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Real Killing Spinors and Holonomy

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Abstract. We give a description of all complete simply connected Riemannian manifolds carrying real Killing spinors. Furthermore, we present a construction method for manifolds with the exceptional holonomy groups G_2 and Spin(7).

1. Introduction

Let M be an n-dimensional complete Riemannian spin manifold. A spinor field ψ is called *Killing spinor* with Killing constant α if for all tangent vectors X the equation $\nabla_X \psi = \alpha \cdot X \cdot \psi$ holds. Here $X \cdot \psi$ denotes the Clifford product of X and ψ. Killing spinors occur in physics, e.g. in supergravity theories, see [11], but they are also of mathematical interest. Friedrich showed that if M is compact and the scalar curvature satisfies $S \geqq S_0 > 0$, $S_0 \in \mathbb{R}$, then for all eigenvalues λ of the Dirac operator the estimate $\lambda^2 \geqq \frac{1}{4}\frac{n}{n-1} S_0$ holds, see [13]. If we have equality in this estimate, then the corresponding eigenspinor is a Killing spinor.

If M carries a Killing spinor, then M is an Einstein manifold with Ricci curvature $\mathrm{Ric} = 4(n-1)\alpha^2$. In particular, we have three distinct cases; α can be purely imaginary, then M is noncompact and we call ψ an imaginary Killing spinor, α can be 0, in this case ψ is a parallel spinor field, and finally α can be real, then M is compact and ψ is called a real Killing spinor. This terminology is somewhat misleading, because a real Killing spinor is not necessarily a real spinor field; we *always* work with complex spinor fields.

Hitchin showed that manifolds with parallel spinor fields can be characterized by their holonomy group, see [28, Th. 1.2 and footnote p. 54]. See also [15] and [35].

Manifolds with imaginary Killing spinors have been classified by Baum in [1–3], shortly later the classification has been extended by Rademacher to generalized imaginary Killing spinors where we allow the Killing "constant" α to be an imaginary function, see [32].

Most results on real Killing spinors known so far are statements for particular (low) dimensions. For example, Friedrich showed in [14] that a complete

4-dimensional manifold with a real Killing spinor is isometric to the standard sphere. The analogous result in dimension 8 is due to Hijazi, see [27]. We will show that in fact in any even dimension $n \neq 6$ only the standard spheres carry real Killing spinors (Theorem 1). In dimension 6 we recover the theorem of Grunewald, see [26], that the manifolds besides the standard sphere carrying real Killing spinors are precisely the nearly Kähler, non-Kähler manifolds (Theorem 2′).

Friedrich and Kath showed in [17] that the complete simply connected 5-dimensional manifolds with real Killing spinors are exactly the standard sphere and the Einstein–Sasaki manifolds. We prove the analogous result in all dimensions $n \equiv 1 \pmod 4$, see Theorem 3′. In dimension $n \equiv 3 \pmod 4$, $n \geqq 11$, we get the standard sphere, the Einstein–Sasaki manifolds and the Sasaki-3-manifolds, according to how many linearly independent real Killing spinors there are (Theorem 4′).

In the remaining dimension $n = 7$ we also get a description of all complete simply connected manifolds with real Killing spinors (Theorem 5′). Parts of this theorem have already been known to Friedrich and Kath, see [18] and [19].

Exceptional holonomy groups. The study of the exceptional dimension 6 provides us with a construction method of Riemannian manifolds with exceptional holonomy group G_2. The recipe is as follows. Take any compact simply connected nearly Kähler, non-Kähler manifold of dimension 6, normalize the metric such that the Ricci curvature is $\mathrm{Ric} = 5$, now the cone over this manifold has holonomy group G_2. Using this method we recover Bryant's first explicit example which is the cone over the complex flag manifold $SU(3)/T^2$, see [8]. Further examples are obtained by taking the cones over $S^3 \times S^3$ and $\mathbb{C}P^3$ with certain non-standard metrics.

Similarly, the cones over certain 7-dimensional manifolds have exceptional holonomy group Spin(7). Examples are the cones over $SO(5)/SO(3)$ and over the squashed 7-sphere. Most of our examples can be found elsewhere in the literature, see [9, 21, and 33], but at least one series of examples of Spin(7)-manifolds is new, namely the cones over the Wallach manifolds. This series is interesting because infinitely many homotopy types occur and there are homeomorphic, non-diffeomorphic examples.

The paper is organized as follows. First we modify the spinor connection because we want to interpret Killing spinors as parallel sections. To do this we have to enlarge the structure group Spin(n) of the spinor bundle to Spin($n+1$). Then we show that this connection is related to the Levi–Civita connection of the cone over the original manifold. Since Killing spinors correspond to fixpoints of the holonomy group of the cone we can use the Berger–Simons classification of possible holonomy groups to see how the cone can look like. In the last sections this information is retranslated into conditions on the original manifold itself.

For facts about Killing spinors the reader can consult [4 or 10], holonomy is explained in [5].

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2. The Modified Connection

Let M be an n-dimensional Riemannian spin manifold. By $P_{SO(n)}(M)$ we denote the bundle of positively oriented orthonormal frames, by $P_{\mathrm{Spin}(n)}(M)$ the spin structure.

