} \cong A_S^{\mathrm{op}} \tag{3.24}$$

with the opposite coalgebra, and

$$A_{S_1 \sqcup S_2} \cong A_{S_1} \times A_{S_2}\,. \tag{3.25}$$

(e) (*Comodule*) *$E(Y)$ is a unitary right $A_{\partial Y}$-comodule. The isometries* (3.22) *and* (3.23) *are compatible with the comodule structure.*

---

[16] The notion of unitarity for coalgebras and comodules is defined in (A.3) and (A.4)

(f) *(Gluing) Suppose $S \subsetneq Y$ is a closed embedded circle and Y^{cut} the manifold obtained by cutting along S. So $\partial Y^{\mathrm{cut}} = \partial Y \sqcup S \sqcup -S$ and we use parametrizations which agree on S and $-S$. Thus $E(Y^{\mathrm{cut}})$ is both a right and left A_S-comodule. Then there is an isometry*

$$E(Y) \cong \frac{1}{\# \Gamma} \cdot \mathrm{Inv}_{A_S} E(Y^{\mathrm{cut}}) \,. \tag{3.26}$$

Since all of our 1-manifolds are parametrized we can identify them with a union of circles. Let

$$A = A_{S^1} \tag{3.27}$$

be the coalgebra attached to the standard circle. Then (3.24) and (3.25) imply that any A_S can be naturally identified with a direct product of copies of A and A^{op}. Also, if E is a hermitian vector space and λ a positive number, then $\lambda \cdot E$ denotes the same underlying vector space with the hermitian product multiplied by a factor of λ. Equation (3.26) is the gluing law which allows one to compute the vector space attached to a surface by cutting and pasting. The isometry (3.26) is compatible with (a)–(c) and (e).

Proof. We only comment on some of the assertions. As a warmup, consider the situation where a finite group G acts on a finite set Z. The space of functions $\mathscr{F}(G)$ is a coalgebra and $\mathscr{F}(Z)$ is a comodule. The coalgebra structure is dual to the multiplication $G \times G \to G$ and the comodule structure is dual to the action $Z \times G \to Z$. The counit is dual to the inclusion of the identity element $1 \to G$. If G has a bi-invariant measure then $L^2(G)$ has a compatible inner product. If Z has a measure preserved by the G action, then $L^2(Z)$ also has a compatible inner product.

Our situation is different in two ways: We have a groupoid and we consider functions with values in complex lines. The latter is not significant, since the lines form a central extension of the groupoid. But the fact that we only have a groupoid, and not a group, means that the coproduct of an element is not naturally defined away from composable elements. In (3.28) we extend by zero. This works.[17] Thus if $a \in A_S$ for some parametrized compact oriented 1-manifold S, then we define the coproduct

$$\Delta a\,([T_1], [T_2]) = \begin{cases} a([T_1] \circ [T_2]), & \text{if } [T_1] \circ [T_2] \text{ is defined ;} \\ 0, & \text{otherwise} \end{cases} \tag{3.28}$$

for $[T_1], [T_2] \in \overline{\mathscr{C}'_{[0,1] \times S}}$. Note that this uses the isometry (3.11). The counit is

$$\varepsilon(a) = \sum_x a\,([T]_{\langle x, e \rangle}) \,, \tag{3.29}$$

where we use the trivialization (3.12). Similarly, if Y is a compact oriented 2-manifold with parametrized boundary, then the comodule structure on $E(Y)$ is defined by

$$\Delta x([Q], [T]) = \begin{cases} x([Q] \circ [T]), & \text{if } [Q] \circ [T] \text{ is defined ;} \\ 0, & \text{otherwise,} \end{cases}$$

[17] In more complicated situations it is not immediately clear what should be done

where $x \in E(Y)$, $[Q] \in \overline{\mathscr{C}'_Y}$, and $[T] \in \overline{\mathscr{C}'_{[0,1]\times\partial Y}}$. Here we use the isometry (3.20). The antiinvolution is

$$\bar{a}([T]) = \overline{a([T]^{-1})}\,.$$

The compatibility of the L^2 inner product with these coproducts, expressed by (A.3) and (A.4), is a simple change of variables. For example, if $a, b, c \in A_S$, then

$$\begin{aligned}(\Delta a, b \otimes c) &= \sum_{[T_1],[T_2]} a([T_1]\circ[T_2])\,\overline{b([T_1])}\,\overline{c([T_2])}\\ &= \sum_{[T_2],[T_3]} a([T_3])\,\overline{c([T_2]^{-1})}\,\overline{b([T_3]\circ[T_2])}\\ &= (a\otimes\bar{c}, \Delta b)\,.\end{aligned}$$

For the gluing law (f), let $\overline{\mathscr{B}_{Y^{\mathrm{cut}}}} \subset \overline{\mathscr{C}'_{Y^{\mathrm{cut}}}}$ denote the subset of bundles which are isomorphic over the two copies of S. We claim that any function $x \in \mathrm{Inv}_{A_S} E(Y^{\mathrm{cut}})$ has support in $\overline{\mathscr{B}_{Y^{\mathrm{cut}}}}$. This follows since by definition

$$x([Q^{\mathrm{cut}}]\circ_S[T]) = x([T]\circ_{-S}[Q^{\mathrm{cut}}]) \tag{3.30}$$

for all $[Q^{\mathrm{cut}}] \in \overline{\mathscr{C}'_{Y^{\mathrm{cut}}}}$ and $[T] \in \overline{\mathscr{C}'_{[0,1]\times S}}$, where we understand the evaluation of x on an undefined composition to vanish. Hence given any $[Q^{\mathrm{cut}}] \in \overline{\mathscr{C}'_{Y^{\mathrm{cut}}}}$ we choose $[T] = [Q^{\mathrm{cut}}]|_{[0,1]\times S}$. Then if $[Q^{\mathrm{cut}}]|_{-S} \ncong [Q^{\mathrm{cut}}]|_S$, the composition $[T]\circ_{-S}[Q^{\mathrm{cut}}]$ is undefined, whence

$$x([Q^{\mathrm{cut}}]) = x([Q^{\mathrm{cut}}]\circ[T]) = 0$$

as claimed. Note that if $[Q^{\mathrm{cut}}]|_{-S} \cong [Q^{\mathrm{cut}}]|_S$ then (3.30) is equivalent to

$$x([T]^{-1}\circ_{-S}[Q^{\mathrm{cut}}]\circ_S[T]) = x([Q^{\mathrm{cut}}])\,.$$

Next, consider the diagram

$$\begin{array}{ccc} & \overline{\mathscr{B}_{Y^{\mathrm{cut}}}} & \\ {\scriptstyle r}\swarrow & & \searrow{\scriptstyle g} \\ \overline{\mathscr{C}'_S} & & \overline{\mathscr{C}'_Y}\end{array}$$

where g is the gluing map (3.6) and r is restriction. From Lemma 2.4(a) we conclude that g is surjective. Furthermore, it follows from part (b) of that lemma that if $[Q^{\mathrm{cut}}] \in g^{-1}([Q])$, then any other element of $g^{-1}([Q])$ is of the form $[T]^{-1}\circ_{-S}[Q^{\mathrm{cut}}]\circ_S[T]$, where $[T] \in \overline{\mathscr{C}'_{[0,1]\times S}}$ restricts over $\{0\}\times S$ to a bundle isomorphic to $[Q^{\mathrm{cut}}]|_S$. Now Theorem 3.2(d) implies that the pullback of $\overline{\mathscr{L}_Y}$ via g is isomorphic to the restriction of $\overline{\mathscr{L}_{Y^{\mathrm{cut}}}}$ to $\overline{\mathscr{B}_{Y^{\mathrm{cut}}}}$. It now follows that pullback via g is an isomorphism $E(Y) \cong \mathrm{Inv}_{S_S} E(Y^{\mathrm{cut}})$.

