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Quantum Knizhnik-Zamolodchikov Equations and Affine Root Systems

Ivan Cherednik

RIMS, Kyoto University, Kyoto 606, Japan*, and
Department of Math., The University of Texas at Austin, USA**

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Abstract. Quantum (difference) Knizhnik-Zamolodchikov equations [S1, FR] are generalized for the R-matrices from [Ch1] with the arguments in arbitrary root systems (and their formal counterparts). In particular, QKZ equations with certain boundary conditions are introducted. The self-consistency of the equations from [FR] and the cross-derivative integrability conditions for the r-matrix KZ equations from [Ch2] are obtained as corollaries. A difference counterpart of the quantum many-body problem connected with Macdonald's operators is defined as an application.

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0. Introduction

In a recent paper [FR], the so-called quantum R-matrices (solutions of the Yang-Baxter equations) were used to introduce certain systems of difference equations. Their quasi-classical limits are the r-matrix Knizhnik-Zamolodchikov equations defined in [Ch2] (see also [Ch3]). To be more precise, the systems of differential equations from the latter are connected with the root systems $(A, B, \ldots, G)$ describing the structure of the arguments. The construction from [FR] corresponds to the r-matrix equations with

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* Permanent address: Math. Dept., University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

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the arguments of type A (the values are in tensor products of representations of a given Lie algebra of any type). The main purpose of this paper is to involve arbitrary root systems and give a uniform proof of the self-consistency of the arising systems of difference equations.

The equation from [FR] in 2×2-matrices and for a special choice of the difference interval coincides with the Smirnov equation [S1] inspired by certain problems from the quantum inverse scattering technique (Faddeev and others). However the approach from [FR] is different. Frenkel and Reshetikhin deduced their system from a q-version of the conformal field theory (for the basic trigonometric R-matrices). See also [S2] where a similar construction was considered for the Yang R-matrices.

We note that the interpretation of the general R-matrix equations from [FR] and the present paper (we call them QKZ) via either QIST or CFT is unknown. As for the r-matrix KZ equations with the arguments of type A, a deduction from the theory of Kac-Moody algebras was obtained in [Ch2, Ch6] for any r. It is not purely formal but gives a way of integrating these equations. For example, a generalization of Schechtman-Varchenko theorem [SV] was found by means of this interpretation (see [Ch6] for details). This approach is connected with the technique of vertex operators for ordinary KZ equations [KZ]. Hopefully (affine) quantum groups could help in integrating QKZ equations as Kac-Moody algebras do for the classical ones.

We will discuss here neither the interpretation nor the integration. Certain formulas (and references) can be found in [S1, FR, S2]. It is worth mentioning that particular cases of QKZ equations (and some other related difference equations) were obtained by Aomoto, Kato, Mimachi as a development of the classic theory of q-special functions (see e.g. [AKM]).

Given an arbitrary quantum R-matrix in the sense of [Ch1] (with the arguments from any fixed root system – see below) we construct QKZ which is a set of difference (or more general) equations. If $R = 1 + hr + o(h)$ for a proper r then the quasi-classical limit ($h \to 0$) of QKZ is the corresponding r-matrix KZ equation from [Ch2, Ch3]. One can obtain a formal version of QKZ by considering arbitrary (pairwise commutative) automorphisms instead of the independent translations of the arguments. The quantum R-matrices for the classical root systems (A, B, C, D) and for G describe certain theories of one-dimensional factorizable particles on a segment with moving endpoints ([Ch1, Ch5]). It makes the definition of QKZ rather visual. As for E, F, such an interpretation is unknown. We note that the corresponding QKZ are closely connected with the so-called monodromy and transfer matrices from the theory of integrable one-dimensional models.

The principal aim of the present paper is to define QKZ. We also give two concrete examples of R-matrices, based on [Ch1] and [Ch3]. The dependence of the arguments is rational and respectively trigonometric, the root system is arbitrary. As an application, we define a difference counterpart of the Calogero quantum many-body problem [C] and prove the commutativity of the arising difference operators ("the integrability"). The comparison with Macdonald's difference operators for the q-Jacobi polynomials seems to be very fascinating (see [M]).

From an abstract point of view, the quantum R-matrix can be introduced as a one-cocycle on the corresponding Weyl group. Our key construction is in extending this cocycle to the affine completion of the Weyl group by means of the lattice of the weights. It gives a more direct way to establish the connection from [Ch2, Ch3] between quantum R-matrices and r-matrix KZ equations.

This paper is organized as follows. We give the necessary properties of affine root systems in Sect. 1. The main purpose is to make the definitions quite constructive. Section 2 contains the main theorems on the R-matrix cocycles on the affine Weyl groups. The definition and the self-consistency of the quantum Knizhnik-Zamolodchikov equations from Sect. 3 result directly from these theorems. We discuss the formal theory of the monodromy representation and consider examples (in particular, the quasi-classical limits). The main application is in constructing difference Calogero operators. Sections 4, 5 are devoted to the classical root systems. We give a geometric interpretation for the corresponding QKZ. A discussion of certain connections with Macdonald's operators can be found in the Appendix.

1. Affine Root Systems

We do not give (complete) proof if the statement is well-known (easily verified by the tables of [B]). The main facts are valid for the non-reduced root systems as well. See [L] for the necessary details (and [B] for the basic properties of affine root systems). As for the dual roots and weights, our notations are different from those of [B]. We use the letters A, B instead of $Q^\vee, P^\vee$.

We fix a euclidean form (v, v') on $\mathbf{R}^n \ni v, v'$ and a root system $\Sigma = \{\alpha\} \subset \mathbf{R}^n$ of type $A_n, B_n, \ldots, G_2$. Given $\tilde{\alpha} = [\alpha, k]$ for $\alpha \in \Sigma$, $k \in \mathbf{Z}$,

$$s_{\tilde{\alpha}}(v) \stackrel{\text{def}}{=} v - ((v, \alpha) + k)\alpha^\vee , \qquad \alpha^\vee = 2\alpha(\alpha, \alpha)^{-1} \tag{1.1}$$

is the orthogonal reflection in the affine hyperplane $(\alpha, v) + k = 0$. The roots are identified with the pairs $[\alpha, 0]$. We will use the Weyl chamber C and the set of positive ($\alpha > 0$) roots Σ_+ with respect to the set $\alpha_1, \ldots, \alpha_n$ of simple roots from the corresponding table of [B]. Let

$$\alpha_0 = [-\theta, 1] , \quad s_0 = s_{\alpha_0} , \quad s_i = s_{\alpha_i} \quad (1 \le i \le n) , \tag{1.2}$$

where $\theta \in \Sigma$ is the maximal positive root.

Later on, $\tilde{\alpha} = [\alpha, k]$ will be considered as vectors in $\mathbf{R}^n \times \mathbf{Z}$ with the natural addition and multiplication by numbers. The action of W on $\tilde{\alpha}$ is via the first component.

The *completed affine root system*, its subset of positive roots and the affine Weyl chamber are as follows:

$$\Sigma^a = \{\tilde{\alpha} = [\alpha, k] \in \Sigma \times \mathbf{Z}\} , \quad \Sigma^a_+ = \{\tilde{\alpha} \in \Sigma^a,\ k > 0 \text{ or } k = 0 < \alpha\} , \tag{1.3}$$

$$C^\alpha = \bigcap_{i=0}^{n} L_i = C \cap L_0 , \quad L_i = L_{\alpha_i} , \quad L_{\tilde{\alpha}} = \{v \in \mathbf{R}^n,\ (\alpha, v) + k > 0\} . \tag{1.4}$$

We use the same notation $\tilde{\alpha} > 0$ for affine positive roots. One has:

$$\Sigma_+ = \{\alpha, C \in L_\alpha\} \subset \Sigma^a_+ = \{\tilde{\alpha}, C^a \subset L_{\tilde{\alpha}}\} , \tag{1.5}$$

$$\Sigma = \Sigma_+ \cup \{-\Sigma_+\} \subset \Sigma^a = \Sigma^a_+ \cup \{-\Sigma^a_+\} . \tag{1.6}$$

The Weyl group W is generated by $\{s_\alpha, \alpha \in \Sigma\}$ and, moreover, by $\{s_i, 1 \le i \le n\}$. The following relations are defining:

$$s_i^2 = 1 \quad (s_i s_j)^m = 1 \quad \text{for} \quad m = 2, 3, 4, 6 , \tag{1.7}$$

where $m = 2$ if α_i and α_j are disconnected (the corresponding indices are not neighbouring) in the *Dynkin graph* Γ. Otherwise, $m = 3, 4, 6$ when $1, 2, 3$ lines respectively connect α_i and α_j in Γ.

The *affine Weyl group* $W^a = \langle s_\alpha, \alpha \in \Sigma^a \rangle$ is generated by $\{s_i, 0 \le i \le n\}$ with the same relations (1.7), where α_0 is identified with the additional vertex of the *completed Dynkin graph* Γ^a (see [B]). The number of the lines between α_0 and α_i in Γ^a gives the order m of $s_i s_0$ or $s_0 s_i$ as above. We note that m coincides with the order of $s_i s_\theta$ or $s_\theta s_i$, the number of the lines (and their direction) is the same as for the pair $\{-\theta, \alpha_i\}$.

Let us introduce the *dual simple roots and fundamental weights* together with the corresponding lattices:

$$a_i = \alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i), \qquad (\beta_i, \alpha_j) = \delta_{ij}, \qquad 1 \le i, j \le n, \tag{1.8}$$

$$A = \bigoplus_{i=1}^{n} \mathbf{Z} a_i = \langle \alpha^\vee, \alpha \in \Sigma \rangle \subset B = \bigoplus_{i=1}^{n} \mathbf{Z} b_i, \tag{1.9}$$

where δ_{ij} is the Kronecker delta. In the sequel, we will use $A_+ = \bigoplus_{i=1}^{n} \mathbf{Z}_+ a_i$, $B_+ = \bigoplus_{i=1}^{n} \mathbf{Z}_+ b_i$ and consider vectors $x \in \mathbf{R}^n$ as the affine shifts (translations)

$$x'(\mathscr{P}) = \mathscr{P} + x, \qquad \mathscr{P} \subset \mathbf{R}^n. \tag{1.10}$$

The group $B \ni b$ acts on the set $\{L_{\tilde\alpha}, \tilde\alpha = [\alpha, k] \in \Sigma^a\}$, the set $\{s_{\tilde\alpha}\}$ and on Σ^a:

$$b'(L_{\tilde\alpha}) = L_{\tilde\beta}, \qquad b' s_{\tilde\alpha} (b')^{-1} = s_{\tilde\beta}, \qquad \tilde\beta = b'(\tilde\alpha) \stackrel{\mathrm{def}}{=} [\alpha, k - (b, a)]. \tag{1.11}$$

The natural action of W on A, B coincides with the action of W on A', B' by conjugations:

$$w b' w^{-1} = w(b)', \qquad w \in W, b \in B. \tag{1.12}$$

Proposition 1.1. a) *The group W^a contains A' and is isomorphic to the semi-direct product of W and A':*

$$a' = s_\alpha s_{\tilde\alpha} \in W^a \quad if \quad a = \alpha^\vee, \tilde\alpha = [\alpha, 1], \qquad \alpha \in \Sigma. \tag{1.13}$$

b) *The group W^b generated by W and B' is the semi-direct product of these groups. As an abstract group, it is generated by $\{s_i, 1 \le i \le n\}$ satisfying* (1.7) *and pairwise commutative $\{b'_i, 1 \le i \le n\}$ with the following defining cross-relations:*

$$s_i b'_j s_i = s_i(b_j)' = (b_j - \delta_{ij} a_i)', \qquad 1 \le i, j \le n. \quad \square \tag{1.14}$$

Let Γ_0 be the subset of the vertices of Γ (identified with α_i, $1 \le i \le n$) which can be obtained from α_0 by automorphisms of Γ^α. This set is empty for E_8, F_4, G_2. We introduce $\Gamma_0^* \subset \Gamma_0 : \Gamma_0^* = \{\alpha_1\}$ for $A, B, E_6, = \{\alpha_n\}$ for $C, D_{2m+1}, E_7, = \{\alpha_1, \alpha_n\}$ for D_{2m}, $1 < m \in \mathbf{N}$, $= \emptyset$ otherwise. The numeration is from [B].

Corollary 1.2. *The group W^b is generated by $\{b'_p, p \in \Gamma_0^*\}$ over W^a (as well as B over A). The following relations (together with* (1.14) *and the commutativity of $\{b'_p\}$) are defining for this extension:*

$$s_0 b'_p s_0 = b'_p - t', \qquad t = \theta^\vee, \qquad (b'_p)^\nu \in A' \subset W^a. \tag{1.15}$$

Here $\nu = n + 1, 4, 3$ respectively for A_n, D_{2m+1}, E_6, and $\nu = 2$ in the remaining cases when $\Gamma_0 \ne \emptyset$.

Definition 1.3. a) *The length* $l = l(w)$ *of* $w \in W^a$ *is the length of the shortest possible (reduced) decomposition*

$$w = s_{i_l} \dots s_{i_2} s_{i_1} \qquad 0 \le i_r \le n,\ l(id) = 0\,. \tag{1.16}$$

b) *The length* $l(w)$ *of* $w \in W^b$ *is the number of elements in the set (see* (1.4)*)*

$$\lambda_w \stackrel{\text{def}}{=} \{\tilde{\alpha} \in \Sigma_+^a,\ w^{-1}(C^a) \not\subset L_{\tilde{\alpha}}\} \tag{1.17}$$

This definition coincides with a) *for* $w \in W^a$.

The proof of the equivalence of a) and b) results from the following explicit description of λ_w. Given a decomposition (1.16),

$$\lambda_w = \{\lambda_{i_1}, s_{i_1}(\lambda_{i_2}), s_{i_1}s_{i_2}(\lambda_{i_3}), \dots, w^{-1}s_{i_l}(\lambda_{i_l})\}\,. \tag{1.18}$$

This can be either extracted from [B] or easily proved by induction on l. □

Proposition 1.4. *The following conditions for* $x, y \in W^b$ *are equivalent:*

$$\text{a) } l(xy) = l(x) + l(y)\,, \qquad \text{b) } \lambda_y \subset \lambda_{xy}\,, \qquad \text{c) } y^{-1}(\lambda_x) \subset \lambda_{xy}\,.$$

If they are imposed then

$$\lambda_{xy} = \lambda_y \cup y^{-1}(\lambda_x)\,, \qquad \lambda_y \cap y^{-1}(\lambda_x) = \emptyset\,.$$

Proof. Let us verify that $\lambda_{xy} \backslash \lambda'_y = y^{-1}(\lambda_x) \cap \Sigma_+$ for $\lambda'_y = \lambda_y \cap \lambda_{xy}$. Indeed, if $\tilde{\alpha} \in \lambda_{xy} \backslash \lambda'_y$, i.e.

$$(xy)^{-1}C^a \not\subset L_{\tilde{\alpha}} \supset y^{-1}C^a \quad \text{for} \quad \tilde{\alpha} > 0\,,$$

then $x^{-1}C^a \not\subset L_{y(\tilde{\alpha})} \supset C^a$ and $y(\tilde{\alpha}) \in \lambda_x$. The converse is clear as well. Hence $\lambda_y \subset \lambda_{xy}$, $y^{-1}(\lambda_x) \subset \lambda_{xy}$, and $\lambda_y \cap y^{-1}(\lambda_x) = \emptyset$ if $l(xy) = l(x) + l(y)$.

