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Large Time Behavior of Classical N-body Systems

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Abstract. Asymptotic properties of solutions of N-body classical equations of motion are studied.

1. Introduction

A system of N classical particles interacting with pair potentials can be described with a Hamiltonian of the form

$$H = \sum_{i=2}^{N} \frac{1}{2m_i} \xi_i^2 + \sum_{i>j=1}^{N} V_{ij}(x_i - x_j) \tag{1.1}$$

defined on the phase space $X \times X'$, where $X = \mathbb{R}^{3N}$ and X' is its conjugate space. Following Agmon [A] it has become almost standard in the mathematically oriented literature to replace (1.1) with an essentially more general class of Hamiltonians, sometimes called generalized N-body Hamiltonians. They are functions on $X \times X'$ of the form

$$H = \frac{1}{2}\xi^2 + \sum_{a \in \mathscr{A}} V^a(x^a), \tag{1.2}$$

where X is a Euclidean space, $\{X^a : a \in \mathscr{A}\}$ is a family of subspaces closed wrt the algebraic sum and containing $\{0\}$, and x^a denotes the orthogonal projection of x onto X^a. It is easy to see that after a change of coordinates any Hamiltonian of the form (1.1) belongs to the class (1.2).

Typical assumptions imposed in the literature on the potential are

$$|\partial^\alpha V^a(x^a)| < c_\alpha \langle x^a \rangle^{-\mu-|\alpha|}, \tag{1.3}$$

where $\mu > 0$. If $\mu > 1$ then we say that the potentials are short range, otherwise they are long range. Note that (1.2) has an obvious quantum analog, which is the

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self-adjoint operator on the Hilbert space $L^2(X)$ obtained from (1.2) by replacing ξ^2 with $-\Delta$.

In the two-body case (which for Hamiltonians of the form (1.2) means that the set $\mathscr{A}$ consists of just two elements) scattering theory is well understood both for classical [Sim, RS, vol. III, He] and quantum systems (see e.g. [Hö, vol. II and IV, De2, Sig, IKi, Pe] and references therein). It is also known that there is a deep analogy between these two cases. We will see that in N-body systems in some aspects this analogy persists, whereas in other aspects quantum systems seem to be better behaved than classical ones.

As far as we know in the literature there are very few rigorous results on the scattering theory of classical N-body systems. One of these results belongs to Hunziker [Hu]. The property that he proved gives a fairly detailed description of the asymptotic motion of classical N-body systems and it is a reasonable candidate for the name "asymptotic completeness" in the classical case; unfortunately, in his proof he had to assume that all the potentials V^a have a compact support in X^a.

On the other hand, scattering theory for quantum N-body systems has been the subject of quite successful research in recent years. One of the first considerable achievements in this area was the proof of the asymptotic completeness of 3-body systems for $\mu > \sqrt{3} - 1$ due to Enss [E1, 2]. Note that $\sqrt{3} - 1 < 1$, hence Enss's proof applies to short range potentials and to a large subclass of long range potentials. Another breakthrough was the proof of the asymptotic completeness in the short range case for any number of particles [SigSof1]. Then a number of papers appeared that clarified various aspects of the propagation of observables in N-body quantum systems [SigSof2, 3, Del1, 2]. A very elegant proof of the asymptotic completeness in the N-body short range case was given by Graf [Graf]. The asymptotic completeness of 4-body long range systems with $\mu = 1$ was first proven in [SigSof4]. Finally, the asymptotic completeness in the long range case with $\mu > \sqrt{3} - 1$ for an arbitrary number of particles was proven in [De3].

This paper is devoted to certain questions about the classical N-body scattering which are closely related to the concept of the asymptotic completeness. The author tries to take methods developed in the quantum case and to apply them in the classical case. It turns out that sometimes analogous results can be shown, almost with no change. In fact, in the classical case some details can be simplified and proofs become less technical. On the other hand, there are statements which can be shown in the quantum setting, whereas we doubt it if one can show their quantum analogs.

Our first result, which is directly inspired by its quantum analog, is the existence of the limit

$$\lim_{t \to \infty} t^{-1} x(t), \tag{1.4}$$

where $x(t)$ is a solution of the equation of motion of an N-body system. In the quantum case this fact follows easily by the methods of [Graf] and was first explicitly stated and proved in [De2] (see also [De3]).