It remains to compare the inner products. We claim that for any $[Q] \in \overline{\mathscr{C}_Y}$,

$$\mu_{[Q]} = \frac{\mathrm{vol}\,(g^{-1}([Q]))}{\#\Gamma}\,. \tag{3.31}$$

For in our situation (2.5) becomes

$$\mu_{[Q]} = \frac{\operatorname{vol}(g^{-1}([Q]))}{\# r(g^{-1}([Q]))\cdot \# \operatorname{Aut}([Q]|_S)} . \tag{3.32}$$

Note that $[Q]|_S$ is only determined up to isomorphism and has no basepoint; nevertheless the number $\#\operatorname{Aut}([Q]|_S)$ is independent of the representative chosen. The factor $\# r(g^{-1}([Q]))$ counts these representatives. Equation (3.31) follows immediately from (3.32) since Γ acts transitively on these representatives (changing the basepoint) with stabilizer the automorphism group. Now consider $x, y \in L^2(\overline{\mathscr{C}'_Y}, \overline{\mathscr{L}_Y})$. Then

$$\begin{aligned}(g^*x, g^*y)_{E(Y^{\mathrm{cut}})} &= \int_{\overline{\mathscr{B}_{Y^{\mathrm{cut}}}}} d\mu([Q^{\mathrm{cut}}])\, x(g([Q^{\mathrm{cut}}]))\, \overline{y(g([Q^{\mathrm{cut}}]))} \\ &= \int_{\overline{\mathscr{C}_Y}} d\mu([Q])\, \#\Gamma \cdot x([Q])\, \overline{y([Q])} \\ &= \#\Gamma \cdot (x, y)_{E(Y)} .\end{aligned}$$

This completes the proof of (3.26).

4. The Modular Functor

Our first task in this section is to construct a *modular functor* [S1] from the quantum Hilbert spaces of Theorem 3.21.[18] We first dualize the coalgebras and comodules of that theorem to obtain the more familiar algebras and modules, and so we make contact with standard structure theory of these objects. (It is also possible to develop an algebra version directly [Q1, Sect. 4].) A key point is semisimplicity (Proposition 4.2) which leads to a set of *labels*. Then from the comodule $E(Y)$ for Y an oriented surface with boundary we derive the vector space (4.7) typically attached to a surface with labeled boundary. We also recover the *Verlinde algebra* (Proposition 4.14). It is important to notice that not only does $E(Y)$ have a richer structure than the vector spaces attached to the surfaces with labeled boundary, but the inner product structure is more natural there. (Compare the gluing laws (3.26) and (4.11).)

We first analyze the coalgebra A (3.27) in more detail. Since algebras are more familiar than coalgebras, we switch to the dual algebra A^*. We have

$$A^* = \bigoplus_{[T]} L^*_{[T]} , \tag{4.1}$$

where $[T]$ ranges over $\mathscr{G}$. If $\xi_1 \in L^*_{[T_1]}$ and $\xi_2 \in L^*_{[T_2]}$, then $\xi_1\xi_2 \in L^*_{[T_1]\circ[T_2]}$ is obtained from (3.11) if $[T_1]\circ[T_2]$ exists and is zero if $[T_1]$ and $[T_2]$ are not composable

[18] The modular functor encodes the structure of *conformal blocks* in rational conformal field theory [MS]

Proposition 4.2. *The algebra A^* is semisimple. Hence*

$$A^* \cong \prod_{\lambda \in \Phi} M_\lambda \tag{4.3}$$

is isomorphic to a direct product of matrix algebras.

The proof imitates the standard proof of Maschke's theorem in the theory of finite group representations.

Proof. It suffices to show that every A^*-module is completely reducible, or equivalently that any surjective morphism of A^*-modules.

$$P_1 \xrightarrow{f} P_2 \tag{4.4}$$

splits. For this suppose that $P_2 \xrightarrow{g} P_1$ is a $\mathbb{C}$-linear splitting of f. For each $[T] \in \mathscr{G}$ fix a nonzero element $\xi_{[T]} \in L^*_{[T]}$ and consider

$$g' = \frac{1}{\#\mathscr{G}} \sum_{[T]} \xi_{[T]} g \xi_{[T]}^{-1} ,$$

where $\xi_{[T]}^{-1} \in L^*_{[T]^{-1}}$ is the unique element such that $\xi_{[T]}^{-1} \otimes \xi_{[T]} \in L^*_{[T]^{-1}} \otimes L^*_{[T]} \cong \mathbb{C}$ is the unit element (cf. (3.12)). Since g is $\mathbb{C}$-linear, the map g' is independent of the choice of $\xi_{[T]}$.We claim that g' is an A^*-module homomorphism which splits (4.4). The calculation that $g'f = \mathrm{id}_{P_1}$ is straightforward. To see that g' commutes with the A^*-action, suppose that $\xi_0 \in L^*_{[T_0]}$. Set

$$\xi'_{[T']} = \xi_0^{-1} \xi_{[T]}, \quad [T'] = [T_0]^{-1} \circ [T] ,$$

when the composition makes sense. Only these compositions appear in the calculation

$$g' \xi_0 = \frac{1}{\#\mathscr{G}} \sum_{[T]} \xi_{[T]} g \xi_{[T]}^{-1} \xi_0 = \frac{1}{\#\mathscr{G}} \sum_{[T']} \xi_0 \xi'_{[T']} g \xi'^{-1}_{[T']}, = \xi_0 g' ,$$

which completes the proof.

Let $\Phi = \{\lambda\}$ be the set of *labels* in (4.3). This set labels the irreducible representations of A^*, or equivalently the irreducible corepresentations E_λ of A. Fix a hermitian structure on E_λ compatible with the A action; it is unique up to a scalar multiple. The set Φ also labels the irreducible corepresentations of A^{op}; then the label λ corresponds to E_λ^*. Now suppose Y is a compact oriented 2-manifold with parametrized boundary. Then using Proposition A.7 we can split the unitary comodule $E(Y)$ according to the irreducible corepresentations of $A_{\partial Y} \cong A \times \cdots \times A \times A^{\mathrm{op}} \times \cdots \times A^{\mathrm{op}}$. In other words, there is an isometry

$$E(Y) \cong \bigoplus_{\lambda} E(Y, \lambda) \otimes E_\lambda , \tag{4.5}$$

where $\lambda = \langle \lambda_1, \ldots, \lambda_k \rangle$ is a labelling of the boundary components,

$$E_\lambda = E_{\lambda_1}^{\pm 1} \otimes \cdots \otimes E_{\lambda_k}^{\pm 1} \tag{4.6}$$

is the corresponding corepresentation of $A_{\partial Y}$, and

$$E(Y, \lambda) = \mathrm{Hom}_{A_{\partial Y}}\left(\frac{E_\lambda}{\dim E_\lambda}, E(Y)\right). \tag{4.7}$$

We use E_λ in (4.6) if the corresponding boundary component is positive, and we use the dual $E_\lambda^{-1} = E_\lambda^*$ if the corresponding boundary component is negative. So $E(Y, \lambda)$ is the inner product space attached to a compact oriented surface with parametrized *labeled* boundary. The assignment

$$\langle Y, \lambda \rangle \mapsto E(Y, \lambda) \tag{4.8}$$

is termed a (reduced) *modular functor* by Graeme Segal [S1]; in the physics literature $E(Y, \lambda)$ is called a *conformal block*. Theorem 3.21 easily implies the following properties of these inner product spaces.

Proposition 4.9. *The assignment* (4.8) *satisfies*:

(a) (*Functoriality*) *If* $f: \langle Y', \lambda' \rangle \to \langle Y, \lambda \rangle$ *is an orientation preserving diffeomorphism which preserves the boundary parametrizations and the labels, then there is an induced isometry*

$$f_*: E(Y', \lambda') \to E(Y, \lambda)$$

and these compose properly.

(b) (*Orientation*) *There is a natural isometry*

$$E(-Y, \lambda) \cong \overline{E(Y, \lambda)}\,.$$

(c) (*Multiplicativity*) *If* $\langle Y, \lambda \rangle = \langle Y_1, \lambda_1 \rangle \sqcup \langle Y_2, \lambda_2 \rangle$ *is a disjoint union, then there is a natural isometry*

$$E(Y_1 \sqcup Y_2, \lambda_1 \sqcup \lambda_2) \cong E(Y_1, \lambda_1) \otimes E(Y_2, \lambda_2)\,. \tag{4.10}$$

(d) (*Gluing*) *Suppose* Y^{cut} *is the manifold obtained from* Y *by cutting along an embedded circle* S. *Let* λ *be a labeling of* ∂Y. *Then there is an isometry*

$$E(Y, \lambda) \cong \bigoplus_{\mu \in \Phi} \frac{\dim E_\mu}{\# \Gamma} \cdot E(Y^{\mathrm{cut}}, \lambda \cup \mu \cup \mu)\,. \tag{4.11}$$

The extra factor in the gluing law (4.11) does not appear in Segal's work [S1], but it does appear in Walker's treatment [Wa] of the $SU(2)$ Chern–Simons theory. We should point out that the inner product in $E(Y, \lambda)$ can be scaled, as can the isomorphism in (4.11). There are choices that would eliminate the extra factor in the gluing law. Our scale choices seem quite natural, nonetheless. The isometry (4.11) is independent of the choice of hermitian structure on E_λ (which is unique up to scale).