Let $\phi: P_{\mathrm{Spin}(n)}(M) \to P_{SO(n)}(M)$ and $\theta: \mathrm{Spin}(n) \to SO(n)$ be the twofold covering maps. Furthermore, let ω_{LC} be the connection 1-form of the Levi–Civita connection of M and $\tilde{\omega}_{\mathrm{LC}}$ its lift to $P_{\mathrm{Spin}(n)}(M)$. The spinor bundle is the bundle $\Sigma M = P_{\mathrm{Spin}(n)}(M) \times_{\rho_n} \Sigma_n$, where $\rho_n: \mathrm{Spin}(n) \to U(\Sigma_n)$ is the complex spinor representation.

Given a local section $\tilde{h}$ of $P_{\mathrm{Spin}(n)}(M)$ we can express the spinor connection on ΣM as

$$\nabla_X [\tilde{h}, \sigma] = [\tilde{h}, \partial_X \sigma + (\rho_n)_* (\tilde{\omega}_{\mathrm{LC}}(d\tilde{h} \cdot X)) \cdot \sigma] \,.$$

If $\iota: SO(n) \to GL(n)$ is the standard representation we can write $TM = P_{SO(n)}(M) \times_\iota \mathbb{R}^n$ and under this identification we have

$$X = [h, \eta(dh \cdot X)]$$

for any local section h of $P_{SO(n)}(M)$. Here η is the solder form.

We view $\mathrm{Spin}(n)$ as sitting in the real Clifford algebra Cl_n. The Lie algebra $\mathbf{spin}(n)$ is the vector subspace of Cl_n spanned by the elements $E_i \cdot E_j$, $1 \leqq i < j \leqq n$, where $E_1, \ldots, E_n$ is the standard basis of $\mathbb{R}^n$. But we can also discover $\mathbf{spin}(n+1)$ sitting in Cl_n; its Lie algebra is $\mathbf{spin}(n+1) = \mathbf{spin}(n) \oplus \mathbb{R}^n$.

Since ρ_n is the restriction of a representation of $\mathbb{C}l_n = Cl_n \otimes \mathbb{C}$ we can also consider $\rho_n | \mathrm{Spin}(n+1)$. If n is even $\rho_n | \mathrm{Spin}(n+1)$ is the spinor representation of $\mathrm{Spin}(n+1)$, if n is odd it is one of the half-spin representations.

Let $\alpha \in \mathbb{R}$. To study Killing spinors we consider the modified connection $\tilde{\nabla}$ on ΣM defined by $\tilde{\nabla}_X \psi = \nabla_X \psi + \alpha \cdot X \cdot \psi$. We want to calculate its connection 1-form. Let $\tilde{h}$ be a local section of $P_{\mathrm{Spin}(n)}(M)$, $h = \tilde{h} \circ \phi$,

$$\begin{aligned} \tilde{\nabla}_X [\tilde{h}, \sigma] &= \nabla_X [\tilde{h}, \sigma] + \alpha \cdot [h, \eta(dh \cdot X)] \cdot [\tilde{h}, \sigma] \\ &= [\tilde{h}, \partial_X \sigma + (\rho_n)_* \circ \theta_*^{-1} (\omega_{\mathrm{LC}}(dh \cdot X) \cdot \sigma + \alpha \cdot (\rho_n)_* (\eta(dh \cdot X)) \sigma] \\ &= [\tilde{h}, \partial_X \sigma + (\rho_n)_* ((\theta_*^{-1} \circ \omega_{\mathrm{LC}} + \alpha \cdot \eta)(dh \cdot X)) \cdot \sigma] \,. \end{aligned}$$

Therefore the "connection 1-form" of $\tilde{\nabla}$ is $\hat{\omega} = \phi^* (\theta_*^{-1} \circ \omega_{\mathrm{LC}} + \alpha \cdot \eta)$. This is a 1-form on $P_{\mathrm{Spin}(n)}(M)$ with values in $\mathbf{spin}(n+1) = \mathbf{spin}(n) \oplus \mathbb{R}^n$.

One easily calculates that $\hat{\omega}$ is $\mathrm{Spin}(n)$-equivariant. We use the embedding $\mathrm{Spin}(n) \subset \mathrm{Spin}(n+1)$ to enlarge the structure group of $P_{\mathrm{Spin}(n)}(M)$ and we get $P_{\mathrm{Spin}(n+1)}(M)$.

Let $b \in P_{\mathrm{Spin}(n)}(M) \subset P_{\mathrm{Spin}(n+1)}(M)$. Then we obtain $T_b P_{\mathrm{Spin}(n+1)}(M) = T_b P_{\mathrm{Spin}(n)}(M) \oplus dL_b \cdot \mathbb{R}^n$, where $L_b: \mathrm{Spin}(n+1) \to P_{\mathrm{Spin}(n+1)}(M)$, $\Lambda \to b \cdot \Lambda$. We define $\hat{\omega}(dL_b \cdot v) := v \in \mathbb{R}^n \subset \mathbf{spin}(n+1)$. We extend $\hat{\omega}$ to a $\mathrm{Spin}(n+1)$-equivariant 1-form $\tilde{\omega}$ on $P_{\mathrm{Spin}(n+1)}(M)$. Clearly, $\tilde{\omega}$ is the connection 1-form of $\tilde{\nabla}$ on $\Sigma M = P_{\mathrm{Spin}(n+1)}(M) \times_{\rho_n} \Sigma_n$.

Similarly, we extend $P_{SO(n)}(M) \subset P_{SO(n+1)}(M)$ and $\phi: P_{\mathrm{Spin}(n+1)}(M) \to P_{SO(n+1)}(M)$. We have $\tilde{\omega} = \theta_*^{-1} \circ \phi^* \omega$, where ω is an $\mathbf{so}(n+1)$-valued 1-form on $P_{SO(n+1)}(M)$. We calculate ω.