The existence of the limit (1.4) enables us to classify the set of all trajectories into natural disjoint categories labelled with elements of $\mathscr{A}$. Namely, if X_a denotes the orthogonal complement of X^a then a solution $x(t)$ of the equation of motion of an N-body system will be called an a-solution if and only if

$$\lim_{t \to \infty} t^{-1} x(t) \in X_a \Big\backslash \bigcup_{X_b \not\supset X_a} X_b \,. \tag{1.5}$$

An a-solution for large time feels mainly the influence of the cluster Hamiltonian

$$H_a := \frac{1}{2}\xi^2 + \sum_{X^b \supset X^a} V^b(x^b),$$

which is simpler than H; the remaining part of interaction $I_a = H - H_a$ acts as a time dependent perturbation which decays with time. The cluster Hamiltonian H_a has the form $\frac{1}{2}\xi_a^2 + H_a$, where H^a does not depend on the "sub-a" variables. Hence in the case of the motion generated by this Hamiltonian "sup-a" and "super-a" variables evolve independently. It is of course no longer true in the case of the full Hamiltonian, nevertheless it is natural to look at these two coordinates of a-solutions separately. (Recall that "super-a" coordinates describe the intracluster motion and the "sub-a" coordinates describe the intercluster motion.)

Our second result gives an estimate on the "super-a" coordinates of an a-solution. It says that

$$|x^a(t)| < ct^{2(2+\mu)^{-1}}. \tag{1.6}$$

Note that a priori we just know that

$$\lim_{t\to\infty} t^{-1}x^a(t) = 0. \tag{1.7}$$

Thus (1.6) is an improvement of (1.7). This estimate is directly inspired by the proof of the asymptotic completeness for the long range N-body quantum problem [De3].

Note that (1.6) cannot be in general improved, as one can easily convince oneself considering a two body Hamiltonian with $V(x) = -|x|^{-\mu}$.

One should note one important difference between classical and quantum systems. In quantum systems two types of states appear: bound states and states from the continuous spectrum. In the classical case if we restrict ourselves to positive time it is natural to distinguish 3 types of solutions:

1) bounded solutions,
2) "almost-bounded solutions," that is unbounded solutions that satisfy

$$\lim_{t\to\infty} t^{-1}x(t) = 0, \tag{1.8}$$

3) "scattering solutions," that is solutions for which

$$\lim_{t\to\infty} t^{-1}x(t) \neq 0. \tag{1.9}$$

Note also, that (1.6) gives an upper bound on "almost-bounded solutions.")

So far classical results (the existence of (1.4) and the bound (1.6)) were close analogs of their quantum counterparts. Note that both of them describe properties of the evolution generated by the full Hamiltonian without reference to some other evolution. When one wants to compare two evolutions, which is the standard approach in the scattering theory, then the analogy between the classical and quantum case becomes much weaker. It is even not clear what property should be called the asymptotic completeness in the classical case. Let us list three candidates to this name.

Property I. *If $x(t)$ is an a-solution then there exists a function $\mathbb{R} \ni t \mapsto y_a(t) \in X_a$ such that $y_a(t)$ is a solution of the equations of motion generated by*

$$h_a := \frac{1}{2}\xi_a^2 + I_a(x_a)$$

and

$$\lim_{t\to\infty} x_a(t) - y_a(t) = 0 .$$

Property II. *We assume additionally that there exists a function* $\mathbb{R} \ni t \mapsto y^a(t) \in X^a$ *such that* $y^a(t)$ *is a solution of the equation of motion generated by* H^a,

$$\lim_{t\to\infty} (x^a(t) - y^a(t)) = 0$$

and

$$\lim_{t\to\infty} t^{-1} y^a(t) = 0 . \tag{1.10}$$

Property III. *As Property II except that* $y^a(t)$ *is bounded.*

Note that Property I is probably too weak to deserve the name of "asymptotic completeness." On the other hand we will show that it is true if $\mu > \sqrt{3}-1$-essentially for the same class of systems for which the asymptotic completeness is known to be true in the quantum case. Note also that in the case of short range systems ($\mu > 1$) we can replace h_a with $\frac{1}{2}\xi_a^2$ in the definition of Property I.