Proof. Equation (4.10) follows immediately from (3.23). For (4.11) we rewrite (3.26) using (4.5) and (A.8):

$$\begin{aligned} E(Y) &\cong \bigoplus_{\lambda} E(Y, \lambda) \otimes E_\lambda \\ &\cong \frac{1}{\# \Gamma} \bigoplus_{\lambda, \mu, \nu} E(Y^{\mathrm{cut}}, \lambda \cup \mu \cup \nu) \otimes E_\lambda \otimes \mathrm{Inv}_{A_S}(E_\mu \otimes E_\nu^*) \\ &\cong \bigoplus_{\lambda, \mu} \frac{\dim E_\mu}{\# \Gamma} \cdot E(Y^{\mathrm{cut}}, \lambda \cup \mu \cup \mu) \otimes E_\lambda\,. \end{aligned} \tag{4.12}$$

Hence both sides of (4.11) are isometric to $\mathrm{Hom}_{A_{\partial Y}}\left(\frac{E_\lambda}{\dim E_\lambda}, E(Y)\right)$, by Proposition A.7, which completes the proof.

Notice that in passing to the last equation in (4.12) we trivialized the complex line $\mathrm{Inv}_{A_S}(E_\mu \otimes E_{\mu^*})$ using the natural duality pairing, which has norm square equal to $\dim E_\mu$.

Next, we introduce a ring structure on $V = \mathbb{Z}[\Phi]$, the free abelian group generated by the label set Φ. A typical element of V is denoted $\sum_\lambda c_\lambda \lambda$, with $c_\lambda \in \mathbb{Z}$. Fix Y_3 a 2-sphere with three open disks removed. Give parametrizations to the three boundary circles so that two are $-$ parametrizations and one is a $+$ parametrizaton. Set

$$N^\lambda_{\mu\nu} = \dim E(Y_3, \lambda \cup \mu \cup \nu)\,, \tag{4.13}$$

where λ labels the $+$ boundary component. Notice that (4.13) is independent of the choice of Y_3. Define multiplication on V by

$$\mu\nu = \sum_\lambda N^\lambda_{\mu\nu}\,\lambda\,.$$

The proof of the next proposition is quite standard.

Proposition 4.14. *V is a commutative associative ring with identity.*

This ring can be used to compute $\dim E(Y, \lambda)$ quite effectively. Its complexification $\mathrm{A}V_{\mathbb{C}} = \mathbb{C}[\Phi]$ is called the *Verlinde algebra.*

We can see the Verlinde algebra as belonging to an auxiliary $1+1$ dimensional field theory. Namely, for a closed oriented 1-manifold S define

$$\tilde{E}(S) = E(S^1 \times S)\,, \tag{4.15}$$

and for a compact oriented 2-manifold Y define

$$\tilde{Z}_Y = E(S^1 \times Y)\,. \tag{4.16}$$

Then it is easy to see that $\tilde{E}$, $\tilde{Z}$ determine a $1+1$ dimensional topological quantum field theory (as defined by the properties in Theorem 2.13). The Verlinde algebra is $V_{\mathbb{C}} = \tilde{E}(S^1)$.

5. Computations

We illustrate the theory of the previous sections with some calculations in the quantum theory. We begin with an arbitrary twisted theory (determined by a cocycle $\hat{\alpha}$) in $2+1$ dimensions. Our first job is to calculate the $SL(2;\mathbb{Z})$ action on the vector space $E(S^1 \times S^1)$ attached to the torus (Proposition 5.8). One consequence is that the factor in the gluing law (4.11) is a matrix element (5.12) of the standard modular transformation S. We then take up the untwisted ($\hat{\alpha} = 0$) theory. We calculate the theory explicitly in $0+1$ and $1+1$ dimensions (Proposition 5.17). We use the $1+1$ dimensional theory to count the representations of a surface group into a finite group (5.19). The structure of the $2+1$ dimensional theory on surfaces with boundary simplifies somewhat, since the central extensions there are trivial. The coalgebra in that theory also obtains a natural Hopf algebra structure, which we compute directly, and the Verlinde algebra can be derived from this Hopf algebra (Proposition 5.25).

Fix a cocycle $\hat{\alpha} \in C^3(B\Gamma; \mathbb{R}/\mathbb{Z})$. Let $E = E(S^1 \times S^1)$ be the inner product space attached to the torus. Then $SL(2; \mathbb{Z})$ acts on E, by (2.15). We will determine the action of the generators

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{5.1}$$

Cut the torus along $\{0\} \times S^1$ to obtain the cylinder $[0, 1] \times S^1$. Note that T defines a diffeomorphism of the cylinder which fixes the boundary. Recall that the space of fields on the cylinder is a groupoid $\mathscr{G} = \{\langle x, g\rangle : x, g \in \Gamma\}$ with composition law (3.10). The vector space of the cylinder is the space of sections A of a line bundle $L \to \mathscr{G}$, and Theorem 3.21 identifies E as the subspace of *central sections*, that is, sections invariant under conjugation. The coalgebra A is the direct sum

$$A = \bigoplus_{x, g} L_{\langle x, g\rangle} ,$$

and the dual algebra is

$$A^* = \bigoplus_{x, g} L^*_{\langle x, g\rangle} ,$$

as in (4.1). Elements of A are then complex-valued functions on A^*. Thus we identify E with the space of complex-valued central functions on the algebra A^*. Since A^* is semisimple (Proposition 4.2) there is a basis of character functions χ_λ, where λ runs over the set Φ of irreducible representations. Each χ_λ is supported on

$$\operatorname{supp} \chi_\lambda = \{\langle x, g\rangle : x \in A, [x, g] = 1\} \tag{5.2}$$

for some conjugacy class $A \subset \Gamma$. (Here $[x, g] = xgx^{-1}g^{-1}$ is the group commutator.) If we fix $x_0 \in A$, then χ_λ is determined by its values on $\{\langle x_0, g\rangle : x_0 g = g x_0\}$, which is (anti)isomorphic to the centralizer C_x of x in Γ. The restriction of χ_λ to this set is a character of $\hat{C}_{x_0}$, the central extension (3.13) of C_{x_0}. The Schur orthogonality relations for these characters is

$$\frac{1}{\#\Gamma} \int_{\mathscr{G}} (\chi_\lambda, \chi_\mu) = \begin{cases} 1, & \lambda = \mu ; \\ 0, & \text{otherwise} . \end{cases} \tag{5.3}$$

The integral is simply a sum over the elements of $\mathscr{G}$, since each element of $\mathscr{G}$ has unit mass. Here $(\cdot, \cdot)$ is the hermitian inner product on the line bundle L. We can rewrite the left-hand side of (5.3) as

$$\sum_{[x]} \sum_{[g]} \frac{1}{\#(C_x \cap C_g)} ((\chi_\lambda)_{\langle x, g\rangle}, (\chi_\lambda)_{\langle x, g\rangle}) ,$$

where the first summation is over representatives of conjugacy classes in Γ and for each x the second summation is over representatives of conjugacy classes in C_x. But $C_g \cap C_x \cong \operatorname{Aut} Q$ is isomorphic to the automorphism group of the bundle $Q \to S^1 \times S^1$ obtained from $\langle x, g\rangle$ by gluing. In view of (2.11) we have shown the following.

