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Statistics of Shocks in Solutions of Inviscid Burgers Equation

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Abstract. The purpose of this paper is to analyze statistical properties of discontinuities of solutions of the inviscid Burgers equation having a typical realization $b(y)$ of the Brownian motion as an initial datum. This case was proposed and studied numerically in the companion paper by She, Aurell and Frisch. The description of the statistics is given in terms of the behavior of the convex hull of the random process $w(y) = \int_0^y (b(\eta) + \eta)\, d\eta$. The Hausdorff dimension of the closed set of those y where the convex hull coincides with w is also studied.

1. General Properties of Solutions of the One-dimensional Inviscid Burgers Equation

Burgers equation is one of the most popular non-linear equations which appears in many concrete physical problems. In this paper we study some properties of solutions of the inviscid Burgers equation having as initial velocity a typical realization of the Brownian motion (as a function of the space variable). This case was proposed in a companion paper by She, Aurell, and Frisch [1] where one can find physical motivations for this case as well as many qualitative arguments and numerical results.

We start with the geometric description of the process of construction of solutions to the inviscid Burgers equation. This theory was already exposed in the pioneering works of Hopf (see [2]) and Burgers (see [3]). We present here a slightly different approach compared with [2] and [3]. The companion paper [1] also begins with this analysis. The notations in the present paper and in [1] are slightly different but it is easy to establish the correspondence between them.

We recall that the one-dimensional Burgers equation without force has the form

$$\partial_t u + u \partial_x u = \mu \partial_x^2 u, \qquad -\infty < x < \infty .$$

Here $\mu > 0$ is the viscosity. The Hopf-Cole substitution $u = -2\mu \frac{\partial_x \varphi}{\varphi}$ (see [1, 3]) shows that φ satisfies the heat equation

$$\partial_t \varphi = \mu \partial_x^2 \varphi .$$

Using this fact one can write down for the solution $u = u_\mu(x, t)$ the explicit expression

$$u_\mu(x,t) = \frac{\int\limits_{-\infty}^{\infty} dy \, \frac{x-y}{t} \exp\left\{ -\frac{1}{2\mu} F(x,y,t) \right\}}{\int\limits_{-\infty}^{\infty} dy \exp\left\{ -\frac{1}{2\mu} F(x,y,t) \right\}} . \tag{1}$$

Here $F(x,y,t) = \frac{(x-y)^2}{2t} + \int\limits_0^y u(\eta;0)\,d\eta$ and $u(\eta;0)$ is the initial datum. The formula (1) works if $\int\limits_0^y u(\eta;0)\,d\eta = o(y^2)$ as $y \to \pm\infty$. The expression of (1) appears often for correlation function in statistical mechanics. In fact the theory of Burgers equation is closely connected with the theory analysing statistical properties of directed polymers.

Hopf in [2] and Burgers in [3] discussed the behavior of solutions in $u_\mu(x;t)$ under the limit transition $\mu \to 0$. In what follows we study the solutions $u_\mu(x;t)$ for a fixed value of t, say $t = 1$. Therefore we often omit t in our future notations. Consider

$$\begin{aligned} M(x) = \min_y F(x,y,1) &= \min_y \left[\frac{(x-y)^2}{2} + \int\limits_0^y u(\eta;0)\,d\eta \right] \\ &= \frac{x^2}{2} + \min_y \left\{ \int\limits_0^y [u(\eta;0) + \eta]\,d\eta - xy \right\}. \end{aligned}$$

Denote $w(y) = \int\limits_0^y [u(\eta;0) + \eta]\,d\eta$. The function $L_w(x) = \min_y\{w(y) - xy\}$ is the Legendre transform of w. We need the simplest properties of this transform. It will be applied below to cases where $u(\eta;0) = 0$ for $\eta < 0$ and $u(\eta;0)$ is continuous for $-\infty < \eta < \infty$. Therefore $w(y) = \frac{y^2}{2}$ for $y < 0$. Introduce the convex hull $C_w(y)$ of w. It is a convex function, and $C_w \le w$. Then C_w is the largest function having the last two properties.

There is also another way of describing C_w. Fix x and take a straight line having the slope x, i.e. a line given on the (y,w)-plane by the equation $w = xy + c$. Then for every x one can find such $c_0(x) = c_0$ that for all $c < c_0$ the lines $w = xy + c$ do not intersect the graph of w while for all $c > c_0$ such intersections arise. For $c = c_0$ the line $w = xy + c_0$ is tangent to the graph of w at one or several points. Put $M(x)$ to be the set of those y where the line $w = xy + c_0$ is tangent to the graph of w. Introduce $m_*(x) = \min\{y \mid y \in M(x)\}$, $m^* = \max\{y \mid y \in M(x)\}$. If $m^*(x) = m_*(x)$ then $C_w(y) = w(y)$ for $y = m_*(x) = m^*(x)$. If $m_*(x) < m^*(x)$

then $C_w(y) = xy + c_0$ for all $m_*(x) \le m^*(x)$. In other words, the graph C_w is a convex curve consisting of straight intervals and a closed set lying outside them. The derivative $F(y) = \dfrac{d}{dy} C_w(y)$ is in general a non-decreasing Cantor devil staircase type function which takes a constant value on each interval where C_w is linear. In these terms the Legendre transform can be written in the form

$$L_w(x) = c_0 .$$

Consider the function $G(x)$ which is the inverse function to $x = F(y)$. However it is not a well-defined object because $G(x)$ is multi-valued for those x where the set of y where $F(y) = x$ is an interval. Since we are interested in a geometric picture it is more convenient to consider $G(x)$ as a continuous curve on the plane which consists of vertical segments for those values of x, where G is discontinuous. Now we can formulate the following theorem by Hopf (see [2]).

Hopf's Theorem. *Let x be such that $M(x)$ consists of one point $y(x) = G(x)$. Then the limit $\lim_{\mu\to 0} u_\mu(x;1) = \lim_{\mu\to 0} u_\mu(x) = u_0(x)$ exists and $u_0(x) = x - G(x)$. If $G(x)$ is an interval of positive length then there exist the limits*

$$u_0^-(x) = \lim_{x'\to -0} u_0(x') = x - m_*(x),$$

$$u_0^+(x) = \lim_{x'\to +0} u_0(x') = x - m^*(x) .$$

In both cases the limits are taken over such x' that $G(x')$ is single-valued.

We can interpret this result as follows. The function $u_0(x)$ is discontinuous for those x where $G(x)$ is multi-valued. At these points there exist the one-sided limits of $u_0(x)$ equal to $x - m_*(x)$ for the left limit and $x - m^*(x)$ for the right limit. This jump is interpreted as a shock and its size is equal to the length of the vertical segment of $G(x)$.

We shall use the following definition.

Definition 1. Cantor-type function $F(y)$ is complete if the union of intervals where F is constant is a set of full Lebesgue measure or, better to say, its complement has Lebesgue measure zero.

Let us prove now the following lemma.

Lemma 1. *If F is complete then $u_0(x)$ is differentiable* a.e. *and* $\dfrac{du_0(x)}{dx} = 1$ a.e.

The proof is simple. Indeed, another way to express the completeness of F is to say that the image under F^{-1} of $R^1 \setminus$ (countable set of x such that $G(x)$ is multi-valued) is a subset of R^1 of the zeroth Lebesgue measure. Put for convenience $G(x) = m^*(x)$ for all x. Then $\lim_{x'\to x} \dfrac{G(x') - G(x)}{x' - x} = 0$ for a.e. x with respect to the Lebesgue measure. Since $u_0(x) = x - G(x)$ for all x except the above mentioned countable set this gives the desired result.

Now we formulate the final conclusions of this section. The limit $u_0(x)$ is a discontinuous function whose discontinuities take place for those x where the equality

$x = F(y)$ holds for a segment on the y-axis of positive length. Outside this countable set $u_0(x) = x - G(x)$. The discontinuities of $u_0(x)$ are always negative, i.e. the limits from the left are bigger than the limits from the right. If the devil's stair-case $F(y)$ is complete then $u_0(x)$ is differentiable a.e. and $\frac{d}{dx} u_0(x) = 1$.