We will prove Property II for systems with potentials that decay faster than any exponential. This property seems to be quite close to our intuition of what the asymptotic completeless should mean. Unfortunately, in the general N-body case we do not know if Property II is true if we relax significantly the assumption of the superexponential decay of the potentials.

Anyway, it is Property III which is probably closest to the intuition of the asymptotic completeless. Unfortunately, it is seldom true due to the presence of almost-bounded trajectories. It is possible to show it if potentials are of compact support [Hu].

2. Notation

In this section we fix notation used in this article. X will denote a Euclidean space. It will have the meaning of the configuration space of an N-body system. $\{X_a : a \in \mathscr{A}\}$ is a certain family of subspaces of X closed wrt the intersection. We will assume that $X_{a_{\min}} := X$ belongs to this family. We will write $a_1 \subset a_2$ iff $X_{a_1} \supset X_{a_2}$ and $b = a_1 \cup a_2$ iff $X_b = X_{a_1} \cap X_{a_2}$. We will write $a_{\max} := \bigcup_{a\in\mathscr{A}} a$. Note that most authors assume that $x_{a_{\max}} = \{0\}$; it will not be necessary to make this assumption. If $a \in \mathscr{A}$ then $\#a$ denotes the maximal number of distinct a_i such that $a = a_n \subset \cdots \subset a_1 = a_{\max}$. We set $N := \max\{\#a : a \in \mathscr{A}\}$. Note that $\#a_{\min} = N$ and $\#a_{\max} = 1$.

The orthogonal complement of X_a in X is denoted X^a. π^a and π_a will stand for the orthogonal projections of X onto X^a and X_a respectively. We will often write x^a and x_a instead of $\pi^a x$ and $\pi_a x$.

There will also be special symbols for the sets

$$Z_a := X_a \Big\backslash \bigcup_{b\not\subset a} X_b \tag{2.1}$$

and

$$Y_a := X \Big\backslash \bigcup_{b\not\subset a} X_b . \tag{2.2}$$

The Euclidean norm of a vector x will be denoted $|x|$. Moreover, $\langle x\rangle := \sqrt{x^2+1}$. If $\varepsilon > 0$ then X_a^ε will denote $\{x : \mathrm{dist}(x, X_a) < \varepsilon\}$. We will write that $f \in \mathscr{F}$ iff f is a function on X and for any $a \in \mathscr{A}$ there exists $\varepsilon > 0$ such that f depends in X_a^ε just on x_a. $\chi(P(x))$ will denote the characteristic function of the set defined by the condition $P(x)$.

The phase space of an N-body system is $X \times X'$. An element of this space will be usually denoted (x, ξ).

We will study the motion described by a Hamiltonian of the form $H = \frac{1}{2}\xi^2 + V(x)$. Such a motion is described by a solution of the equation

$$\ddot{x}(t) = -\nabla V(x(t)) . \tag{2.3}$$

We will call (2.3) the equation of motion generated by the Hamiltonian H (the e.m.g. by H).

We assume that for every $a \in \mathscr{A}$ we are given a function $V^a \in C^1(X^a)$ such that $\lim_{|x^a|\to\infty} V^a(x^a) = 0$. We set

$$V(x) := \sum_{a\in\mathscr{A}} V^a(x^a)$$

and

$$V_a(x) := \sum_{b\subset a} V^b(x^b) .$$

We define $H := \frac{1}{2}\xi^2 + V(x)$ and $H_a := \frac{1}{2}\xi^2 + V_a(x)$. Clearly, $H = H_{a_{\max}}$. Note that $H_a = \frac{1}{2}\xi_a^2 + H^a$, where $H^a := \frac{1}{2}(\xi^a)^2 + V_a(x)$. We define $e_a := \inf V_a(x^a)$, and

$$E_a := \liminf_{|x^a|\to\infty} V_a(x^a) = \min\{e_b : b \subsetneq a\} .$$

We set $I_a := V - V_a$. We also define $h_a := \frac{1}{2}\xi_a^2 + I_a(x_a)$.

3. Main Results

Our first result says that every trajectory of an N-body system possesses an asymptotic velocity. Note that this result has a quantum analog [De2, 3] and is inspired by [Graf].