Lemma 5.4. *The characters χ_λ form an orthonormal basis of E.*

We have already noted (3.12) the triviality of $L_{\langle x,e\rangle}$ for any $x \in \Gamma$. Let $l_{\langle x,e\rangle} \in L_{\langle x,e\rangle}$ and $l^*_{\langle x,e\rangle} \in L^*_{\langle x,e\rangle}$ be the trivializing basis elements. We also claim that there is also a trivialization

$$L_{\langle x,x\rangle} \cong \mathbb{C}\,, \tag{5.5}$$

and corresponding trivializing elements $l_{\langle x,x\rangle} \in L_{\langle x,x\rangle}$ and $l^*_{\langle x,x\rangle} \in L^*_{\langle x,x\rangle}$. To see this we first note

$$L_{\langle x,x\rangle} \otimes L_{\langle x,x\rangle} \cong L_{\langle x,x^2\rangle} \tag{5.6}$$

by (3.11). Now the diffeomorphism $T\colon [0,1]\times S^1 \to [0,1]\times S^1$, defined in (5.1), pulls a bundle with holonomy $\langle x,x\rangle$ back to a bundle with holonomy $\langle x,x^2\rangle$, and so by functoriality (Theorem 3.2) gives an isometry

$$L_{\langle x,x\rangle} \cong L_{\langle x,x^2\rangle}\,. \tag{5.7}$$

We obtain (5.5) from (5.6) and (5.7).

We claim that $l_{\langle x,x\rangle}$ is a central element of $\hat{C}_x$, where $\hat{C}_x$ is the central extension of C_x defined in (3.13). For suppose $g \in C_x$ and $\hat{l} \in L_{\langle x,g\rangle}$ is an element of unit norm, so also $\hat{l} \in \hat{C}_x$. Let $P_{x,x,g} \to S^1 \times S^1 \times S^1$ be the Γ bundle with holonomy x, x, g around the three generating circles. Then Proposition 3.14 implies that the commutator of $l_{\langle x,x\rangle}$ and $\hat{l}$ in $\hat{C}_x$ is the exponential of $\bar{F}^*[\alpha]([S^1\times S^1\times S^1])$, where $F\colon P_{x,x,g} \to E\Gamma$ is a classifying map. In other words, the commutator is the classical action evaluated on $P_{x,x,g}$. But $P_{x,x,g}$ is the pullback of the bundle $\check{T}_{\langle x,g\rangle} \to S^1 \times S^1$ by the map

$$\begin{gathered} S^1\times S^1\times S^1 \to S^1\times S^1 \\ \langle t_1,t_2,t_3\rangle \mapsto \langle t_1+t_2,t_3\rangle\,. \end{gathered}$$

from which it follows easily that the action is 1.

Now we can compute the action of $SL(2;\mathbb{Z})$ on E. Let ρ_λ denote the representation whose character is χ_λ. By the preceding argument and Schur's lemma $\rho_\lambda(l^*_{\langle x,x\rangle})$ is a scalar, if χ_λ is supported on C_x.

Proposition 5.8. *The elements $T, S \in SL(2;\mathbb{Z})$, defined in* (5.1), *act on E as follows. The basis $\{\chi_\lambda\}$ diagonalizes T:*

$$T_*\chi_\lambda = \rho_\lambda(e^*_{\langle x,x\rangle})\chi_\lambda\,, \tag{5.9}$$

where x is chosen as in (5.2). *Also,*

$$\begin{aligned} S_*\chi_\lambda &= \sum_\mu \frac{1}{\#\Gamma}\int_{\mathscr{G}} (S_*\chi_\lambda, \chi_\mu)\chi_\mu \\ &= \frac{1}{\#\Gamma}\sum_\mu\sum_{x,g} ((\chi_\lambda)_{\langle g^{-1},x\rangle}, (\chi_\mu)_{\langle x,g\rangle})\chi_\mu\,. \end{aligned} \tag{5.10}$$

Note that since χ_λ is a central function,

$$\begin{aligned} \chi_\lambda(l^*_{\langle gxg^{-1},gxg^{-1}\rangle}) &= \chi_\lambda((l^*_{\langle x,g\rangle})^{-1}\, l^*_{\langle x,x\rangle}\,(l^*_{\langle x,g\rangle})) \\ &= \chi_\lambda(l^*_{\langle x,x\rangle})\,, \end{aligned}$$

where $l^*_{\langle x,g\rangle} \in L^*_{\langle x,g\rangle}$ is any nonzero element. Hence (5.9) is well-defined. Formulas (5.9) and (5.10) agree with the results in [DVVV].

Proof. The diffeomorphism T acts on the cylinder, commuting with the boundary parametrizations, and the induced action on the set of bundles $\mathscr{G}$ is

$$T^*\langle x, g\rangle = \langle x, gx\rangle .$$

To compute the effect on the character, we can suppose $g \in C_x$ commutes with x. Then

$$\langle x, gx\rangle = \langle x, x\rangle \cdot \langle x, g\rangle$$

in the groupoid $\mathscr{G}$. Using (3.11) and the trivialization (5.5) we easily derive (5.9).

The gluing law in Theorem 3.2 allows us to identify the induced action of S on $\overline{\mathscr{C}'_{S^1\times S^1}}$ with an action on the subset of the groupoid $\mathscr{G}$ consisting of pairs $\langle x, g\rangle$ of commuting elements. That action is

$$S^*\langle x, g\rangle = \langle g^{-1}, x\rangle .$$

Note that this gives an isometry

$$L_{\langle g-1, x\rangle} \cong L_{\langle x, g\rangle} \cdot \tag{5.11}$$

Now (5.10) follows from the Schur orthogonality relations (5.3). Of course, the first line of (5.10) holds for *any* element of $SL(2; \mathbb{Z})$.

There is a distinguished representation $1 \in \Phi$. Its character χ_1 is supported on $\{\langle e, g\rangle\} \cong C_e = \Gamma$. Now by (5.11) and (3.12) we have a trivialization of $L_{\langle e, g\rangle} \cong L_{\langle g-1, e\rangle}$. Then

$$(\chi_1)_{\langle e, g\rangle} = e_{\langle e, g\rangle}$$

is the corresponding trivializing element. (This character corresponds to the representation of $\hat{C}_e \cong \Gamma \times \mathbb{T}$ which is trivial on the Γ factor and is standard on the $\mathbb{T}$ factor.) Now from (5.10) we compute the matrix element

$$\begin{aligned}(S_*\chi_\lambda, \chi_1) &= \frac{1}{\#\Gamma}\sum_{g,x}((\chi_\lambda)_{\langle g^{-1}, x\rangle}, (\chi_1)_{\langle x, g\rangle})\\ &= \frac{1}{\#\Gamma}\sum_{g}((\chi_\lambda)_{\langle g^{-1}, e\rangle}, l_{\langle e, g\rangle})\\ &= \frac{1}{\#\Gamma}\sum_{g\in \operatorname{supp}\chi_\lambda}\chi_\lambda(l^*_{\langle g, e\rangle})\\ &= \frac{\dim E_\lambda}{\#\Gamma},\end{aligned} \tag{5.12}$$

where E_λ is the representation with character χ_λ. This is exactly the factor which occurs in the gluing law (4.11).[19] To avoid confusion, we point out that if $\langle x, g\rangle \in \operatorname{supp}\chi_\lambda$, and there are k elements in the conjugacy class of x, then $\dim E_\lambda = k \cdot \dim \bar{E}_\lambda$, where $\bar{E}_\lambda$ is the irreducible representation of $\hat{C}_x$ with character $\chi_\lambda|_{\hat{C}_x}$.

We turn now to the untwisted ($\hat{\alpha} = 0$) theory. The quantum theory described in Sect. 2 makes sense in arbitrary dimensions (and the manifolds do not have to be

[19] Walker [Wa] uses this factor (defined as this matrix element of S) in his construction of the Chern–Simons theory with gauge group $SU(2)$

oriented). Formulas (2.6)–(2.8) can be worked out more explicitly. First, for any closed, connected d-manifold Y we use (2.7) and (1.1) to identify

$$E(Y) = L^2(\mathrm{Hom}(\pi_1 Y, \Gamma)/\Gamma)\,,$$

where the mass of a representation $\gamma \in \mathrm{Hom}(\pi_1 Y, \Gamma)$ is

$$\mu_{[\gamma]} = \frac{1}{\# C_\gamma}\,. \tag{5.13}$$

Here C_γ is the centralizer of the image of γ in Γ and $[\gamma]$ denotes the set of conjugates of a representation γ. For a closed, connected $(d+1)$-manifold X we compute from (2.6):

$$\begin{aligned} Z_X &= \sum_{[\beta] \in \mathrm{Hom}(\pi_1 X, \Gamma)/\Gamma} \frac{1}{\# C_\beta} \\ &= \sum_{[\beta] \in \mathrm{Hom}(\pi_1 X, \Gamma)} \frac{1}{\# C_\beta}\, \#\,[\beta] \\ &= \frac{1}{\# \Gamma}\, \#\,\mathrm{Hom}(\pi_1 X, \Gamma)\,. \end{aligned} \tag{5.14}$$

If X is a compact, connected $(d+1)$-manifold with nonempty connected boundary, and $\gamma \in \mathrm{Hom}\,(\pi_1 \partial X, \Gamma)$, then from (2.8) we deduce

$$Z_X(\gamma) = \#\,(\iota^*)^{-1}\,(\gamma)\,,$$

where $\iota^*\colon \mathrm{Hom}(\pi_1 X, \Gamma) \to \mathrm{Hom}(\pi_1 \partial X, \Gamma)$ is the restriction map.