Let us suppose that $\lambda_y \subset \lambda_{xy}$ and check that $y^{-1}(\lambda_x) \subset \Sigma_+$. If $\tilde{\alpha} \in \lambda_x$ and $y^{-1}(\tilde{\alpha}) < 0$, then $\lambda_y \ni L_{-\tilde{\alpha}} = -L_{\tilde{\alpha}} \notin \lambda_{xy}$. This contradiction proves the equivalence of a) and b). As for c), it is equivalent to b) for $y = x^{-1}$, $x = y^{-1}$ [since $\lambda_{x^{-1}} = -x(\lambda_x)$] and therefore to a) because $l(x) = l(x^{-1})$. □

Let w_0 be the *longest element* in W relative to the above length, $w_0\{J\}$ the longest element in the subgroup $W\{J\} \subset W$ generated by $\{s_i,\ 1 \le i \le n,\ i \notin J\}$ for $J = \{j\}$, $\{j, j'\}$, $1 \le j,\ j' \le n$. These elements are involutive. We introduce

$$\sigma_j = w_0\{j\}w_0\,, \qquad \tau_j = b'_j\sigma_j \quad \text{for} \quad \alpha_j \in \Gamma_0\,. \tag{1.19}$$

Proposition 1.5. a) *The sets* $S = \{id, \sigma_j\} \subset W$, $T = \{id, \tau_j\} \subset W^b$, $\alpha_j \in \Gamma_0$, *are subgroups. They are generated by* $\{\sigma_p\}$ *and* $\{\tau_p\}$ *for* $\alpha_p \in \Gamma^a$ *and are isomorphic to* B/A *with respect to the maps* $\sigma_j \to b_j \leftarrow \tau_j$. *The group* W^b *is the semi-direct product of* W^a *and* T.

b) *The elements of* S *preserve the set* $\{-\theta, \alpha_1, \dots, \alpha_n\}$ *and may be embedded into* $\mathrm{Aut}(\Gamma^a)$ *after the identification of* $-\theta$ *with the vertex corresponding to* α_0. *The group* T *leaves the set* $\{\alpha_0, , \dots, \alpha_n\}$ *invariant and induces the same subgroup in* $\mathrm{Aut}(\Gamma^a)$ *as* S. *In particular,*

$$\sigma_j(-\theta) = \alpha_j = \tau_j(\alpha_0) \qquad T = \{w \in W^b,\ l(w) = 0\}\,.$$

The multiplicities of $\alpha_j \in \Gamma_0$ *in arbitrary* $\alpha \in \Sigma_+$ *are 0 or 1.*

Proof. It follows from the tables of [B] that σ_j for $\alpha_j \in \Gamma_0$ takes the set $\{\alpha_i, 1 \le i \le n\}$ except one element onto $\{\alpha_i, i \ne j, 1 \le i \le n\}$. The image β of the excluded element is a root having the same scalar products with α_i, $1 \le i \ne j \le n$ as $-\theta$. Hence $\beta = -\theta + mb_j$ for a proper $m \in \mathbf{Z}$. The tables show that m has to be zero for E_6, E_7. As for A, B, C, D, it follows from the direct description of σ_j. The invariance of $\{-\theta, \alpha_1, \dots, \alpha_n\}$ and the formula $\sigma_j(-\theta) = \alpha_j = \tau_j(\alpha_0)$ result from the same considerations. It gives the corresponding properties of T. The statement about the multiplicities is clear, since α_j are of multiplicity one in θ and the latter is maximal. □

We will describe the following λ-sets for later reference.

Proposition 1.6. *In the above notations,*
a) $\lambda_{\sigma_i} = \{\alpha \in \Sigma_+, (b_i, \alpha) \ne 0\}, 1 \le i \le n;$
b) $\lambda_{b'} = \{\tilde\alpha = [\alpha, k] \in \Sigma_+^a, \alpha \in \Sigma_+, (b, \alpha) > k \ge 0\}$ *for* $b \in B_+ = \bigoplus_{i=1}^n b_i;$
c) $\lambda_{\sigma_j^{-1}} = \lambda_{b'_j}$ *if* $\alpha_j \in \Gamma_0;$
d) $\lambda_{t'} = \lambda_{s_\theta} \cup [\theta, 1]$ *for* $t = \theta^\vee;$
e) $s_\theta = w_0\{1, n\}w_0$ *for* A_n *and* $s_\theta = \sigma_{i_0}$
for the other types, where α_{i_0} *is the unique vertex joined with* α_0 *in* Γ^a.

Proof. The right-hand side of a) belongs to λ_{σ_i} [see (1.18)]. The cardinality of λ_{σ_i} (the length of σ_i) is equal to the order of Σ_+ minus the number of positive roots written without α_i. It gives a). Assertion b) is valid because $\tilde\alpha \in \lambda_{b'}$ iff $-b \notin$ closure $(L_{\tilde\alpha})$, where $\tilde\alpha \in \Sigma_+^a$, $b \in B_+$. The coincidence of the sets from c) follows from a), b). Statement d) is clear, since $t' = s_0 s_\theta$ [apply (1.18)]. As for e), it can be checked by the tables of [B] (cf. the proof of Proposition 1.5). □

2. R-Matrices

We fix an arbitrary $\mathbf{C}$-algebra $\mathscr{F}$. Our aim is to introduce $\mathscr{F}$-valued (abstract) non-affine $\mathbf{R}$-matrices like in [Ch1] (see also [Ch3], Proposition 3.3) and then to extend them to affine ones. We use the notations from Sect. 1. Let us denote $\mathbf{R}\alpha + \mathbf{R}\beta \subset \mathbf{R}^n$ by $\mathbf{R}\langle\alpha, \beta\rangle$ for $\alpha, \beta \in \Sigma$.

Definition 2.1 a) *A set* $R = \{R_\alpha \in \mathscr{F}, a \in \Sigma_+\}$ *is an R-matrix if*

$$R_\alpha R_\beta = R_\beta R_\alpha\,, \tag{2.1}$$

$$R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha\,, \tag{2.2}$$

$$R_\alpha R_{\alpha+\beta} R_{\alpha+2\beta} R_\beta = R_\beta R_{\alpha+2\beta} R_{\alpha+\beta} R_\alpha\,, \tag{2.3}$$

$$R_\alpha R_{3\alpha+\beta} R_{2\alpha+\beta} R_{3\alpha+2\beta} R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_{3\alpha+2\beta} R_{2\alpha+\beta} R_{3\alpha+\beta} R_\alpha\,, \tag{2.4}$$

under the assumption that $\alpha, \beta \in \Sigma_+$ *and*

$$\mathbf{R}\langle\alpha, \beta\rangle \cap \Sigma = \{\pm\gamma\}, \gamma \text{ runs over all the indices} \tag{2.5}$$

in the corresponding identity.
b) *An affine R-matrix* $R^a = \{\tilde R_{\tilde\alpha} \in \mathscr{F}, \tilde\alpha \in \Sigma_+^a\}$ *has to obey the same relations for* $\tilde\alpha, \tilde\beta \in \Sigma_+^a$ *with the condition that*

$$\mathbf{R}\langle\tilde\alpha, \tilde\beta\rangle \cap \Sigma^a = \{\pm\tilde\gamma\}\,, \tag{2.5^a}$$

where $\{\tilde\gamma\}$ *is the set of the indices in the corresponding relation* ((2.1)–(2.4)) *for* $\tilde\alpha, \tilde\beta$).

c) *A closed R-matrix (or a closure of the above R) is a set* $\{R_\alpha \in \mathscr{F}, \alpha \in \Sigma\}$ *(extending R and) satisfying relations* (2.1)–(2.4) *for arbitrary (maybe negative)* $\alpha, \beta \in \Sigma$ *such that the corresponding condition* (2.5) *is fulfilled. Affine closed R-matrices are defined in the same manner.* □

The condition (2.5) for identity (2.1) means that

$$(\alpha, \beta) = 0 \quad \text{and} \quad \mathbf{R}\langle\alpha, \beta\rangle \cap \Sigma = \{\pm\alpha, \pm\beta\}, \tag{2.6}$$

i.e. there exists $w \in W$ such that $\alpha = w(\alpha_i)$, $\beta = w(\alpha_i)$ for simple $\alpha_i \neq \alpha_j$ ($1 \leq i$, $j \leq n$) disconnected in Γ (check the equivalence). The same holds true for R^a, when $0 \leq i$, $j \leq n$, and for the closed counterparts of R, R^a as well.

The corresponding assumptions for (2.2)–(2.4) give that α, β are the simple roots of a certain two-dimensional root subsystem in Σ (or Σ^a) of type A_2, B_2, G_2. Here α, β stay for α_1, α_2 in the notations from the figure of the systems of rank 2 from [B]. One can represent them as follows: $\alpha = w(\alpha_i)$, $\beta = w(\alpha_j)$ for a proper w from W (or from W^a) and joined (neighbouring) α_i, α_j.

Definition 2.2. a) *Let us suppose that the R-matrix from Definition 2.1 is closed and*

$$R_\alpha R_\beta = R_\beta R_\alpha \textit{ for long roots such that } (\alpha, \beta) = 0\,. \tag{2.7}$$

In the case of G_2, *we add conditions* (2.2) *and* (2.7) *respectively for long* α, β *and when* α *is short but* β *is long. We call such an R-matrix extensible.*
b) *If the group* $A \ni a$ *(see* (1.9)*) operates on the algebra* $\mathscr{F} \ni f$ *(written* $f \to a(f)$*) and*

$$a(R_\alpha) = R_\alpha \quad \textit{whenever} \quad (a, \alpha) = 0,\ a \in A,\ \alpha \in \Sigma\,, \tag{2.8}$$

then the extensible R-matrix is called to be of a-type. If

$$b(R_\alpha) = R_\alpha \quad \textit{whenever} \quad (b, \alpha) = 0,\ b \in B,\ \alpha \in \Sigma \tag{2.9}$$

for a certain action of B (see (1.9)*) on* $\mathscr{F}$, *then it is of b-type.* □

We note that condition (2.1) does not result in (2.7) since α, β in the latter are not supposed to satisfy (2.6). However in the most interesting examples, (2.1) holds true for arbitrary orthogonal roots without any limitations.

We will use the following formal notations for R, R^a and their closed counterparts: ${}^w c = c$ for $c \in \mathbf{C}$,

$$^w(cR_{\tilde\alpha}) = cR_{w(\alpha)}\,, \qquad ^w(cR_{\tilde\alpha}R_{\tilde\beta}) = cR_{w(\tilde\alpha)}R_{w(\tilde\beta)}, \ \dots\,, \tag{2.10}$$

where w is from W, W^a or W^b, the roots $\tilde\alpha, \tilde\beta$ are from the corresponding system. We do not assume here that either W or its affine extension acts on $\mathscr{F}$.

Mathematically, it is convenient to introduce the *root algebra* $\mathscr{T}$ generated by $\{R_\alpha,\ \alpha \in \Sigma\}$ considered as independent variables satisfying the relations from Definitions 2.1, 2.2, a). Then (2.10) can be uniquely extended to an action of W on $\mathscr{T}$. The algebra $\mathscr{T}^a$ and the universal action of W^a on it can be defined in the same way.

Theorem 2.3. a) *If R is an R-matrix then there exists a unique set* $\{R_w,\ w \in W\}$ *satisfying the (cocycle) relations*

$$R_{xy} = ^{y^{-1}} R_x R_y\,, \qquad R_{s_i} = R_i \stackrel{\text{def}}{=} R_{\alpha_i} \qquad R_{id} = 1\,, \tag{2.12}$$

where $1 \leq i \leq n$, $x, y \in W$, *and* $l(xy) = l(x) + l(y)$. *Given* w *and an arbitrary reduced decomposition* (1.16) *with* $1 \leq i_r \leq n$,

$$R_w = {}^{w^{-1}s_{i_l}}R_{i_l} \ldots {}^{s_{i_1}s_{i_2}}R_{i_3}\, {}^{s_{i_1}}R_{i_2} R_{i_1} \,. \tag{2.13}$$

The same holds true for R^a *if* $x, y \in W^a$ *and* $0 \leq i, i_r \leq n$.
b) *Given invertible* R *(or* R^a*), let* $\bar{R}$ *(or* $\bar{R}^a$*) be its unique extension as follows:*

$$R_{-\tilde{\alpha}} \stackrel{\text{def}}{=} R_{\tilde{\alpha}}^{-1}\,, \qquad \tilde{\alpha} \in \Sigma_+ \ (or\ \Sigma_+^a)\,. \tag{2.14}$$

Then $\bar{R}$ *is a closure of* R. *Moreover, it satisfies relations* (2.12), (2.13) *where* x, y *are arbitrary and the decomposition of* w *is not assumed to be reduced. The same holds true for* R^a.
c) *Let us suppose that a closed R-matrix is of a-type (Definition 2.2) and put (see* (1.11)*)*

$$ {}^{a'}R_\alpha = R_{\tilde{\alpha}} \stackrel{\text{def}}{=} a(R_\alpha)\,, \quad if \ \ \tilde{\alpha} = a'(\alpha) = [\alpha, -(a, \alpha)] \tag{2.15}$$

for arbitrary $\alpha \in \Sigma$, $a \in A$ *with respect to the action of* A *on* $\mathscr{F}$ $(f \to a(f))$. *Then* $R_{\tilde{\alpha}}$ *are well-defined (depend on the corresponding scalar products* (a, α) *only) and form a closed affine R-matrix. If* (2.14) *is valid for* $\{R_\alpha\}$ *the same is true for* $\{R_{\tilde{\alpha}}\}$.

Proof. The roots $\{\lambda_{i_1}, s_{i_1}(\lambda_{i_2}), s_{i_1}s_{i_2}(\lambda_{i_3}), \ldots, w^{-1}s_{i_l}\lambda_{i_l})\}$ [see (1.18)], which are the indices of the R-factors in (2.13), appear positive since (1.16) is reduced. The group W is a Coxeter group [see (1.7)], the right-hand side of (2.13) does not depend on the choice of decomposition (1.16) due to Definition 2.1. Hence $\{R_w\}$ may be defined by (2.13) and satisfy (2.8). The uniqueness of this set is clear because (2.13) results from (2.12). This reasoning can be applied to R^a as well.

As for b), the transitivity of the action of W (or W^a) on the set of Weyl chambers and a direct consideration of the root systems of rank 2 prove that $\bar{R}$ (or $\bar{R}^a$) is a closure of R (R^a). Relations (2.12), (2.13) for $\bar{R}, \bar{R}^a$ follow from a). Let us verify c).