Theorem 3.1. *Assume that for every $a \in \mathscr{A}$ and some $\mu > 0$,*

$$|\nabla V^a(x^a)| \le c\langle x^a\rangle^{-1-\mu} . \tag{3.1}$$

Let $x(t)$ be a solution of the e.m.g. by H. *Then there exists*

$$\lim_{t\to\infty} t^{-1}x(t) . \tag{3.2}$$

If this limit belongs to Z_a then it equals

$$\lim_{t\to\infty} \dot{x}_a(t) . \tag{3.3}$$

Denote 3.2 by p_a^+ and set $E := H(x(t), \xi(t))$. (Clearly, E does not depend on t.) Then

$$E \ge \frac{1}{2}(p_a^+)^2 + e_a .$$

The configuration space X is the disjoint union of sets Z_a. Hence the condition

$$\lim_{t\to\infty} t^{-1}x(t) \in Z_a \tag{3.4}$$

separates the set of all trajectories into distinct categories labelled with elements of $\mathscr{A}$. Clearly, for a solution satisfying (3.4),

$$\lim_{t\to\infty} t^{-1}x^a(t) = 0\,. \tag{3.5}$$

It turns out that (3.5) can be improved, which is the subject of our next result. Also this result has a quantum analog, which is an important step in the proof of the asymptotic completeness of quantum N-body long range systems [De3].

Theorem 3.2. *Let $x(t)$ be a solution of the* e.m.g. by H *such that (3.4) holds. Let p_a^+ and E be defined as in Theorem 3.1. If*

$$E < \frac{1}{2}(p_a^+)^2 + E_a\,,$$

then $x^a(t)$ is bounded. Otherwise the following estimates are true.
a) *If for any $b \in \mathscr{A}$ $|\nabla V^b(x^b)| < c_1\langle x^b\rangle^{-\mu}$, then*

$$|x^a(t)| < ct^{2(2+\mu)^{-1}}\,. \tag{3.6}$$

b) *If for any $b \in \mathscr{A}$ there exists $\theta > 0$ such that $|\nabla V^b(x^b)| < \sigma e^{-\theta\langle x^b\rangle}$, then*

$$|x^a(t)| < c(1 + \ln t)\,. \tag{3.7}$$

c) *If for any $b \in \mathscr{A}$ V^b is compactly supported, then $x^a(t)$ is bounded.*

The above theorem gives some information on the behavior of "internal" coordinates of a trajectory. It turns out that one can say a lot more about the "external" coordinates. We will show that they are close to a solution of the e.m.g. by the Hamiltonian h_a.

Note that in the following theorem two borderline values of μ appear: $\sqrt{3}-1$ and 1. In the quantum case the first one is the borderline for the validity of the proof of the asymptotic completeness given in [De3] and $\mu = 1$ is the borderline for the existence of usual wave operators. Statements about classical systems given in Theorem 3.3 are however much more modest.

Theorem 3.3. *Suppose that $x(t)$ is a solution of the* e.m.g. by H *that satisfies (3.4). Let p_a^+ and E be defined as in Theorem 3.1. Suppose that for any $b \in \mathscr{A}$ $|\partial^\alpha V^b(x_b)| < c\langle x^b\rangle^{-|\alpha|-\mu}$ for $|\alpha| = 1,2$.*
a) *If y_a is any solution of the* e.m.g. by h_a *such that $\lim_{t\to\infty} \dot{y}_a(t) = p_a^+$ then we have:*

$$x_a(t) - y_a(t) = O(t^{\max(0,-\mu+2(2+\mu)^{-1})})\,. \tag{3.8}$$

b) *Suppose that $a = a_{\min}$ or $E < \frac{1}{2}(p_a^+)^2 + E_a$ or $\mu > \sqrt{3}-1$. Then there exists a unique solution $\tilde{y}_a(t)$ of the* e.m.g. by h_a *such that*

$$\lim_{t\to\infty}(x_a(t) - \tilde{y}_a(t)) = 0\,. \tag{3.9}$$

Moreover, $\lim_{t\to\infty} \frac{d}{dt}\tilde{y}_a(t) = p_a^+$.
c) *Let $\mu > 1$. Then there exists a unique $\tilde{y}_a^+ \in X_a$ such that*

$$\lim_{t\to\infty}(x_a(t) - \tilde{y}_a^+ - tp_a^+) = 0\,. \tag{3.10}$$

In the case of quantum N-body systems the theorem on the asymptotic completeness [SigSof1, 4, Graf, De3] gives a fairly deep description of scattering states. In the classical case it is even not clear what should be the definition of the notion to be called the asymptotic completeness. Our next theorem proposes such a definition and states that it is satisfied if potentials decay faster than any exponential.

