Commun. Math. Phys. 148, 487–502 (1992)

Communications in
Mathematical
Physics

The Index of the Scattering Operator on the Positive Spectral Subspace

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Received July 13, 1991; in revised form February 21, 1992

Abstract. We construct the scattering operator for a spinor field in a time dependent background by the Dyson expansion. Then we show that the restriction of the scattering operator to the positive spectral subspace (with respect to a reference Hamiltonian) is Fredholm. The computation of the index of this restriction is reduced to the index computation for an elliptic pseudodifferential operator of order zero. We obtain the index in terms of a cohomological formula by means of the Atiyah-Singer index theorem.

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1. Introduction

The scattering operator Ω_- describes how the time evolution of a field governed by a time dependent Hamiltonian $H(t)$ behaves in comparison with an evolution given by a constant reference Hamiltonian H_0. The operator Ω_- maps the space of incoming states with respect to H_0 to the incoming states of $H(t)$. $\Omega_-\phi =: \psi$ is given, roughly speaking, by the condition that $\phi(t) \approx \psi(t)$ as t tends to $-\infty$. $\psi(t)$, $\phi(t)$ are the backward time evolutions of $\psi = \psi(0)$, $\phi = \phi(0)$ with respect to $H(t)$, H, respectively. The scattering operator exists if H_0 and $H(t)$ are close enough for small t. Analogously, one defines Ω_+ using the forward evolutions. The scattering matrix is then defined as $\Omega := \Omega_+^* \Omega_-$.

Let P be the projection onto the positive spectral subspace of H_0. For constructing a second quantized theory it is interesting (see [5]) to know how much of the negative spectrum of H_0 is mapped by Ω into the positive and vice versa. This is measured by the index of $P\Omega P$ on the image of P. If H_0 has non-absolute continuous spectrum one has to restrict the considerations to the absolute continuous subspace.

Under some assumptions we show the Fredholm property of $P_{\mathrm{ac}} P \Omega P P_{\mathrm{ac}}$. Here we use the following simple fact: If U is an unitary operator modulo compact operators, i.e. $UU^* - 1$, $U^*U - 1$ are compact, and $[U, P]$ is compact then PUP is a Fredholm operator on $\operatorname{im} P$ with a parametrix PU^*P because of

$$PU^*PPUP = PU^*UP + PU^*[P, U]P = P + \text{compact}, \tag{1}$$

$$PUPPU^*P = PUU^*P + P[U, P]U^*P = P + \text{compact}. \tag{2}$$

As in [5] we introduce families of gauge transformations $W_\pm(t)$ defined for $t \geq T_+$, $t \leq T_-$, respectively, which measure the behaviour of $H(t)$ for large $|t|$. This allows us to express the index in question by the index of a simpler operator

$$\operatorname{ind}_{\operatorname{im} P_{\mathrm{ac}} P} P_{\mathrm{ac}} P \Omega P P_{\mathrm{ac}} = -\operatorname{ind}_{\operatorname{im} P} PWP, \tag{3}$$

where $W := W_+^*(T_+) W_-(T_-)$. This formula is proved by a deformation argument using a Dyson expansion representation of the scattering matrix Ω.

The abstract setting described above will arise from the following geometric objects:

- M – noncompact complete Riemannian spin manifold;
- $S(M)$ – spinor bundle;
- E, ∇^E – Hermitian vector bundle with connection;
- $H_0 := D_E$ – twisted Dirac operator on $L^2(M, S(M) \otimes E)$;
- $\{\imath\Gamma(t) \in \Gamma(M, \operatorname{End}(E))\}_{t \in \mathbb{R}}$ – smooth family of selfadjoint operators;
- $\{\nabla^E(t)\}_{t \in \mathbb{R}}$ – smooth family of connections in E;
- $D(t)$ – twisted Dirac operator with respect to $\nabla^E(t)$;
- $H(t) := D(t) - \imath\Gamma(t)$.

We will formulate our Assumptions 1 ... 6 as boundedness and support conditions on these objects.

In Sect. 2 we will show how these things relate to the Dirac equation on a pseudo-Riemannian manifold $M \times \mathbf{R}$.

In view of (3) we have to compute

$$\operatorname{ind}_{\operatorname{im} P} PWP = \operatorname{ind}(1 - P + WP).$$

The index of operators of such type on compact manifolds has been considered by many authors. It could be interpreted as the pairing of the $K_1(C^\infty(M))$ class represented by W with the cyclic cocycle given by D_E, as spectral flow or as a KK-product. If M is compact then $1 - P + WP$ is an elliptic pseudodifferential operator and its index can be computed from its symbol by the Atiyah-Singer formula. If the dimension $n=2k+1$ of M is odd then the result is

$$\operatorname{ind}(1-P+WP)=(-1)^k\mathbf{CS}(W)\hat{\mathbf{A}}(M)[M], \tag{4}$$

where $\mathbf{CS}(W)$ is the Chern-Simons class associated to W. Let

$$\tilde{E}\to M\times\mathbf{S}^1$$

be the bundle obtained by glueing together the ends of the cylinder $M\times I$ and identifying the fibres by

$$\mathrm{pr}_1^* E_{|M\times\{0\}} \xrightarrow{W} \mathrm{pr}_1^* E_{|M\times\{1\}},$$

where $\mathrm{pr}_1: M\times I\to M$ is the projection onto the first factor. Then $\mathbf{CS}(W) = \mathrm{pr}_{1*}\mathbf{ch}(\tilde{E})$. It is represented by the differential form

$$CS(W):=\frac{\imath}{2\pi}\operatorname{Tr} W^*\nabla W\int_0^1 \exp\left(\frac{\imath(t-t^2)}{2\pi}W^*\nabla W W^*\nabla W\right)dt,$$

$$CS(W)_{2r-1}=2\frac{\imath^r}{(2\pi)^r}\frac{r!}{(2r)!}\operatorname{Tr}([W^*\nabla W]^{2r-1}).$$

Italic letters denote the differential forms while the corresponding cohomology classes are indicated by bold ones. For a similar computation see [2].