The theory in $0+1$ dimensions is completely trivial. The inner product space $E(pt) \cong \frac{1}{\#\Gamma} \cdot \mathbb{C}$ and $Z_{[0,1]}\colon \frac{1}{\#\Gamma} \cdot \mathbb{C} \to \frac{1}{\#\Gamma} \cdot \mathbb{C}$ is the identity map. From (5.14) we have $Z_{S^1} = 1$. Notice that this is consistent with the gluing law (2.17) applied to $[0, 1]$.

In $1+1$ dimensions the inner product space $E = E(S^1)$ of the circle carries extra structure. Let Y_3 denote S^2 with three disks removed. Parametrize the boundary circles so that one is positively oriented and the remaining two are negatively oriented. Notice that

$$Z_{Y_3}\colon E \otimes E \to E \tag{5.15}$$

is independent of the parametrizations, since the space of orientation preserving diffeomorphisms of S^1 is connected. (We use the inner product on E to write Z_{Y_3} in this form – cf. Theorem 2.13(b).) It is easy to verify from the axioms that (5.15) defines an algebra structure on E which is commutative and associative. Furthermore, if Y_1 denotes a disk with positively parametrized boundary, then $Z_{Y_1} \in E$ is an identity element for this multiplication. The algebra E has an involution $a \mapsto \bar{a}$ induced by any orientation reversing diffeomorphism of S^1, and it is unitary in the sense dual to (A.3):

$$(a, bc) = (a\bar{c}, b) = (\bar{b}a, c)$$

for all $a, b, c \in E$. It now follows that E is a semisimple algebra, and so is isomorphic to a direct product $\mathbb{C} \times \cdots \times \mathbb{C}$. Up to isomorphism this unitary algebra is determined by the norm squares $\lambda_i^2 > 0$ of the identity element in each factor of the direct product.

The preceding discussion applies to any $1 + 1$ dimensional unitary topological field theory (in the sense of Theorem 2.13). Conversely, such theories can be constructed from unitary commutative, associative algebras with unit. We compute the partition function $Z_{Y(g)}$ of a closed, oriented surface of genus g in terms of this algebra:

$$Z_{Y(g)} = \sum (\lambda_i^2)^{1-g} . \tag{5.16}$$

These considerations apply to the Verlinde algebra (cf. (4.15)–(4.16)); in that context (5.16) appears in [V].

Returning to the untwisted theory attached to a finite group we have the following.

Proposition 5.17. *In the untwisted $1 + 1$-dimensional theory $E = E(S^1)$ is naturally identified with the character ring $\mathscr{F}_{\mathrm{cent}}(\Gamma)$ of complex-valued central functions on Γ. The multiplication is by convolution. The hermitian structure is*

$$(f_1, f_2) = \frac{1}{\#\Gamma} \sum_{x \in \Gamma} f_1(x)\overline{f_2(x)}, \quad \chi_1, \chi_2 \in \mathscr{F}_{\mathrm{cent}}(\Gamma) . \tag{5.18}$$

Finally,

$$Z_{Y(g)} = \frac{1}{\#\Gamma} \# \operatorname{Hom}(\pi_1 Y(g), \Gamma) = (\#\Gamma)^{2g-2} \sum_i (\dim E_i)^{2-2g} , \tag{5.19}$$

where i runs over the irreducible representations E_i of Γ.

Formula (5.19), which counts the number of representations of a surface group in a finite group, can also be derived using standard methods in finite group theory [Se]. Here we derive it by chopping $Y(g)$ into a union of "pairs of pants" and annuli. This is a simple illustration of how gluing laws are used in topological quantum field theory to compute global invariants from local computations.

Proof. The equivalence classes $\overline{\mathscr{C}_{S^1}}$ of bundles over the circle correspond to conjugacy classes in Γ, and for $g \in \Gamma$ the conjugacy class of g is weighted by $1/\# C_g$, according to (5.13). Now (5.18) follows immediately. To compute the multiplication (5.15) consider Y_3 with a basepoint on each boundary component. Then, as in Fig. 1, the set of equivalence classes $\overline{\mathscr{C}'_{Y_3}}$ is in 1:1 correspondence with the set of 4-tuples $x_1, g_1, x_2, g_2 \in \Gamma$. In other words, there is a 1:1 correspondence

$$\overline{\mathscr{C}'_{Y_3}} \leftrightarrow \mathscr{G} \times \mathscr{G} . \tag{5.20}$$

The path integral over Y_3 is defined in (2.12). Then that fact that E consists of central functions implies that the multiplication (5.15) is the convolution

$$(f_1 * f_2)(x) = \sum_{x_1 x_2 = x} f_1(x_1) f_2(x_2) .$$

Note that the factor of $(\#\Gamma)^2$ which comes from the summation over g_1, g_2 is canceled by the factor in the inner product (5.18). The characters of the irreducible

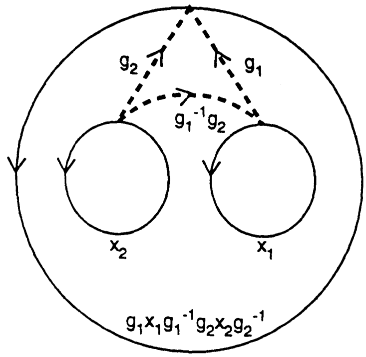

Fig. 1. The bundle over Y_3 corresponding to $\langle x_1, g_1\rangle \times \langle x_2, g_2\rangle \in \mathscr{G} \times \mathscr{G}$

representations, suitably normalized, are commuting idempotents in $\mathscr{F}_{\mathrm{cent}}(\Gamma)$, by the Schur orthogonality relations. Hence

$$\lambda_i^2 = \left(\frac{\dim E_i}{\#\Gamma}\right)^2 ,$$

from which (5.19) follows immediately given (5.16).

The modular structure of Sects. 3 and 4 has an analog here: Consider bundles over the interval with basepoints over the endpoints. The reader should check the assertions in Theorem 3.21 and Proposition 4.9 in this 1 + 1 dimensional theory. They all reduce to standard facts about finite groups. The basic coalgebra A is simply the coalgebra of functions on Γ. The twisted theory in 1 + 1 dimensions can also be made quite explicit. Namely, a cocyle $\hat{\alpha} \in C^2(B\Gamma\colon \mathbb{R}/\mathbb{Z})$ determines a central extension

$$1 \to \mathbb{T} \to \hat{\Gamma} \to \Gamma \to 1 .$$

Then the basic algebra $A_{\hat{\alpha}}$ in the corresponding twisted quantum theory is the coalgebra of functions twisted by the cocycle $\hat{\alpha}$. The representations of this coalgebra correspond to representations of $\hat{\Gamma}$ which are standard on the central $\mathbb{T}$. The reader may wish to work out this twisted case in detail.