Theorem 3.4. *Suppose that for every $b \in \mathscr{A}$ $\nabla^2 V^b(x^b)$ is bounded and for every $\theta > 0$ there exists σ such that $|\nabla V^a(x^a)| < \sigma e^{-\theta|x^a|}$. Then the following statement are true.*
a) *For any solution $y(t)$ of the* e.m.g. by H_a *such that*

$$\lim_{t\to\infty} t^{-1}y(t) \in Z_a \tag{3.11}$$

there exists a unique solution $x(t)$ of the e.m.g. by H *such that for any $\theta > 0$*

$$\lim_{t\to\infty} e^{\theta t}(x(t) - y(t)) = 0 \tag{3.12}$$

and

$$\lim_{t\to\infty} t(\dot{x}(t) - \dot{y}(t)) = 0\,. \tag{3.13}$$

b) *For any solution $x(t)$ of the* e.m.g. by H *such that*

$$\lim_{t\to\infty} t^{-1}x(t) \in Z_a\,, \tag{3.14}$$

there exists a unique solution $y(t)$ of the e.m.g. by H_a *such that for any $\theta > 0$,*

$$\lim_{t\to\infty} e^{\theta t}(x(t) - y(t)) = 0 \tag{3.15}$$

and

$$\lim_{t\to\infty} t(\dot{x}(t) - \dot{y}(t)) = 0\,. \tag{3.16}$$

Note that all the solutions of e.m.g. by H_a are of the form $y(t) = y^a(t) + y_a^+ + tp_a^+$, where $y^a(t)$ is a solution of the e.m.g. by H^a. If $y(t)$ satisfies (3.11) then $y^a(t)$ is a bounded or almost-bounded solution. If potentials are compactly suported then there are no almost-bounded solutions, as follows from Theorem 3.2c. Thus in this case all the trajectories can be asymptotically decomposed into a bounded intracluster motion and a free intercluster motion – which is probably the most intuitive candidate for the definition of the asymptotic completeness. Hence for compactly supported potentials Theorem 3.4 reduces to the result proved by Hunziker [Hu].

4. Special Observables

In this section we describe the construction of certain special observables. Actually, one could work here with the observables from [De3], which were used there in the quantum case. But in the classical case we do not need them to be differentiable, which makes their construction easier. (Because of their nondifferentiability we will have to deal with derivatives in the distributional sense, which causes no additional problems.)

Let $\varrho_1, \dots, \varrho_N > 0$ be a sequence of positive numbers. We define

$$Q_a(x) := \begin{cases} 1 & \text{if } x_a^2 + p_{\#a} > x_b^2 + \varrho_{\#b} \text{ for all } b \neq a\,, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

$$R(x) := \frac{1}{2} \max\{x_a^2 + \varrho_{\#a} : a \in \mathscr{A}\} \tag{4.2}$$

$$= \sum_{a\in\mathscr{A}} \frac{1}{2}(x_a^2 + \varrho_{\#a}) Q_a(x)\,. \tag{4.3}$$

The following proposition describes some properties of R.

Proposition 4.1. *R is a continuous convex function. Moreover:*
a) $\frac{1}{2}(x^2 + c_1) < R(x) < \frac{1}{2}(x^2 + c_2)$ *for some* $c_1, c_2 > 0$,
b) $\nabla R(x) = \sum_{a\in\mathscr{A}} x_a Q_a(x)$,
c) $\nabla^2 R(x) \geq \sum_{a\in\mathscr{A}} \pi_a Q_a(x)$,
d) *for any* $\xi \in X$

$$\xi \nabla^2 R(x) \xi - 2\nabla R(x)\xi + 2R(x) \geq \sum_{a\in\mathscr{A}} Q_a(x)\,|\xi_a - x_a|^2\,, \tag{4.4}$$

e) *if we choose appropriately* $\varrho_1, \dots, \varrho_N$ (e.g. *if* $\varrho_j := \varepsilon M^{-j}$ *for large enough* M) *then* $R \in \mathscr{F}$.

For the proof of this proposition we refer the reader to [De3]. In fact, it is straightforward, maybe except for e). Note also that this proposition is closely related to the construction of [Graf].