The elements $R_{\tilde{\alpha}}$ are well-defined by virtue of (2.8). Arbitrary elements $\tilde{\alpha}, \tilde{\beta} \in \Sigma^a$ satisfying (2.5^a) can be represented as $\tilde{\alpha} = w(\alpha_i)$, $\tilde{\beta} = w(\alpha_j)$ for $(1 \leq i \neq j \leq n)$ and a proper $w \in W^a$. If $ij \neq 0$ then the corresponding relation [see (2.1)–(2.4)] is valid because R is an R-matrix. Let us suppose that $ij = 0$ and the root system is not of type G_2.

If α_i and α_j are connected in Γ^α then the conjugation by another suitable w will give a pair $ij \neq 0$. These pairs have been already considered. Otherwise one may use the conjugations by $t' \in A$, $t = \theta^\vee$ and s_θ to reduce the problem to the case when $\alpha = \theta$, $\beta = \alpha_i$, $(\theta, \alpha_i) = 0$. It immediately results in condition (2.7). The system G_2 has to be treated separately. We arrive at the remaining relations from Definition 2.2.

The compatibility of the construction from b) and that from c) is clear. □

One may reformulate c) in a formal way by means of the algebras $\mathscr{T}$ and $\mathscr{T}^a$ (see above). We obtain that $\mathscr{T}^a$ is the universal extension of $\mathscr{T}$ equipped with an action of A satisfying relations (2.8).

In the next sections we will use the following generalization of this theorem involving a character (a homomorphism) $\chi : T \simeq B/A \to \mathbf{C}^*$.

Theorem 2.4. a) *Given a closure of an R-matrix R of b-type, let the set* $\{R_w, w \in W^a\}$ *be from Theorem 2.3,* c). *Then it can be uniquely extended to the set* $\{R_w, w \in$

$W^b\}$ *satisfying relations* (2.12) *for arbitrary* $x, y \in W^b$ $(l(xy) = l(x)+l(y))$ *by means of the pairwise equivalent conditions*

$$R_{b'_j} = \chi(\tau_j)R_{\sigma_j^{-1}} \Leftrightarrow R_{\tau_j} = \chi(\tau_j) \quad for \quad \alpha_j \in \Gamma_0 \,. \tag{2.16}$$

b) *Arbitrary* R_w *in the above set is the product of* $R_{\tilde{\alpha}}$ *when* $\tilde{\alpha}$ *runs over* λ_w *(see Proposition 1.6). It does not depend on the choice of the closure of* R *if* λ_w *contains no elements* $\tilde{\alpha} = [\alpha, k]$ *with* $\alpha < 0$ *(we call such* $w \in W^b$ *dominant). Moreover, the elements* R_w *for non-dominant* w *vanish if the following closure of* R *is taken:*

$$R^+ = \{R_\alpha, R_{-\alpha} \stackrel{\text{def}}{=} 0 \text{ if } \alpha \in \Sigma_+\} \,. \tag{2.17}$$

If the closure $\bar{R}$ *is from Theorem 2.3,* b), *then formulas* (2.12) *are valid for any* $x, y \in W^b$.

c) *The elements from* $B'_+ = \bigoplus_{i=1}^{n} \mathbf{Z}b'_i$ *are dominant. Given* $b, c \in B_+$, $l((b+c)') = l(b') + l(c')$ *and*

$$R_{b'+c'} = {}^{-c}R_{b'}R_{c'} = {}^{-b}R_{c'}R_{b'} \,, \tag{2.18}$$

$$R_{\theta'} = R_{[\theta,1]}R_{s_\theta} \,, \qquad R_{b'_i} = \mathscr{P}_i R_{\sigma_i^{-1}} \,, \qquad 1 \le i \le n \,, \tag{2.19}$$

where $\mathscr{P}_i$ *is a product of* $R_{\tilde{\alpha}}$ *for all* $\tilde{\alpha} = [\alpha, k]$ *such that* $\alpha \in \Sigma_+$, $(b_i, \alpha) > k > 0$.

Proof. If $w \in W^a$ then R_w is a product of $R_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \lambda_w$ in a certain order [see (1.18) and (2.13)]. Arbitrary $w \in W^b$ has a representation $w = \tau_j\hat{w}$ for an appropriate $\alpha_j \in \Gamma_0$, $\hat{w} \in W^a$. It gives the uniqueness of $\{R_w, w \in W^b\}$ and the statements from b). As for c), formulas (2.17), (2.18) result from a) and the relations

$$\lambda_{b'} \subset \lambda_{b'+c'} \,, \qquad \lambda_{\sigma_i^{-1}} \subset \lambda_{b'_i} \tag{2.20}$$

(see Proposition 1.4, b) and Proposition 1.6, c)). The equivalence of the two formulas from (2.16) is clear since $b'_j = \tau_j\sigma_j^{-1}$ and $l(\tau_j) = 0$. The existence of $\{R_w, w \in W^b\}$ follows directly from

Lemma 2.5. *As an abstract group,* W^b *is generated by* W^a *and* T *with the relations*

$$\tau_j s_i = s_{i'}\tau_j \,, \quad where \quad \alpha_{i'} = \tau_j(\alpha_i), \, 0 \le i, i' \le n, \, \alpha_j \in \Gamma_0 \,, \tag{2.21}$$

(see Proposition 1.5). □

Given $W^b \ni w = \tau_j\hat{w}$, $\hat{w} \in W^a$, put $R_w = \chi(\tau_j)R_{\hat{w}}$. It gives (2.16). Let us prove (2.12) for $x = \tau_j\hat{x}$, $y = \tau_p\hat{y} \in W^b$ satisfying the condition $l(xy) = l(x) + l(y)$. Here $\hat{x}, \hat{y} \in W^a$, $\alpha_j, \alpha_p \in \Gamma_0$. One has:

$$l(x) = l(\hat{x}) \,, \qquad l(y) = l(\hat{y}) \,, \qquad l(\hat{z}\hat{y}) = l(\hat{z}) + l(\hat{y})$$

for $\hat{z} \stackrel{\text{def}}{=} \tau_p^{-1}\hat{x}\tau_p \in W^a$, since $l(\tau_j) = l(\tau_p) = 0$. Hence,

$$R_{xy} = R_{\tau_j\tau_p}R_{\hat{z}\hat{y}} = \chi(\tau_j\tau_p)^{\hat{y}^{-1}}R_{\hat{z}}R_{\hat{y}} = {}^{y^{-1}}R_xR_y$$

because ${}^{\tau_p}R_{\hat{z}} = R_{\hat{x}}$ [use (2.21), (2.13), and (2.10)]. □

We will show that the constructions of the above theorems are compatible with the embeddings of the Dynkin graphs. Let Γ be a connected subgraph of $\hat{\Gamma}$ with $m > n$ vertices (representing the simple roots $\hat{\alpha}_1, \ldots, \hat{\alpha}_m$). Then every α_i $(1 \le i \le n)$

coincides with a proper $\hat{\alpha}_{i^*}$ $(1 \le i^* \le m)$. We will fix the notation $i \to i^*$ for the corresponding map $\Gamma \to \hat{\Gamma}$. The set $\hat{\Sigma}$ contains Σ in the natural way, $\hat{\Sigma}_+ \supset \Sigma_+$.

Proposition 2.6. a) *Given a closed R-matrix for Σ, the set*

$$\{\hat{R}_{\hat{\alpha}} = R_\alpha \text{ when } \hat{\alpha} \in \Sigma,\ \hat{R}_{\hat{\alpha}} = 1, \text{ elsewhere} \tag{2.22}$$

is a closed R-matrix for $\hat{\Sigma} \ni \hat{\alpha}$. If the initial R-matrix R is of b-type so is its extension (2.22) *for the following action of $\hat{B}$:*

$$\hat{b}_{i^*}(f) = b_i(f), \qquad \hat{b}_j(f) = f \quad \text{if} \quad j \ne i^* \quad \text{for} \quad 1 \le i \le n. \tag{2.23}$$

b) *Let us suppose that the above character χ on B/A is the restriction of a certain character $\hat{\chi}$ on $\hat{B}/\hat{A}$ with respect to the embedding $b_i \to b(i) \stackrel{\text{def}}{=} \hat{b}_{i^*} : B \to \hat{B}$. Then $\hat{R}_w$ constructed as in Theorem 2.3 for $\hat{R}$ coincides with R_w for $w \in W^a \subset \hat{W}^a$ and $R_{b'_i} = \hat{R}_{b'(i)} \stackrel{\text{def}}{=} \hat{R}_{\hat{b}'_{i^*}}$ for $1 \le i \le n$.*

Proof. Let α, β be from $\hat{\Sigma}$. Given a certain formula (2.1)–(2.4), if there exists an index γ from $\hat{\Sigma}\backslash\Sigma$, then not more than one index (among all the indices in the considered formula) can be from Σ. So the validity of the required relation is clear in this case [see (2.22)]. Otherwise $\{\gamma\} \subset \Sigma$ and we arrive at the relations for R only. The compatibility of Definition 2.2 with extension (2.22) follows directly from (2.23). As for b), the coincidence of R_w and $\hat{R}_w$ for $w \in W^a$ is evident. The description of the λ-sets for b'_i and $b'(i) \stackrel{\text{def}}{=} \hat{b}'_{i^*}$ gives the other statement (see Proiposition 1.6, b) and Theorem 2.4, b)). □

For instance, let $\chi = \hat{\chi} = 1$ and $\Gamma \subset \hat{\Gamma}$ be the natural embedding of Γ for E_6 or E_7 into $\hat{\Gamma}$ of type E_8. Then $\hat{A} = \hat{B}$ and $\hat{W}^a = \hat{W}^b$. Therefore Theorem 2.4 is equivalent to Theorem 2.3 for E_8 (in contrast with $E_{6,7}$). There is no need to guess formula (2.16) in this case. However the latter for E_6, E_7 and arbitrary given R can be deduced from Theorem 2.4 applied to $\hat{R}$ constructed by means of Proposition 2.6. Let us check it.

One has $\lambda_{\sigma_i^{-1}} \subset \lambda_{\sigma^{-1}(i)} \subset \lambda_{b'(i)}$, where $\sigma^{-1}(i) \stackrel{\text{def}}{=} \hat{\sigma}_{i^*}^{-1}$, $\hat{\sigma}_j$ is defined by (1.19) for $\hat{W}$ $(\sigma_i \in W \in \hat{W})$, $1 \le i \le n$. In particular, $\hat{R}_{b'(i)} = R_{\sigma_i^{-1}}$ when $i = 1, 6$ for E_6 and $i = 7$ for E_7. We have arrived at formula (2.16) for $E_{6,7}$.

The same deduction of Theorem 2.4 from Theorem 2.3 for a suitable bigger $\hat{\Gamma}$ may be applied for other root systems. Roughly speaking, formula (2.16) is necessary to ensure the compatibility of the above construction with the embeddings of Dynkin graphs.

3. The Definition of QKZ

We fix a $\mathscr{F}^*$-valued R-matrix R ($\mathscr{F}^*$ is the group of invertible elements in $\mathscr{F}$). In this section, $\hat{R} = \{R_\alpha,\ \alpha \in \Sigma\}$ is the closure of R from Theorem 2.3, b). Let us suppose it to be of b-type in the sense of Definition 2.2. We will denote its *affine completion* (2.15) from Theorem 2.3, c) by $\bar{R}^a$. It satisfies (2.14). Let $\{R_w,\ w \in W^b\}$ be the set from Theorem 2.4 [defined by (2.12) for arbitrary $x, y \in W^b$]. In particular, ${}^w R_w = R_{w^{-1}}^{-1}$.

Definition 3.1. *The quantum Knizhnik-Zamolodchikov equation* (QKZ) *is one of the following equivalent systems of relations for an element $\Phi \in \mathscr{F}$:*

$$\text{a) } b_i^{-1}(\Phi) = R_{b_i'}\Phi,\ 1 \le i \le n; \tag{3.1}$$

$$\text{b) } b^{-1}(\Phi) = R_{b'}\Phi \text{ for any } b \in B; \tag{3.2}$$

c) *the cocycle $\{R_{b'},\, b' \in B\}$ is the coboundary of Φ.* □

We will not use c) in this paper and give this condition to connect our definition with the regular terminology from the theory of cohomologies of abstract groups only. The equivalence of a), b) follows from (2.18). Actually these relations are nothing else but the self-consistency conditions for (3.2). The special choice of the closure (Theorem 2.3, c)) ensures the validity of (2.18) for arbitrary $b, c \in B$ (not only from B_+).

Theorem 3.2. a) *Let us assume that the action of B is extended to an action of $W^b \ni w$ on the algebra $\mathscr{F} \ni f$ ($f \to w(f)$) by $\mathbf{C}$-automorphisms and*

$$w(R_\alpha) = R_{w(\alpha)} \text{ for arbitrary } \alpha \in \Sigma,\, w \in W. \tag{3.3}$$

Then $w(R_{\tilde\alpha}) = R_{w(\tilde\alpha)}$ for $w \in W^b$, $\tilde\alpha \in \Sigma^a$ and $R_{\tilde\alpha}$ from $\bar R^a$ (see above). In particular, $w(R_{\tilde\alpha}) = {}^wR_{\tilde\alpha}$, where the latter is from (2.10), *and $\bar R^a$ is unitary: $R_{-\tilde\alpha}^{-1} = s_{\tilde\alpha}(R_{\tilde\alpha})$.*

b) *The $\mathbf{C}$-linear homomorphisms*

$$\varrho(f) \stackrel{\text{def}}{=} R_{w^{-1}}^{-1}(w(f)) = w(R_w f), \qquad w \in W^b, \tag{3.4}$$

of $\mathscr{F}$ considered as a linear space form a representation:

$$\varrho_x \circ \varrho_y = \varrho_{xy} \quad \text{for} \quad x, y \in W^b. \tag{3.5}$$

Given a solution Φ and $w \in W^b$, (3.3) *implies that $\varrho_w(\Phi)$ satisfies* (3.2) *as well.*

c) *In the above setup, $\varrho_{wb'}(\Phi) = \varrho_w(\Phi)$ for $b \in B$. If Φ is invertible then*

$$T_w = T_w(\Phi) \stackrel{\text{def}}{=} w^{-1}(\Phi^{-1})R_w\Phi = T_{wb'} \quad \text{for} \quad w \in W^b \tag{3.6}$$

and any b. Moreover, $\{T_w\}$ satisfy (2.12)*:*

$$T_{xy} = y^{-1}(T_x)T_y \quad \text{for} \quad x, y \in W^b \tag{3.7}$$

and belong to $\mathscr{F}^B \stackrel{\text{def}}{=} \{\in \mathscr{F},\, b(f) = f\}$.