If the dimension of M is even then this index is zero.

If M is noncompact our assumptions assure that $1-P+WP$ is (modulo compact operators) a very special elliptic pseudodifferential operator, the symbol of which is 1 at the infinity of M. We show in Sect. 4.1 that there is a pseudodifferential operator A^+ on a compact manifold M^+ containing a large subset of M such that

$$\operatorname{ind}(1-P+WP)=\operatorname{ind}A^+.$$

Moreover, A^+ has the same symbol as $1-P^+ + W^+P^+$. Here P^+ is the positive spectral projection with respect to the twisted Dirac operator on M^+, and W^+ is some extension of $\tilde{W}(0)$, where $\{\tilde{W}(\tau)\}_{\tau=0}^1$ is a deformation of W such that $1-\tilde{W}(0)$ has compact support and $\tilde{W}(1)=W$. Hence the index of A^+ is given by the formula (4). Examining the supports one obtains

$$\operatorname{ind}(1-P+WP)=(-1)^k CS(\tilde{W})\hat{A}(M)[M].$$

The main theorem of this paper, proved in Sect. 4.3, is

Theorem 1.1. *Suppose the Assumptions* 1 ... 6 *(see Sects. 3.1, 3.2, and 4.2) hold. Then*

$$\operatorname{ind}_{\operatorname{im}P} P_{\mathrm{ac}}P\Omega P P_{\mathrm{ac}} = -(-1)^k\mathbf{CS}(\tilde{W})\hat{\mathbf{A}}(M)[M],$$

where $\tilde{W}:=\tilde{W}(0)$ *and* $\dim(M)=2r+1$.

Note that $\mathbf{CS}(\tilde{W})$ is a cohomology class with compact support.

The special case $M=\mathbf{R}^3$ was considered in [5]. The present work was intended to extend the results of [5] to more general situations. In contrast to this reference, where a global pseudodifferential calculus and the Fedosov formula has been employed to compute the index, our method relies essentially on Rellich's theorem and the index formula of Atiyah-Singer.

We are grateful to T. Matsui for the discussion of a preliminary version of that paper and pointing out a technique used in Lemma 4.6. We thank the referee for showing us [2].

2. Geometrical Setting

2.1. The Spin Bundle. The scattering operators $\Omega_\pm$ considered in this article are associated to a scattering process of a spinor field in a time dependent background field. The spinor field satisfies the Dirac equation on a pseudo-Riemannian spin manifold $\tilde{M}$ of signature 1. We assume that $\tilde{M}$ has a decomposition into a product $\tilde{M}=\mathbf{R}\times M$ of the time axis $\mathbf{R}$ and a Riemannian spin manifold M. Then the Dirac equation can be viewed as an equation for a time dependent spinor field on M. First we present the structure of the spinor bundle $S(\tilde{M})$ of $\tilde{M}$.

Since the index under consideration will be trivial if $\dim M$ is even we restrict ourselves to manifolds M of odd dimension $n=2m+1$. Let $\pi:\tilde{M}\to M$ be the projection onto the second factor. Then the spinor bundle of $\tilde{M}$ admits a decomposition

$$S(\tilde{M})=\pi^*S(M)\oplus\pi^*S(M)=S^+\oplus S^-, \tag{5}$$

where π^* denotes the pull back. This identification respects the connection and the Hermitian metric. The decomposition into the plus and minus part is the usual one on even dimensional manifolds with the associated involution

$$\tau=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The tangent vectors of $\tilde{M}$ act by Clifford multiplication fibrewise on $S(\tilde{M})$. According to the decomposition (5) $X\in T\tilde{M}$ acts as the matrix

$$\begin{pmatrix}0 & X\\ X & 0\end{pmatrix}$$

and the unit vector in time direction ∂_t acts as

$$\begin{pmatrix}0 & \imath\\ -\imath & 0\end{pmatrix}.$$

We note that the square of ∂_t is 1. For definitions and proofs see [1].

2.2. Twisted Dirac Operators. The background field mentioned above is a connection $\tilde{\nabla}^{\tilde{E}}$ on a Hermitian vector bundle $\tilde{E}$ over $\tilde{M}$. We assume that $\tilde{E}$ is the pull back of a Hermitian vector bundle E over M. Then $\tilde{\nabla}^{\tilde{E}}$ gives rise to a family $\nabla^E(t)$ of connections of E and a family of antihermitian endomorphisms

$$\Gamma(t):=\left(\tilde{\nabla}^{\tilde{E}}_{\partial_t}-\frac{\partial}{\partial t}\right)_{|t}\in\Gamma(M,\mathrm{End}(E)).$$

Every connection on $\tilde{E}$, together with the Levi-Civita connection in $S(\tilde{M})$, induces a connection $\nabla^{\otimes}$ in $S(\tilde{M})\otimes\tilde{E}$ and a family of connections $\nabla^{\otimes}(t)$ in $S(M)\otimes E$. Furthermore, the Clifford bundle structure of the spinor bundle extends to the tensor product via the action on the first factor.

Let $D(t)$ be the twisted Dirac operator on $\Gamma(M, S(M)\otimes E)$ given locally in terms of an orthonormal frame of M by

$$D(t) := \sum_{i=1}^{n} X_i \nabla^{\otimes}_{X_i}(t).$$

Then the twisted Dirac operator $D_{\tilde{E}}$ on $\Gamma(\tilde{M}, S(\tilde{M})\otimes\tilde{E})$ is given by

$$D_{\tilde{E}} = \begin{pmatrix} 0 & \imath \\ -\imath & 0 \end{pmatrix} \left[\frac{\partial}{\partial t} + \Gamma(t)\right] + \begin{pmatrix} 0 & D(t) \\ D(t) & 0 \end{pmatrix}$$

according to the decomposition (5). We are interested in the space of solutions of $D_{\tilde{E}}\phi = 0$, where ϕ is a section of $S^+\otimes\tilde{E}$. Identifying sections of this bundle with time dependent sections of $S(M)\otimes E$ we get the equation:

$$\imath\frac{\partial}{\partial t}\phi = D(t)\phi - \imath\Gamma(t)\phi. \tag{6}$$

Below we will compare (6) with a time independent equation. Let ∇^E be a connection on E. Then it induces a connection $\nabla^{\tilde{E}}$ on $\tilde{E}$ and repeating the above construction we obtain the corresponding equation

$$\imath\frac{\partial}{\partial t}\phi = D_0\phi,$$

where D_0 is the twisted Dirac operator on $S(M)\otimes E$ associated to ∇^E. In the special case $M = \mathbf{R}^3$ considered in [5] E is the trivial bundle and ∇^E is the flat connection. In general there is no canonical choice of ∇^E. One has to choose it appropriately just by hand. In order to get through the analysis we will add further analytic assumptions to this geometric framework which we will describe later.