Next we define $r(x) := \sqrt{2R(x)}$; (see [Ya] for a similar construction).

Proposition 4.2. *$r(x)$ is a continuous convex function.*

Proof. (See also [De3]). The positivity of the left-hand side of (4.4) implies the following inequality:

$$\nabla^2 R(x) 2R(x) - \nabla R(x) \nabla R(x) \geq 0\,.$$

This implies immediately

$$\nabla^2 r(x) \geq 0\,.$$

QED

Now suppose that a function $\mathbb{R}_+ \ni t \mapsto w(t) \in \mathbb{R}_+$ has been fixed. We set

$$R_t(x) := w(t)^2 R\left(\frac{x}{w(t)}\right) \tag{4.5}$$

and

$$r_t(x) := w(t) r\left(\frac{x}{w(t)}\right)\,. \tag{4.6}$$

We also define

$$\begin{aligned} B_t(x) &:= \left(\frac{d}{dt} + \xi \frac{d}{dx}\right) \frac{R_t(x)}{t} \\ &= t^{-1} w(t) \xi \nabla R\left(\frac{x}{w(t)}\right) - t^{-2} w^2(t) R\left(\frac{x}{w(t)}\right) \\ &\quad + t^{-1} \dot{w}(t) w(t) \left(2R\left(\frac{x}{w(t)}\right) - \frac{x}{w(t)} \nabla R\left(\frac{x}{w(t)}\right)\right)\,, \end{aligned} \tag{4.7}$$

and

$$b_t(x) := \left(\frac{d}{dt} + \xi \frac{d}{dx}\right) r_t(x)$$
$$= \xi \nabla r\left(\frac{x}{w(t)}\right) + \dot{w}(t)\left(r\left(\frac{x}{w(t)}\right) - \frac{x}{w(t)} \nabla r\left(\frac{x}{w(t)}\right)\right). \tag{4.8}$$

The observables $\frac{R_t(x)}{t}$ and $r_t(x)$ are approximately convex along the trajectories (modulo terms which decay with time). This remarkable property is expressed by the following identities:

$$\left(\frac{d}{dt} + \xi \frac{d}{dx} - \nabla V(x) \frac{d}{d\xi}\right)^2 \frac{R_t(x)}{t}$$
$$= \left(\frac{d}{dt} + \xi \frac{d}{dx} - \nabla V(x) \frac{d}{d\xi}\right) B_t$$
$$= t^{-1} C_t + t^{-1} \ddot{w}(t) w(t) \left(2R\left(\frac{x}{w(t)}\right) - \frac{x}{w(t)} \nabla R\left(\frac{x}{w(t)}\right)\right)$$
$$- t^{-1} w(t) \nabla V(x) \nabla R\left(\frac{x}{w(t)}\right), \tag{4.9}$$

where

$$C_t = \left(\xi - \dot{w}(t) \frac{x}{w(t)}\right)^2 \nabla^2 R\left(\frac{x}{w(t)}\right)$$
$$- 2t^{-1} w(t)\left(1 - t\frac{\dot{w}(t)}{w(t)}\right)\left(\xi - \dot{w}(t) \frac{x}{w(t)}\right) \nabla R\left(\frac{x}{w(t)}\right)$$
$$+ 2t^{-2} w^2(t)\left(1 - \frac{t\dot{w}(t)}{w(t)}\right)^2 R\left(\frac{x}{w(t)}\right)$$
$$\geq \sum_{a \in \mathscr{A}} Q_a\left(\frac{x}{w(t)}\right)\left(\xi_a - \frac{x_a}{t}\right)^2 \geq 0. \tag{4.10}$$

Note that the second term on the right-hand side of (4.9) is $O(t^{-1}\ddot{w}(t)w(t))$ and the last term is $O(t^{-1} \sup\{|\nabla V^a(x^a)| : |x^a| > cw(t)\})$ for some $c > 0$. If $w(t) = t^\delta$ and the potentials satisfy (3.1) then these terms are $O(t^{-3+2\delta})$ and $O(t^{-1-\delta\mu})$ respectively.