2. The Case Studied in the Companion paper [1]

The motivation for this paper was to explain some numerical results obtained by She, Aurell, and Frisch [1]. Among many cases considered by these authors there was the case of $u(x;0) = b(x)$, where $b(x)$ is a Brownian trajectory for $x \geq 0$ and $u(x;0) = 0$ for $x < 0$. According to the theory described in Sect. 1 we have to construct the random process $w(y)$, where $w(y) = \frac{y^2}{2}$ for $y < 0$ and $w(y) = \int_0^y (b(\eta) + \eta)\, d\eta$ for $y > 0$ and to study its convex hull C_w which is a non-linear and non-local functional of b. It is clear that for some $y_0 = y_0(b) < 0$ the convex hull C_w coincides with $\frac{y^2}{2}$ for $y < y_0$. Therefore $F(y) = y$ for $y < y_0$ and for such y the function F is not a devil's stair-case.

Theorem 1. *With probability* 1 *the devil's stair-case $x = F(y)$ is complete on the semiline $y > 0$.*

Proof. Fix $\bar{y}$ and consider the tangent line $\Gamma_{\bar{y}}$ to the graph of $w(y)$ at $y = \bar{y}$ given by the equation $w = w(\bar{y}) + (y - \bar{y})\ (b(\bar{y}) + \bar{y})$. We shall say that $\bar{y}$ is a special point for $b = \{b(y)\}$ if one can find a neighborhood U of $\bar{y}$ depending on b and such that in this neighborhood the graph of w lies above $\Gamma_{\bar{y}}$, i.e. if

$$w(y) \geq w(\bar{y}) + (y - \bar{y})(b(\bar{y}) + \bar{y}), \qquad y \in U\,.$$

We shall show that for any $\bar{y}$ the probability that it is a special point is equal to zero. Let us derive from this statement the assertion of the theorem.

Fix $Y > 0$ and consider the probability space $(C_Y, \mathscr{F}_Y, P) \times ([0,Y], \mathscr{F}, l) = (\Omega, \mathscr{D}, P)$. Here C_Y is the space of continuous functions defined on the segment $[0,Y]$, equal to zero at $y = 0$. $\mathscr{F}_Y$ is the Borel σ-algebra of the space C_Y, P is the standard Wiener measure defined on $\mathscr{F}_Y$. Further $\mathscr{F}$ is the Borel σ-algebra of the segment $[0,Y]$ and l is the normed length. Introduce the subset $A \subset \Omega$ consisting of such pairs (b, y) that y is a special point for b. It is easy to see that $A \in \mathscr{D}$. The above mentioned statement implies

$$P(A) = \int_0^Y dl(\bar{y})\, P\{\bar{y} \text{ is a special point for } b\} = 0\,,$$

and by Fubini's theorem

$$0 = P(A) = \int dP(b)\, l(\{\bar{y} \mid \bar{y} \text{ is a special point for } b\})$$

which gives $l(\{\bar{y} \mid \bar{y}$ is a special point for $b\}) = 0$ for a.e. b.

In order to prove the main statement we shall show that the probability that for some $\alpha = \alpha(b) > 0$ and all x, $0 \le x \le \alpha$,

$$w(\bar{y}+x) - w(\bar{y}) = \int\limits_{\bar{y}}^{\bar{y}+x} (b(\eta)+\eta)\, d\eta \ge (b(\bar{y})+\bar{y})x$$

is zero. Rewrite the last inequality in the form

$$\int\limits_0^x (b_1(\eta)+\eta)\, d\eta \ge 0, \qquad 0 \le x \le \alpha\,,$$

where $b_1(\eta) = b(\eta+\bar{y}) - b(\bar{y})$. It is clear that $b_1(\eta)$ has the same distribution as $b(\eta)$. Put $b_2(\eta) = b_1(\eta)+\eta$. Girsanov's theorem (see [4]) says that the probability measure corresponding to the process $b_2(\eta)$ on any finite interval of η is equivalent tot he Wiener measure.

Denote by $P_+(P_-)$ the probability with respect to the Wiener measure that there exists $\alpha_1 = \alpha_1(b) > 0$ such that $\int_0^x b(\eta)\, d\eta \ge 0$ (≤ 0) for all x, $0 \le x \le \alpha_1(b)$. By symmetry, $P_+ = P_-$. Remark now that the event whose probability we study belongs to the σ-algebra depending on the behaviour of the process in one point and therefore by the "$0-1$" law can take only the values 1 or 0. Since in our case $P_+ = P_-$ and $P_+ + P_- \le 1$ it can take only the *zeroth* value. Due to the absolute continuity of the measure corresponding to b_2 to the Wiener measure, this probability for $b_2(\eta)$ is also zero. Q.E.D.

Remark. The proof given above was shown to me by M. Yor (private communication). My original proof was more complicated.

Return now to the function F and introduce the closed set $S(b)$ of all $y > 0$ lying outside the union of intervals where F is constant. In other words, $S(b)$ consists of such $\bar{y} \in S(b)$ that the tangent line $w = (b(\bar{y})+\bar{y})(y-\bar{y}) + w(\bar{y})$ intersects the graph of the function $w(y) = \int\limits_0^y (b(\eta)+\eta)\, d_\eta$ only at the point $(\bar{y}, w(\bar{y}))$. In the next sections we study the fractal properties of $S(b)$.

The main result of our studies is the following theorem.

Main Theorem. *With probability* 1 *the Hausdorff dimension of* $S(b)$ *is equal to* $\frac{1}{2}$.

The proof of this theorem is based upon the estimations of probabilities of small fragments of C_w which we derive in the next section.

3. Estimations of Probabilities of Small Fragments of C_w

Consider on the plane (y, w) two vertical lines $y = a_1$, $y = a_2$, $0 < a_1 < a_2$, and two strips $\Pi_1 = \{(y,w) \mid\ | y - a_1 | \le \delta_1(a_2 - a_1)\}$, $\Delta_2 = \{(y,w) \mid\ | y - a_2 | \le \delta_2(a_2 - a_1)\}$. In what follows $a_2 - a_1$ will tend to zero while all δ_j will remain fixed but small. Take also a straight line Γ given by the equation $w = \beta y + \beta_1 = l(y)$ and such that the point $(0,0)$ lies above Γ. Introduce the parallelograms

$$\Pi_1 = \left\{(y,w)\big|\, |y-a_1| \le \delta_1(a_2-a_1),\ \ |w-l(y)| \le \delta_3(a_2-a_1)^{3/2}\right\},$$
$$\Pi_2 = \left\{(y,w)\big|\, |y-a_2| \le \delta_2(a_2-a_1),\ \ |w-l(y)| \le \delta_3(a_2-a_1),^{3/2}\right\}$$

We need also the segments

$$\begin{aligned}
&\Gamma_0 \subset \Gamma,\\
&\Gamma_0 = \{(y,w) \mid a_1+\delta_1(a_2-a_1) \le y \le a_2-\delta_2(a_2-a_1), (y,w)\in\Gamma\},\\
&\Gamma_{01} = \Big\{(y,w) \mid a_1-\delta_1(a_2-a_1) \le y \le a_1+\delta_1(a_2-a_1),\\
&\qquad w = l(y) - \frac{\delta_3}{2}(a_2-a_1)^{3/2}\Big\}\\
&\Gamma_{02} = \Big\{(y,w) \mid a_2-\delta_2(a_2-a_1) \le y \le a_2+\delta_2(a_2-a_1),\\
&\qquad w = l(y) - \frac{\delta_3}{2}(a_2-a_1)^{3/2}\Big\},
\end{aligned}$$

the ray

$$\Gamma^- = \{(y,w) \mid y \le a_1-\delta_1(a_2-a_1), w = l_-(y)\},\ \ l_-(y) = \beta^- y + \beta_1^-,$$

whose continuation passes through the points

$$(a_1-\delta_1(a_2-a_2), l(a_1-\delta_1(a_2-a_1))) \in \Gamma$$

and

$$\left(a_2-\delta_2(a_2-a_1), l(a_2-\delta_2(a_2-a_1)) - \delta_3(a_2-a_1)^{3/2}\right)$$

and the ray

$$\Gamma^+ = \{(y,w) \mid y \ge a_2+\delta_2(a_2-a_1), w = l_+(y)\},\ \ l_+(y) = \beta^+ y + \beta_1^+,$$