3. Construction of the Scattering Operators

3.1. The Propagator. The operators $H_0 := D_0$ and $H(t) := D(t) - \imath\Gamma(t)$, $t\in\mathbf{R}$, introduced in 2.2 are essentially selfadjoint on $L^2(M, S(M)\otimes E)$ with domain $C_c^\infty(M, S(M)\otimes E)$. The closures of these operators are denoted by the same symbols.

The operator family $\{H(t)\}$ generates a unique propagator $U(t,s)$ (cf. [7, Theorem 4.4.1]) if the domains of $H(t)$ are independent of t. The following assumption assures that $\mathrm{dom}(H(t)) = \mathrm{dom}(H_0)$.

Assumption 1. *The endomorphism-valued one-form $\check{\nabla}^{\tilde{E}} - \nabla^{\tilde{E}}$ satisfies*

$$\sup_{\tilde{M}} \|\check{\nabla}^{\tilde{E}} - \nabla^{\tilde{E}}\| < \infty.$$

The propagator is then characterized by

1. $U(t,s)$ is unitary and strongly continuous in t and s.
2. $U(t,s)U(s,u) = U(t,u)$, $t,s,u\in\mathbf{R}$.
3. $U(t,s)\,\mathrm{dom}\,H_0 = \mathrm{dom}\,H_0$.
4. $\imath\frac{\partial}{\partial t}U(t,s)\phi = H(t)U(t,s)\phi$ for $\phi\in\mathrm{dom}\,H_0$.
5. $-\imath\frac{\partial}{\partial s}U(t,s)\phi = U(t,s)H(s)\phi$ for $\phi\in\mathrm{dom}\,H_0$.

3.2. Asymptotic Constants. Representing the propagator $U(t,s)$ by a Dyson expansion as in [5] we have to adapt Lemma 3.4 (loc. cit.) to our situation. Let $W \in \Gamma(M, U(E))$ be a gauge transformation satisfying

$$|W-1| \in C_0(M), \tag{7}$$

where $C_0(M)$ are the continuous functions on M vanishing at infinity. W acts as a unitary multiplication operator on $L^2(M, S(M) \otimes E)$. Let P_{ac} be the projection onto the absolute continuous spectral subspace of H_0.

Lemma 3.1. *If the gauge transformation* W *satisfies* (7), *then*

$$s-\lim_{t\to\infty} e^{-\imath t H_0} W e^{\imath t H_0} P_{ac} = P_{ac}\,.$$

The analogous result holds for $t \to -\infty$.

Proof. Let $\chi_R(H_0)$ be the spectral projection of H_0 with respect to the interval $[-R, R]$, $R>0$. Then by Rellich's theorem $(W-1)\chi_R(H_0)$ is compact. Let $\phi \in \operatorname{im} P_{ac}$. For every $\varepsilon>0$ we can choose $R>0$ such that $\|(1-\chi_R(H_0))\phi\| \leqq \varepsilon/2$. Moreover, we have

$$w-\lim_{t\to\infty} e^{\imath t H_0}\phi = 0\,.$$

It follows

$$\lim_{t\to\infty} (W-1)\chi_R(H_0) e^{\imath t H_0}\phi = 0\,,$$

and hence

$$\lim_{t\to\infty} \|e^{-\imath t H_0} W e^{\imath t H_0}\phi - \phi\| \leqq \varepsilon\,.$$

Since ε can be made arbitrary small it follows

$$\lim_{t\to\infty} e^{-\imath t H_0} W e^{\imath t H_0}\phi = \phi\,.$$

This proves the lemma. □

3.3. Dyson Expansion of the Propagator. Having the propagator $U(t,s)$, we define the scattering operators $\Omega_\pm$ and the scattering matrix Ω by

Definition 3.2.

$$\begin{aligned} \Omega_\pm &:= s-\lim_{t\to\pm\infty} U(0,t) e^{-\imath t H_0} P_{ac}\,, \\ \Omega &:= \Omega_+^* \Omega_-\,, \end{aligned} \tag{8}$$

provided the limits exist. We are going to prove the existence of the scattering operators by Dyson expansions. In order to obtain a norm convergent Dyson expansion of the propagator along the lines indicated in [5] we require that for large $|t|$ the connections $\tilde{\nabla}^{\tilde{E}}$ and ∇^E are nearly gauge equivalent. More precisely, there should be some fixed times $T_- < T_+$ and smooth families of gauge transformations $W_\pm(t)$ of E defined for $t \geq T_+$ and $t \leq T_-$, respectively, such that the following holds.

Assumption 2.

$$\sup_M \|\check{V}^{\tilde{E}} - W_\pm^* V^{\tilde{E}} W_\pm\|_{|t} \leqq G(t),$$

where $G(t)$ is a bounded continuous integrable function defined outside of $[T_-, T_+]$ vanishing at infinity.

Assumption 3. $W_\pm(t) - 1 \in C_0(M, U(E))$ *for all* $t \notin (T_-, T_+)$.

Assumption 4. $\lim_{t \to \pm\infty} W_\pm(t) = \bar{W}_\pm$ *uniformly on* M.