Finally, we take up the modular structure in the *untwisted* 2 + 1 dimensional theory. Consider once more Y_3, as shown in Fig. 1, with two boundary circles negatively oriented, one boundary circle positively oriented, and basepoints on each of the boundary circles. Quantizing (5.20) we find a vector space isomorphism

$$E(Y_3) \cong A \otimes A , \tag{5.21}$$

where $A = E([0, 1] \times S^1)$ is the unitary coalgebra of the untwisted theory. Recall that this is just the set of functions on $\mathscr{G}$. Now $E(Y_3)$ has a right A-comodule structure Δ coming from the positively oriented boundary circle. Combining with the counit (3.29) we obtain a map

$$m\colon A \otimes A \xrightarrow{\Delta} A \otimes A \otimes A \xrightarrow{\varepsilon \otimes \varepsilon} A .$$

More explicitly, if $a, b \in A$ are functions on $\mathscr{G}$, then

$$m(a, b)_{\langle x, g\rangle} = \sum_{x_1 x_2 = x} a_{\langle x_1, g\rangle} b_{\langle x_2, g\rangle} .$$

Also, there is a canonical element $\mathbf{1} \in A$ as follows. First, if Y_1 is the disk then $E(Y_1) \cong \mathbb{C}$ is the trivial comodule. But we can write Y_1 as the union of Y_1 and $Y_2 = [0, 1] \times S^1$, and by (3.26) this gives an isometry

$$\mathbb{C} \boxtimes A \cong \mathbb{C} .$$

From this we determine $\mathbf{1} \in A$:

$$\mathbf{1}_{\langle x, g\rangle} = \begin{cases} 1, & x = e; \\ 0, & \text{otherwise} . \end{cases}$$

Proposition 5.22. *The multiplication m and unit* **1** *render the coalgebra A a Hopf algebra.*

This Hopf algebra is the dual of the "quantum double" of Γ considered in [DPR]. We leave the verification of the Hopf algebra axioms to the reader.

The algebra structure can be used to define tensor products of comodules of A. Namely, if E_1, E_2 are right comodules, with coproducts Δ_1, Δ_2, then the composition

$$E_1 \otimes E_2 \xrightarrow{\Delta_1 \otimes \Delta_2} E_1 \otimes A \otimes E_2 \otimes A \to E_1 \otimes E_2 \otimes A \otimes A \xrightarrow{1 \otimes 1 \otimes m} E_1 \otimes E_2 \otimes A \tag{5.23}$$

is a coproduct on $E_1 \otimes E_2$. We denote this comodule by $E_1 \otimes_A E_2$. In particular, we can take tensor products of the irreducible corepresentations. Define nonnegative integers $\tilde{N}_{\mu\nu}^{\lambda}$ by

$$E_\mu \otimes_A E_\nu \cong \bigoplus_{\lambda} \tilde{N}_{\mu\nu}^{\lambda} E_\lambda . \tag{5.24}$$

Proposition 5.25. *We have*

$$\tilde{N}_{\mu\nu}^{\lambda} = N_{\mu\nu}^{\lambda} ,$$

where $N_{\mu\nu}^{\lambda}$ is defined in (4.13).

In other words, this tensor product on corepresentations reproduces the Verlinde algebra.

Proof. Recall that $E(Y_3)$ is a right $A^{\mathrm{op}} \times A^{\mathrm{op}} \times A$-comodule. Further, the isomorphism (5.21) is defined in terms of the identification (5.20). Under this isomorphism $A \otimes A$ becomes a right $A^{\mathrm{op}} \times A^{\mathrm{op}} \times A$-comodule. The right $A^{\mathrm{op}} \times A^{\mathrm{op}}$-comodule structure is simply the natural *left* $A \times A$-comodule structure on $A \otimes A$; the right A-comodule structure is that of the tensor product $A \otimes_A A$, as defined by (5.23). From (4.7), (4.13), and (A.9) we compute

$$\begin{aligned} N_{\mu\nu}^{\lambda} &= \dim E(Y_3, \lambda \cup \mu \cup \nu) \\ &= \dim \mathrm{Hom}_{A^{\mathrm{op}} \times A^{\mathrm{op}} \times A} (E_\mu^* \otimes E_\nu^* \otimes E_\lambda, A \otimes A) \\ &= \dim \mathrm{Hom}_A(E_\lambda, E_\mu \otimes_A E_\nu) . \end{aligned}$$

But this last expression is exactly $\tilde{N}^{\lambda}_{\mu\nu}$, from the definition (5.24).

A much more complete treatment of the Hopf algebra structure in both the untwisted and twisted theories appears in [F5].

A. Appendix: Coalgebras and Comodules

Some basic definitions may be found in [MM, Sect. 2]. *A coalgebra A* over $\mathbb{C}$ is a complex vector space endowed with a *comultiplication*

$$\Delta\colon A \to A \otimes A \tag{A.1}$$

and a *counit*

$$\varepsilon\colon A \to \mathbb{C}\,. \tag{A.2}$$

The comultiplication is required to be coassociative. A (right) *comodule E* is a complex vector space with a coaction

$$\Delta\colon E \to E \otimes A$$

which is compatible with (A.1) and (A.2).

A natural example is the following. Suppose G is a finite group acting on a finite set X. Then the vector space $\mathscr{F}(G)$ of complex-valued functions on G is a coalgebra and the space of functions $\mathscr{F}(X)$ is a comodule. The comultiplication is dual to the group multiplication $G \times G \to G$ and the coaction is dual to the group action $X \times G \to X$. The counit is $\varepsilon(f) = f(1)$, where $1 \in G$ is the identity element.

Now suppose $a \mapsto \bar{a}$ is an antiinvolution of A. Then A is *unitary* if A is endowed with a hermitian inner product which satisfies

$$(\Delta a, b \otimes c) = (a \otimes \bar{c}, \Delta b) = (\bar{b} \otimes a, \Delta c) \tag{A.3}$$

for all $a, b, c \in A$. A right comodule E is *unitary* if it has a hermitian inner product which satisfies

$$(\Delta x, y \otimes a) = (x \otimes \bar{a}, \Delta y) \tag{A.4}$$

for all $a \in A$, $x, y \in E$.

Suppose E_R is a right A-comodule and E_L is a left A-comodule. Then the *cotensor product* $E_R \boxtimes_A E_L$ is the vector subspace of $E_R \otimes_{\mathbb{C}} E_L$ annihilated by $\Delta_R \otimes \mathrm{id}$–$\mathrm{id} \otimes \Delta_L$. It is not a comodule, but rather is simply a vector space. If E_R, E_L are unitary, then $E_R \boxtimes_A E_L$ inherits the subspace inner product. More generally, if E has both a left A-comodule structure Δ_L and a right A-comodule structure Δ_R, then we define

$$\mathrm{Inv}_A(E) \subset E$$

to be the subspace annihilated by $\Delta_L - P\Delta_R$, where $P\colon E \otimes_{\mathbb{C}} A \to A \otimes_{\mathbb{C}} E$ is the natural isomorphism.

The dual A^* of a coalgebra is an algebra. In finite dimensions the dual of an algebra is a coalgebra, so we can pass freely between the two. Recall that an algebra (over $\mathbb{C}$) is *simple* if it is isomorphic to a matrix algebra, and it is *semisimple* if it is isomorphic to a direct product of matrix algebras. If

$$A^* \cong \prod_{\lambda \in \Phi} M_\lambda \tag{A.5}$$

is a finite direct product of matrix algebras M_λ, then there are commuting idempotents $a_\lambda^* \in A^*$ which correspond to the identity matrix in M_λ. Now each M_λ has a unique nonzero irreducible representation E_λ^* (up to isomorphism). Under the isomorphism (A.5) we view E_λ^* as an irreducible representation of A^*. Then if E^* is any representation of A^*, there is an isomorphism

$$\bigoplus_{\lambda \in \Phi} \mathrm{Hom}_{A^*}(E_\lambda^*, E^*) \otimes E_\lambda^* \to E^*$$

$$\bigoplus_\lambda (f_\lambda \otimes e_\lambda^*) \mapsto \sum_\lambda f_\lambda(e_\lambda^*) \,. \tag{A.6}$$

Now suppose that A^* is a unitary algebra, i.e., has a hermitian inner product which satisfies the dual of (A.3):

$$(a^*, b^* c^*) = (a^* \overline{c^*}, b^*) = (\overline{b^*} a^*, c^*) \,.$$

Then an easy argument with the a_λ^* shows that the images of M_λ in A^* under the isomorphism (A.5) are orthogonal. Since each matrix algebra M_λ has a unique unitary structure up to scaling, we may assume that (A.5) is an isometry. The irreducible module E_λ^* also has a unique unitary structure up to scaling. Fix unitary structures on the E_λ^*. Now suppose that E^* is unitary structure up to scaling. Fix unitary structures on the E_λ^*. Now suppose that E^* is a unitary A^*-module, and $f, f' \in \mathrm{Hom}_{A^*}(E_\lambda^*, E^*)$. Then for $e^*, e^{*\prime} \in E_\lambda^*$ we set

$$h(e^*, e^{*\prime}) = (f(e^*), f'(e^{*\prime}))_{E^*} \,.$$

This defines a new unitary structure on E_λ^*. So by uniqueness,

$$h(e^*, e^{*\prime}) = \lambda \cdot (e^*, e^{*\prime})_{E_\lambda^*}$$

for some constant λ. Let $\{e_i^*\}$ be an orthonormal basis of E_λ^*. Then the inner product (f, f') in $\mathrm{Hom}_{A^*}(E_\lambda^*, E^*)$ is

$$\begin{aligned}(f, f') &= \sum_{e_i^*} (f(e_i^*), f'(e_i^*))_E \,.\\ &= \lambda \cdot \dim E_\lambda^* \,.\end{aligned}$$

Using this we see that (A.6) is an isometry if we scale the inner product on E_λ^* in $\mathrm{Hom}_{A^*}(E_\lambda^*, E^*)$ by a factor $1/\dim E_\lambda^*$. Dually, we have proved the following.