Consider the diagram (Cl_{n+1}^0 is the even part of Cl_{n+1}):

$$\begin{array}{ccc} E_k & \to & E_k \cdot E_{n+1} \\ \cap & & \cap \\ Cl_n & \overset{\cong}{\to} & Cl_{n+1}^0 \\ \cup & & \cup \\ \mathbf{spin}(n) \oplus \mathbb{R}^n & \overset{\cong}{\to} & \mathbf{spin}(n+1) \\ & & \downarrow \theta_* \\ & & \mathbf{so}(n+1) \end{array}$$

If we restrict θ_* to $\mathbf{spin}(n)$ we get again $\theta_*: \mathbf{spin}(n) \to \mathbf{so}(n)$. But how does θ_* look like on the $\mathbb{R}^n$-part?

Let $A_k = (a_{ij}) \in \mathbf{so}(n+1)$ where

$$a_{ij} = \begin{cases} 1, & \text{if } i = k, j = n+1 \\ -1, & \text{if } i = n+1, j = k \\ 0, & \text{otherwise.} \end{cases}$$

We get $\theta_*^{-1} A_k = -\frac{1}{4}\sum_{ij} a_{ij} E_i E_j = -\frac{1}{2} E_k E_{n+1} \cong -\frac{1}{2} E_k$. Therefore $\theta_* E_k = -2A_k$. Thus ω has the form

$$\omega = \begin{pmatrix} \omega_{\mathrm{LC}} & -2\alpha\eta \\ 2\alpha\eta^t & 0 \end{pmatrix}.$$

3. The Cone $\bar{M}$

We consider the warped product $\bar{M} = M \times_{r^2} \mathbb{R}^+$, in other words $\bar{M}$ is the cone over M carrying the metric $\langle \cdot, \cdot \rangle_{\bar{M}} = r^2 \langle \cdot, \cdot \rangle_M + dr^2$. We pull back the bundle $P_{SO(n+1)}(M)$ via the projection $\bar{M} \to M$ and obtain $\bar{P}_{SO(n+1)}(\bar{M})$. This is not the same bundle as $P_{SO(n+1)}(\bar{M})$ but it is equivalent to it. The map

$$\bar{P}_{SO(n+1)}(\bar{M}) \times \mathbb{R}^n \to T\bar{M} ,$$

$$(X_1, \ldots, X_n, \partial_r) \times v \to \left(\frac{1}{r} X_1, \ldots, \frac{1}{r} X_n, \partial_r\right) \cdot v ,$$

induces an isomorphism of Riemannian vector bundles $T\bar{M} \cong \bar{P}_{SO(n+1)}(\bar{M}) \times_\iota \mathbb{R}^{n+1}$. We calculate the connection 1-form $\bar{\omega}$ on $\bar{P}_{SO(n+1)}(\bar{M})$ for the Levi–Civita connection $\bar{\nabla}$ of the metric $\langle \cdot, \cdot \rangle_{\bar{M}}$. Using the formula $\nabla_\xi [h, v] = [h, \partial_\xi v + \bar{\omega}(dh \cdot \xi) \cdot v]$ for the local section $h = (X_1, \ldots, X_n, \partial_r)$ and the formulas for warped products (see [31, p. 206])

$$\bar{\nabla}_r \partial_r = 0 ,$$

$$\bar{\nabla}_r X = \bar{\nabla}_X \partial_r = \frac{1}{r} X ,$$

$$\bar{\nabla}_X Y = \nabla_X Y - r\langle X, Y \rangle_M \cdot \partial_r ,$$

where X and Y denote vector fields on M viewed as vector fields on $\bar{M}$, one gets

$$\bar{\omega}(dh \cdot \partial_r) = 0 ,$$

$$\bar{\omega}(dh \cdot X) \cdot E_{n+1} = \eta(dh \cdot X) ,$$

$$\bar{\omega}(dh \cdot X) \cdot E_k = \omega_{\mathrm{LC}}(dh \cdot X) \cdot E_k - \eta_k(dh \cdot X) \cdot E_{n+1}, \quad 1 \leqq k \leqq n .$$

Thus

$$\bar{\omega} = \begin{pmatrix} \omega_{LC} & \eta \\ -\eta^t & 0 \end{pmatrix}.$$

We see that for $\alpha = -\frac{1}{2}$, $\bar{\omega}$ equals the connection 1-form ω of the preceding section.

If we change the orientation of $\bar{M}$, we get the local section $h = (X_1, \ldots, X_n, -\partial_r)$ and obtain

$$\bar{\omega} = \begin{pmatrix} \omega_{\mathrm{LC}} & -\eta \\ \eta^t & 0 \end{pmatrix}.$$

Now for $\alpha = \frac{1}{2}$ we get $\bar{\omega} = \omega$.