Then the families of selfadjoint operators

$$H_1(t) := W_\pm^*(t) H_0 W_\pm(t) - i W_\pm^*(t) \frac{\partial}{\partial t} W_\pm(t)$$

for $t \geqq T_+$ and $t \leqq T_-$, respectively, satisfy the assumptions for the existence of a propagator denoted by $U_\pm(t,s)$ for $t, s \geqq T_+$ or $t, s \leqq T_-$. Under the Assumptions 1 ... 4 the operators H_0 and $H(t)$ are close enough that we can prove the existence of the scattering operators (8). The proof yields an explicit representation which will be useful later for the index computation.

Proposition 3.3. *If the Assumptions 1 ... 4 are satisfied then the scattering operators $\Omega_\pm$ exist, are partial isometries and can be represented by*

$$\Omega_\pm = U(0, T_\pm) V_\pm W_\pm^*(T_\pm) e^{-iT_\pm H_0} P_{ac}, \tag{9}$$

where

$$V_\pm := T - \exp\left\{ \int_{T_\pm}^{\pm\infty} X_\pm(u) du \right\}$$

with

$$X_\pm := i U_\pm(T_\pm, t) \left\{ H(t) + i W_\pm^*(t) \frac{\partial}{\partial t} W_\pm(t) - W_\pm^*(t) H_0 W_\pm(t) \right\} U_\pm(t, T_\pm). \tag{10}$$

Proof. We will only consider Ω_+ since the proof is analogous for Ω_-. First we check that the time ordered exponential

$$V_+ := 1 + \int_{T_+}^{\infty} X_+(t_1) dt_1 + \int_{T_+}^{\infty} \int_{t_1}^{\infty} X_+(t_2) X_+(t_1) dt_2 dt_1$$
$$+ \int_{T_+}^{\infty} \int_{t_1}^{\infty} \int_{t_2}^{\infty} X_+(t_3) X_+(t_2) X_+(t_1) dt_3 dt_2 dt_1 + \dots$$

is well defined. Since $X_+(t)$ is bounded it is sufficient that

$$\int_{T_+}^{\infty} \|X_+(t)\| dt < \infty. \tag{11}$$

But this follows from Assumption 2. Actually, in terms of a local orthonormal frame $\{X_i\}_{i=1}^n$ of TM we have

$$H(t) - W_+^*(t) H_0 W_+(t) + \imath W_+^*(t) \frac{\partial}{\partial_t} W_+(t)$$

$$= \sum_{i=1}^n X_i(\check{V}^{\tilde{E}}_{X_i} - W_+^* V^{\tilde{E}}_{X_i} W_+)(t) - \imath(\check{V}^{\tilde{E}}_{\partial t} - W_+^* V^{\tilde{E}}_{\partial t} W_+)(t). \tag{12}$$

Since $U(t,s)_+$ is unitary we obtain from (12) and Assumption 2,

$$\|X_+(t)\| \leqq CG(t).$$

This implies the claim (11). It remains to verify (9). We write

$$U(0,t)e^{-\imath tH_0}P_{\mathrm{ac}} = U(0,T_+)[U(T_+,t)e^{-\imath tH_0}P_{\mathrm{ac}}].$$

Thus it is enough to compute

$$s-\lim_{t\to\infty} U(T_+,t)e^{-\imath tH_0}P_{\mathrm{ac}}.$$

It is easy to see that

$$U_+(t,s) = W_+^*(t) e^{\imath(s-t)H_0} W_+(s),$$

since the right-hand side satisfies the conditions $1,\ldots,5$ characterizing the propagator (cf. 3.1) with $H_1(t)$ instead of $H(t)$. Hence we have

$$U(T_+,t)e^{-\imath tH_0}P_{\mathrm{ac}} = [U(T_+,t)U_+(T_+,t)^*] W_+(T_+)^* \times e^{-\imath T_+ H_0}[e^{\imath tH_0} W_+(t) e^{-\imath tH_0} P_{\mathrm{ac}}].$$

The term $U(T_+,t)U_+(T_+,t)^*$ satisfies the differential equation

$$\frac{\partial}{\partial t} U(T_+,t)U_+(T_+,t)^* = U(T_+,t)U_+(T_+,t)^* X_+(t)$$

and is therefore given by a time-ordered exponential converging in the uniform topology to V_+ as $t\to\infty$. Since $W_+(t)$ converges uniformly to $\bar{W}_+$ we have $\bar{W}_+ - 1 \in C_0(M, \mathrm{End}(E))$. Applying now Lemma 3.1 to $\bar{W}_+$ and using Assumption 4 one obtains

$$s-\lim_{t\to\infty} e^{\imath tH_0} W_+(t) e^{-\imath tH_0} P_{\mathrm{ac}} = P_{\mathrm{ac}}.$$

This proves the proposition. □

4. An Index Theorem for the Scattering Operators

The aim of this section is to prove that under some conditions, the restriction of the scattering operator to the positive spectrum of H_0 is a Fredholm operator, and to relate its index to a Toeplitz-type elliptic pseudodifferential operator on M. We show how the index of this elliptic pseudodifferential operator is related to an index problem on a compact manifold M^+ containing a large open subset of M.

4.1. An Index Lemma. In this section we show how one can compute the index of certain pseudodifferential operator A of zero order. We assume that $M = K \cup U$, where K is compact and that there exist smooth functions χ, χ_1 satisfying $\operatorname{supp}\chi, \operatorname{supp}\chi_1 \subset K$, $\chi_1\chi = \chi$ such that $A = \chi_1 A\chi + (1-\chi)$ and A is elliptic in the local sense. This means that A is elliptic, localized near the diagonal and it is the identity on U. We say that A has a constant symbol at infinity. There is a parametrix B of A of the same type, i.e. $B = \chi_1 B\chi + (1-\chi)$.