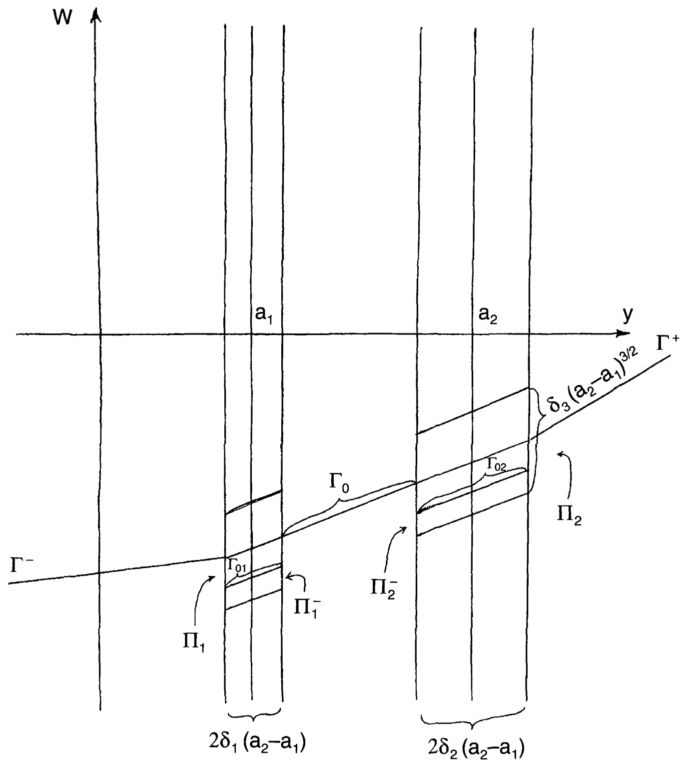

Fig. 1

whose continuation passes through the points

$$\big(a_1+\delta_1(a_2-a_1), l((a_1+\delta_1(a_2-a_1))-\delta_3(a_2-a_1)^{3/2}\big)$$

and

$$\big(a_2+\delta_2(a_2-a_1), l(a_2+\delta_2(a_2-a_1))\big)\in\Gamma\,.$$

Let also

$$\Pi_1^- = \big\{(y,w)\big|\,|y-a_1|\le\delta_1(a_2-a_1), -\delta_3(a_2-a_1)^{3/2} \le w-l(y)\le -\tfrac{1}{2}\,\delta_3(a_2-a_1)^{3/2}\big\}\,,$$

$$\Pi_2^- = \big\{(y,w)\big|\,|y-a_2|\le\delta_2(a_2-a_1), -\delta_3(a_2-a_1)^{3/2} \le w-l(y)\le -\tfrac{1}{2}\,\delta_3(a_2-a_1)^{3/2}\big\}\,,$$

All these segments, rays, parallelograms, strips are drawn in Fig. 1.

Lemma 2. *If* $\delta_1,\delta_2\le$ const *then any straight line passing through a point inside* Π_1^- *and through a point* Π_2^- *lies below* Γ^+ *and* Γ^-.

Geometrically the statement of the lemma is obvious. Remark also that the whole construction is defined as soon as Γ, Π_1, Π_2 are given. Thus they can be considered as determining parameters. Return now to the process $w(y)=\int_0^y(b(\eta)+\eta)\,d\eta$ and take another small number $\delta_4>0$.

Definition 2. A realization w has a right behavior (with respect to our construction) if

A_1) for

$$y \leq a_1 - \delta_1(a_2 - a_1)$$

the graph of w lies above Γ_-, i.e.

$$w(y) > l_-(y) \qquad \text{for all such} \quad y;$$

A_2)

$$\begin{aligned} l(a_1 - \delta_1(a_2 - a_1)) &< w(a_1 - \delta_1(a_2 - a_1)) \\ &< l(a_1 - \delta_1(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}; \end{aligned}$$

$$\begin{aligned} \beta \geq w'(a_1 - \delta_1(a_2 - a_1)) &= b(a_1 - \delta_1(a_2 - a_1)) + (a_1 - \delta_1(a_2 - a_1)) \\ &\geq \beta - \delta_4(a_2 - a_1)^{1/2}; \end{aligned}$$

moreover,

$$w'(a_1 - \delta_1(a_2 - a_1)) \leq \beta^-;$$

A_3) for all $a_1 - \delta_1(a_2 - a_1) \leq y \leq a_1 + \delta_1(a_2 - a_1)$

$$w(y) > l(y) - \delta_3(a_2 - a_1)^{3/2};$$

and there is a non-empty subset of such y that

$$w(y) < l(y) - \frac{\delta_3}{2}(a_2 - a_1)^{3/2};$$

A_4)

$$\begin{aligned} l(a_1 + \delta_1(a_2 - a_1)) &< w(a_1 + \delta_1(a_2 - a_1)) \\ &< l(a_1 + \delta_1(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}; \end{aligned}$$

A_5) for all $a_1 + \delta_1(a_2 - a_1) \leq y \leq a_2 - \delta_2(a_2 - a_1)$

$$w(y) > l(y);$$

A_6)

$$\begin{aligned} l(a_2 - \delta_2(a_2 - a_1)) &< w(a_2 - \delta_2(a_2 - a_1)) \\ &< l(a_2 - \delta_2(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}; \end{aligned}$$

A_7) for all $a_2 - \delta_2(a_2 - a_1) \leq y \leq a_2 + \delta_2(a_2 - a_1)$

$$w(y) > l(y) - \delta_3(a_2 - a_1)^{3/2};$$

and there is a non-empty open set of such y that

$$w(y) < l(y) - \frac{\delta_3}{2}(a_2 - a_1)^{3/2};$$

A_8)

$$\begin{aligned} l(a_2 + \delta_2(a_2 - a_1)) &< w(a_2 + \delta_2(a_2 - a_2)) \\ &< l(a_2 + \delta_2(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}; \end{aligned}$$

$$\begin{aligned} \beta \leq w'(a_2 + \delta_2(a_2 - a_1)) &= b(a_2 + \delta_2(a_2 - a_1)) + (a_2 + \delta_2(a_2 - a_1)) \\ &\leq \beta + \delta_4(a_2 - a_1)^{1/2}; \end{aligned}$$

moreover,

$$w'(a_2 + \delta_2(a_2 - a_1)) > \beta^+ .$$

A$_9$) for all $y \geq a_2 + \delta_2(a_2 - a_1)$ the graph of w lies above Γ_+, i.e.

$$w(y) \geq l_+(y) \qquad \text{for all such} \quad y .$$

Figure 2 shows the right behavior. Also one can easily see some symmetry in the properties (A$_1$)–(A$_9$). The reasons for our scaling will become clear from further estimations.