Proof. Relations (2.10) result directly from (3.3) and (2.15). Formula (3.5) is equivalent to (2.12). Given a solution Φ, let $b \in B$, $\hat b = w^{-1}bw \in B$. One has:

$$\begin{aligned} b^{-1}(w(R_w\Phi)) &= b^{-1}w(R_w)wb^{-1}(\Phi) = b^{-1}w(R_w)w(R_{\hat b'}\Phi) \\ &= w(\hat b^{-1}(R_w)R_{\hat b'}\Phi) = w(R_{w\hat b'}\Phi) = w(R_{b'w}\Phi) = R_{b'}w(R_w\Phi). \end{aligned}$$

Hence $\varrho_w(\Phi)$ satisfies (3.2). Relations (3.7) formally follow from (3.6) and (2.12) (here Φ may be absolutely arbitrary). The B-invariance of each T_w and the equalities $T_w = T_{wb'}$ result from (3.2) and b). □

We will call $T = \{T_w,\, w \in W\}$ the *monodromy cocycle*. We notice that the restriction of T onto W is enough to reconstruct its values for any $w \in W^b$ because of c). See e.g. [Ch2, Ch4] for the discussion of the classic definition of the monodromy representation applied to KZ equations. The analogous notions based on the theory of difference equations are considered in [AKM] and [FR]. In certain contrast with the monodromy of KZ, that of QKZ is trivial if $\mathscr{F}$ is of matrix type. This property was pointed out by Smirnov for the solutions of his equation and seems to be a rather general feature of difference equations:

Corollary 3.3. *Let $\mathscr{F}^B$ be the matrix algebra $M_N(\mathscr{K})$ for a field $\mathscr{K}$ with a faithful action of W by C-automorphisms compatible with the action of W on $\mathscr{F}^B : w(kf) = w(k)w(f)$ for $w \in W$, $k \in \mathscr{K}$, $f \in \mathscr{F}$. We impose relations* (3.3). *Given an arbitrary invertible solution Φ, T is a coboundary, i.e. there exists $F \in \mathscr{F}^* = GL_N(\mathscr{K})$ such that*

$$T_w = w^{-1}(F^{-1})F\,, \qquad w \in W\,. \tag{3.8}$$

The element $\tilde{\Phi} = \Phi F^{-1}$ satisfies (3.2) *and has the trivial monodromy cocycle $\tilde{T} = T(\Phi)$:*

$$\varrho_w(\tilde{\Phi}) = \tilde{\Phi} \quad for \quad w \in W^b\,. \tag{3.9}$$

Proof. Indeed, W acts on $\mathscr{F}^B = M_N(\mathscr{K})$ since B' is a normal subgroup of W^b. The group W is finite. Hence we may apply a proper version of Hilbert theorem 90 $(H^1(W, GL_N(\mathscr{K})) = \{id\})$. □

The corollary does not mean that (3.9) should be imposed from the very beginning without any reservation. Sometimes it is more convenient to consider Φ with non-trivial $T(\Phi)$ (e.g. for obtaining classical solutions as limits of quantum ones). Now we will briefly describe the procedure of *quasi-classical degeneration* of QKZ.

Let us fix a *W-invariant set* $\{\kappa_\alpha \in \mathbf{C}, \alpha \in \Sigma\}$. The invariance means that $\kappa_\alpha = \kappa_{w(\alpha)}$ for arbitrary $w \in W$, $\alpha \in \Sigma$. The assumptions will be as those for Definition 3.1. We suppose that the action of B on $\mathscr{F}$ and a given R-matrix R continuously depend on small $h \in \mathbf{C}$:

$$b(f) - f = h\partial_b(f) + o(h)\,, \qquad b \in B,\, f \in \mathscr{F}\,, \tag{3.10}$$
$$R_\alpha = 1 + hr_\alpha + o(h) \quad \text{for} \quad \alpha \in \Sigma,\, r_\alpha \in \mathscr{F}\,. \tag{3.11}$$

Here ∂_b is a $\mathbf{C}$-linear endomorphism of $\mathscr{F}$, which has to be a $\mathbf{C}$-derivative of $\mathscr{F}$ because $f \to b(f)$ is its homomorphism as an algebra.

Proposition 3.4. *If $\Phi = \Phi(h)$ is a continuous solution of (3.2) (relative to h), then $\phi = \Phi(0)$ satisfies the system*

$$\partial_b(\phi) = \sum_\alpha \kappa_\alpha(b, \alpha)r_\alpha\phi\,, \quad where \quad \alpha \in \Sigma_+,\, b \in B\,, \tag{3.12}$$

of "differential" equations.

Proof. One has: $\partial_b(\phi) = r_b\phi$, where $R_{b'} = 1 + hr_b + o(h)$. Let us apply Theorem 2.4, b). We see that $r_b = \sum_\alpha \kappa_\alpha(b, \alpha)r_\alpha$. □.

The connection of Eq. (3.12) and quantum R-matrices was establishes in [Ch2] (see also [Ch3], Proposition 3.3). The cross-derivative integrability conditions for (3.12) were deduced from the r-matrix quadratic relations for $\{r_\alpha\}$, which are the quasi-classical limit of the identities from Definitions 2.1, 2.2. We note that the r-matrix relations are better to consider independently without any reference to quantum R-matrix ones because there are (many) examples when r has no quantum deformations $R(h)$. We do not supply this paper with the r-matrix relations (see the mentioned papers). The above way of getting (3.12) from QKZ (when B acts by the shifts of the arguments of type A) coincides, in fact, with the corresponding reasoning from [FR].

Examples. First, we put down system (3.1) in the cases A_2, B_2, G_2. Let $\alpha = \alpha_1$, $\beta = \alpha_2$, $a = b_1$, $b = b_2$ (see Definition 2.1). One has the following systems:

$$a^{-1}(\Phi) = R_{\alpha+\beta}R_{\alpha}\Phi\,,\ \ b^{-1}(\Phi) = R_{\alpha+\beta}R_{\beta}\Phi\,; \tag{3.13}$$

$$a^{-1}(\Phi) = R_{\alpha+2\beta}R_{\alpha+\beta}R_{\alpha}\Phi\,,\ \ b^{-1}(\Phi) = R_{[\alpha+2\beta,1]}R_{\alpha+\beta}R_{\alpha+2\beta}R_{\beta}\Phi\,; \tag{3.14}$$

$$\begin{aligned} a^{-1}(\Phi) = {} & R_{[3\alpha+2\beta,2]}R_{[3\alpha+\beta,2]}R_{[2\alpha+\beta,1]}R_{[3\alpha+2\beta,1]}R_{[3\alpha+\beta,1]} \\ & \times R_{\alpha+\beta}R_{3\alpha+2\beta}R_{2\alpha+\beta}R_{3\alpha+\beta}R_{\alpha}\Phi\,, \end{aligned} \tag{3.15a}$$

$$b^{-1}(\Phi) = R_{[3\alpha+\beta,1]}R_{3\alpha+\beta}R_{2\alpha+\beta}R_{3\alpha+2\beta}R_{\alpha+\beta}R_{\beta}\Phi\,. \tag{3.15b}$$

We remind that $R_{[\gamma,k]} = c(R_\gamma)$ for $\gamma \in \Sigma_+$, $c \in B$ if $(c,\gamma) = -k$. Imposing relations (3.3) under the assumption that W^b operators on $\mathscr{F}$, one obtains:

$$\begin{gathered} R_{\alpha+\beta} = s_\beta(R_\alpha) = s_\alpha(R_\beta) \quad \text{for } A_2\,, \\ R_{\alpha+\beta} = s_\alpha(R_\beta)\,, \qquad R_{2\alpha+\beta} = s_\beta(R_\alpha) \quad \text{for } B_2\,, \\ R_{\alpha+\beta} = s_\beta(R_\alpha)\,, \qquad R_{2\alpha+\beta} = s_{\alpha+\beta}(R_\alpha)\,, \\ R_{3\alpha+\beta} = s_\alpha(R_\beta)\,, \qquad R_{3\alpha+2\beta} = s_{3\alpha+\beta}(R_\beta) \end{gathered}$$

in the case of G_2. We notice that the order of the R-factors in (3.15a) is rather intricate. Looking at this formula one can imagine how complicated the appropriate expressions for F_4, E_{6-8} should be! The number of the R-factors in the fourth equation (for b_4) in the case E_8 is equal to 270 (the multiplicity of α_4 in the sum 2ϱ of all positive roots).

The following examples of R-matrices are from [Ch3], Lemma 3.5. Let us introduce the algebras $\mathbf{C}[y] = \mathbf{C}[y_1, y_2, \ldots, y_n]$ of polynomials in $y = (y_1, \ldots, y_n) \in \mathbf{C}^n$ and $\mathbf{C}[\bar{Y}] = \mathbf{C}[Y_1^{\pm}, Y_2^{\pm}, \ldots, Y_n^{\pm}]$ for $Y_i^{\pm} = \exp(\pm y_i)$. We identify the roots $\alpha \in \Sigma$ with the corresponding linear combinations y_α substituting $y_1, \ldots, y_n$ instead of $\alpha_1, \ldots, \alpha_n$ ($y_{\alpha_i} = y_i$ and so on). It gives the following action of W^b on $\mathbf{C}[y]$ and $\mathbf{C}[\bar{Y}]$:

$$w(y_a) = y_{w(a)}\,, \qquad b(y_\alpha) = b'(y_\alpha) \stackrel{\text{def}}{=} y_\alpha - (b,\alpha)h\,, \qquad w \in W\,, \tag{3.16}$$

$$w(Y_\alpha) = Y_{w(\alpha)}\,, \qquad b(Y_\alpha) = b'(Y_\alpha) \stackrel{\text{def}}{=} Y_\alpha \exp\{-(b,\alpha)h\} \qquad b \in B\,. \tag{3.17}$$

Here $h \in \mathbf{C}$ is supposed to be fixed.

Let $\mathscr{F}_0$ be the algebra of endomorphisms of $\mathbf{C}[y]$ considered as a vector space, $\mathscr{F}_1 \stackrel{\text{def}}{=} \mathrm{End}_{\mathbf{C}}\, \mathbf{C}[\bar{\mathbf{Y}}]$. We denote the composition of endomorphisms by "$\circ$". The group W^b acts on $\mathscr{F}_0, \mathscr{F}_1$ by conjugations with respect to the natural map $W^b \to \mathscr{F}_{0,1}$. We identify $\mathbf{C}[y]$ or $\mathbf{C}[\bar{Y}]$ with the corresponding subalgebras in $\mathscr{F}_0, \mathscr{F}_1$ ($y_\alpha(p) = y_\alpha p$, $Y_\alpha(p) = Y_\alpha p$ for $p \in \mathbf{C}[y]$, $\mathbf{C}[\bar{Y}]$). Let us fix W-invariant sets $\{\kappa_\alpha\} \subset \mathbf{C}$, $\{q_\alpha\} \subset \mathbf{C}^*$ (see above).

Proposition 3.5. *The sets $g = \{g_\alpha\} \subset \mathscr{F}_0$, $G = \{G_\alpha\} \subset \mathscr{F}_1$ for $\alpha \in \Sigma_+$ and*

$$g_\alpha = 1 + \kappa_\alpha y_\alpha^{-1} \circ (1 - s_\alpha)\,, \tag{3.18a}$$

$$G_\alpha = q_\alpha + (q_\alpha - q_\alpha^{-1})(Y_\alpha - 1)^{-1} \circ (1 - s_\alpha) \tag{3.18b}$$

are R-matrices with the values in $\mathscr{F}_0, \mathscr{F}_1$. Moreover,

$$(g_\alpha \circ s_\alpha)^2 = 1\,, \qquad (G_\alpha \circ s_\alpha - q_\alpha) \circ (G_\alpha \circ s_\alpha + q_\alpha^{-1}) = 0 \tag{3.19}$$

and the closure $\bar{g} = \{g_\alpha, g_{-\alpha} \stackrel{\text{def}}{=} g_\alpha^{-1}\}$ *satisfies* (3.3), *i.e.* $\bar{g}$ *can be defined by* (3.18a) *for all* $\alpha \in \Sigma$. □

We note that $\{g_\alpha, G_\alpha\}$ are actually from certain smaller algebras. They commute with the action of the algebra Sym of symmetric (W-invariant) polynomials in y or $\bar{Y}$ by multiplications. Hence, $\{g_\alpha, G_\alpha\} \subset \mathrm{End}_{\mathrm{Sym}}$ and they act in arbitrary quotient-spaces of $\mathbf{C}[y]$ or $\mathbf{C}[\bar{Y}]$ by ideals generated by symmetric polynomials.

Next we will use $\{g_\alpha\}$ to introduce a difference counterpart of the Calogero quantum many-body problem (see [C] and [Ch4] for the definitions and references). A similar construction can be made for $\{G_\alpha\}$ (the proof is somewhat different). We obtain difference operators of Sutherland type [Su]. See the Appendix for certain connections with Macdonald's construction [M].

Let $\mathbf{C}(y)$ be the field of rational functions in $y_1, y_2, \ldots, y_n$, $\mathscr{F}_0'$ the subalgebra of $\mathrm{End}_{\mathbf{C}}\, \mathbf{C}(y)$ generated by W^b and $\mathbf{C}(y)$. Arbitrary $f \in \mathscr{F}_0'$ can be uniquely represented as follows:

$$f = \sum_w f_w \circ w\,, \quad \text{where} \quad f_w \in \mathscr{D} \stackrel{\text{def}}{=} \mathbf{C}(y) \circ B' \subset \mathscr{F}_0' \qquad w \in W\,. \tag{3.20}$$

We denote $b' \circ g_{b'}$, by $\varrho_{b'}$, where $g_{b'} \stackrel{\text{def}}{=} R_{b'}$ for $R = g$ [see (3.4)].

Theorem 3.6. *Given an arbitrary finite* W*-invariant set* $X \subset B$ *and* $m \in \mathbf{N}$, *let*

$$L_X^m = \sum_x (\varrho_{x'} - 1)^m = \sum_w D_w \circ w\,, \quad \Delta_X^m = \sum_w D_w\,, \quad x \in X\,, \tag{3.21}$$

where $D_w \in \mathscr{D}$, $w \in W$ *(see* 3.20)*). Then*

$$w \circ \Delta_X^m \circ w^{-1} = \Delta_X^m\,, \quad \Delta_X^m \circ \Delta_Z^l = \Delta_Z^l \circ \Delta_X^m \tag{3.22}$$

for arbitrary $w \in W$, $l \in \mathbf{N}$ *and an invariant set* $Z \subset B$.

Proof. First, (3.5) results in

$$\varrho_w \circ L_X^m \circ \varrho_{w^{-1}} = L_X^m\,, \quad L_X^m \circ L_Z^l = L_Z^l \circ L_X^m\,. \tag{3.23}$$

Let us check that $L_X^m(\mathrm{Sym}) \subset \mathrm{Sym}$. Indeed,

$$\{\varrho_w(p) = p \text{ for all } w \in W \text{ and } p \in \mathbf{C}[y]\} \Leftrightarrow \{p \in \mathrm{Sym}\}$$

because it is true for $\{\varrho_{s_i} = s_i \circ g_{\alpha_i},\ 1 \le i \le n\}$ generating $\{\varrho_w\}$. Hence, the relation $\varrho_w(L_X^m(p)) = L_X^m(\varrho_w(p)) = L_X^m(p)$ for $p \in \mathrm{Sym}$ gives the desired inclusion. We obtain that $\Delta_X^m(\mathrm{Sym}) \subset \mathrm{Sym}$ and relations (3.22) are valid for the restrictions of Δ_X^m and Δ_Z^l onto Sym coinciding with those for L_X^m and L_Z^l. To deduce (3.22) from its restriction on Sym we note that the operators on each side of these relations are from $\mathscr{D}$ and use

Lemma 3.7. *If* $D \in \mathscr{D}$ *and* $D(p) = 0$ *for arbitrary* $p \in \mathrm{Sym}$, *then* $D = 0$.