The computation of the index of an elliptic operator A with constant symbol at infinity can be reduced to the index computation for an elliptic operator A^+ on a compact manifold M^+. Let X be a relatively compact subset of M containing $\operatorname{supp}\chi_1$. It is possible to find a compact manifold M^+, a suitable vector bundle E^+ and measure $d\mu^+$ on M^+ such that X can be viewed as a subset of M^+, E^+ extends $E_{|X}$ and $d\mu_{|X} = d\mu^+_{|X}$. The operator $A^+ := \chi_1 A\chi + (1-\chi)$ is an elliptic pseudodifferential operator on M^+, where 1 here means the identity operator on M^+.

Lemma 4.1. *The operators A and A^+ have same index.*

Proof. We employ the analytic index formula to compare these indices. Set

$$R_1 := BA - 1, \qquad R_2 := AB - 1,$$
$$R_1^+ := B^+A^+ - 1, \qquad R_2^+ := A^+B^+ - 1.$$

The analytic index formula implies

$$\operatorname{ind} A = \operatorname{Tr} R_1^{n+1} - \operatorname{Tr} R_2^{n+1},$$
$$\operatorname{ind} A^+ = \operatorname{Tr}(R_1^+)^{n+1} - \operatorname{Tr}(R_2^+)^{n+1}.$$

The operators $R_1^{n+1}, \ldots$ belong to the trace class and can be represented as integral operators with continuous kernels $r_1(x,y), \ldots$ and their traces are given by

$$\operatorname{Tr} R_1^{m+1} = \int_M \operatorname{tr} r_1(x,x)\, d\mu(x), \ldots,$$

where tr denotes the fibrewise trace. But it is easy to see from the definitions that $r_1 = r_1^+$ and $r_2 = r_2^+$ on $X \times X$ and all kernels vanish outside of this set. This proves the lemma. □

The index theorem of Atiyah and Singer gives the tool to compute the index of A^+.

4.2. The Positive Spectral Projection. In this section we investigate properties of the projection P onto the positive spectral subspace of H_0. First we show how it is localized near the diagonal. This eventually allows us to apply the result of Sect. 4.1 to the index computation. The rest of this section is a preparation of the main result stated in the next section.

In the present section assume for simplicity $H := H_0$. Let $E_H(\ )$ be the spectral family of H. For $\varepsilon > 0$ and $R < \infty$ with $\varepsilon < R$ we set

$$Q := Q(R,\varepsilon) := E_H([-R,R] \setminus [-\varepsilon,\varepsilon])..$$

Lemma 4.2. *For $\phi \in \operatorname{im} Q$ we have*

$$P\phi = \frac{\phi}{2} + \lim_{S\to\infty} \frac{1}{2\pi} \int_{-S}^{S} (H + \imath\lambda)^{-1} \phi\, d\lambda \tag{13}$$

$$= \frac{\phi}{2} + \frac{1}{\pi} \int_0^\infty H[H^2 + \lambda^2]^{-1} \phi\, d\lambda. \tag{14}$$

The integrals are strongly convergent.

Proof. The claim is a consequence of the identity

$$\theta(x)=\frac{1}{2}+\lim_{S\to\infty}\frac{1}{2\pi}\int_{-S}^{S}(x+\imath\lambda)^{-1}d\lambda \quad \text{if} \quad x\neq 0 \tag{15}$$

and the spectral theorem. □

Note that P is given by these formulas on a dense set in $L^2(M,S(M)\otimes E)$.

Let $K_1\subset K\subset M$ be compact subsets such that K contains a neighbourhood of K_1 and $\chi,\chi_1\in C^\infty(M)$, $\operatorname{supp}(\chi_1)\subset K_1$, $\operatorname{supp}(\chi)\subset M\backslash K$.

Lemma 4.3. *The composition $\chi_1 P\chi$ is compact.*

Proof. Let $l(x)\in C^\infty(\mathbf{R})$ be such that $l(x)=0$ if $x\in[-1,1]$ and $\operatorname{supp}(1-l)\in[-2,2]$. We define $L=l(H)$ via the spectral theorem. Then

$$\chi_1 P\chi=\chi_1 PL\chi+\chi_1 P(1-L)\chi.$$

Since $\chi_1 P(1-L)\chi$ is compact by Rellich's theorem it is enough to consider $\chi_1 PL\chi$.

Using the second representation of P in Lemma 4.2,

$$\chi_1 PL\chi=\frac{1}{2}\chi_1 L\chi+\frac{1}{\pi}\int_0^\infty \chi_1 H(H^2+\lambda^2)^{-1}L\chi d\lambda.$$

But

$$\chi_1 L\chi=\chi_1\chi L+\chi_1[L,\chi]=\chi_1[L-1,\chi]$$

is compact by Rellich's theorem. Using

$$H[H^2+\lambda^2]^{-1}=\int_0^\infty He^{-t[H^2+\lambda^2]}dt$$

we find

$$\int_0^\infty \chi_1 H(H^2+\lambda^2)^{-1}L\chi d\lambda=\left(\int\int_{I_1}+\int\int_{I_2}\right)\chi_1 HLe^{-t[H^2+\lambda^2]}\chi dt d\lambda, \tag{16}$$

where

$$I_1:=\{(t,\lambda)\in[1,\infty)\times[0,\infty)\},$$
$$I_2:=\{(t,\lambda)\in[0,1]\times[0,\infty)\}.$$

There are constants $C_1<\infty$, $c_2>0$ such that

$$\|\chi_1 HLe^{-t[H^2+\lambda^2]}\chi\|\leqq C_1 e^{-c_2 t[1+\lambda^2]}, \qquad \forall(t,\lambda)\in I_1.$$

Furthermore,

$$\begin{aligned}\int\int_{I_2}\chi_1 HLe^{-t[H^2+\lambda^2]}\chi dt d\lambda&=\sqrt{\pi}\int_0^1 \chi_1 HLe^{-z^2H^2}\chi dz\\&=\sqrt{\pi}\int_0^1\chi_1 He^{-z^2H^2}\chi dz\\&\quad+\sqrt{\pi}\int_0^1\chi_1 H(L-1)e^{-z^2H^2}\chi dz.\end{aligned}$$