For some fixed base point let $\mathrm{Hol}(\bar{M}) \subset SO(n+1)$ be the holonomy group of $\bar{M}$. This is the holonomy group of the principle bundle $\bar{P}_{SO(n+1)}(\bar{M})$ with the connection $\bar{\omega}$. Let $\tilde{\mathrm{H}}\mathrm{ol}(M) \subset \mathrm{Spin}(n+1)$ be the holonomy group of $P_{\mathrm{Spin}(n+1)}(M)$ with the connection $\tilde{\omega}$. We have shown

Lemma 1. *Let M be simply connected (such that all holonomy groups are connected). Then if $\alpha = -\frac{1}{2}$, $\mathrm{Hol}(\bar{M}) = \theta(\tilde{\mathrm{H}}\mathrm{ol}(M))$ and $\tilde{\mathrm{H}}\mathrm{ol}(M)$ is the identity component of $\theta^{-1}(\mathrm{Hol}(\bar{M}))$. If $\alpha = \frac{1}{2}$ the same is true if we change the orientation of $\bar{M}$.*

For a proof of the following lemma see [20, Prop. 3.1].

Lemma 2. *Let M be compact and simply connected. If $\mathrm{Hol}(\bar{M})$ is reducible, then $\bar{M}$ is flat and therefore M is isometric to the standard sphere.*

4. The Classification

Let M be a compact simply connected Riemannian spin manifold of dimension n carrying a Killing spinor ψ with real Killing constant $\alpha \neq 0$, i.e. $\nabla_X \psi = -\alpha \cdot X \cdot \psi$ for all tangent vectors X. Rescaling the metric if necessary we can assume that $\alpha = \pm\frac{1}{2}$. Killing spinors are parallel sections for the modified connection $\tilde{\nabla}$ of the second section. Therefore they correspond to fixpoints of the holonomy group $\tilde{\mathrm{H}}\mathrm{ol}(M) \subset \mathrm{Spin}(n+1)$ under the spinor representation if n is even and under one half-spin representation if n is odd. If n is even the number of linearly independent Killing spinors with constant $\alpha = \frac{1}{2}$ is the same as that for $\alpha = -\frac{1}{2}$. For n odd switching from $\alpha = \frac{1}{2}$ to $\alpha = -\frac{1}{2}$ corresponds to changing the orientation of $\bar{M} = M \times_{r^2} \mathbb{R}^+$. This is equivalent to conjugating $\mathrm{Hol}(\bar{M}) \subset SO(n+1)$ by the matrix $\begin{pmatrix} 1_n & 0 \\ 0 & -1 \end{pmatrix} \in O(n+1)$. For $\tilde{\mathrm{H}}\mathrm{ol}(M)$ this means that we interchange the half-spin representations. We use the classification theorem of Berger and Simons (see [5, p. 300]) and Lemma 2 to see which $\mathrm{Hol}(\bar{M})$ (and thus which $\tilde{\mathrm{H}}\mathrm{ol}(M)$) can occur. Since the Ricci curvature of M is $\mathrm{Ric} = 4 \cdot (n-1) \cdot \alpha^2 = n-1$ the manifold $\bar{M}$ is Ricci-flat and cannot have the holonomy of a symmetric space of rank $\geqq 2$.

Definition. *We say that M is of type (p, q) if M carries exactly p linearly independent Killing spinors with Killing constant $\alpha = \frac{1}{2}$ and exactly q linearly independent Killing spinors with Killing constant $\alpha = -\frac{1}{2}$ or vice versa.*

For example, the standard sphere is of type $(2^{[n/2]}, 2^{[n/2]})$. Now the remaining possibilities for $\mathrm{Hol}(\bar{M})$ are (compare also [35])

n = dimension of M	$\mathrm{Hol}(\bar{M})$	type
n arbitrary	trivial	S^n
$n+1 = 2m$, m odd	$SU(m)$	$(1, 1)$
$n+1 = 4m$	$SU(2m)$	$(2, 0)$
$n+1 = 4m$	$Sp(m)$	$(m+1, 0)$
$n+1 = 8$	$\mathrm{Spin}(7)$	$(1, 0)$
$n+1 = 7$	G_2	$(1, 1)$

We deduce

Theorem 1. *Let M be a complete Riemannian spin manifold of dimension n carrying a Killing spinor with Killing constant $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If n is even, $n \neq 6$, then M is isometric to the standard sphere.*

Proof. If M carries a Killing spinor, so does its universal cover $\tilde{M}$. In particular, $\tilde{M}$ is compact. The only nontrivial holonomy group for $n+1$ odd in the above list is G_2 for $n = 6$. Thus $\tilde{M}$ is isometric to the standard sphere S^n. For n even the only quotients of S^n are non-orientable (and hence non-spin) projective spaces $\mathbb{R}P^n$. Thus $M = \tilde{M} = S^n$.

The same proof also shows.

Theorem 2. *Let M be a 6-dimensional complete simply connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are two possibilities:*

(i) $M = S^6$
(ii) *M is of type* $(1, 1)$ *and* $\bar{M} = M \times_{r^2} \mathbb{R}^+$ *has holonomy* G_2.

Remarks. 1. Theorem 1 has already been known in dimension 4 (Friedrich) and 8 (Hijazi), see [4] and [27].
2. We can use Theorem 2 to construct examples with holonomy group G_2, see Sect. 7.

Looking again at the list of possible holonomy groups we can immediately deduce the following three theorems.