Proof. One has: $D = \sum_{r=1}^{d} f_r \circ c_r$, where $f_r \in \mathbf{C}(y)$, $c_r \in B$, $c_r \neq c_s$ for $1 \le r, s \le d$. There exists $y^0 = (y_1^0, \ldots, y_n^0) \in \mathbf{R}^n$ such that $w(a) - a \notin h\{w(c_r) - c_s,\ 1 \le r, s \le d\}$ for $a = \sum_{i=1}^{n} y_i^0 \alpha_i$ and any $w \in W$, $w \neq id$ or, equivalently,

$$\{w(y^0) - y^0,\ w \neq id\} \cap h\{w(z^r) - z^s,\ 1 \le r, s \le d\} = \emptyset$$

for $z^r = ((c_r, b_1), \ldots, (c_r, b_n))$. If $D(p)(y) = \sum_{r=1}^{d} f_r(y)p(y - hz^r) = 0$ for arbitrary $p \in \mathrm{Sym}$, $y \in \mathbf{R}^n$, then $\sum_{r=1}^{d} \phi_r(y)p(y^0 - hz^r) = 0$ for $\phi_r = f_r(y^0)$. We may assume that $\phi_{r_0} \neq 0$ for a certain r^0. However the above conditions ensure that the numbers $p(y^0 - hz^1), \ldots, p(y^0 - hz^d)$ can be made arbitrary for suitable $p \in \mathrm{Sym}$. □□

This theorem is a "quantization" of the corresponding Dunkl-Heckman way to introduce Calogero-Sutherland operators (see [Ch4] for the references and details). The latter can be obtained from $\{h^{-m}\Delta_X^m\}$ when $\kappa_\alpha = hk_\alpha$, $h \to 0$ after a certain conjugation. The properties of $\{\Delta_X^m\}$ and their trigonometric counterparts will not be discussed here. We only mention that the equivalence of KZ equations and the Calogero-Sutherland problem (established by Matsuo and the author – see [Ch4]) has a quantum analogue (see [Ch7]).

The examples of Proposition 3.5 were of constant type. Now we are going to consider a functional generalization of (3.18). We introduce the linear functions $v_\alpha = (\alpha, v)$, $v \in \mathbf{R}^n$, $\alpha \in \Sigma$, the coordinates $v_i = (\alpha_i, v)$, $1 \leq i \leq n$, the field $\mathbf{C}(v)$ of rational functions in $v = (v_1, \ldots, v_n)$ and the field $\mathbf{C}(V)$ of rational functions in $V = (\ldots, V_i = \exp(v_i), \ldots)$. Let

$$\mathscr{F} = \mathscr{F}_0(v) = \mathscr{F}_0 \otimes \mathbf{C}(v) \quad \text{or} \quad \mathscr{F} = \mathscr{F}_1(V) = \mathscr{F}_0 \otimes \mathbf{C}(V)$$

and the action of W^b on $\mathscr{F} \ni f$ be as follows:

$$\tilde{w}(f)(v) = \tilde{w} \circ f(\tilde{w}^{-1}(v)) \circ \tilde{w}^{-1}, \qquad (wb')(v) = w(v) + hb, \tag{3.24}$$

where $W^b \ni \tilde{w} = wb'$ for $w \in W$, $b \in B$, $\tilde{w}$ are considered as elements of $\mathscr{F}_{0,1}$ due to (3.16).(3.17). We fix a W-invariant set $\{k_\alpha\} \subset \mathbf{C}$, $\alpha \in \Sigma$.

Proposition 3.8. *In the notations of Proposition 3.5, the sets*

$$\tilde{g}_\alpha = \psi_\alpha^{-1}(v_\alpha)(g_\alpha + k_\alpha v_\alpha^{-1} s_\alpha), \qquad \psi_\alpha(z) = 1 + k_\alpha z^{-1}, \tag{3.25}$$

$$\tilde{G}_\alpha = \Psi_\alpha^{-1}(V_\alpha)(G_\alpha + (q_\alpha - a_\alpha^{-1})(V_\alpha - 1)^{-1} s_\alpha), \quad \textit{where}$$

$$\Psi_\alpha(z) = q_\alpha + (q_\alpha - q_\alpha^{-1})(z-1)^{-1}, \qquad \alpha \in \Sigma,\ z \in \mathbf{C} \tag{3.26}$$

are closed (and unitary) R-matrices satisfying (3.3) *for the above action of W (see* (3.24)*).* □

The proposition follows from [Ch3], Proposition 1.2 (and is connected with certain identities of [L]). As for (3.25), this formula is, in fact, from [Ch1], where there are other examples (e.g. with an elliptic dependence on the arguments $\{v_i\}$). Many R-matrices with the arguments of type A can be found in the papers of the last decade.

4. Particles on a Segment

We will give a graphic interpretation of the above constructions for the classical root systems of types A, B, C, D. Let us consider a rather big segment $[lr] \subset \mathbf{R}$ with the moving left endpoint:

$$l(t) = l_0 + \tan(\delta) < r = \text{const}, \qquad \delta \leq 0 \ (|\delta| \text{ is sufficiently small}). \tag{4.1}$$

A particle is represented by a point $x \in [lr]$ moving with constant velocity:

$$x(t) = x_0 + t\tan(\phi), \qquad -\pi/2 < \phi < \pi/2. \tag{4.2}$$

We put $\sigma(t) = -$ if the continuation of line (4.2) backwards for the values $t' < t$ intersects first the line $r(t) = r$. In this case, the particle is moving from right to left and its angle is negative (the converse is true if δ is infinitesimal). Otherwise $\sigma(t) = +$. We will call σ the *sign of the particle*.

The symbol $A_\sigma(\phi, x) = A_\sigma(\phi, x)_t$ means by definition that we place a particle with the angle ϕ and the sign σ in the position x at the moment t. We suppose that the particles are reflected in the endpoints as follows. If $A_{\sigma'}(\phi', x')'_t$ for $t' < t$ is the symbol of the same particle just before the previous reflection then

$$\phi' = (\sigma + 1)\delta - \phi \quad \text{and} \quad \sigma' = -\sigma\,. \tag{4.3}$$

This rule corresponds to the behaviour of the rapidities $i\phi$ in the relativistic theory. We prefer to use angles instead of rapidities in this paper. Here and further we assume that (4.2) is fulfilled during the considered interval of time, i.e. transformation (4.3) preserves the above inequality for ϕ.

Let us take n particles of different types $1, 2, \ldots, n$ or, equivalently, assign a number to each particle. To distinguish them we use the symbols $A^1, A^2, \ldots, A^n$. Particles never change their types. Given a certain permutation $I = (i_1, i_2, \ldots, i_n)$ of $(1, \ldots, n)$, a set of angles $u = (u_1, \ldots, u_n)$ in a general position and a set of the signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, the corresponding set of the particles will be described by the *multi-symbol*

$$A^I_\varepsilon(u, X) = A^{i_1}_{\varepsilon_1}(u_1, x_1) \ldots A^{i_n}_{\varepsilon_n}(u_n, x_n) \quad \text{if } X = (x_1 < x_2 < \ldots < x_n)\,. \tag{4.4}$$

Note that the ordering of the A-factors (and the numeration of u, ε, X) is due to the positions of the particles at the considered moment t and has nothing to do with their initial numbers (types). The order of the latter at t is described by I. Multi-symbol (4.4) may be connected only with the particles in a general position. Later on, we assume that the particle move independently (are transparent for each other) and the velocities (angles) are in a general position. So the multi-symbols are well-defined for almost all t.

Turning to the quantum scattering, let us fix an algebra $\mathscr{F}_0$ (e.g. the tensor power $M_N^{\otimes n}$ of a certain matrix algebra M_N). To introduce the scattering of two intersecting particles we set

$$A^j_\tau(\psi, x') A^i_\sigma(\phi, y')_{\text{in}} = R^{\sigma\tau}_{ij}(\phi - \psi) A^i_\sigma(\phi, x) A^j_\tau(\psi, y)_{\text{out}}\,, \tag{4.5}$$

where the out-state (the suffix is "out") is at the moment t, the in-state is at $t' < t$ (written "in") and between t' and t these two particles (and only they) intersected once. There are no reflections during the considered interval of time. Here R is a function of one variable with the values in $\mathscr{F}_0$.

This writing means that the scattering "matrix" $R^{\pm\pm}_{ij}$ depends only on the types, the difference of the angles and the signs (and does not depend on the other particles). By the way, the combination of the signs $\sigma\tau = +-$ is impossible because of the plain geometric reasons. The considered particles are neighbouring in the complete set of particles (between t' and t). So their symobls stand side by side in (4.4) and dropping the remaining A-terms cannot lead to confusion.

To describe the scattering for the reflections we set

$$A^i_{-\sigma}(\phi', x')_{\text{in}} = Q^\sigma_i((\sigma + 1)\delta - 2\sigma\phi) A^i_\sigma(\phi, x)_{\text{out}}\,, \tag{4.6}$$

where ϕ' is from (4.3) and the particle in the out-state has been reflected in the endpoint r or l if $\sigma = -$ or $\sigma = +$ respectively. Here the values of the function

$Q_i^{\pm}$ are in $\mathscr{F}_0$ as well and do not depend on the remaining particles (of type $j \neq i$). We leave out the unnecessary A-factors. Of course we suppose that there are no other collisions (intersections or reflections) between the in-state and the out-state. To ensure the desired reflection, symbol (4.6) of the considered particle has to be the first ($\sigma = +$) or the last ($\sigma = -$) in $A_\varepsilon^I(u, X)$.

One may omit the coordinates x, x', y, y' in the above symbols because the scattering matrices [see (4.5), (4.6)] do not depend on them. As for more complicated processes, dropping the coordinates of the particles involved is not safe. Certain changes of the initial positions (even preserving the multi-symbols of the in-out-states) alter the picture of the intermediate collisions during the considered interval of time. However it can be done if the following conditions are imposed.

Given $A_\varepsilon^I(u) = A_{\varepsilon_1}^{i_1}(u_1) \ldots A_{\varepsilon_n}^{i_n}(u_n)$ for $I = (i_1, i_2, \ldots, i_n)$, $u = (u_1, \ldots, u_n)$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ as the out-state and the corresponding $A'_{\text{in}} = A_{\varepsilon'}^{I'}(u')_{\text{in}}$, we postulate that

$$A_{\varepsilon'}^{I'}(u')_{\text{in}} = \mathfrak{R}_{II'}^{\varepsilon\varepsilon'}(u, u') A_\varepsilon^I(u)_{\text{out}}, \tag{4.7}$$

where the $\mathscr{F}_0$-valued function $\mathfrak{R}$ is the chronological product of the R, Q-matrices over the intermediate "elementary collisions" (intersections and reflections) and does not depend on the positions of the particles.

Here everything is in a general position so intersections of three particles (or more) and simultaneous reflections of two or more particles are not considered. We note that the R, Q-factors are the same for any initial position of the particles but their order depends on the latter. If the algebra $\mathscr{F}_0$ were commutative there would be nothing to check. We will not discuss a geometric description of the corresponding conditions in full detail. When $\mathbf{R}$ is taken as the space ($l = -\infty$, $r = +\infty$), these relations are due to Yang, Baxter, Zamolodchikov and other physicists (see e.g. [ZZ]). The identities for the general case were introduced in [Ch1] and [Ch5]. We reformulate the main postulate above in an algebraic way.

Let us introduce the following free $\mathscr{F}_0$-module $\mathscr{A} = \sum_{I,u,\varepsilon} \mathscr{F}_0 A_\varepsilon^I(u)$, where the generators $A_\varepsilon^I(u)$ are considered as independent letters (symbols) with the indices I, u, ε (u is continuous). Given meromorphic $\mathscr{F}_0$-valued functions $\{R_{ij}^{\pm\pm}(\phi),\ Q_i^{\pm}(\phi)\}$ for $\phi \in \mathbf{C}$, $1 \leq i \neq j \leq n$, we define the quotient-module $\bar{\mathscr{A}}$ by imposing the *R-relations*

$$A_{\varepsilon'}^{I'}(u') = R_{i_p\ i_{p+1}}^{\varepsilon_p\ \varepsilon_{p+1}}(u_p - u_{p+1}) A_\varepsilon^I(u), \qquad u_p - u_{p+1} < 0, \qquad 1 \leq p < n, \tag{4.8}$$

$$u' = s_p(u), \qquad \varepsilon' = s_p(\varepsilon), \qquad I' = s_p(I), \quad \text{where} \quad s_p = (p\,p+1) \in \mathbf{S}_n \tag{4.9}$$

are the adjacent transpositions, and the *Q-relations*

$$A_{\varepsilon'}^{I}(u') = Q_{i_p}^{\varepsilon_p}(\tilde{u}_p) A_\varepsilon^I(u), \qquad \tilde{u}_p = (\varepsilon_p + 1)\delta - 2u_p < 0, \qquad p = 1, n, \tag{4.10}$$

where $\varepsilon_1 = +$, $\varepsilon_n = -$, $u' = t_p^{\varepsilon_p}(u)$, $\varepsilon' = t_p^{\varepsilon_p}(\varepsilon)$ for the automorphisms

$$\begin{gathered} t_p^\sigma : u_p \to u'_p = (\sigma + 1)\delta - u_p, \\ \varepsilon_p \to -\varepsilon_p, \qquad u_j \to u_j, \qquad \varepsilon_j \to \varepsilon_j, \qquad j \neq p, \end{gathered} \tag{4.11}$$

defined for arbitrary $1 \leq p \leq n$, $\sigma = \pm$. In these formulas, u is from a certain connected neighbourhood of $0 \in \mathbf{C}^n$.

Definition 4.1. *The scattering "matrices" $\mathfrak{R}$ do not depend on the initial positions of the particles (the scattering theory is factorizable) iff the images of the following elements*

$$A_\varepsilon^I(u) \quad for \quad \delta < u_1 < u_2 < \ldots < u_n < 0\,, \qquad u \in \mathbf{R}^n\,,$$

are $\mathscr{F}$-linearly independent in $\bar{\mathscr{A}}$ for all the indices I, ε, and u as above. $\square$.

Let us put down explicitly the corresponding fundamental relations (see [Ch1, Ch5]). We should note that the above definition provides the validity of these relations only for rather small (and ordered in a special way) $u = (u_i)$. However R, Q are supposed to be meromorphic. So the relations hold good for any complex $\{u_i\}$ such that the R, Q-factors involved are well-defined. The arguments in all the formulas below are complex numbers from a certain domain.