Theorem 2. *Let $U = U(\Gamma, \Pi_1, \Pi_2, \delta_4)$ be the event consisting of such b that w has a right behavior (see Definition 2). Then for any $b \in U$ the graph of C_w contains an interval which has endpoints inside Π_1 and Π_2.*

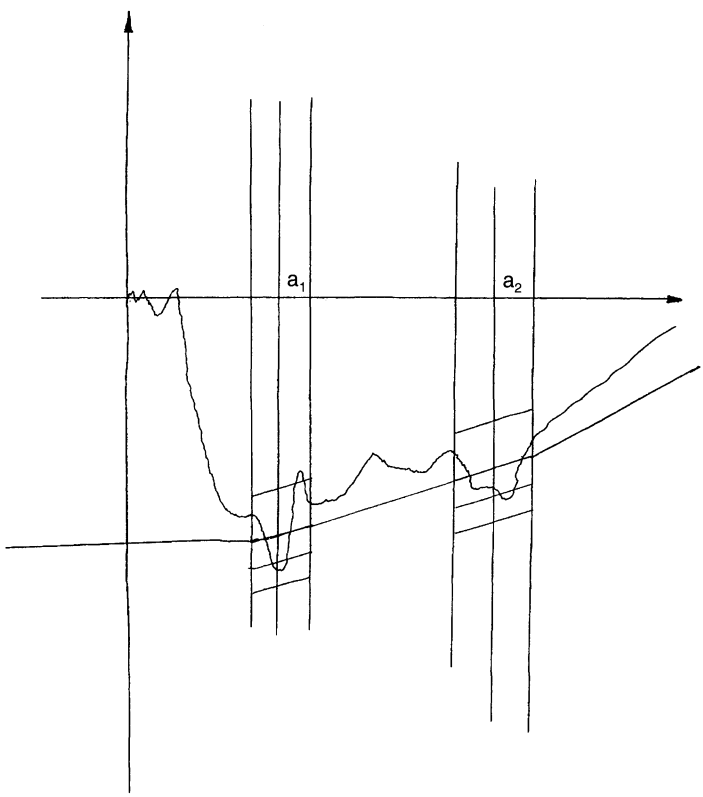

Fig. 2

Proof is simple. Consider the straight line $\Gamma_{\beta,c}$ given by the equation $w = \beta y + c$ for $c < 0$, $|c|$ is large, such that $\Gamma_{\beta,c}$ does not intersect the graph of w. Start steadily to increase c. Then there will appear such $\bar{c}$ that for all $c < \bar{c}$ there are no such intersections and $\bar{c}$ is the upper bound of c having this property. The straight line $\Gamma_{\beta,\bar{c}}$ is tangent to the graph of w at one or several points. If among these points there are points inside Π_1 as well as inside Π_2 then our statement is proven.

Suppose that all common points of the graph w and $\Gamma_{\beta,\bar{c}}$ lie inside Π_1, the case of points inside Π_2 is considered in the same way. For $\beta' > \beta$ sufficiently close to β take the analogous line $\Gamma_{\beta',\bar{c}(\beta')}$. Then all common points of w and $\gamma_{\beta',\bar{c}(\beta')}$ lie inside Π_1. One can find $\bar{\beta} > \beta$ such that for all β', $\beta < \beta' < \bar{\beta}$ we shall have the same property while for $\bar{\beta}$ the straight line $\Gamma_{\bar{\beta},c(\bar{\beta})}$ will have common points inside Π_1 and outside Π_1. From the right behavior (A_4)–(A_6) it follows easily that these points lie inside Π_1^- and Π_2^-.

Lemma 1 and (A_1)–(A_9) imply that there are no common points of w and $\Gamma_{\bar{\beta},c(\bar{\beta})}$ outside the strip $a_1 - \delta_1(a_2 - a_1) \leq y \leq a_2 + \delta_2(a_2 - a_1)$, Q.E.D.

One of our main estimations is given in the following theorem.

Theorem 3. *Let δ_j, $1 \leq j \leq 4$, be sufficiently small, and a_1, β be fixed. Then for all sufficiently small $a_2 - a_1$ the probability (with respect to the Wiener measure)*

$$P(U) \geq F(\delta_1, \delta_2, \delta_3, \delta_4, a_1) \cdot (a_2 - a_1)^{1/2} Q(\delta_1, \delta_2, \beta, \beta_1, a_1, a_2)\,,$$

where $F(\delta_1, \delta_2, \delta_3, \delta_4, a_1)$ is a positive constant and $Q(\delta_3, \delta_4, \beta, \beta_1, a_1, a_2)$ is the probability that

$$\begin{aligned}
&b(a_1 - \delta_1(a_2 - a_1)) \in (-(a_1 - \delta_1(a_2 - a_1))) - \delta_4(a_2 - a_1)^{1/2};\\
&\qquad \big(-(a_1 - \delta_1(a_2 - a_1)) + \delta_4(a_2 - a_1)^{1/2}\big)\,;\\
&w(a_1 - \delta_1(a_2 - a_1)) \in (l(a_1 - \delta_1(a_2 - a_1))) - \delta_3(a_2 - a_1)^{3/2}\,;\\
&\qquad l\big(a_1 - \delta_1(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}\big)\,.
\end{aligned}$$

It is clear that for any compact set of values of β, β_1 the probability Q is proportional to $(a_2 - a_1)^2 \cdot \delta_3 \cdot \delta_4$.

Proof. Denote by z_1, z_2, z_3, z_4 the values of $b(a_1 - \delta_1(a_2 - a_1))$, $w(a_1 - \delta_1(a_2 - a_1))$, $b(a_2 + \delta_2(a_2 - a_1))$, $w(a_2 + \delta_2(a_2 - a_1))$ respectively, and introduce also their dimensionless values through the rescaling

$$\begin{aligned}
z_1 &= \beta + Z_1(a_2 - a_1)^{1/2}\,,\\
z_2 &= l(a_1 - \delta_1(a_2 - a_1)) + Z_2(a_2 - a_1)^{3/2}\,,\\
z_3 &= \beta + Z_3(a_2 - a_1)^{1/2}\,,\\
z_4 &= l(a_2 + \delta_2(a_2 - a_1)) + Z_4(a_2 - a_1)^{3/2}\,.
\end{aligned}$$

In the case of the right behavior, i.e. $b \in U$, we have $0 < Z_1 < \delta_4$, $-\delta_3 < Z_2 < \delta_3$, $0 < Z_3 < \delta_4$, $-\delta_3 < Z_4 < \delta_3$. We need also some rescaling of y, i.e. we put $y = a_1 + (a_2 - a_1)Y$.

Using the fact that the pair $(b(y), w(y))$ is a two-dimensional Markov process we can write

$$P(U) = \int dz_1\, dz_2\, dz_3\, dz_4 \cdot p(0, 0; z_1, z_2; a_1 - \delta_1(a_2 - a_1)) \cdot p(z_1, z_2; z_3', z_4;$$

$$
\begin{aligned}
& a_1-\delta_1(a_2-a_1), a_2+\delta_2(a_2-a_1)) \\
& \quad \cdot P(A_1 \mid b(a_1-\delta_1(a_2-a_1))=z_1, w(a_1-\delta_1(a_2-a_1))=z_2)) \\
& \quad \cdot P\{A_3, A_4, A_5, A_6, A_7 \mid b(a_1-\delta_1(a_2-a_1))=z_1, \\
& w(a_1-\delta_1(a_2-a_1)=z_2, \\
& b(a_2+\delta_2(a_2-a_1))=z_3, w(a_2+\delta_2(a_2-a_1))=z_4\} \\
& \quad \cdot P\{A_9 \mid b(a_2+\delta_2(a_2-a_1))=z_3, w(a_2+\delta_2(a_2-a_1))=z_4\}. \qquad (2)
\end{aligned}
$$

The domain of integration is determined by A_2 and A_8 and was written down in terms of the dimensionless parameters $Z_j; p(u_1, u_2; v_1, v_2; s, t)$ is the transition density of the process (b, w) from the initial state u_1, u_2 at the moment of time s to the final state v_1, v_2 at the moment t, the written probabilities describe the conditional probabilities of the corresponding properties A_j.