Theorem 3. *Let M be a complete simply connected Riemannian spin manifold of dimension n with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n = 2m-1$, $m \geqq 3$ odd, then there are two possibilities:*

(i) $M = S^n$
(ii) *M is of type* $(1, 1)$ *and* $\bar{M}$ *is Kähler.*

Theorem 4. *Let M be a complete simply connected Riemannian spin manifold of dimension n with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n = 4m-1$, $m \geqq 3$, then there are three possibilities:*

(i) $M = S^n$,
(ii) *M is of type* $(2, 0)$ *and* $\bar{M}$ *is Kähler, but not hyperkähler,*
(iii) *M is of type* $(m+1, 0)$ *and* $\bar{M}$ *is hyperkähler.*

Theorem 5. *Let M be a 7-dimensional complete simply connected Riemannian spin manifold with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. Then there are four possibilities:*

(i) $M = S^7$,
(ii) *M is of type* $(1, 0)$ *and* $\bar{M}$ *has holonomy group* Spin(7),
(iii) *M is of type* $(2, 0)$ *and* $\bar{M}$ *is Kähler, but not hyperkähler,*
(iv) *M is of type* $(3, 0)$ *and* $\bar{M}$ *is hyperkähler.*

In the theorems we have additional information about the geometry of $\bar{M}$ if M is not the standard sphere. We are going to translate this into conditions on the geometry of M itself.

5. The Case $\mathrm{Hol}(\bar{M}) = SU(m)$

The condition $\mathrm{Hol}(\bar{M}) = SU(m)$ means that $\bar{M}$ is Kähler (and Ricci-flat), i.e. there is a parallel complex structure J on $\bar{M}$. We identify M with $M \times \{1\} \subset \bar{M}$ and define

$$X := J(\partial_r) ,$$
$$\eta(V) := \langle X, V \rangle ,$$
$$\phi := -\nabla X .$$

Lemma 3. *X is a Killing vector field of length $|X| \equiv 1$.*

Proof.

$$\begin{aligned}\langle \nabla_V X, W \rangle &= \langle \bar{\nabla}_V X + \langle V, X \rangle \partial_r, W \rangle \\ &= \langle J \bar{\nabla}_V \partial_r, W \rangle \\ &= \langle J(V), W \rangle\end{aligned}$$

is skew symmetric in V and W.

We recall the definition of a Sasaki structure. If X is a vector field, η a 1-form, and ϕ a tensor field of type (1, 1), then (ϕ, X, η) is called a Sasaki structure if

(i) (ϕ, X, η) form a metric contact structure,
(ii) X is a Killing vector field, and
(iii) $(\nabla_V \phi)(W) = \langle V, W \rangle X - \eta(W) V$.

It is easy to check that (ϕ, X, η) as defined above form a Sasaki structure. Conversely, given a Sasaki structure on M we define on $\bar{M}$:

$$J(r\partial_r) := X ,$$
$$J(X) := -r\partial_r ,$$
$$J(V) := -\phi(V) \quad \text{for } V \perp X, r\partial_r .$$

Again, it is a simple playing with the definitions to see that J defines a Kähler structure on $\bar{M}$. We note

Lemma 4. *There is a* 1-1-*correspondence between Kähler structures on $\bar{M}$ and Sasaki structures on M.*

Therefore we can replace Theorem 3 by

Theorem 3′. *Let M be a complete simply connected Riemannian spin manifold of dimension n with Killing spinor for $\alpha = \frac{1}{2}$ or $\alpha = -\frac{1}{2}$. If $n = 2m - 1$, $m \geqq 3$ odd, then there are two possibilities:*

(i) $M = S^n$
(ii) *M is of type* (1, 1) *and M is an Einstein–Sasaki manifold.*

Conversely, if M is a complete simply connected Einstein–Sasaki spin manifold of dimension as above, then M carries Killing spinors for $\alpha = \frac{1}{2}$ and for $\alpha = -\frac{1}{2}$.

Proof. To prove the converse one has simply to observe that the Ricci curvature of an Einstein–Sasaki manifold is $\mathrm{Ric} = n - 1$. Thus $\bar{M}$ is Ricci-flat. Now the Sasaki structure yields a Kähler structure on $\bar{M}$, therefore its holonomy is reduced to $SU(m)$.

Remark. In dimension 5 this theorem has been shown by T. Friedrich and I. Kath, see [17]. They also gave a different proof of the "converse" part of Theorem 3′ for arbitrary dimension $n = 2m - 1$, m odd, see [19].

6. The Case $\mathrm{Hol}(\bar{M}) = Sp(m)$

The condition $\mathrm{Hol}(\bar{M}) = Sp(m)$ means that $\bar{M}$ is hyperkähler, i.e. there exist parallel complex structures I, J, and K such that $IJ = -JI = K$. On M identified with $M \times \{1\} \subset \bar{M}$ we define three Sasaki structures by

$$X_1 := I(\partial_r)\,,$$
$$X_2 := J(\partial_r)\,,$$
$$X_3 := -K(\partial_r)\,.$$

Recall that three Sasaki structures are said to form a Sasaki-3-structure if the following conditions hold:

(i) X_1, X_2, X_3 are orthonormal.
(ii) $[X_1, X_2] = 2X_3, [X_2, X_3] = 2X_1, [X_3, X_1] = 2X_2$.
(iii) $\phi_3\phi_2 = -\phi_1 + \eta_2 \otimes X_3, \quad \phi_2\phi_3 = \phi_1 + \eta_3 \otimes X_2\,,$

$$\phi_1\phi_3 = -\phi_2 + \eta_3 \otimes X_1, \quad \phi_3\phi_1 = \phi_2 + \eta_1 \otimes X_3\,,$$
$$\phi_2\phi_1 = -\phi_3 + \eta_1 \otimes X_2, \quad \phi_1\phi_2 = \phi_3 + \eta_2 \otimes X_1\,.$$

Here η_i is the dual form of X_i and $\phi_i = -\nabla X_i$. It is easy to check that the three Sasaki structures coming from I, J, and K form a Sasaki-3-structure.