Proposition A.7. *Suppose A is a finite dimensional unitary semisimple coalgebra, and $\{E_\lambda\}_{\lambda \in \Phi}$ a representative set of irreducible corepresentations. Fix unitary structures on each E_λ. Then for any unitary comodule E the map*

$$\bigoplus_{\lambda \in \Phi} \mathrm{Hom}_A\left(\frac{E_\lambda}{\dim E_\lambda}, E\right) \otimes E_\lambda \to E$$

$$\bigoplus_\lambda (f_\lambda \otimes e_\lambda) \mapsto \sum_\lambda f_\lambda(e_\lambda) \,. \tag{A.8}$$

is an isometry.

(Recall that $\frac{E_\lambda}{\dim E_\lambda}$ denotes the space E_λ with the inner product scaled by the factor $1/\dim E_\lambda$.)

If E is a right A-comodule, then E^* has a left A-comodule structure determined by the formula

$$\langle e, \Delta_{E^*} e^* \rangle = (\Delta_E e, e^* \rangle \in A, \quad e \in E, \quad e^* \in E^* .$$

If E is unitary, then so is E^*. In this way $\{E_\lambda^*\}$ is a representative list of the irreducible left A-comodules. Equivalently, it is a representative list of the irreducible right comodules for the opposite coalgebra A^{op}.

From (A.5) it follows that a semisimple coalgebra A decomposes as

$$A \cong \bigoplus_{\lambda \in \Phi} E_\lambda^* \otimes E_\lambda . \tag{A.9}$$

We can interpret (A.9) as an equation for right, left, or bi A-comodules.

Finally, for $\lambda, \mu \in \Phi$ we have the formula

$$\mathrm{Inv}_A(E_\lambda \otimes E_\mu^*) = E_\lambda \boxtimes_A E_\mu^* \cong \begin{cases} \dim E_\lambda \cdot \mathbb{C} & \text{if } \lambda = \mu; \\ 0, & \text{if } \lambda \neq \mu . \end{cases}$$

The factor $\dim E_\lambda$ is the norm square of the canonical element (the duality pairing) in $E_\lambda \otimes E_\lambda^*$.

B. Appendix: Integration of Singular Cocycles

Fix an integer d, and suppose that X is a compact oriented $(d+1)$-manifold with boundary. Then if α is a differential form of degree $d+1$ on X, the integral $\int_X \alpha$ is well-defined. Notice that $d\alpha = 0$ since any $(d+2)$-form on X vanishes. If instead we consider a (real-valued) singular cocycle α, then the integral $\int_X \alpha$ is well-defined if X is closed. For in this case X has a fundamental class $[X] \in H_{d+1}(X)$ and the integral is the pairing of $[X]$ with the cohomology class represented by α. If $\partial X \neq 0$ we must work a little harder, essentially because α may not vanish on degenerate chains. (By contrast differential forms vanish on degenerate chains.) Our constructions in this appendix keep track of these degeneracies. Notice that we only define integration of *closed* cochains, i.e., cocycles. We work first with manifolds of arbitrary dimensions d and $d+1$, though there is a generalization to CW complexes. Then we specialize to $d = 2$ and extend the theory to surfaces with boundary by fixing some standard choices on the boundary.

Proposition B.1. *Let Y be a closed oriented d-manifold and $\alpha \in C^{d+1}(Y; \mathbb{R}/\mathbb{Z})$ a singular cocycle. Then there is a metrized "integration line" $I_{Y,\alpha}$ defined. If X is a compact oriented $(d+1)$-manifold, $i\colon \partial X \hookrightarrow X$ the inclusion of the boundary, and $\alpha \in C^{d+1}(X; \mathbb{R}/\mathbb{Z})$ a cocycle, then*

$$\exp\left(2\pi i \int_X \alpha\right) \in I_{\partial X, i^*\alpha}$$

is defined and has unit norm. These lines and integrals satisfy:

(a) (*Functoriality*) *If* $f\colon Y' \to Y$ *is an orientation preserving diffeomorphism, then there is an induced isometry*

$$f_*\colon I_{Y',f^*\alpha} \to I_{Y,\alpha}$$

and these compose properly. If $F\colon X' \to X$ *is an oreintation preserving diffeomorphism, then*

$$(\partial F)_*\left[\exp\left(2\pi i \int_X F^*\alpha\right)\right] = \exp\left(2\pi i \int_X \alpha\right).$$

(b) (*Orientation*) *There is a natural isometry*

$$I_{-Y,\alpha} \cong \overline{I_{Y,\alpha}}\,,$$

and

$$\exp\left(2\pi i \int_{-X} \alpha\right) = \overline{\exp\left(2\pi i \int_X \alpha\right)}.$$

(c) (*Additivity*) *If* $Y = Y_1 \sqcup Y_2$ *is a disjoint union, then there is a natural isometry*

$$I_{Y_1 \sqcup Y_2, \alpha_1 \sqcup \alpha_2} \cong I_{Y_1,\alpha_1} \otimes I_{Y_2,\alpha_2}\,.$$

If $X = X_1 \sqcup X_2$ *is a disjoint union, then*

$$\exp\left(2\pi i \int_{X_1 \sqcup X_2} \alpha_1 \sqcup \alpha_2\right) = \exp\left(2\pi i \int_{X_1} \alpha_1\right) \otimes \exp\left(2\pi i \int_{X_2} \alpha_2\right).$$

(d) (*Gluing*) *Suppose* $j\colon Y \hookrightarrow X$ *is a closed oriented codimension one submanifold and* X^{cut} *is the manifold obtained by cutting* X *along* Y. *Then* $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$. *Suppose* $\alpha \in C^{d+1}(X; \mathbb{R}/\mathbb{Z})$ *is a singular* $(d+1)$*-cocycle on* S, *and* $\alpha^{\text{cut}} \in C^{d+1}(X^{\text{cut}}; \mathbb{R}/\mathbb{Z})$ *the induced cocycle on* X^{cut}. *Then*

$$\exp\left(2\pi i \int_X \alpha\right) = \mathrm{Tr}_{Y,j^*\alpha}\left[\exp\left(2\pi i \int_{X^{\text{cut}}} \alpha^{\text{cut}}\right)\right], \tag{B.2}$$

where $\mathrm{Tr}_{Y,j^*\alpha}$ *is the contraction*

$$\mathrm{Tr}_{Y,j^*\alpha}\colon I_{\partial X^{\text{cut}},\alpha^{\text{cut}}} \cong I_{\partial X,i^*\alpha} \otimes I_{Y,j^*\alpha} \otimes \overline{I_{Y,j^*\alpha}} \to I_{\partial X,i^*\alpha}$$

using the hermitian metric on $I_{Y,j^*\alpha}$.

(e) (*Stokes' Theorem* I) *Let* $\alpha \in C^{d+1}(W; \mathbb{R}/\mathbb{Z})$ *be a singular cocycle on a compact oriented* $(d+2)$*-manifold* W. *Then*

$$\exp\left(2\pi i \int_{\partial W} \alpha\right) = 1\,. \tag{B.3}$$

(f) (*Stokes' Theorem* II) *A singular d-cochain* $\beta \in C^d(Y; \mathbb{R}/\mathbb{Z})$ *on* Y *determines a trivialization*

$$I_{Y,\delta\beta} \cong \mathbb{C}\,.$$

A singular d-cochain $\beta \in C^d(X; \mathbb{R}/\mathbb{Z})$ *on* X *satisfies*

$$\exp\left(2\pi i \int_X \delta\beta\right) = 1$$

under this isomorphism.