Given pairwise distinct $1 \le i, j, k, l, \le n$, and arbitrary $\sigma, \tau, \zeta, \xi \in \{\pm\}$,

$$[R_{ij}^{\sigma\tau}(\phi),\, R_{kl}^{\zeta\xi}(\psi)] = 0 = [R_{ij}^{\sigma\tau}(\phi),\, Q_k^{\zeta}(\psi)] = [Q_i^+(\phi),\, Q_j^-(\psi)]\,, \tag{4.12}$$

$$R_{ij}^{\sigma\tau}(\phi)R_{ik}^{\sigma\zeta}(\phi+\psi)R_{jk}^{\tau\zeta}(\psi) = R_{jk}^{\tau\zeta}(\psi)R_{ik}^{\sigma\zeta}(\phi+\psi)R_{ij}^{\sigma\tau}(\phi)\,, \tag{4.13}$$

$$\begin{aligned} &R_{j\ i}^{+\,+}(\phi)Q_i^-(2\phi+\psi)R_{i\ j}^{-\,+}(\phi+\psi)Q_j^-(\psi) \\ &\quad = Q_j^-(\psi)R_{j\ i}^{-\,+}(\phi+\psi)Q_i^-(2\phi+\psi)R_{i\ j}^{-\,-}(\phi)\,, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &R_{j\ i}^{-\,-}(\phi)Q_j^+(2\phi+\psi)R_{i\ j}^{-\,+}(\phi+\psi)Q_i^+(\psi) \\ &\quad = Q_i^+(\psi)R_{j\ i}^{-\,+}(\phi+\psi)Q_j^+(2\phi+\psi)R_{i\ j}^{-\,-}(\phi)\,. \end{aligned} \tag{4.15}$$

We call that $R_{i\ j}^{+\,-}$ do not exist and notice that there are no identities involving $Q^\pm$ for the coinciding signs.

To calculate the scattering matrices $\mathfrak{R}$ for arbitrary collisions we introduce the group Ω generated by the symmetric group $\mathbf{S}_n \ni w$ and $t_i^\pm$, $1 \le i \le n$ with the following relations:

$$(t_i^\sigma) = 1 \qquad t_i^\sigma t_j^\tau = t_j^\tau t_i^\sigma \quad \text{if} \quad \sigma i + \tau j \ne 0\,, \qquad w t_i^\sigma w^{-1} = t_{w(i)}^\sigma\,, \tag{4.16}$$

where $1 \le i, j \le n$, $\sigma, \tau \in \{\pm\}$. Formulas (4.9), (4.11) define a faithful action of Ω on $\mathbf{R}^n \ni u$. This group is isomorphic to the affine Weyl group W^a of type C_n (or A_1 if $n = 1$). Identify the above s_p with s_p from W^a (Sect. 1) for $1 \le p < n$, t_n^- with s_n and t_1^+ with s_0 to check this.

We will use this identification and a certain version of notations (2.10). Let

$$R_p^I(u,\varepsilon) \overset{\text{def}}{=} R_{i_p\ i_{p+1}}^{\varepsilon_p\ \varepsilon_{p+1}}(u_p - u_{p+1})\,, \qquad 1 \le p \le n\,, \tag{4.17a}$$

$$R_0^I(u,\varepsilon) \overset{\text{def}}{=} Q_{i_1}^{\varepsilon_1}(\tilde{u}_1) \quad \text{if} \quad \varepsilon_1 = +\,, \tag{4.17b}$$

$$R_n^I(u,\varepsilon) \overset{\text{def}}{=} Q_{i_n}^{\varepsilon_n}(\tilde{u}_n) \quad \text{if} \quad \varepsilon_n = -\,. \tag{4.17c}$$

The values of R_0^I or R_n^I are not defined for the opposite signs of $\varepsilon_{1,n}$. We do not impose any other conditions on $u = (u_1, \ldots, u_n)$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $I = (i_1, \ldots, i_n)$ [cf. (4.8), (4.10)]. Setting $t_p^\sigma(I) \overset{\text{def}}{=} I$ for any p, σ, we obtain an action of Ω on $\{\varepsilon\}$, $\{I\}$ [see (4.9) and (4.11)]. Finally, given $\omega \in \Omega$, we put ${}^\omega c = c$ for $c \in \mathbf{C}$,

$$\begin{aligned} {}^{\omega^{-1}}R_p^I(u,\varepsilon) &= R_p^{\omega(I)}(\omega(u),\omega(\varepsilon)) \\ {}^{\omega^{-1}}(R_p^I(u,\varepsilon)R_p^J(v,\gamma)) &= R_p^{\omega(I)}(\omega(u),\omega(\varepsilon))R_p^{\omega(J)}(\omega(v),\omega(\gamma)) \\ &\qquad \text{for other } \{v,\gamma,J,q\}, \text{ etc.} \end{aligned} \tag{4.18}$$

We come back to the geometric pictures. The arguments $\{u_i\}$ are the angles again, inequalities (4.8), (4.10) are fulfilled. One can describe an arbitrary elementary collision by
a) either s_p, $1 \leq p < n$, when the points with the coordinates x_p, x_{p+1} are intersecting,
b) or $s_0 = t_1^+$ when the point x_1 is reflected in the endpoint r,
c) or $s_n = t_n^-$ when x_n is reflected in l. We remind that the points $x_1 < \ldots < x_n$ (the positions of the particles) are numbered at the moment right after the corresponding collision.

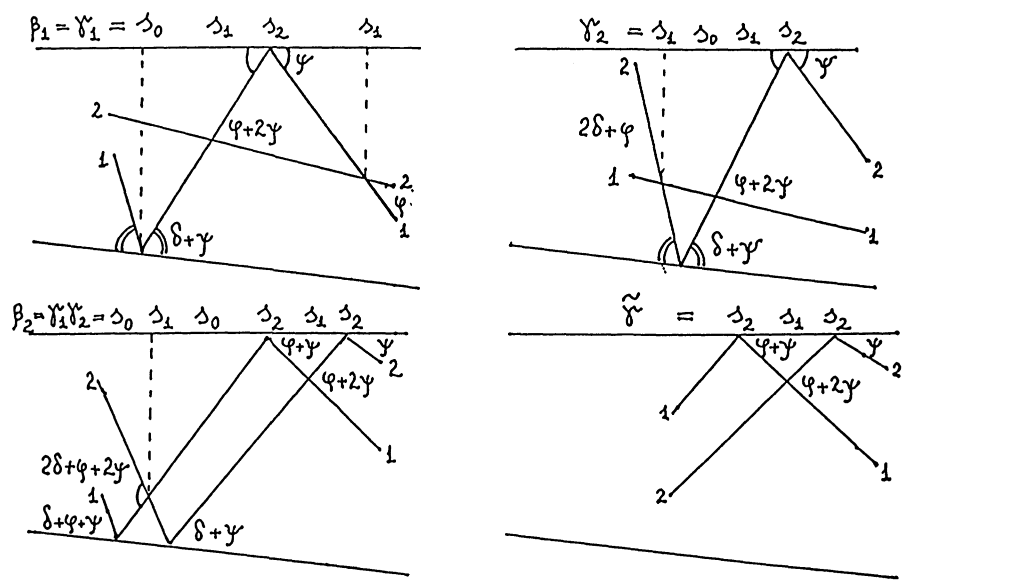

Fig. 1. Basic transformations for QKZ with two particles

Given in-out-states A'_{in}, A_{out} and a certain picture of the lines between them, we arrive at the corresponding set of the elements $s_{p_l}, \ldots, s_{p_2}, s_{p_1}$ (see Fig. 1). Here $0 \leq p_r \leq n$, $1 \leq r \leq l$. In particular, s_{p_1} describes the last collision just before A_{out}, s_{p_l} is assigned to the first collision (the intersection or the reflection) right after A'_{in}. We introduce

$$\omega = s_{p_l} \ldots s_{p_2} s_{p_1} \in \Omega \tag{4.19}$$

and claim (see [Ch1, Ch5]) that
a) this product depends on the multi-symbols A'_{in}, A_{out} only (the concrete choice of the initial positions of the particles does not matter),
b) (4.19) is a reduced decomposition of ω with respect to $\{s_i, 0 \leq i \leq n\}$ in the sense of Definition 1.3 [in particular, l is equal the length $l(\omega)$ of ω],
c) arbitrary reduced decomposition of ω is associated with a proper set of the initial positions (i.e. with a certain picture of the lines between A'_{in} and A_{out}),
d) moreover, the element ω can be uniquely determined by the vector $\omega(u) = (\omega(u_1), \omega(u_2), \ldots, \omega(u_n))$ for generic $u_1, \ldots, u_n, \delta$.

Later on we will take $I = I_0 \stackrel{\text{def}}{=} (1, 2, , \ldots, n)$ considering

$$A_{\text{out}} = A^{I_0}\varepsilon(u) = A^1_{\varepsilon_1}(u_1) \ldots A^n_{\varepsilon_n}(u_n)$$

as the out-state, and use the simplified notations

$$R_p = R_p^{I_0}(u, \varepsilon), \qquad 0 \le p \le n. \tag{4.20}$$

The angles are supposed to be in general position ($|\delta|$ is rather small).

Theorem 4.2. *Let $\omega \in \Omega$ correspond to a certain process $A'_{\text{in}} = \mathfrak{R} A_{\text{out}}$ and relations* (4.12)–(4.15) *be imposed. Then*

$$\mathfrak{R} \stackrel{\text{def}}{=} \mathscr{R}_w = {}^{w^{-1}s_{p_l}} R_{p_l} \dots {}^{s_{p_1}s_{p_2}} R_{p_3} {}^{s_{p_1}} R_{p_2} R_{p_1} \tag{4.21}$$

does not depend on the positions of the particles, i.e. on the choice of reduced decomposition (4.19). *Moreover,*

$$\mathscr{R}_{xy} = {}^{y^{-1}} \mathscr{R}_x \mathscr{R}_y \quad \textit{if} \quad l(xy) = l(x) + l(y), \qquad x, y \in \Omega. \tag{4.22}$$

b) *The arguments (angles) of the R, Q-factors obtained in* (4.21) *by applying* (4.18) *are negative. They are pairwise distinct for generic $u_1, \dots, u_n, \delta$ and belong to Σ_+^a of type C_n (or A_1 for $n = 1$) with respect to the **Z**-homomorphism defined by the relations*

$$\alpha_p = u_p - u_{p+1}, \qquad 1 \le p < n, \qquad \alpha_n = 2u_n, \qquad \alpha_0 = 2(\delta - u_1). \tag{4.23}$$

This homomorphism is compatible with the action of Ω on $\{u_p\}$ and that of W^a on $\{\alpha_\imath\}$.
c) *Let us assume that $u_1 < u_2 < \dots < u_n < \delta$ and $\delta \to 0$ (i.e. δ is infinitesimal). Then $\varepsilon = \varepsilon_0 \stackrel{\text{def}}{=} (-, -, \dots, -)$ and the above angles constitute the set of all roots $\tilde{\alpha} = [\alpha, k] \in \Sigma^a$ such that $\alpha > 0$, $k \ge 0$ (see* (4.23) *and Sect. 1).*

Proof. These statements are easy to check geometrically (see [Ch5]). A formal algebraic deduction of a) from (4.12)–(4.15) is the same as for Theorem 2.3. Assertion a) gives that the arguments in (4.21) coincide with the indices of the R-factors from (2.13) for R_ω (in the case of C_n or A_1). This implies b), c). □

5. QKZ with Reflection

Now let us turn to QKZ. We keep the notations from Sect. 4 and fix $I = I_0$, $\varepsilon = \varepsilon_0$ (see Theorem 4.2): $A_{\text{out}} = A_{\varepsilon_0}^{I_0}(u)$. The condition $\varepsilon = \varepsilon_0$, implies that $\{u_1, \dots, u_n\} < 0$. Let us introduce certain special elements $\omega \in \Omega$ and the corresponding R-matrices $\mathscr{R}_\omega$.

Given $1 \le i \le n$, we suppose that the i^{th} particle from the out-state intersected the other one (each two times) and was reflected in the both endpoints with no other interactions (intersections or reflections) of the particles. Let us denote the corresponding element ω by γ_i. It takes $u_{\text{out}} = (u_1, \dots, u_n)$ to $u_{\text{in}} = (u_1, \dots, u_{i-1}, 2\delta + u_i, u_{\imath+1}, \dots, u_n)$. This description is equivalent to the above geometric explanation. We remind that Ω acts on the angles (types, signs) of the out-state (not on those of the in-state). Having associated the corresponding group element with this process, one can forget about the geometric pictures and use the formal machinery from Theorem 4.2. However the graphic interpretation is very convenient to see that

$$\begin{aligned}\mathscr{R}_{\gamma_i} = {} & R_{ii-1}^{--}(2\delta + u_i - u_{i-1}) \dots R_{i\,1}^{--}(2\delta + u_i - u_1) Q_i^{+}(2\delta + 2u_\imath) \\ & \times R_{1\,i}^{-+}(u_i + u_1) \dots R_{n\,i}^{-+}(u_i + u_n) Q_i^{-}(2u_i) \\ & \times R_{i\,n}^{--}(u_i - u_n) \dots R_{ii+2}^{--}(u_i - u_{i+2}) R_{ii+1}^{--}(u_i - u_{\imath+1}).\end{aligned} \tag{5.1}$$

Here and further $R_{i\,i}^{-+} = 1$ by definition. Geometrically, the required chain of interactions takes place if $\{u_j,\, j \neq i\}$ "almost" coincide and u satisfies the conditions $u_i < u_j$ for each $j \neq i$ (see Fig. 1, where $n = 2$, $\phi = u_1 - u_2$, $\psi = u_2$).

The next element $\tilde{\gamma} \in \Omega$ is the transition

$$(u_1, \ldots, u_n)_{\text{out}} \to (-u_n, -u_{n-1}, \ldots, -u_1)_{\text{in}}\,.$$

Assume that $\{u_i\}$ are close enough to each other to draw the picture. The formal definition of the corresponding R-matrix is as follows:

$$\mathscr{R}_{\tilde{\gamma}} = \prod_{i=1}^{n} (R_{1\,i}^{-+}(u_i + u_1) \ldots R_{i-1\,i}^{-+}(u_i + u_{i-1}) Q_i^{-}(2u_i))\,, \tag{5.2}$$

where the index i increases from left to right ($\mathscr{R}_{\tilde{\gamma}} = Q_{1\,2}^{-+} \ldots$).

The last set of elements $\{\beta_i,\, 1 \leq i \leq n\} \subset \Omega$ geometrically correspond to the conditions $\{u_1, \ldots, u_i\} < \{u_{i+1}, \ldots, u_n\}$, where the angles in each of these two groups "almost" coincide (see Fig. 1). The transformations are as follows:

$$\beta_i = \gamma_1 \ldots \gamma_i : (u_1, \ldots, u_n)_{\text{out}} \to (2\delta + u_1, \ldots, 2\delta + u_i, u_{i+1}, \ldots, u_n)_{\text{in}}\,. \tag{5.3}$$

Here the corresponding R-matrices are:

$$\begin{aligned}\mathscr{R}_{\beta_i} = &\prod_{j=1}^{i} (R_{j\,1}^{-+}(2\delta + u_j + u_1) \ldots R_{j\,j-1}^{-+}(2\delta + u_j + u_{j-1}) Q_j^{+}(2\delta + 2u_j)) \\ &\times [R_{i+1\,1}^{-+}(u_{i+1} + u_1) \ldots R_{n\,1}^{-+}(u_n + u_1)] \ldots \\ &[R_{i+1\,i}^{-+}(u_{i+1} + u_i) \ldots R_{n\,i}^{-+}(u_n + u_i)] \\ &\times \prod_{j=1}^{i} (R_{1\,j}^{-+}(u_j + u_1) \ldots R_{j-1\,j}^{-+}(u_j + u_{j-1}) Q_j^{-}(2u_j)) \\ &\times [R_{1\,n}^{--}(u_1 - u_n) \ldots R_{1\,i+1}^{--}(u_1 - u_{i+1})] \ldots \\ &[R_{i\,n}^{--}(u_i - u_n) \ldots R_{i\,i+1}^{--}(u_i - u_{i+1})]\,.\end{aligned}$$

We note that the elements $\{\gamma_i, \beta_j,\, 1 \leq i, j \leq n\}$ do not change $I = I_0$ and $\varepsilon = \varepsilon_0$. They act as certain shifts of the arguments $\{u_i\}$. Hence they are commutative. We arrive at the following identities, where the arguments can be arbitrary real or complex (from a certain domain).