By the finite propagation speed method [4] we have $C_3, C_5 < \infty$ and $c_4 > 0$ such that

$$\|\chi_1 H e^{-z^2H^2}\chi\| \leqq C_3 e^{-c_4/z^2}, \qquad \forall z \in [0,1],$$
$$\|\chi_1 H(L-1)e^{-z^2H^2}\chi\| \leqq C_5, \qquad \forall z \in [0,1].$$

Thus (16) converges with respect to the operator norm. Since the heat operator is smoothing and χ_1 restricts to a compact set, the integrand is compact by Rellich's theorem. □

Lemma 4.4. *Let $W \in C^\infty(M, U(E))$ be a gauge transformation such that $W-1$ has compact support. Then $1+(W-1)P$ is (modulo compact operators) an elliptic pseudodifferential operator with constant symbol at infinity and principal symbol $\sigma_A(x,\xi) = 1+(W(x)-1)p(x,\xi)$, where $p(x,\xi)$ is the projection onto the positive spectral subspace of the Clifford multiplication with $\imath\xi$.*

Proof. Let $\chi_2 \in C_c^\infty(M)$ be such that $\operatorname{supp}\chi_2 \subset K$. Employing the representation (14), the fact that $(H^2+\lambda^2)^{-1}$ is a parameter-depending pseudodifferential operator for large λ and that χ, χ_2 restrict to compact sets, one shows that there is a pseudodifferential operator (in the local sense) B such that $B - \chi_2 P\chi$ is compact. For the symbol of B we get $\sigma_B(x,\xi) = \chi_2(x)p(x,\xi)\chi(x)$. In fact, one can take

$$B := \frac{\chi_2\chi}{2} + \frac{\chi_2}{\pi}\int_R^\infty H(H^2+\lambda^2)^{-1}d\lambda\chi \tag{17}$$

(R large enough). By Rellich's theorem $\chi_2 P\chi - B$ is compact. To see this note that

$$\chi_2 P\chi - B = \frac{1}{\pi}\int_0^R \chi_2 H(H^2+\lambda^2)^{-1}\chi d\lambda .$$

Let $Q := E(\mathbf{R}\backslash[-1,1])$. Then

$$\frac{1}{\pi}\int_0^R \chi_2 HQ(H^2+\lambda^2)^{-1}\chi d\lambda$$

converges in the operator norm and the integrand is compact. Moreover,

$$\frac{1}{\pi}\int_0^R \chi_2 H(1-Q)(H^2+\lambda^2)^{-1}\chi d\lambda = \frac{\chi_2}{\pi}(1-Q)\arctan\left(\frac{R}{H}\right)(1-E(\{0\}))\chi .$$

But $\chi_2(1-Q)$ is compact and

$$\left\|\arctan\left(\frac{R}{H}\right)(1-E(\{0\}))\chi\right\| \leqq \frac{\pi}{2} .$$

Let $\chi \in C_c^\infty(M)$ with $\chi_{|K} = 1$. Then $1+(W-1)P$ equals $1+(W-1)P\chi$ modulo compact operators, which in turn has a pseudodifferential approximation A. It turns out that A is elliptic with constant symbol at infinity. The principal symbol of A is $\sigma_A(x,\xi) = 1+(W(x)-1)p(x,\xi)$. □

Lemma 4.5. *Let $F \in C^\infty(M, \operatorname{End}(E)) \cap C_0(M, \operatorname{End}(E))$. Then $[P,F]$ is compact.*

Proof. It is enough to show this for F with compact support, say in K_1. Let K be compact containing a neighbourhood of K_1 and $\chi \in C^\infty(M)$ with $\operatorname{supp}\chi \subset K$, $\chi_{|K_1} = 1$. Then by Lemma 4.3 $FP(1-\chi) \sim 0$, $(1-\chi)PF \sim 0$, hence $[P,F]$

$\sim \chi PF - FP\chi$ ($\sim$ denotes equality modulo compact operators). As in Lemma 4.4 this is (modulo compact operators) a pseudodifferential operator which is compactly supported and has order -1. □

Lemma 4.6. *Let $Y(t)$ be a continuous family of sections in $C_0(M, \mathrm{End}(S(M)\otimes E))$. Then for $s<t$ the commutator*

$$\left[P, \int_s^t e^{\imath rH} Y(r) e^{-\imath rH} dr\right]$$

is compact.

Proof. Using an approximation argument one can assume that $Y(t)$ is a smooth family of smooth sections with compact support in $K \subset M$. Let $E_H(\;)$ be the spectral family of H. For some $a>0$ we set $Q := E_H[a,\infty)$, $R := E_H(-\infty, -a] + E_H[a,\infty)$. Note that the recombinations $R\frac{1}{|H|}R$ and $R\frac{1}{H}R$ are well defined. We will apply Rellich's theorem and Lemma 4.3 several times. We have

$$(P-Q)Y(t) \sim 0, \quad (1-R)Y(t) \sim 0,$$
$$Q = \frac{1}{2} R\left(1 + \frac{H}{|H|}\right) R.$$

Here we have used that the spectral projection of H onto a bounded interval is smoothing. Then

$$\begin{aligned}
\left[P, \int_s^t e^{\imath rH} Y(r) e^{-\imath rH} dr\right] &= \int_s^t e^{\imath rH}[P, Y(r)] e^{-\imath rH} dr \\
&\sim \int_s^t e^{\imath rH}[Q, Y(r)] e^{-\imath rH} dr \\
&= \frac{1}{2}\int_s^t e^{\imath rH}\left[R\frac{H}{|H|}R, Y(r)\right] e^{-\imath rH} dr \\
&\quad + \frac{1}{2}\int_s^t e^{\imath rH}[(1-R), Y(r)] e^{-\imath rH} dr \\
&\sim \frac{1}{2}\int_s^t e^{\imath rH}\left[R\frac{H}{|H|}R, Y(r)\right] e^{-\imath rH} dr .
\end{aligned}$$