Conversely, given a Sasaki-3-structure on M we define on $\bar{M}$:

$$I(r\partial_r) := X_1, \quad I(X_1) := -r\partial_r, \quad I(V) := -\phi_1(V) \quad \text{for } V \perp X_1, r\partial_r\,,$$
$$J(r\partial_r) := X_2, \quad J(X_2) := -r\partial_r, \quad J(V) := -\phi_2(V) \quad \text{for } V \perp X_2, r\partial_r\,,$$
$$K(r\partial_r) := -X_3, \quad K(X_3) := r\partial_r, \quad K(V) := -\phi_3(V) \quad \text{for } V \perp X_3, r\partial_r\,.$$

Now it is simple to check that $IJ = -JI = K$. We note

Lemma 5. *There is a* 1–1-*correspondence of hyperkähler structures on* $\bar{M}$ *and Sasaki*-3-*structures on* M.

From Theorem 4 we get

Theorem 4′. *Let* M *be a complete simply connected Riemannian spin manifold of dimension* n *with Killing spinor for* $\alpha = \frac{1}{2}$ *or* $\alpha = -\frac{1}{2}$. *If* $n = 4m - 1, m \geqq 3$, *then there are three possibilities*:

(i) $M = S^n$.

(ii) M *is of type* (2, 0) *and* M *is an Einstein–Sasaki manifold, but does not carry a Sasaki*-3-*structure.*

(iii) M *is of type* $(m + 1, 0)$ *and* M *carries a Sasaki*-3-*structure.*

Conversely, if M *is a complete simply connected Riemannian spin manifold with a Sasaki*-3-*structure,* $M \neq S^n$, *then* M *is of type* $(m + 1, 0)$. *If* M *is a complete simply connected Einstein–Sasaki spin manifold which does not carry a Sasaki*-3-*structure, then* M *is of type* (2, 0).

Proof. To prove the converse one simply has to recall that a manifold with Sasaki-3-structure is automatically Einstein with Ricci curvature $\mathrm{Ric} = n - 1$, compare [4].

Again, there is a different proof of the second part of this theorem due to T. Friedrich and I. Kath, see [19].

7. The Case $\mathrm{Hol}(\bar{M}) = G_2$

Lemma 6. *Let V be a 7-dimensional oriented Euclidean vector space, $\phi \in \Lambda^3 V^*$ a 3-form, $E_0 \in V$, $|E_0| = 1$. Then the following assertions are equivalent:*

(i) *For every $X \in V$, $|X| = 1$, the restriction of the interior product $i_X\phi$ to the 6-dimensional orthogonal complement $X^\perp$ defines a complex structure on it and $X^* \wedge i_X\phi \wedge i_X\phi = 6 \cdot \mathrm{vol}$. Here X^* is the 1-form dual to X.*

(ii) *The assertion of (i) holds for $X = E_0$ and for all $X \perp E_0$.*

(iii) *One can extend E_0 to a positively oriented orthonormal basis $E_0, E_1, \ldots, E_6$ such that*

$$\phi = \omega^{012} + \omega^{034} + \omega^{056} + \omega^{135} - \omega^{146} - \omega^{236} - \omega^{245} .$$

Here $\omega^0, \ldots, \omega^6$ are the dual basis for $E_0, \ldots, E_6$ and $\omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k$.

Proof. (i) → (ii) is clear and (ii) → (iii) is simple linear algebra. To prove (iii) → (i) one first considers the case $X = E_0$ and then one uses that G_2 acts transitively on the unit sphere in V and leaves fixed the form ϕ, compare [8, p. 539].

If ϕ satisfies the above conditions we will call ϕ *nice*. Since G_2 is precisely the stabilizer of a nice 3-form, see [8], the condition $\mathrm{Hol}(\bar{M}) \subset G_2$ is equivalent to the existence of a parallel nice 3-form on $\bar{M}$. For our cone $\bar{M} = M \times_{r^2} \mathbb{R}^+$, M 6-dimensional, only $\mathrm{Hol}(\bar{M}) = \{1\}$ and $\mathrm{Hol}(\bar{M}) = G_2$ are possible (Lemma 2). So in this case $\mathrm{Hol}(\bar{M}) = G_2$ is equivalent to the existence of precisely one parallel nice 3-form.

Now let ϕ be a parallel nice 3-form on the cone $\bar{M}$. We identify M with $M \times \{1\} \subset \bar{M}$ and define the almost complex structure J by

$$\langle X, JY \rangle := \phi(\partial_r, X, Y) .$$

Elementary calculation shows

$$|(\nabla_X J)(Y)|^2 = |X|^2 |Y|^2 - \langle X, Y \rangle^2 - \langle JX, Y \rangle^2 .$$

In particular, $(\nabla_X J)(X) = 0$, but $\nabla_X J \neq 0$. A complex structure J satisfying the above equation is called a nearly Kähler structure of constant type 1.