We study the inner factor

$$
\begin{aligned}
& P\{A_3, A_4, A_5, A_6, A_7 \mid b(a_1-\delta_1(a_2-a_1)=z_1, \\
& \quad w(a_1-\delta_1(a_2-a_1))=z_2, b(a_2+\delta_2(a_2-a_1))=z_3, \\
& \quad w(a_2+\delta_2(a_2-a_1))=z_4\}.
\end{aligned}
$$

Under the described rescaling and the rescaling of the Wiener process

$$
\begin{aligned}
& b(y)=b(a_1-\delta_1(a_2-a_1))+\sqrt{a_2-a_1} B(Y), \\
& w(y)=w(a_1-\delta_1(a_2-a_1))+(a_2-a_1)^{3/2} \cdot W(y),
\end{aligned}
$$

we see that the properties (A_2)–(A_7) are expressed only in terms of the dimensionless variables and the rescaled processes B, W. Therefore for all values of z_1, z_2, z_3, z_4 under consideration

$$
\begin{aligned}
& P\{A_3, A_4, A_5, A_6, A_7 \mid b(a_1-\delta_1(a_2-a_1))=z_1, w(a_1-\delta_1(a_2-a_1))=z_2, \\
& \quad b(a_2+\delta_2(a_2-a_1))=z_3, w(a_2+\delta_2(a_2-a_1))=z_4\} \\
& \qquad = F_1(Z_1, Z_2; Z_3, Z_4, \delta_1, \delta_2, \delta_3, \delta_4)>0.
\end{aligned}
$$

The most crucial part is the estimation of

$$
P\{A_1 \mid b(a_1-\delta_1(a_2-a_1))=z_1, w(a_1-\delta_1(a_2-a_1)=z_2\}
$$

and

$$
P\{A_9 \mid b(a_2+\delta_2(a_2-a_1))=z_3, w(a_2+\delta_2(a_2-a_1)=z_4\}.
$$

Consider first the last probability. Now it is better to change slightly the rescaling and to consider

$$
\begin{aligned}
& b(y)=z_3+(a_2-a_1)^{1/2} B_1(Y-(1+\delta_2)), \\
& w(y)=z_4+z_3(y-(a_2+\delta_2(a_2-a_1)))+(a_2-a_1)^{3/2} W_1(Y-(1+\delta_2)), \\
& \quad Y \geq 1+\delta_1.
\end{aligned}
$$

The probability distribution of the processes $B_1(Y)$, $W_1(Y)$ does not depend on $(a_2 - a_1)$. Write down the equation for $\Gamma_+(y)$ in the form $l_+(y) = \beta_1^+ + \beta^+ y$, $y \geq a_2 + \delta_2(a_2 - a_1)$. Then the inequality

$$w(y) \geq \beta_1^+ + \beta^+ y, \quad y \geq a_1 + \delta_2(a_2 - a_1) \tag{3}$$

can be rewritten as follows. Since

$$\begin{aligned} w(y) &= \int_0^y (b(\eta) + \eta)\, d\eta \\ &= w(a_2 + \delta_2(a_2 - a_1)) \\ &\quad + b(a_2 + \delta_2(a_2 - a_1)) \cdot (y - (a_2 + \delta_2(a_2 - a_1))) \\ &\quad + (a_2 + \delta_2(a_2 - a_1)) \cdot (y - (a_2 + \delta_2(a_2 - a_1))) \\ &\quad + \int_0^{y-(a_2+\delta_2(a_2-a_1))} (b_1(\eta) + \eta)\, d\eta \\ &= z_4 + (z_3 + a_2 + \delta_2(a_2 - a_1))(Y - (1 + \delta_2)) \\ &\quad + \int_0^{(a_2-a_1)(Y-(1-\delta_2))} (b_1(\eta) + \eta)\, d\eta\,, \end{aligned}$$

where $b_1(\eta) = b(\eta + a_2 + \delta_2(a_2 - a_1)) - z_3$, the inequality (3) takes the form

$$\begin{aligned} &z_4 - (\beta_1^+ + \beta^+(a_2 + \delta_2(a_2 - a_1))) + (z_3 + (a_2 + \delta_2(a_2 - a_1)) - \beta^+) \\ &\quad \cdot (a_2 - a_1) \cdot (Y - (1 + \delta_2)) \\ &\quad + (a_2 - a_1)^{3/2} \cdot \int_0^{Y-(1+\delta_2)} (b_2(\eta) + (a_2 - a_1)^{1/2}\eta)\, d\eta > 0\,, \end{aligned} \tag{4}$$

for all Y such that $Y \geq 1 + \delta_2$, $b_2(\eta) = (a_2 - a_1)^{1/2} b_1((a_2 - a_1)^{-1}\eta)$ and has the same distribution as the initial Brownian motion. Now we see that all terms in (4) are of order $(a_2 - a_1)^{3/2}$. Indeed,

$$\begin{aligned} z_4 - (\beta_1^+ + \beta^+(a_2 + \delta_2(a_1 + a_2))) &= Z_4(a_2 - a_1)^{3/2}\,, \\ z_3 + (a_2 + \delta_2(a_2 - a_1) - \beta^+) &= z_3 + a_2 + \delta_2(a_2 - a_1) - \beta + (\beta - \beta^+) \\ &= (a_2 - a_1)^{1/2}(Z_3 + C) = (a_2 - a_1)^{1/2} Z_3^{(1)}\,, \end{aligned}$$

where C depends only on δ_1, δ_2, δ_3.

All these relations explain the reason for our rescaling. Thus we come to the dimensionless expression of (3) and (4)

$$Z_4 + Z_3^{(1)}(Y - (1 + \delta_2)) + \int_0^{Y-(1+\delta_2)} (b_2(\eta) + (a_2 - a_1)^{1/2}\eta) \geq 0\,. \tag{5}$$

It follows from (A_8) that $Z_3^{(1)} \geq 0$, $Z_4 > 0$. The probability (5) was in fact estimated in [5] and it was shown in [5] that it is not less than $\mathrm{const}(a_2 - a_1)^{1/4}$ (one should put $\sigma = (a_2 - a_1)^{1/2}$ in Theorem 7 in [5] where const depends on δ_j, $1 \leq j \leq 4$.

The estimation of the conditional probability of (A_1), provided that $w(a_1 - \delta_1(a_2 - a_1)$, $b(a_1 - \delta_1(a_2 - a_1)$ are given, is done in a similar way (see Theorem 7′ in [5]). It is also bounded from below by const $(a_2 - a_1)^{1/4}$.

The probability Q arises from the integration over z_1, z_2 in (2). Thus the theorem is proven.

Remark. The function $F(\delta_1, \delta_2, \delta_3, \delta_4, a_1)$ shows in fact some dependence between δ_4 and δ_1, δ_2. The meaning of this dependence is quite clear. If δ_3 is relatively large and we integrate in (2) over a domain of large values of z_2 and z_3 then it becomes highly probable that $w(y)$ intersects the low side of Π_1 or Π_2 and thus the conditional probability of the right behavior inside the interval $a_1 + \delta_1(a_2 - a_1)$, $a_2 - \delta_2(a_2 - a_1)$ becomes small.

Now we are going to obtain a similar estimate from above. Assume that as above the strips Π_1, Π_2, the line Γ and two parallelograms Π_1, Π_2 are given (see Fig. 3).

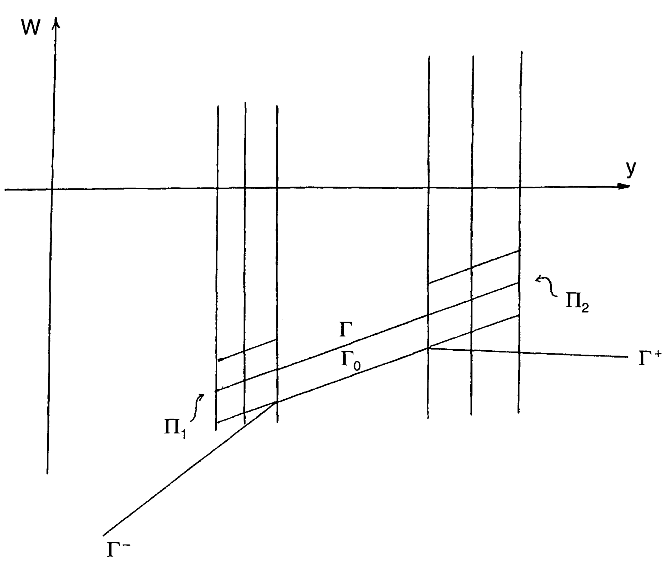

Fig. 3

Remark that for the fixed a_1, a_2, Γ the determining parameters for the whole construction are only $\delta_1, \delta_2, \delta_3$. Also we have to assume that $(0, 0)$ lies above Γ.