We compute

$$\begin{aligned}
\int_s^t e^{\imath rH} R\frac{H}{|H|} R Y(r) e^{-\imath rH} dr &= \int_s^t \frac{d}{\imath dr}(e^{\imath rH}) R\frac{1}{|H|} R Y(r) e^{-\imath rH} dr \\
&= \imath\left(e^{\imath sH} R\frac{1}{|H|} R Y(s) e^{-\imath sH} - e^{\imath tH} R\frac{1}{|H|} R Y(t) e^{-\imath tH}\right) \\
&\quad + \imath \int_s^t e^{\imath rH} R\frac{1}{|H|} R\frac{d}{dr}(Y(r)) e^{-\imath rH} dr \\
&\quad + \int_s^t e^{\imath rH} R\frac{1}{|H|} R Y(r) H e^{-\imath rH} dr \\
&\sim \int_s^t e^{\imath rH} R\frac{1}{|H|} R Y(r) H e^{-\imath rH} dr .
\end{aligned}$$

Note that $R\frac{1}{H}R$ and $R\frac{1}{|H|}R$ raise the Sobolev order and their composition with the multiplication with a bounded, compactly supported function is compact. Hence

$$\left[P, \int_s^t e^{irH} Y(r) e^{-irH} dr\right] \sim \frac{1}{2} \int_s^t e^{irH}\left[R\frac{1}{|H|}R, Y(r)\right] He^{-irH} dr.$$

It remains to show that

$$\left[R\frac{1}{|H|}R, Y(r)\right] H$$

is compact. Let $\chi, \chi_1 \in C_c^\infty(M)$ with $\chi_{|K}=1$, $\chi_1 Y(t) = Y(t)$, and $\chi\chi_1 = \chi_1$. We write

$$\left[R\frac{1}{|H|}R, Y(r)\right] H = \chi R\frac{1}{|H|}RY(t)H - Y(t)R\frac{1}{|H|}R\chi H \tag{18}$$

$$+(1-\chi)R\frac{1}{|H|}RY(t)H \tag{19}$$

$$+Y(t)R\frac{1}{|H|}R(1-\chi)H. \tag{20}$$

(18) is compactly supported. Hence one can use pseudodifferential calculus in order to show the compactness of (18). Now we investigate (19),

$$\begin{aligned}
&(1-\mathrm{x})R\frac{1}{|H|}RY(t)H \\
&= (1-\chi)R\frac{1}{|H|}R[Y(t), H] + (1-\chi)R\frac{H}{|H|}RY(t) \\
&\quad \left(\text{replace } R\frac{H}{|H|}R \text{ by } 2Q \text{ and note that } (1-\chi)RY(t) \text{ is compact}\right) \\
&\sim (1-\chi)R\frac{1}{|H|}R\chi_1 HR\frac{1}{H}R[Y(t), H] + 2(1-\chi)QY(t) \\
&\quad (\text{replace } Q \text{ by } P \text{ and use Lemma 4.3}) \\
&\sim (1-\chi)R\frac{H}{|H|}R\chi_1 R\frac{1}{H}R[Y(t), H] + (1-\chi)R\frac{1}{|H|}R[\chi_1, H]R\frac{1}{H}R[Y(t), H] \\
&\sim 2(1-\chi)Q\chi_1 R\frac{1}{H}R[Y(t), H] \\
&\sim 0.
\end{aligned}$$

Note that $[Y(t), H]$ is a differential operator of first order, $R\frac{1}{H}R[Y(t), H]$ is bounded and that $[H, \chi_1]$ bounded. This proves the compactness of (19). The compactness of (20) follows from

$$\begin{aligned}
Y(t)R\frac{1}{|H|}R(1-\chi)H &= Y(t)R\frac{1}{|H|}R[(1-\chi), H] + Y(t)R\frac{H}{|H|}R(1-\chi) \\
&\sim 2Y(t)Q(1-\chi) \sim 0. \quad \square
\end{aligned}$$

4.3. The Index of the Scattering Operators. In this section we show that the index of the scattering operators $\Omega_\pm$ on the positive absolute continuous spectral subspace of H_0 is well defined and can be computed in terms of an integral of a differential form over M. For this we need two further assumptions. Let $Y(t):=H(t)-H_0$. It is a smooth family of bundle endomorphisms of $S(M)\otimes E$.

Assumption 5.

$$Y(t)\in C^\infty(M,\mathrm{End}(S(M)\otimes E))\cap C_0(M,\mathrm{End}(S(M)\otimes \mathrm{End}(E)))\,.$$

The family is operator norm continuous.

Assumption 6. *Let $R:=1-P_{\mathrm{ac}}$. For any $f\in C_0^\infty(M)$ the composition fR is compact.*

This is true if, for example, the non-absolute continuous spectrum of H_0 is contained in a bounded interval. Assumption 6 is fulfilled if M is euclidean at infinity and the connection ∇^E is flat outside of a compact set. Then it follows by the results of [3] that H_0 has absolute continuous spectrum outside zero. Another example is the hyperbolic space $\mathbf{H}^n$, where (E,∇^E) is the flat bundle.

Lemma 4.7. *Suppose the Assumption 5. Then for $s<t$ the commutator $[P,U(s,t)]$ is compact.*

Proof. We employ the representation of $U(s,t)$ as a time-ordered exponential

$$U(s,t)=e^{-\imath tH_0}T\!-\!\exp\left(\int_s^t e^{\imath rH_0}Y(r)e^{-\imath rH_0}dr\right)e^{\imath sH_0}\,.$$

Hence it is enough to show the compactness of

$$\left[P,\int_s^t e^{\imath rH_0}Y(r)e^{-\imath rH_0}dr\right],$$

but this is true by Lemma 4.6. □

Proposition 4.8. *Assume the Assumptions 1...6. Then $[P,\Omega_\pm]$ and $[R,\Omega_\pm]$ are compact. Moreover, there are norm continuous families $\Omega^\lambda_\pm$, $\lambda\in[0,1]$, joining $\Omega_\pm$ and $W^*_\pm(T_\pm)P_{\mathrm{ac}}$ such that $[P,\Omega^\lambda_\pm]$ and $[R,\Omega^\lambda_\pm]$ are compact, too.*