Conversely, let J be a nearly Kähler structure of constant type 1 on M, then we define on $\bar{M}$

$$\phi(\partial_r, X, Y) = -\phi(X, \partial_r, Y) = \phi(X, Y, \partial_r) := r^2 \langle X, JY \rangle_M ,$$

$$\phi(X, Y, Z) := r^3 \langle Y, (\nabla_X J)(Z) \rangle_M .$$

Here X, Y, and Z always denote vectors tangent to M. It is easy to see that ϕ is alternating. To prove that ϕ is parallel the only a bit more difficult part is to show that the components $(\bar{\nabla}_X \phi)(Y, Z, W)$, all entries tangent to M, vanish. Here one has to use formula (2.9) of [25] and Lemma 7(i) of [4, p. 132]. To show that ϕ is nice one uses characterization (ii) of Lemma 6 with $E_0 = \partial_r$. We have

Lemma 7. *There is a* 1–1-*correspondence of parallel nice* 3-*forms on* $\bar{M}$ *and nearly Kähler structures of constant type* 1 *on* M.

We remark that by Theorem 5.2 of [25] a 6-dimensional nearly Kähler manifold which is not Kähler is automatically of constant type 1 (after possibly rescaling the metric) and Einstein with Ricci curvature Ric = 5. We replace Theorem 2 by

Theorem 2′. *Let* M *be a* 6-*dimensional complete simply connected Riemannian spin manifold with Killing spinor for* $\alpha = \pm\frac{1}{2}$. *Then there are two possibilities*:

(i) $M = S^6$
(ii) M *is of type* (1, 1) *and* M *is nearly Kähler of constant type* 1.

Conversely, if a complete simply connected Riemannian manifold $M \neq S^6$ *is nearly Kähler, not Kähler, then* M *is of type* (1, 1).

This theorem was first shown by Grunewald, see [26].

We have obtained a way to construct Riemannian manifolds with holonomy group G_2, namely

Corollary. *Let* $M \neq S^6$ *be simply connected nearly Kähler of constant type* 1, *then* $\bar{M} = M \times_{r^2} \mathbb{R}^+$ *has holonomy group* G_2.

Example 1. The complex flag manifold $F_{1,2} = SU(3)/T^2$. Let $T^2 = S(U(1) \times U(1) \times U(1)) \subset SU(3)$ be the maximal torus. $F_{1,2} = SU(3)/T^2$ can be naturally identified with the set of pairs (l, p), where $l, p \subset \mathbb{C}^3$ are complex linear subspaces of dimension 1 and 2 resp. with $l \subset p$. The normal metric induced by the biinvariant metric of $SU(3)$ gives $F_{1,2}$ the structure of a Riemannian 3-symmetric space, compare [24]. The automorphism of $SU(3)$ of order 3 which induces the almost complex structure is given by conjugation with the matrix $\begin{pmatrix} \xi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\xi} \end{pmatrix}$, where $\xi \in \mathbb{C}$, $\xi^3 = 1$, $\xi \neq 1$. Since the metric is normal it is naturally reductive, hence $F_{1,2}$ is nearly Kähler by [24, Prop. 5.6]. But $F_{1,2}$ with this metric is not Kähler. Thus $F_{1,2}$ is of type (1, 1) and if the metric is scaled such that Ric = 5, then the cone over $F_{1,2}$ has holonomy group G_2. This example was first discovered by Bryant and can be found in [8]; it is the first explicit example of a Riemannian manifold with holonomy G_2.

Example 2. $S^3 \times S^3$. For a compact simply connected simple Lie group G the product $G \times G$ can be given the structure of a Riemannian 3-symmetric space. To do this one writes $G \times G = G \times G \times G/G$, where G is diagonally imbedded in $G \times G \times G$, see [24]. If g is the biinvariant metric of G, then $g \times g \times g$ is biinvariant for $G \times G \times G$ and we give $G \times G$ the corresponding normal metric. The metric on $G \times G$ is naturally reductive, but it is not the product metric $g \times g$. The automorphism of $G \times G \times G$ which induces the almost complex structure is given by $(g_1, g_2, g_3) \to (g_3, g_1, g_2)$. We take $G = SU(2) \approx S^3$. Thus $S^3 \times S^3$ is nearly Kähler and not Kähler because the second Betti number vanishes. Therefore $S^3 \times S^3$ is of type (1, 1) and the cone over it has holonomy G_2 if the metric is scaled appropriately.

Example 3. The complex projective space $\mathbb{C}P^3$. We write $\mathbb{C}P^3 = SP(2)/Sp(1) \times U(1)$ and give $\mathbb{C}P^3$ the normal metric induced by the biinvariant metric of $Sp(2)$. Conjugation with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$, $\xi \in \mathbb{C}$, $\xi^3 = 1$, $\xi \neq 1$, gives rise to an automorphism of $Sp(2)$ inducing an almost complex structure of $\mathbb{C}P^3$. Of course, this is *not* the usual integrable complex structure. We have made $\mathbb{C}P^3$ into a Riemannian 3-symmetric space which is nearly Kähler because the metric is naturally reductive. But with this metric $\mathbb{C}P^3$ is not Kähler, see [36]. Again we conclude that this $\mathbb{C}P^3$ is of type (1, 1) and the cone over it has holonomy G_2.

Direct proofs that these three examples are of type (1, 1) can be found in [4, pp. 141–148].