Theorem 4. *Suppose that a_1, a_2, Γ are given. Then for all sufficiently small $\delta_1, \delta_2, \delta_3$ the probability P that C_w has a segment whose left endpoint belongs to Π_1 while the right endpoint belongs to Π_2 satisfies the inequality*

$$P \leq F_1(\delta_1, \delta_2, \delta_3, a_1, \Gamma)(a_2 - a_1)^{1/2} \cdot (a_2 - a_1)^2 \cdot \delta_1 \delta_3$$

for all sufficiently small $(a_2 - a_1)$. *Here* $F_1(\delta_1, \delta_2, \delta_3, a_1, \Gamma)$ *is a positive constant.*

Proof. We use the same notation $l(y) = \beta y + \beta_1$ for the straight line Γ. Introduce also the ray Γ^+ passing through the points

$$\left(a_1 + \delta_1(a_2 - a_1), l(a_1 + \delta_1(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}\right)$$

and

$$(a_2 - \delta_2(a_2 - a_1),\ l(a_2 - \delta_2(a_2 - a_1)) - \delta_3(a_2 - a_1)^{3/2}),\\ y \geq a_2 - \delta_2(a_2 - a_1),$$

and the ray Γ^- passing through the points

$$\left(a_1 + \delta_1(a_2 - a_1),\ l(a_1 + \delta_1(a_2 - a_1)) - \delta_3(a_2 - a_1)^{3/2}\right)$$

and

$$\left(a_2 - \delta_2(a_2 - a_1),\ l(a_2 - \delta_2(a_2 - a_1)) + \delta_3(a_2 - a_1)^{3/2}\right),\\ y \leq a_1 + \delta_1(a_2 - a_1).$$

Also Γ_0 is the straight segment given by the equation

$$w = l(y) - \delta_3(a_2 - a_1)^{3/2}, a_1 + \delta_1(a_2 - a_1) \leq y \leq a_2 - \delta_2(a_2 - a_1).$$

Geometrically it is also clear that if C_w has a segment with the endpoints in Π_1 and Π_2 then w lies above Γ^+ for $y \geq a_2 - \delta_2(a_2 - a_1)$, above Γ^- for $y \leq a_1 + \delta_1(a_2 - a_1)$ and above Γ_0 for $a_1 + \delta_1(a_2 - a_1) \leq y \leq a_2 - \delta_2(a_2 - a_1)$. Now we denote $z_1', z_2', z_1'', z_2'', z_3', z_4', z_3'', z_4''$ the values of $b(a_1 - \delta_1(a_2 - a_1))$, $w(a_1 - \delta_1(a_2 - a_1))$, $b(a_1 + \delta_1(a_2 - a_1))$, $w(a_1 + \delta_1(a_2 - a_1))$, $b(a_2 - \delta_2(a_2 - a_1))$, $w(a_2 - \delta_2(a_2 - a_1))$, $b(a_2 + \delta_2(a_2 - a_1))$, $w(a_2 + \delta_2(a_2 - a_1))$, respectively. Using again the Markov property of the two-dimensional random process $b(y)$, $w(y)$ we can write

$$\begin{aligned} P \leq \int & p(0, 0; z_1', z_2'; 0, a_1 - \delta_1(a_2 - a_1)) \cdot p(z_1', z_2'; z_1'', z_2''; a_1 - \delta_1(a_2 - a_1), \\ & a_1 + \delta_1(a_2 - a_1)) \cdot p(z_1'', z_2''; z_3', z_4'; a_1 + \delta_1(a_2 - a_1), a_2 - \delta_2(a_2 - a_1)) \\ & \cdot p(z_3', z_4'; z_3'', z_4''; a_2 - \delta_2(a_2 - a_1), a_2 + \delta_2(a_2 - a_1)) \\ & \cdot P_1(z_1', z_2') \cdot P_2(z_1', z_2', z_1'', z_2'') \cdot P_3(z_1'', z_2'', z_3', z_4') \\ & \cdot P_4(z_3', z_4'; z_3'', z_4''; a_2 - \delta_2(a_2 - a_1), a_2 + \delta_2(a_2 - a_1)) \\ & \cdot P_5(z_3'', z_4'') \prod_{i=1}^{4} dz_1' \, dz_i''. \end{aligned}$$

Here $P_1(z_1', z_2')$ is the conditional probability that $w(y)$ lies above Γ^- for $y \leq a_1 - \delta_1(a_2 - a_1)$ under the conditions $b(a_1 - \delta_1(a_2 - a_1)) = z_1'$, $w(a_1 - \delta_1(a_2 - a_1)) = z_2'$; $P_2(z_1', z_2', z_1'', z_2'')$ is the conditional probability that $w(y)$ lies above the corresponding part of Γ_0 and intersects the upper side of Π_1 under the conditions $b(a_1 - \delta_1(a_2 - a_1)) = z_1'$, $w(a_1 - \delta_1(a_2 - a_1)) = z_2'$, $b(a_1 + \delta_1(a_2 - a_1)) = z_1''$,

$w(a_1 + \delta_1(a_2 - a_1)) = z_2''$, $P_3(z_1'', z_2'', z_3', z_4')$ is the conditional probability that $w(y)$ lies above Γ_0 for $a_1 + \delta_1(a_2 - a_1) \le y \le a_2 - \delta_2(a_2 - a_1)$ under the conditions $b(a_1 + \delta_1(a_2 - a_1)) = z_1''$, $w(a_1 + \delta_2(a_2 - a_1)) = z_2''$, $b(a_2 + \delta_2(a_2 - a_1)) = z_3'$, $w(a_2 - \delta_2(a_2 - a_1)) = z_4'$; P_4 is the analogous conditional probability with respect to Π_2 as P_2; P_5 is the conditional probability that $w(y)$ lies above Γ^+ for $y \ge a_2 + \delta_2(a_2 - a_1)$ under the conditions $b(a_2 + \delta_2(a_2 - a_1)) = z_3''$, $w(a_2 - a_1)) = z_4''$.

Introduce again the dimensionless values:

$$
\begin{aligned}
z_1' &= \beta + Z_1'(a_2 - a_1)^{1/2},\\
z_2' &= l(a_1 - \delta_1(a_2 - a_1)) + Z_2'(a_2 - a_1)^{3/2},\\
z_1'' &= \beta + Z_1''(a_2 - a_1)^{1/2},\\
z_2'' &= l(a_1 + \delta_1(a_2 - a_1)) + Z_2''(a_2 - a_1)^{3/2},\\
z_3' &= \beta + Z_3'(a_2 - a_1)^{1/2},\\
z_4' &= l(a_2 - \delta_2(a_2 - a_1)) + Z_4'(a_2 - a_1)^{3/2},\\
z_3'' &= \beta + Z_3''(a_2 - a_1)^{1/2},\\
z_4'' &= l(a_2 + \delta_2(a_2 - a_1)) + Z_4''(a_2 - a_1)^{3/2}.
\end{aligned}
$$

For $Z_j', Z_j'' = O(1)$, $1 \le j \le 4$, we can use the same arguments as above. In particular, $P_1(z_1', z_2') = O(1) \cdot (a_2 - a_1)^{1/4}$, $P_5(z_3', z_4') = O(1) \cdot (a_2 - a_1)^{1/4}$ (see Theorem 7, 7′ in [5] with $\sigma = (a_2 - a_1)^{1/2}$).

In order to estimate $P_2(z_1', z_2', z_3', z_4')$ consider two cases:

In the first case, $c \le z_2' \le l(a_1 - \delta_1(a_2 - a_1)) + 2\delta_3(a_2 - a_1)^{3/2}$, where c is the vertical coordinate of the intersection Γ^- and $y = a_1 - \delta_1(a_2 - a_1)$. It is easy to see that $P_2(z_1', z_2'; z_1'', z_2'')$ decays faster than exponentially as a function of Z_2'. In the second case $z_2' \ge l(a_1 - \delta_1(a_2 - a_1)) + 2\delta_3(a_2 - a_1)^{3/2}$ and the conditional probability decays faster than exponentially also as a function Z_1'. More exact estimations of the remainder terms which together with the estimate const $(a_2 - a_1)^2\delta_1 \cdot \delta_3$ of the integral over z_1', z_2' lead to the statement of the theorem. They will be given in another publication.