Theorem 5.1. a) *In the above notations,* $l(\beta_i\beta_j) = l(\beta_i) + l(\beta_j)$ *for* $1 \leq i, j \leq n$ *and*

$$\mathscr{R}_{\beta_i\beta_j}(u) = \mathscr{R}_{\beta_i}(\beta_j(u))\mathscr{R}_{\beta_j}(u) = \mathscr{R}_{\beta_j}(\beta_i(u))\mathscr{R}_{\beta_i}(u)\,. \tag{5.4}$$

b) *If the following (unitarity) conditions*

$$R_{i\,j}^{--}(\phi) R_{j\,i}^{--}(-\phi) = 1 \quad \textit{for any } \phi, i \neq j\,, \tag{5.5}$$

are imposed, then $R_{i\,j}^{++}(\phi) R_{j\,i}^{++}(-\phi) = 1$ *due to* (4.14) *(or* (4.15)*) and*

$$\mathscr{R}_{\gamma_i\gamma_j}(u) = \mathscr{R}_{\gamma_i}(\gamma_j(u))\mathscr{R}_{\gamma_j}(u) = \mathscr{R}_{\gamma_j}(\gamma_i(u))\mathscr{R}_{\gamma_i}(u)\,, \tag{5.6}$$

$$\mathscr{R}_{\beta_i}(u) = \mathscr{R}_{\gamma_i}(\gamma_{i-1} \ldots \gamma_1(u)) \ldots \mathscr{R}_{\gamma_2}(\gamma_1(u))\mathscr{R}_{\gamma_1}(u)\,, \qquad 1 \leq i, j \leq n\,. \tag{5.7}$$

c) *Let us assume that*

$$R_{i\,j}^{--}(\phi) = R_{j\,i}^{++}(\phi)\,, \qquad R_{i\,j}^{-+}(\phi) = R_{j\,i}^{-+}(\phi)\,, \qquad Q_i^{+}(\phi) = Q_i^{-}(\phi) \tag{5.8}$$

for arbitrary ϕ, $i \neq j$. We introduce the automorphism

$$\beta_0 = \gamma_0 : (u_1, \ldots, u_n)_{\text{out}} \to (u_1 + \delta, \ldots, u_n + \delta)_{\text{in}}$$

and put $\mathscr{R}_{\beta_0}(u) = \mathscr{R}_{\gamma_0}(u) \stackrel{\text{def}}{=} \mathscr{R}_{\tilde{\gamma}}(u)$. *Note that* $\gamma_0 \neq \Omega$. *The relations* (5.4) *are valid for* $0 \leq i, j \leq n$. *The same holds true for* (5.6) *if conditions* (5.5) *are fulfilled. One has*

$$\mathscr{R}_{\beta_n}(u) = \mathscr{R}_{\gamma_0}(\gamma_0(u))\mathscr{R}_{\gamma_0}(u) = \mathscr{R}_{\gamma_n}(\gamma_{n-1} \cdots \gamma_1(u)) \ldots \mathscr{R}_{\gamma_1}(u)\,, \tag{5.9}$$

where the second equality is valid for unitary $R_{i\,j}^{-\,-}$ *only.*

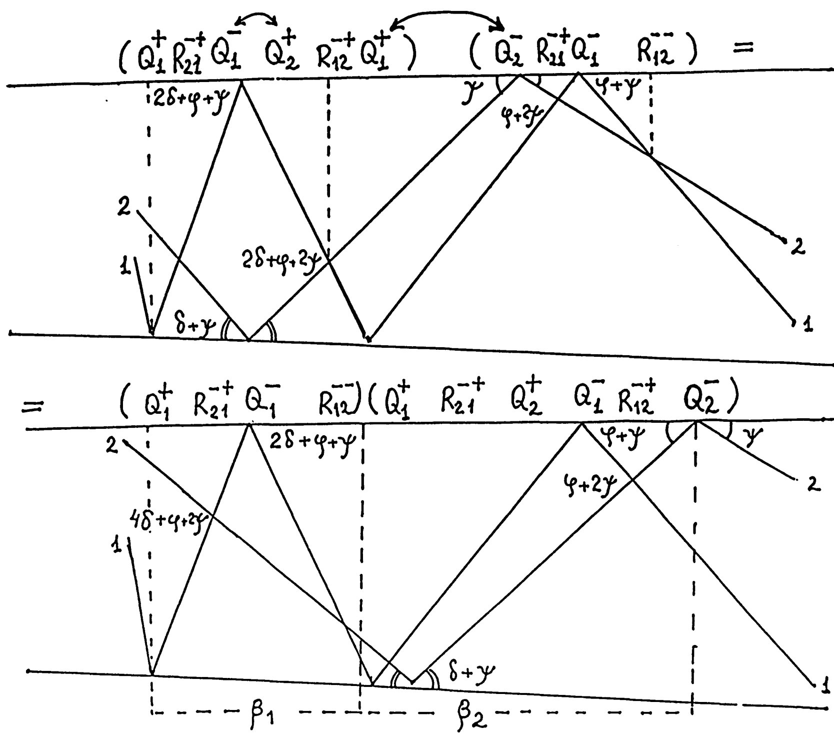

Fig. 2. A graphic of the "commutativity" of $\mathscr{R}_{\beta_1}$ and $\mathscr{R}_{\beta_2}$

Proof. The relation $l(\beta_i\beta_j) = l(\beta_i) + l(\beta_j)$ is clear geometrically (the conditions for u which ensure the corresponding processes are pairwise compatible for different β_i – see Fig. 2). It can be deduced from Proposition 1.6 as well (cf. Theorem 2.4, c)). Formula (5.4) results from Theorem 4.2. The particular case of this formula when $n = 2$ is in Fig. 2. This reasoning does not work for $\{\gamma_i\}$ since $l(\gamma_i\gamma_j) \neq l(\gamma_i) + l(\gamma_j)$ for $i \neq j$. Indeed, there is no graphic representation of $\gamma_i\gamma_j$ extending that of γ_i. However $\mathscr{R}_{\gamma_i\gamma_j}(u)$ can be obtained from $\mathscr{R}_{\gamma_i}(\gamma_j(u))\mathscr{R}_{\gamma_j}(u)$ by transformations (4.12)–(4.15) together with cancellations of certain pairs $R_{k\,l}^{-\,-}(\phi)R_{l\,k}^{-\,-}(-\phi)$ (see Fig. 2). To check c) we add γ_0 to Ω (in the group of automorphisms of $\mathbf{R}^n$ or $\mathbf{C}^n$) and extend $\mathscr{R}_\omega$ to this bigger group by the relation $\mathscr{R}_{\gamma_0\tilde{\gamma}^{-1}} = 1$ (cf. Theorem 2.4). Another (and the most convenient) way is by means of the correponding pictures. □

Definition 5.2. *The* QKZ *equation (or simply* QKZ*) with reflection is the following self-consistent system for a $\mathscr{F}_0$-valued function $\Phi(u)$:*

$$\Phi(\beta_i(u)) = \mathscr{R}_{\beta_i}(u)\Phi(u)\,, \qquad 1 \le i \le n\,. \tag{5.10}$$

If conditions (5.5) *are imposed, then (5.10) is equivalent to the system*

$$\Phi(\gamma_i(u)) = \mathscr{R}_{\gamma_i}(u)\Phi(u)\,, \qquad 1 \le i \le n\,. \tag{5.11}$$

One may add the equation

$$\Phi(\gamma_0(u)) = \mathscr{R}_{\gamma_0}(u)\Phi(u)$$

to these systems if relations (5.8) *(with* (5.5) *for* (5.11)*) are valid.* □

To discuss the connections with QKZ from Sect. 3 we postulate the following symmetries [see (5.8)]:

$$R_{i\ j}^{-\,-}(\phi) = R_{j\ i}^{+\,+}(\phi) \stackrel{\mathrm{def}}{=} R_{ij}(\phi)\,, \qquad R_{i\ j}^{-\,+}(\phi) = R_{j\ i}^{-\,-}(\phi) \stackrel{\mathrm{def}}{=} \hat{R}_{ij}(\phi)\,. \tag{5.12}$$

The QKZ equations for the classical root systems of type A_{n-1}, B_n, C_n, D_n correspond to the following four reductions of the above systems.

(A) Let us suppose that $\hat{R}_{ij} = 1 = Q_i^- = Q_i^+$ for $1 \le i \ne j \le n$. Then R has to satisfy the commutativity relations $[R_{ij}(\phi), R_{kl}(\psi)] = 0$ for the pairwise distinct indices [see (4.12)] and the Yang-Baxter equations (4.13) without the signs. System (5.11) in this case is, in fact, from [FR]. We should mention that it is self-consistent only for unitary R [see (5.5) in contrast with (5.10)].

(B) Let $Q^+ = 1$, $[R_{ij}(\phi), \hat{R}_{ij}(\phi)] = 0$. Relations (4.12)–(4.14) are also imposed. Then (4.15) is fulfilled identically. Here (like in the previous case) $\mathscr{R}_{\beta_i}$ correspond to $R_{b_i'}$ determined for B_n when $1 \le i \le n$.

(C) We assume that $Q_i^-(\phi) = Q_i^+(\phi)$. This case was considered in Theorem 5.1, c). The relations (4.14) and (4.15) coincide. System (5.9) with $0 \le i \le n$ is a particular case of (3.1) for C_n. Substitute $R_{b_n'}$ for r_{β_0} and $R_{b_i'}$ for $\mathscr{R}_{\beta_i}$, when $1 \le i < n$, to see it.

(D) This case is the intersection of B and C: $Q^+ = 1 = Q^-$, $[R_{ij}(\phi), \hat{R}_{ij}(\psi)] = 0$. Here $R_{b_n'}$ corresponds to $\mathscr{R}_{\beta_0}$, $R_{b_{n-1}'}$ to $\mathscr{R}_{s_0\gamma_n^{-1}s_0\gamma_0}$, $R_{b_i'}$ to $\mathscr{R}_{\beta_i}$ if $1 \le i < n-1$; $R_{b_j'}$ are defined for the root system D_n.

To be more precise, systems A, B, C, D can be obtained from (3.2) when the algebra of $\mathscr{F}_0$-valued functions of u with the above action of B by the shifts of the arguments is considered as $\mathscr{F}$ from Definition 2.2. To connect $\mathscr{R}_\omega$ and R_ω (e.g. $\mathscr{R}_{\beta_i}$ and $R_{b_i'}$) we replace $R_{ij}(u)$ by $R_{\gamma_i-\gamma_j}$, $\hat{R}_{ij}(u)$ by $R_{\gamma_i+\gamma_j}$, $Q_i^-(u)$ by either R_{γ_i} for B or $R_{2\gamma_i}$ for C. Here we identify the union of $\{\gamma_i \pm \gamma_j,\ 1 \le i < j \le n\}$ and $\{c\gamma_i,\ 1 \le i \le n\}$ with the set Σ_+ of positive roots either for B_n ($c=1$) or C_n ($c=2$) or D_n ($c=0$); $\{\gamma_i - \gamma_j,\ 1 \le i < j \le n\} = \Sigma_+$ in the case of A_{n-1} (see [B]).

As a certain application, we will prove the main property of the monodromy matrix for the particles with reflection (it has nothing to do with the monodromy cocycle discussed in Sect. 4 and below). The definition is as follows:

$$\begin{aligned}\mathscr{T}_1(u) = \mathscr{T}_1^{3\cdot n}(u) &\stackrel{\mathrm{def}}{=} (Q_1^+(2\delta + 2u_1)R_{1\ 2}^{-\,+}(u_1+u_2))^{-1}\mathscr{R}_{\gamma_1}(u)\\ &= R_{3\ 1}^{-\,+}(u_1+u_3)\dots R_{n\ 1}^{-\,+}(u_1+u_n)Q_1^-(2u_1)\\ &\quad\times R_{1\ n}^{-\,-}(u_1-u_n)\dots R_{1\ 3}^{-\,-}(u_1-u_3)\,,\end{aligned} \tag{5.13}$$

where (5.5) is imposed. Then (5.6) for $i = 1$, $j = 2$ gives the relation:

$$\begin{aligned} &R_{2\,1}^{++}(u_1 - u_2)\mathscr{T}_1^{3\cdot n}(u)R_{1\,2}^{-+}(u_1 + u_2)\mathscr{T}_2^{3\cdot n}(u) \\ &= \mathscr{T}_2^{3\cdot n}(u)R_{2\,1}^{-+}(u_1 + u_2)\mathscr{T}_1^{3\cdot n}(u)R_{1\,2}^{--}(u_1 - u_2)\,, \end{aligned} \tag{5.14}$$

where $\mathscr{T}_2^{3\cdot n}$ is defined by the right-hand side of the same formula (5.13) for the index 2 instead of 1 in all expressions.

In case A (when $R^{-+} = 1 = Q^{\pm}$, $R_{i\,j}^{--} = R_{ij} = R_{j\,i}^{++}$), (5.14) is the well-known formula for the monodromy matrix (due to Yang, Baxter, Faddeev and others – see e.g. [F]). Usually, $\mathscr{F} = M_{N_1} \otimes \ldots \otimes M_{N_n}$ for $N_1, \ldots, N_n \in \mathbf{N}$ and $R_{ij}(\phi)$ take values in $M_{N_i} \otimes M_{N_j}$. Then (5.14) immediately results in the commutativity relation $[T_1(u), T_2(u)] = 0$ for the *transfer matrix* $T_i(u) = \mathrm{Sp}_i(\mathscr{T}_i(u))$. Here Sp_i is the trace for the i_{th} component M_{N_i}, $i = 1, 2$, the function T_i depends on $u_i, u_3, \ldots, u_n$, and takes its values in the tensor product of the component $3, \ldots, n$.

Relations (5.14) in case D were considered by Sklyanin [S]. They play an important role in the recent paper [O] by Olshansky devoted to the construction of Yangians (see [D]) for relation (4.14). See [CG] for some interpretation of the latter in terms of open strings, [KS] about a generalization of the Pasquier and Saleur approach to the Hamiltonian of the XXZ-model with certain linear terms (via the same relation) and a recent Noumi paper on q-symmetric spaces [N]. We will not discuss these and other applications here. However one point is worth mentioning.