8. The Case $\mathrm{Hol}(\bar{M}) = \mathrm{Spin}(7)$

Lemma 8. *Let V be an 8-dimensional oriented Euclidean vector space, $\Phi \in \Lambda^4 V^*$ a 4-form, $E_0 \in V$, $|E_0| = 1$. Then the following assertions are equivalent.*

(i) *For every unit vector $X \in V$ the restriction of the interior product $i_X \Phi$ to the 7-dimensional orthogonal complement $X^\perp$ is equal to a nice 3-form ϕ on $X^\perp$ and the restriction of Φ itself is $*_7\phi$ where $*_7$ denotes the Hodge star operator on $X^\perp$.*

(ii) *The assertion* (i) *holds $X = E_0$.*

(iii) *Φ can be written as $\Phi = \omega^0 \wedge \phi + *_7\phi$, where ϕ is a nice 3-form on $E_0^\perp$ and ω^0 is the 1-form dual to E_0.*

Proof. The implications (i) → (ii) and (ii) → (iii) are clear. To show (iii) → (i) prove it first for $X = E_0$ and recall that Spin(7) acts transitively on the unit sphere in V while it leaves fixed the form Φ, see [8, p. 545].

If a 4-form satisfies the above conditions we call it a nice 4-form. Since $\mathrm{Spin}(7) \subset SO(8)$ may be defined as the stabilizer of a nice 4-form, see [8], the condition $\mathrm{Hol}(\bar{M}) \subset \mathrm{Spin}(7)$ is equivalent to the existence of a parallel nice 4-form on $\bar{M}$.

Let Φ be a parallel nice 4-form on $\bar{M}$. We define on M:

$$\phi(X, Y, Z) := \Phi(\partial_r, X, Y, Z) .$$

One easily calculates

$$(\nabla_W \phi)(X, Y, Z) = \Phi(W, X, Y, Z) = (*\phi)(W, X, Y, Z) .$$

Hence ϕ is a nice 3-form on M with $\nabla\phi = *\phi$.

Conversely, given such a 3-form on M we can define on $\bar{M}$:

$$\Phi(\partial_r, X, Y, Z) := r^3 \phi(X, Y, Z) ,$$

$$\Phi(W, X, Y, Z) := r^4 (\nabla_W \phi)(X, Y, Z) .$$

Then Φ is a parallel nice 4-form.

Lemma 9. *There is a 1–1-correspondence between parallel nice 4-forms on $\bar{M}$ and nice 3-forms ϕ on M satisfying $\nabla\phi = *\phi$.*

Theorem 5′. *Let M be a 7-dimensional complete simply connected Riemannian spinor manifold with Killing spinor for $\alpha = \pm\frac{1}{2}$. Then there are four possibilities:*

(i) *$M = S^7$.*

(ii) *M is of type* (1, 0) *and M carries a nice 3-form ϕ with $\nabla\phi = *\phi$, but not a Sasaki structure.*

(iii) *M is of type* (2, 0) *and M carries a Sasaki structure, but not a Sasaki-3-structure.*

(iv) *M is of type* (3, 0) *and M carries a Sasaki-3-structure.*

Conversely, if a 7-dimensional complete simply connected Riemannian spin manifold M carries a nice 3-form ϕ *with* $\nabla\phi = *\phi$, *but not a Sasaki structure, then M is of type* (1, 0). *If M is an Einstein–Sasaki manifold without Sasaki-3-structure, then M is of type* (2, 0). *If* $M \neq S^7$ *carries a Sasaki-3-structure, then M is of type* (3, 0).

Remark. One can replace the concept of nice 3-forms ϕ with $\nabla\phi = *\phi$ by the notion of nearly parallel vector cross products A via the definition $\phi(X, Y, Z) = \langle X, A(Y, Z)\rangle$, compare [22] and [4, ch. 4.6].

As for G_2 we obtain a construction method for examples of Riemannian manifolds with holonomy group Spin(7).

Corollary. *If M is a 7-dimensional complete simply connected Riemannian spin manifold of type* (1, 0), *then* $\bar{M} = M \times_{r^2} \mathbb{R}^+$ *has holonomy group* Spin(7).

Example 1. $SO(5)/SO(3)$. We consider $M = SO(5)/SO(3)$ where the inclusion $SO(3) \subset SO(5)$ is induced by the representation of $SO(3)$ on the 5-dimensional space of harmonic polynomials homogeneous of degree 2 in 3 variables. M is an isotropy irreducible homogeneous space. Bryant shows in [8] that the cone over M has holonomy Spin(7). Therefore M is of type (1, 0).

Example 2. The squashed 7-sphere. If we consider the Hopf fibration $S^7 \to S^4$ with fiber S^3 the canonical variation of the metric yields a second Einstein metric besides the metric of constant curvature. In [11] it is shown that this squashed 7-sphere is of type (1, 0). Thus after rescaling the metric such that Ric = 6 the cone over this S^7 has holonomy Spin(7).

Example 3. The Wallach manifolds. We consider the Wallach manifolds $N_{k,l} = SU(3)/S^1$ where the inclusion $S^1 \to SU(3)$ is given by $z \to \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & z^{-k-l} \end{pmatrix}$.

It is shown in [4, p. 115, Th. 12] that for $k \neq 1$ or $l \neq 1$ there exist two Einstein metrics on $N_{k,l}$ such that $N_{k,l}$ is of type (1, 0), see also [34] and [11, pp. 89–90]. Thus the cones over the $N_{k,l}$'s provide a series of examples with holonomy Spin(7). This series is interesting because infinitely many homotopy types occur ($H^4(N_{k,l}; \mathbb{Z})$ is cyclic of order $k^2 + kl + l^2$) and Kreck and Stolz showed that there are examples of homeomorphic non-diffeomorphic $N_{k,l}$'s (for example $N_{-56788,5227}$ and $N_{-42652,61213}$), see [29].

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