Theorem 5. *Let two intervals be given* $I_1 = \{y : |y - a_1| \le \delta_1(a_2 - a_1)\}$, $I_2 = \{y : |y - a_2| \le \delta_2(a_2 - a_1)\}$, $a_1 \ge$ const, *and the corresponding vertical strips* $\Delta_1 = \{(y, w) \mid y \in I_2\}$, $\Delta_2 = \{(y, w) \mid y \in I_2\}$. *Then the probability* P *that* C_w *has a segment whose endpoints lie inside* Δ_1 *and* Δ_2 *respectively satisfies the inequalities*

$$
F_3(\delta_1, \delta_2, a_1)(a_2 - a_1)^{1/2} \le P \le F_4(\delta_1, \delta_2, a_1)(a_2 - a_1)^{1/2},
$$

where F_3, F_4 *are positive constants.*

Proof. The estimation from below follows easily from Theorem 3 by summation over parallelograms Π_1, Π_2. In order to get the estimation from above cover the vertical line passing through a_1 by equal intervals U_j of the length $\delta_3(a_2 - a_1)^{3/2}$.

and cover the axis of angles by equal intervals Φ_s of the length $\delta_4(a_2 - a_1)^{1/2}$. In both cases the coverings are chosen so that each point is covered by at most two elements of the covering. Having the centers u_j of U_j and β_s of Φ_s we can construct the corresponding parallelograms Π_1, Π_2. If C_w has a segment with the endpoints inside Δ_1 and Δ_2 then these points lie inside at least one pair Π_1, Π_2 if δ_3 and δ_4 are sufficiently small and δ_4 is smaller than δ_3, i.e. $\delta_4 \ll \delta_3$.

The estimation of P follows from the estimation of Theorem 4 by summation over j and s.

We use the results of Theorem 3–5 in the next section for the estimation of the Hausdorff dimension of $S(b)$. However they are of more general importance because they describe some statistical properties of small shocks in solutions of the Burgers equation. One can find in [1] numerical results which are in a perfect agreement with the estimations of Theorem 4 and 5.

4. The Hausdorff Dimension of the Set $S(b)$

Take a realization

$$w(y) = \int_0^y (b(\eta) + \eta)\, d\eta\,,$$

its convex hull C_w and the closed set $S(b)$ of such $\bar{y}$ that the tangent line $w = (b(\bar{y}) + \bar{y})(y - \bar{y}) + w(\bar{y})$ intersects the graph of $w(y)$ only at the point $\bar{y}, w(\bar{y})$. In this section we study the Hausdorff dimension of $S(b) \cap [a', a'']$ for any segment $[a', a'']$, $0 < a' < a'' < \infty$.

We begin with the estimation of the fractal dimension from above, which is usually simpler. Our arguments are based upon the following lemma. Let S be a closed subset of $[a', a'']$, $O = [a', a''] \setminus S$ be its open complement. Assume that the Lebesgue measure $l(S) = 0$ and O_j are open connected components of O. Denote by N_k the number of those O_j, for which $\frac{1}{2^{k+1}} \leq l(O_j) \leq \frac{1}{2^k}$, $k = 0, 1, 2, \ldots$

Lemma 3. *If for some $c, 0 < c < 1$, and any $\delta > 0$ the numbers $N_k \leq 2^{k(c+\delta)}$ for all sufficiently large k, then the Hausdorff dimension $d(S) \leq c$.*

Proof. Fix δ and take all sufficiently large k. We define $\frac{1}{3 \cdot 2^k}$ coverings of S in the following way. Denote

$$S^{(k)} = [a', a''] \setminus O^{(k)},$$

where $O^{(k)} = \bigcup_{l(O_j) \geq \frac{1}{2^k}} O_j$. The set $S^{(k)}$ is the union of closed segments. Two neighboring open components of each segment consisting of deleted segments O_j have lengths not less than $\frac{1}{2^k}$. Cover each segment of $S^{(k)}$ by intervals of the length $\frac{1}{3 \cdot 2^k}$ in such a way that each point is covered by at most two segments. Denote the intervals of our $\frac{1}{3 \cdot 2^k}$ covering by $U_s^{(k)}$. Then for any $c_1 > c + \delta$,

$$\sum_s [l(U_s^{(k)})]^{c_1} = \frac{1}{(3 \cdot 2^k)^{c_1 - 1}} \sum_s l(U_s^{(k)})$$
$$\leq \frac{2}{(3 \cdot 2^k)^{c_1 - 1}} \left[l(S^{(k)}) + \frac{2}{3 \cdot 2^k} \sum_{j=1}^{k} N_j \right]. \tag{6}$$

The last term appears because we must take into account the lengths of parts of $U_s^{(k)}$ belonging to $O^{(k)}$. Since $l(S) = 0$,

$$l(S^{(k)}) \leq \sum_{O_j \not\subset O^{(k)}} l(O_j) \leq \sum_{p \geq k} \frac{1}{2^p} N_p \leq \frac{\text{const}}{2^{k(1-c-\delta)}}.$$

Also

$$\frac{1}{2^k} \sum_{j=1}^{k} N_j \leq \frac{\text{const}}{2^{k(1-c-\delta)}}.$$

This yields (see (6))

$$\sum_s [l(U_s^{(k)})]^{c_1} \leq \frac{\text{const}}{2^{k(c_1 - c - \delta)}} \to 0$$

as $k \to \infty$. Therefore $d(S) \leq c_1$ and $d(S) \leq c$ because δ is arbitrary. The Lemma is proven.

Return to our set $S(b) \cap [a', a'']$. Lemma 4 shows that in order to estimate its Hausdorff dimension from above we have to estimate the numbers N_k.

Lemma 4. *For any $\delta > 0$ with probability 1,*

$$N_k \leq 2^{k\left(\frac{1}{2} + \delta\right)}$$

for all sufficiently large k.

Corollary. *With probability 1 the Hausdorff dimension $d(S(b) \cap [a', a'']) \leq \frac{1}{2}$.*

Corollary follows directly from Lemma 3 and 4.

Proof of Lemma 4. Fix k and choose a sufficiently large M. Decompose the segment $[a', a'']$ onto equal segments V_j of the length $\frac{1}{2^k M}$ and consider the pairs V_{j_1}, V_{j_2} such that the distance between their centers is

$$\frac{p}{M} \cdot \frac{1}{2^k}, \quad \frac{M-1}{2} \leq p \leq M.$$

Let also $C_{j_1 j_2}$ be the event consisting of such Brownian trajectories $b(\eta)$, such C_w has an interval whose endpoints have projections in V_{j_1} and V_{j_2}. If $\chi_{j_1,j_2}(b)$ is the indicator of the event $C_{j_1 j_2}$ then

$$N_k \leq \sum \chi_{C_{j_1 j_2}} .$$

It follows from Sect. 3 that the expectation $E\chi_{C_{j_1 j_2}} \leq \frac{\text{const}}{2^{\frac{k}{2}}}$ and thus

$$EN_k \leq \text{ const} \cdot M^2 \cdot 2^{\frac{k}{2}} .$$

From Chebyshev's inequality

$$P\{N_k \geq 2^{\frac{k}{2}(1+\delta)}\} \leq \text{ const}\, M^2 \cdot \frac{2^{\frac{k}{2}}}{2^{\frac{k}{2}(1+\delta)}} = \text{const}\, M^2 \cdot 2^{-\frac{k}{2}\delta} .$$

Therefore for any $\delta > 0$ the series

$$\sum P\{N_k \geq 2^{\frac{k}{2}(1+\delta)}\} < \infty .$$

In view of the Borel-Cantelli lemma for a.e. b one can find $k_0(b)$ such that for all $k \geq k_0(b)$ we shall have $N_k < 2^{\frac{k}{2}(1+\delta)}$, Q.E.D.

The estimation of the Hausdorff dimension from below is based upon Frostman's lemma. For the convenience of a reader we give its formulation adapted to our case.