Given $R_{k\,l}^{-+}, R_{k\,l}^{--}, R_{k\,l}^{++}, Q_k^-$ for $1 \le k, l \le n{+}m$ satisfying relations (4.12)–(4.14), we can construct other Q. Let us use the notations $1', 2', \ldots, m'$ for $n{+}1, \ldots, n{+}m$ and put $\tilde{Q}_i^- = \mathscr{T}_i^{1'\cdot m'}(u_i; u_{1'}, \ldots, u_{m'})$, where $1 \le i \le n$, I_0 is replaced by $(i, 1', \ldots, m')$. Then $R_{i\,j}^{\pm\pm}$, $\tilde{Q}_i^-$ for $1 \le i, j \le n$ obey all the relations (4.12)–(4.13). Here $\{u_{1'}, \ldots, u_{m'}\}$ are considered as some extra parameters. The same can be done to produce new $\tilde{Q}^+$ by means of the counterpart of (5.13) for Q^+ instead of Q^-.

Let us specialize the definition of the monodromy representation for the considered systems [see (3.6)]. Given a certain action of $\mathbf{S}_n$ on $\mathscr{F}_0$, we impose (5.12) and assume that there are four $\mathscr{F}_0$-valued functions R, $\hat{R}$, Q, $\bar{Q}$ of $\phi \in \mathbf{C}$ such that

$$R_{ij}(\phi) = w(R(\phi))\,, \quad \hat{R}_{ij}(\phi) = w(\hat{R}(\phi)) \quad \text{if} \quad w = (i, j, \ldots)\,, \tag{5.15}$$

$$Q_i^-(\phi) = w(Q(\phi)) \qquad Q_i^+(\phi) = w(\bar{Q}(\phi)) \quad \text{if} \quad w = (i, \ldots)\,. \tag{5.16}$$

Here $1 \le i, j \le n$ and we use the so-called one-line notations for $w \in \mathbf{S}_n$.

Relations (4.12)–(4.15) for $i = 1$, $j = 2$, $k = 3$, $l = 4$ imply those for the other indices because of (5.15)–(5.16). We postulate them together with the unitary condition $R_{12}(\phi)R_{21}(-\phi) = 1$ [see (5.5)], ensuring the equivalence of (5.4) and (5.6). Let Φ be an invertible $\mathscr{F}_0$-valued solution of the latter. If $Q = \bar{Q}$, $[R(\phi), \bar{R}(\psi)] = 0$ (cases C or D) than $i = 0, 1, \ldots, n$. Otherwise $1 \le i \le n$.

Corollary 5.3. a) *The monodromy functions*

$$T_w(u) \stackrel{\mathrm{def}}{=} w^{-1}(\Phi^{-1}(w(u))R_w(u)\Phi(u) \quad \textit{for} \quad w \in \mathbf{S}_n \tag{5.17}$$

are 2δ-periodic (i.e. $T_w(\gamma_i(u)) = T_w(u)$, $i = 1, \ldots, n$) and, moreover, satisfy the additional relation $T_w(\gamma_0(u) = T_w(u)$ in cases C, D. *The cocycle conditions $T_{xy} = y^{-1}(T_x)T_y$ are valid for any $x, y \in \mathbf{S}_n$ (see* 3.7)).

b) *Let the algebra $\mathscr{F}_0$ be semi-simple and finite-dimensional. Then there exists a $\mathscr{F}_0^*$-valued 2δ-periodic function $F(u)$ ($F_w(\gamma_0(u) = F_w(u)$ for* C, D*) such that the transformed solution $\tilde{\Phi} = \Phi F$ of* (5.4)–(5.6) *has the trivial monodromy: $\tilde{T}_w = 1$, for $w \in \mathbf{S}_n$.* □

We conclude this paper with the following general remark. One can extend the construction of QKZ to any group G. Let us fix a set $\{s_1, s_2, \dots, s_n\}$ of its generators (maybe $n = \infty$). Given an algebra $\mathscr{F}$ with an action of G, we need a $\mathscr{F}$-valued R-matrix that is a set $\{R_w,\, w \in G\}$ satisfying the conditions

$$R_x y = {}^{y^{-1}}R_x R_y \quad \text{if} \quad l(xy) = l(x) + l(y)\,,$$

where the length is defind relative to $\{s_i\}$. Then an arbitrary set $\{a_1, a_2, \dots, a_m\} \in G$ of pairwise commutative elements will give the corresponding QKZ if $l(a_i a_j) = l(a_i) + l(a)$ for $1 \le i, j \le m$. One finds a lot of examples. For instance, coboundaries $\{R_w = {}^{w^{-1}}FF^{-1},\, w \in G\}$ are R-matrices in the above sense for any $F \in \mathscr{F}^*$.

Practically, it is important to consider "homogeneous" G and R (this point could be argued). The main requirement is as follows. Let ${}^wR_s = R_{s_i}$ for arbitrary $w \in G$, $1 \le i \le n$ such that $ws_i = s_i w$ and $l(ws_i) = l(w) + l(s_i)$. The problem of getting R-matrices of this kind is much more delicate. All the examples are connected with remarkable mathematical (and physical) structures.

It was known for a time that the R-matrices in common use are cohomologically trivial, i.e. are coboundaries (see [Ch1]). Now we realize that the corresponding F are very important for many reasons. As for the basic examples, these F appear to be certain quantum counterparts of the n-point functions from the conformal field theory. They are closely connected with the representation theory of Kac-Moody algebras (and their q-deformations), the theory of q-special functions and (last but not the least) with integrable lattice models.

Appendix: Macdonald's Operators

We will apply the construction of Theorem 3.6 to the trigonometric R-matrix in the case A_{n-1}. In accordance with the notations from Sect. 5,

$$\Sigma = \{\alpha_{ij} \stackrel{\text{def}}{=} \gamma_i - \gamma_j,\, 1 \le i \ne j \le n\}\,, \qquad \alpha_i = \alpha_{ii+1}\,, \qquad 1 \le i < n\,. \tag{A.1}$$

Let us fix $q, \xi \in \mathbf{C}^*$. We introduce the field $\mathbf{C}(Z) = \mathbf{C}(Z_1, \dots, Z_n)$ of rational functions in $Z_1, \dots, Z_n$ equiped with the natural action of $\mathbf{S}_n$ [$s_{ij} \stackrel{\text{def}}{=} (ij) \in \mathbf{S}_n$ transpose Z_i and Z_j]. Let

$$\Gamma_i(Z_j) = \delta_{ij} Z_j \xi\,, \qquad 1 \le i, j \le n\,. \tag{A.2}$$

The elements $\{Z_i\}$ are identified with the corresponding linear operators $Z_i(p) = Z_i p$ for $p \in \mathbf{C}(Z)$, $1 \le i \le n$.

Formula (3.18b) can be rewritten as follows:

$$G_{ij} = G_{ij}(q) = q + (q - q^{-1})(Z_i Z_j^{-1} - 1)^{-1} \circ (1 - s_{ij})\,, \qquad 1 \le i \ne j \le n\,. \tag{A.3}$$

One has [see (3.19)]:

$$G_{ij}^{-1} = G_{ji} - (q - q^{-1}) s_{ij} = G_{ij}(q^{-1})\,, \qquad 1 \le i \ne j \le n\,. \tag{A.4}$$

Proposition 3.5 and formula (5.4) result in

Corollary A.1. *The operators (cf.* (5.1)*)*

$$\mathscr{R}_i = \prod_{j=1}^{i-1} G_{ji}^{-1} \circ \Gamma_i \circ \prod_{j=i+1}^{n} G_{ij} \in \mathrm{End}_{\mathbf{C}}\, \mathbf{C}(Z) \tag{A.5}$$

are pairwise commutative for $1 \leq i \leq n$. $\square$

Let us introduce the operators

$$\begin{gathered} \mathscr{L}_m = \sum_{i=1}^{n} \mathscr{R}_i^m , \qquad \mathscr{M}_k = \sum_I \mathscr{R}_{i_1} \ldots \mathscr{R}_{i_k} , \\ I = \{1 \leq i_1 < i_2 < \ldots < i_k \leq n\} , \end{gathered} \tag{A.6}$$

where $m \in \mathbf{N}$, $1 \leq k \leq n$. We can represent them as follows [see (3.21)]:

$$\mathscr{L}_m = \sum_w D_w(m) \circ w \qquad \mathscr{M}_k = \sum_w D'_w(k) \circ w , \qquad w \in \mathbf{S}_n , \tag{A.7}$$

where $D_w(m), D_w(k) \in \mathscr{D} \overset{\text{def}}{=} \mathbf{C}(Z) \circ \mathbf{C}[\Gamma] \in \mathrm{End}_{\mathbf{C}}(Z)$ for the algebra $\mathbf{C}[\Gamma] = \mathbf{C}[\Gamma_1, \ldots , \Gamma_n]$ of polynomials in $\Gamma_1, \ldots , \Gamma_n$.

Corollary A.2. *The following operators*

$$L_m = \sum_w D_w(m) , \qquad M_k = \sum_w D'_w(k) \in \mathscr{D} , \quad w \in \mathbf{S}_n , \tag{A.8}$$

are pairwise commutative and belong to $\mathscr{D}^{\text{inv}} \overset{\text{def}}{=} \{D \in \mathscr{D}, w \circ D \circ w^{-1} = D\}$, *where* $w \in \mathbf{S}_n$. *If* ξ *is not a root of unity, then* $\{L_k\}$ *(or* $\{M_k\}$*) are algebraically independent over* $\mathbf{C}(Z)$ *for* $1 \leq k \leq n$ *and generate the commutative subalgebra* $\{D \in \mathscr{D}^{\text{inv}}, [D, L_2] = 0\}$. *Moreover,*

$$\begin{aligned} L_2 &= \sum_i l_i \circ \Gamma_i^2 + \sum_{i<j} l_{ij} \circ \Gamma_i \Gamma_j , \qquad 1 \leq i, j \leq n , \\ l_i &= q^{2(1-n)} \prod_{r \neq i} (Z_i \xi q^2 - Z_r)(Z_i q^2 - Z_r)(Z_i \xi - Z_r)^{-1}(Z_i - Z_r)^{-1} , \\ l_{ij} &= q^{2(1-n)}(q^2 - 1)(\xi + 1)(q^2 - \xi) Z_i Z_j (Z_i \xi - Z_j)^{-1}(Z_j \xi - Z_i)^{-1} \\ &\quad \times \prod_{r \neq i,j} (Z_i q^2 - Z_r)(Z_j q^2 - Z_r)(Z_i - Z_r)^{-1}(Z_j - Z_r)^{-1} , \\ &\quad 1 \leq r \leq n . \quad \square \end{aligned} \tag{A.9}$$

Conjecture A.3*. *The operators* M_k *for* $1 \leq k \leq n$ *coincide with Macdonald's operators (see e.g.* [M]*):*

$$\begin{gathered} M_k = \sum_I m_I \Gamma_I , \quad \textit{where} \quad \Gamma_I = \Gamma_{i_1} \ldots \Gamma_{i_k} \\ I = \{1 \leq i_1 < i_2 < \ldots < i_k \leq n\} , \\ m_I = q^{k(k-n)} \prod_{r,j} (Z_{i_r} q^2 - Z_j)(Z_{i_r} - Z_j)^{-1} , \\ \textit{for} \quad j \notin I, 1 \leq r \leq k . \quad \square \end{gathered} \tag{A.10}$$

* Note added in proof. Proved.

The conjecture was checked for $m = 1, 2$ and for arbitrary m when $n \leq 5$ (by computer). The above construction can be extended to the $\mathscr{R}$-operators from Sect. 5 in a natural way. One arrives at a certain family of invariant scalar difference operators of BC-type depending on four parameters. We will give some formulas for the coefficients of the (pairwise commutative invariant) counterparts of $\{M_k\}$ in this case without going into detail. They are verified for small n only. We consider the rational case (like in Theorem 3.6) for the sake of simplicity. Let $\delta, \kappa, \kappa_1, \kappa_2 \in \mathbf{C}$. Then

$$\begin{aligned}\tilde{M}_k = \sum_{p,\varepsilon,I} m_I^\varepsilon \gamma_I^\varepsilon\,, \quad \gamma_I^\varepsilon = (\gamma_{i_1})^{\varepsilon_1} \ldots (\gamma_{i_p})^{\varepsilon_p}\,, \quad \varepsilon = \{\varepsilon_r,\, 1 \leq r \leq p\}\,, \quad \varepsilon_r = \pm 1\,,\\
I = \{1 \leq i_1 < i_2 < \ldots < i_p \leq n\}\,, \quad 0 \leq p \leq k\,,\\
\gamma_i(z_j) = z_j + 2\delta\delta_{ij}\,, \quad 1 \leq i, j \leq n\,, \qquad \text{(A.11)}\\
m_I^\varepsilon(p = k) = \prod_r ((z_{i_r}\varepsilon_r + \kappa_1)(z_{i_r}\varepsilon_r + \kappa_2)(z_{i_r}\varepsilon_r + \delta)^{-1}(z_{i_r}\varepsilon_r)^{-1}\\
\times \prod_{0<s<r} \{(z_{i_r}\varepsilon_r + z_{i_s}\varepsilon_s + 2\kappa)(z_{i_r}\varepsilon_r + z_{i_s}\varepsilon_s + 2\kappa + 2\delta)\\
\times (z_{i_r}\varepsilon_r + z_{i_s}\varepsilon_s)^{-1}(z_{i_r}\varepsilon_r + z_{i_s}\varepsilon_s + 2\delta)^{-1}\}\\
\times \prod_{j \notin I} \{(z_{i_r}\varepsilon_r - z_j + 2\kappa)(z_{i_r}\varepsilon_r + z_j + 2\kappa)\\
\times (z_{i_r}\varepsilon_r - z_j)^{-1}(z_{i_r}\varepsilon_r + z_j)^{-1}\})\,.\end{aligned}$$

If either $\kappa_1 = \delta$ or $\kappa_2 = \delta$, then (conjecturally) $m_I^\varepsilon = 0$ when $p = k - 1$. The formulas for $m_I^\varepsilon, p < k - 1$ seem rather complicated.

We mention that L_2 from (A.9) is expected to be radical part of the Laplace operator on the q-symmetric space $(GL_n(\mathbf{R})/O_n(\mathbf{R}))_q$ for a certain choice of ξ. As for L_1, the corresponding statement was checked by Noumi. The trigonometric counterparts of operators (A.11) should coincide with the invariant q-operators of BC-type for suitable values of the parameters and be connected with a recent Koornwinder construction [K]. The same conjectures can be put forward for arbitrary q-symmetric spaces.

We will conclude this appendix with the following formula for the "quasi-classical" limit of $L' \stackrel{\text{def}}{=} L_2 - 2L_1 + n$. Let $\xi = \exp(h)$, $q = \exp(h\kappa)$. Then

$$\lim_{h \to 0}(h^2 L') = \sum_{i=1}^{n} \partial_i^2 + 2K \sum_{i<j} (Z_i + Z_j)(Z_i - Z_j)^{-1}(\partial_i - \partial_j) + c_n\,, \qquad \text{(A.12)}$$

where $\partial_i = Z_i \partial/\partial Z_i$ and the constant term c_n is equal to $2\binom{n+1}{3}k^2$.

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