Proof. We give the constructions and leave the reader to verify the properties. Consider the category $\mathscr{C}_Y$ whose objects are oriented cycles $y \in C_d(Y)$ which represent the fundamental class $[Y] \in H_d(Y)$. A morphism $y \xrightarrow{a} y'$ is a chain $a \in C_{d+1}(Y)$ such that $y' = y + \partial a$. Define a functor $\mathscr{F}_{Y,\alpha}\colon \mathscr{C}_Y \to \mathscr{L}$ by $\mathscr{F}_{Y,\alpha}(y) = \mathbb{C}$ for each object y and $\mathscr{F}_{Y,\alpha}(y \xrightarrow{a} y')$ acts as multiplication by $e^{2\pi i\alpha(a)}$. Now if $y \xrightarrow{a} y$ is an automorphism, then $\partial a = 0$. Since $H_{d+1}(Y) = 0$, there is a $(d+2)$-chain b with $\partial b = a$. Hence $\alpha(a) = \alpha(\partial b) = \delta\alpha(b) = 0$. Therefore, the functor $\mathscr{F}_Y$ has no holonomy, so defines the desired line $I_{Y,\alpha}$ of invariant sections.

For X, choose a chain $x \in C_{d+1}(X)$ which represents the fundamental class $[X] \in H_{d+1}(X, \partial X)$. Then $\partial x \in C_d(\partial X)$ is closed and represents the fundamental class $[\partial X] \in H_d(\partial X)$. Consider the section

$$\partial x \mapsto e^{2\pi i\alpha(x)} \tag{B.4}$$

of the functor $\mathscr{F}_{\partial X, i^*\alpha}$. If x' is another chain representing $[X]$ with $\partial x = \partial x'$, then $x' = x + \partial c$ for some $c \in C_{d+1}(X)$. But then $\alpha(\partial c) = \delta\alpha(c) = 0$, so that (B.4) is well-defined. A similar check shows that (B.4) is an invariant section of $\mathscr{F}_{\partial X, i^*\alpha}$, so determines an element of unit norm in $I_{\partial X, i^*\alpha}$ as desired.

We generalize these constructions[20] in the case $d = 2$. Fix once and for all the standard oriented circle $S^1 = [0,1]/0 \sim 1$ and the standard cycle $s \in C_1(S^1)$ which represents the fundamental class $[S^1]$. (Thus s is the identity map $[0,1] \to [0,1]$ followed by the quotient map onto S^1.) The following proposition generalizes the construction of integration lines to surfaces with boundary.

Proposition B.5. *Let Y be a compact oriented 2-manifold, and suppose that each component $(\partial Y)_i$ of the boundary is endowed with a fixed parametrization $S^1 \to (\partial Y)_i$ (which may or may not preserve the orientation). Suppose $\alpha \in C^3(Y; \mathbb{R}/\mathbb{Z})$ is a singular cocycle. Then there is a metrized line $I_{Y,\alpha}$ defined. If $\partial Y = \emptyset$, then this is the line defined in Proposition B.1. These lines satisfy properties* (a)[21], (b), (c) *of that proposition and in addition satisfy*:

(d) (*Gluing*) *Suppose $S \hookrightarrow Y$ is a closed embedded 1-manifold and Y^{cut} the manifold obtained by cutting along S. Then $\partial Y^{\text{cut}} = \partial Y \sqcup S \sqcup -S$ and we use parametrizations which agree on S and $-S$. Let α be a 3-cocycle on Y and α^{cut} the induced cocycle on Y^{cut}. Then there is an isometry*

$$I_{Y,\alpha} \cong I_{Y^{\text{cut}},\alpha^{\text{cut}}} \tag{B.6}$$

These isometries compose properly under successive gluings.

Proof. The construction is similar to that in Proposition B.1. Using the boundary parametrizations we construct from s and $-s$ a cycle $z \in C_1(\partial Y)$ which represents the fundamental class $[\partial Y]$. We take $\mathscr{C}_Y$ to be the category whose objects are oriented cycles $y \in C_2(Y)$ which represent the fundamental class $[Y, \partial Y]$ and satisfy $\partial y = z$. A morphism $y \xrightarrow{a} y'$ is a chain $a \in C_3(Y)$ with $y' = y + \partial a$. Notice that $\mathscr{C}_Y$ is connected. The rest of the construction $I_{Y,\alpha}$ is as before.

[20] There is a simpler version of what follows for $d = 1$. In that case there is no need to fix a standard cycle s nor to parametrize the boundary. The analogue of Proposition B.5 holds, now for gluings of intervals. We use the $d = 1$ version in the proof of Proposition 3.14

[21] The diffeomorphisms should commute with the boundary parametrizations

For the gluing law we proceed as follows. The gluing map $g\colon Y^{\text{cut}} \to Y$ induces a map $g_*\colon C_2(Y^{\text{cut}}) \to C_2(Y)$ on chains, and this in the turn induces a map $g_*\colon \mathscr{C}_{Y^{\text{cut}}} \to \mathscr{C}_Y$ since the boundaries of the chains along S and $-S$ cancel out under gluing. Then g_* extends in an obvious way to a functor, and $\mathscr{F}_{Y^{\text{cut}}, \alpha^{\text{cut}}} = \mathscr{F}_{Y,\alpha} \circ g_*$. Thus g_* induces the desired isometry on the space of invariant sections.

Suppose $\beta \in C^3(S^1; \mathbb{R}/\mathbb{Z})$ is a singular cocycle. We also use "β" to denote the induced cocycles on $[0,1] \times S^1$ and $S^1 \times S^1$, obtained by pullback from the second factor. Glue two cylinders together to form a single cylinder. Then (B.6) leads to an isomorphism

$$I_{[0,1]\times S^1,\beta} \otimes I_{[0,1]\times S^1,\beta} \cong I_{[0,1]\times S^1,\beta}\,. \tag{B.7}$$

There is a unique trivialization

$$I_{[0,1]\times S^1,\beta} \cong \mathbb{C} \tag{B.8}$$

which is compatible with (B.7). Gluing the two ends of $[0,1] \times S^1$ together and applying (B.6) we obtain a trivialization

$$I_{S^1 \times S^1,\beta} \cong \mathbb{C} \tag{B.9}$$

compatible with (B.8) under gluing.

Finally, we observe that the gluing law in Proposition B.1(d) extends to a 3-manifold glued along part of its boundary.

Proposition B.10. *Let X be a compact oriented 3-manifold and Y a compact oriented 2-manifold with parametrized boundary. Suppose $Y \subsetneq X$ is an embedding which restricts to an embedding $\partial Y \subsetneq \partial X$. Let X^{cut} be the "manifold with corners" obtained by cutting X along Y. Then $\partial X^{\text{cut}} = \partial X \sqcup Y \sqcup -Y$, where the union is over $\partial Y \sqcup -\partial Y$. Suppose $\alpha \in C^{d+1}(X; \mathbb{R}/\mathbb{Z})$ is a 3-cocycle, with α^{cut} the induced cocycle on X^{cut}, and α the restriction of α to Y. Then*

$$\exp\left(2\pi i \int_X \alpha\right) = \mathrm{Tr}_{Y,\alpha}\left[\exp\left(2\pi i \int_{X^{\text{cut}}} \alpha^{\text{cut}}\right)\right],$$

where $\mathrm{Tr}_{Y,\alpha}$ is the contraction

$$\mathrm{Tr}_{Y,\alpha}\colon I_{\partial X^{\text{cut}},\alpha^{\text{cut}}} \cong I_{\partial X,\alpha} \otimes I_{Y,\alpha} \otimes \overline{I_{Y,\alpha}} \to I_{\partial X,\alpha} \tag{B.11}$$

using the hermitian metric on $I_{Y,\alpha}$.

There is a canonical way to "straighten the angle" [CF] to make X^{cut} a smooth manifold with boundary. In particular, X^{cut} has a (relative) fundamental class, which is needed to define the integration lines and the integrals. Note that (B.11) uses the isomorphism (B.6). We have also implicitly used (B.6) when we cut ∂X along ∂Y. The proof of Proposition B.10 is straightforward once the definitions are clear.

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