Frostman's lemma (see [6]). Assume that one can find a finite measure μ concentrated on $S(b) \cap [a', a'']$ and such that for some $t > 0$,

$$\iint \frac{d\mu(y_1)\, d\mu(y_2)}{|y_1 - y_2|^t} < \infty .$$

Then the Hausdorff dimension $d(S(b) \cap [a', a'']) \geq t$.

We need some extra notations. Let O_{kj} be such components of O that

$$\frac{1}{2^k} \leq l(O_{k,j}) < \frac{1}{2^{k-1}} .$$

Also $S_j^{(k)}$ are closed components of $S(k)$. Introduce the measure μ_k which is the normed uniform measure on $S(k)$. Then for some subsequence $\{k_j\}$ the measures μ_{k_j} converge weakly to a limit which we shall denote by μ. Our purpose now is to show that for any $t < \frac{1}{2}$ the integral

$$\iint \frac{d\mu(y_1)\, d\mu(y_2}{|x - y|^t} < \infty .$$

Certainly it is sufficient to show that for any $\delta > 0$

$$\iint\limits_{|x-y|>\delta} \frac{d\mu(x)\, d\mu(y)}{|x-y|^t} \le A\,,$$

where a constant A does not depend on δ. In our case

$$\max_j \operatorname{diam}(S_j^{(i)}) = p_i \to 0 \quad \text{as} \quad i \to \infty\,.$$

We shall show that

$$I_m^{(i)} = \sum_{j_1<j_2} \iint\limits_{x\in S_{j_1}^{(i)},\, y\in S_{j_2}^{(i)}} \frac{d\mu_m(x)\, d\mu_m(y)}{|x-y|^t} \le A\,,$$

for all sufficiently large i. (It needs some extra efforts to show that it is sufficient for our purposes.) The last sum can be written in the following way:

$$I_m^{(i)} = \sum_{p=1}^{i} \sum_j \sum_{S_{j_1}^{(i)}, S_{j_2}^{(i)} \subset S_j^{(p)}} \iint\limits_{x\in S_{j_1}^{(i)},\, y\in S_{j_2}^{(i)}} \frac{d\mu_m(x)\, d\mu_m(y)}{|x-y|^t}\,.$$

Now we remark that different $S_{j_1}^{(i)}$, $S_{j_2}^{(i)}$ lying inside a segment $S_j^{(p)}$ are separated form each other by at least one interval $O_\gamma^{(p-1)}$. Therefore

$$|x-y| \ge \frac{\text{const}}{2^p} \quad \text{for } x \in S_{j_1}^{(i)},\ \ y \in S_{j_2}^{(i)}$$

and

$$I_m^{(i)} \le \text{const} \sum_{p=1}^{i} 2^{pt} \sum_j (\mu_m(S_j^{(p)}))^2\,. \tag{8}$$

We shall need the following three statements which we shall formulate as separate lemmas.

Lemma 5. *For* a.e. b *and any* $\delta > 0$ *one can find* $k_0(b,\delta)$ *such that*

$$N_k \ge 2^{\frac{k}{2}(1+\delta)}$$

for all $k \ge k_0(b,\delta)$.

Lemma 6. *For* a.e. b *and any* $\beta > 0$ *one can find such* $i_0(b,\beta)$ *that for all* $i \ge i_0(b,\beta)$ *the length of each* $S_j^{(i)}$ *is not more than* $\frac{1}{2})^{i(1-\beta)}$.

The next lemma gives a possibility to compare $\mu_k(S_j^{(i)})$ with the length $l(S_j^{(i)})$. Denote by $N_p(S_j^{(i)})$ the number of intervals $O_{p,k} \subset S_j^{(i)}$, $p > 1$, and write $N_p(S_j^{(i)}) = 2^{\frac{p}{2}} \cdot l(S_j^{(i)}) 2^{i/2(1+\varepsilon)} \gamma_p(S_j^{(i)})$, where $\varepsilon > 0$ will be chosen later.

Since $O_j^{(p)} \subset S_i^{(p-1)}$ for some i and $\cap_p \cup_{S_k^{(p)}} \subset S_j^{(i)}$ has measure zero we conclude that for each p the length of the union over k of all $S_k^{(p)} \subset S_j^{(i)}$ is equal to the sum

$$\sum_{p_1 > p} \sum_{O_{p_1,l} \subset S_j^{(i)}} l(O_{p_1,l}) \leq \text{const} \cdot l(S_j^{(i)}) \cdot 2^{\frac{i}{2}}(1+\varepsilon) \sum_{p_1 > p} 2^{-\frac{1}{2}p_1} \cdot \gamma_{p_1}(S_j^{(i)})$$

$$= \text{const}\ l(S_j^{(i)}) \cdot 2^{\frac{i}{2}(L+\varepsilon)} \cdot 2^{-\frac{p}{2}} \cdot \sum_{p_1 > p} 2^{-\frac{(p_1-p)}{2}} \gamma_{p_1}(S_j^{(i)}) .$$

From Lemma 5 it follows that

$$\mu_p(S_j^{(i)}) \leq \frac{1}{\sum_j l(O_{p+1,j})} \cdot \text{const} \cdot 2^{\frac{i}{2}(1+\varepsilon)} \cdot l(S_j^{(i)}) \cdot 2^{-\frac{p}{2}} \cdot \sum_{p_1 > p} 2^{-\frac{(p_1-p)}{2}} \gamma_{p_1}(S^{(i)})$$

$$\leq \text{const} \cdot 2^{\frac{i}{2}\left(1+\frac{\varepsilon}{2}\right)} l(S_j^{(i)}) \cdot \sum_{p_1 > p} 2^{\frac{(p_1-p)}{2}} \gamma_{p_1}(S_j^{(i)}) .$$

Putting this inequality in (8) and using Lemmas 5 and 6 we get

$$I_k^{(i)} \leq \text{const} \sum_{q=1}^{i} 2^{qt} \cdot 2^{\frac{1}{2}q\left(1+\frac{\varepsilon}{2}\right)} \cdot \sum_j \mu_k(S_j^{(q)}) l(S_j^{(q)}) \sum_{p_1 > p} 2^{-\frac{(p_1-p)}{2}} \gamma_{p_1}(S_j^{(q)})$$

$$\leq \text{const} \sum_{p_1 > p} 2^{-\frac{(p_1-p)}{2}} \sum_{q=1}^{l} 2^{tq} \cdot 2^{\frac{q}{2(1+\varepsilon)}} \cdot 2^{-\frac{p}{1-\beta}} \cdot \sum_j \mu_p(S_j^{(q)}) \gamma_{p_1}(S_j^{('q)}) .$$

Choose β and ε so that

$$2^t \cdot 2^{\frac{2+\varepsilon}{4}} \cdot 2^{-1+\beta} = \delta < 1 .$$

Then

$$I_k^{(i)} \leq \text{const} \sum_{q=1}^{i} \delta^q \sum_{p_1 > p} 2^{-\frac{(p_1-p)}{2}} \sum_j \mu_p(S_j^{(q)}) \cdot \gamma_{p_1}(S_j^{(q)}) .$$

Lemma 7. *There exists such a constant B depending on ε such that for any $p, p_1 > p$,*

$$E\left(\sum_j \mu_k(S_j^{(p)}) \gamma_{k_1}(S_j^{(p)}\right) \leq B .$$

From Lemma 7 it follows that

$$E\left(\sum_{p_1>p} 2^{-\frac{(p_1-p)}{2}} \sum_j \mu_p(S_j^{(q)}) \cdot \gamma_{p_1}(S_j^{(q)})\right) \leq \text{ const } B\,.$$

Using Fatou's lemma we can find infinite subsequences $\{l_j\}$ and $\{\bar{p}_j\}$ that

$$\lim_{s\to\infty} \sum_{q=1}^{l_s} \delta^q \sum_{p_1>\bar{p}_s} 2^{-\frac{(p_1-\bar{p}_s)}{2}} \sum_j \mu_{\bar{p}_s}(S_j^{(q)}) \cdot \gamma_{p_1}(S_j^{(q)})$$

is finite. This gives a desired result.

The proofs of Lemmata 5, 6, 7 are straightforward but lengthy. They will be published elsewhere.

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