Proof. We employ the representation (9) of $\Omega_\pm$. We will consider only Ω_+ since the proof is analogous for Ω_-. Note that we are free to replace T_+ by $S_+>T_+$ in (9). But

$$\lim_{S_+\to\infty}\|1-V_+\|=0\,, \tag{21}$$

where $V_+=V_+(S_+)$. Since $[P,U(0,S_+)]$ is compact by Lemma 4.7 and $[P,W^*_+(S_+)]$ is compact by Lemma 4.5 we have

$$[P,\Omega_+]=U(0,S_+)[P,V_+]W^*_+(S_+)e^{-\imath S_+H_0}P_{\mathrm{ac}}+K(S_+)\,,$$

where $K(S_+)$ is compact. Because of (21) taking the limit $S_+\to\infty$ one obtains the compactness of $[P,\Omega_+]$.

Using the representations of the terms in Ω_+ by norm convergent Dyson expansions one gets the compactness of $[R,\Omega_+]$ by Assumption 6.

The family Ω^λ_+ is constructed replacing $X_+(t)$, $Y(t)$ by $\lambda X_+(t)$, $\lambda Y(t)$, respectively. □

Now we can prove the main theorem of this section.

Theorem 4.9. *If the Assumptions 1...6 are satisfied then the operator $P(\Omega_\pm + R)P$ is Fredholm on* $\operatorname{im} P$ *and*

$$\operatorname{ind}_{\operatorname{im} P} P(\Omega_\pm + R)P = \operatorname{ind}(1 + (W_\pm^*(T_\pm) - 1)P) = -(-1)^{\frac{n-1}{2}} CS(\tilde{W}_\pm)\hat{A}(M)[M],$$

where $\tilde{W}_\pm$ is a deformation of $W_\pm(T_\pm)$ defined below such that $CS(\tilde{W}_\pm)$ has compact support,

$$CS(W) := \frac{\imath}{2\pi} \operatorname{Tr} W^* \nabla W \int_0^1 \exp\left(\frac{\imath(t-t^2)}{2\pi} W^* \nabla W W^* \nabla W\right) dt,$$

$$CS(W)_{2r-1} = 2 \frac{\imath^r}{(2\pi)^r} \frac{r!}{(2r)!} \operatorname{Tr}([W^* \nabla W]^{2r-1})$$

is the Chern-Simons form, $\hat{A}(M)$ the form representing the $\hat{\mathbf{A}}$-class and $[M]$ is the fundamental cycle of M.

Proof. By Proposition 4.8, because $\Omega_\pm R = 0$, the sum $\Omega_\pm + R$ is unitary modulo compact operators and $[P, \Omega_\pm]$ is compact. Hence by (1) and (2) $\Omega_\pm + R$ is Fredholm on $\operatorname{im} P$. By Proposition 4.8

$$\begin{aligned}\operatorname{ind}_{\operatorname{im} P} P(\Omega_\pm + R)P &= \operatorname{ind}(1 - P + P(W_\pm^*(T_\pm)P_{ac} + R)P) \\ &= \operatorname{ind}(1 - P + PW_\pm^*(T_\pm)P + P(1 - W_\pm^*(T_\pm))RP) \\ &= -\operatorname{ind}(1 + (W_\pm(T_\pm) - 1)P).\end{aligned}$$

Thus it remains to compute the index of $(1 + (W-1)P)$, where $W := W_\pm(T_\pm)$. We construct a deformation $\tilde{W}(\tau)$ of W such that $\tilde{W}(\tau) - 1 \in C_0^\infty(M, U(E))$, $\tilde{W}(1) = W$ and $\tilde{W}(0) - 1$ has compact support. Let $M = M_1 \cap U$ be a decomposition of M into a compact subset M_1 and an open subset $U := M \setminus M_1$, V be a neighbourhood of M_1 and $\theta \in C^\infty(M)$ with $\theta = 1$ on M_1, $\operatorname{supp}\theta \in V$ and $\sup_{\operatorname{supp}(1-\theta)} |W-1|$ small enough. Let $\tilde{W}(\tau)$ be the unitary part of $(\theta W + (1-\theta)(1-\tau) + (1-\theta)\tau W)$ with respect to the polar decomposition. Since $(1 + (\tilde{W}(\tau) - 1)P)$ is an operator norm continuous family of Fredholm operators we have

$$\operatorname{ind}(1 + (W-1)P) = \operatorname{ind}(1 + (\tilde{W}(0) - 1)P).$$

The theorem follows now from Lemma 4.4, Lemma 4.1, and (4). □

Theorem 4.10. *The index of the scattering matrix Ω on the positive absolute continuous spectral subspace of H_0 is*

$$\begin{aligned}\operatorname{ind}_{\operatorname{im} P_{ac} P} P\Omega P &= (-1)^{\frac{n-1}{2}} (CS(\tilde{W}_+) - CS(\tilde{W}_-))\hat{A}(M)[M] \\ &= -(-1)^{\frac{n-1}{2}} \mathbf{CS}(W_+^* W_-)\hat{\mathbf{A}}(M)[M].\end{aligned}$$

Proof. This follows from

$$\begin{aligned}\operatorname{ind}_{\operatorname{im} P_{ac} P} P\Omega P &= \operatorname{ind}_{\operatorname{im} P} P(\Omega + R)P \\ &= \operatorname{ind}_{\operatorname{im} P} P(\Omega_+^* \Omega_- + R)P \\ &= \operatorname{ind}_{\operatorname{im} P} P(\Omega_+ + R)^*(\Omega_- + R)P \\ &= \operatorname{ind}_{\operatorname{im} P} P(\Omega_- + R)P - \operatorname{ind}_{\operatorname{im} P} P(\Omega_+ + R)P. \quad \square\end{aligned}$$

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Communicated by H. Araki