Here are analogous identities for r_t:

$$\left(\frac{d}{dt} + \xi \frac{d}{dx} - \nabla V(x) \frac{d}{d\xi}\right)^2 r_t = \left(\frac{d}{dt} + \xi \frac{d}{dx} - \nabla V(x) \frac{d}{d\xi}\right) b_t$$
$$= w^{-1}(t) c_t + \ddot{w}(t)\left(r\left(\frac{x}{w(t)}\right) - \frac{x}{w(t)} \nabla r\left(\frac{x}{w(t)}\right)\right)$$
$$- \nabla V(x) \nabla V r\left(\frac{x}{w(t)}\right), \tag{4.11}$$

where

$$c_t = \left(\xi - \dot{w}(t) \frac{x}{w(t)}\right)^2 \nabla^2 r\left(\frac{x}{w(t)}\right) \geq 0. \tag{4.12}$$

Note that the second term on the right-hand side of (4.11) is $O(\ddot{w}(t))$ and the last term is $O(\sup\{|\nabla V^a(x^a)| : |x^a| > cw(t)\})$ for some $c > 0$. If $w(t) = t^\delta$ and the potentials satisfy (3.1) then these terms are $O(t^{-2+\delta})$ and $O(t^{-\delta(1+\mu)})$ respectively.

5. Existence of Asymptotic Velocity

This section is devoted to the proof of Theorem 3.1 Essentially all the arguments used in this proof are parallel to the ones used in the proof of its quantum analog [Graf, De2, 3]. At some points the commutativity of observables in the classical mechanics allows for some simplification.

In what follows $x(t)$ is an arbitrary solution of the e.m.g. by H and $\xi(t) = \dot{x}(t)$. We start with a simple lemma about the boundedness of the velocity.

Lemma 5.1. *There exists c such that* $|\xi(t)| \leq c$ *and* $|x(t)| \leq c(1+t)$.

Proof. V is bounded and H is constant on a trajectory. Hence $\frac{1}{2}\xi^2(t) = H(x(t), \xi(t)) - V(x(t))$ is bounded. Moreover

$$\frac{d^2}{dt^2}\frac{1}{2}x^2 = \xi^2(t) - x\nabla V(x) \leq c\,.$$

Hence $x^2 \leq c(1+t)^2$. QED

The next two propositions are analogs of basic propagation estimates of the Graf approach [Graf, De2, 3].

Proposition 5.2. *Let* $a \in \mathscr{A}$, $1 > \delta > 0$ *and* $\varepsilon > 0$. *Then*

$$\int_1^\infty t^{-1}\chi(x : \text{ for every } b \not\subset a\, |x^b| > \varepsilon t^\delta)\left(\frac{x_a(t)}{t} - \xi_a(t)\right)^2 dt < \infty\,. \tag{5.1}$$

Proof. Consider the observable $B^t(x, \xi)$ constructed in Sect. 4, where we set $w(t) := t^\delta$. Let $B(t) := B_t(x(t), \xi(t))$ Note that $B(t)$ is uniformly bounded. Now

$$\begin{aligned} c &\geq \int_{t_1}^{t_2} \frac{d}{dt}B(t)\,dt \\ &\geq \int_{t_1}^{t_2} \sum_{a\in\mathscr{A}} t^{-1}Q_a\left(\frac{x(t)}{t^\delta}\right)\left(\frac{x_a(t)}{t} - \xi_a(t)\right)^2 dt \\ &\quad - \int_{t_1}^{t_2} c(t^{-3+2\delta} + t^{-1-\delta\mu})\,dt\,. \end{aligned} \tag{5.2}$$

Now the second integral on the right-hand side of (5.2) is uniformly bounded, and if we choose the parameters $\varrho_1, \ldots, \varrho_N$ appropriately then the first integral will dominate the integral in (5.1). QED

Proposition 5.3. *Let* $a \in \mathscr{A}$ *and* $\varepsilon > 0$. *Then*

$$\int_1^\infty t^{-1}\chi(x : \text{ for every } b \not\subset a\, |x^b| > \varepsilon t)\left|\frac{x_a(t)}{t} - \xi_a(t)\right| dt < \infty\,. \tag{5.3}$$

Proof. We may suppose that $|x(t)| \leq c_0 t$. Choose $J \in C_0^\infty(X) \cap \mathscr{F}$ such that $\operatorname{supp} J \subset Y_a$ and $J = 1$ on

$$\{x : \text{ for every } b \not\subset a\, |x^b| > \varepsilon,\ |x| \leq c_0\}.$$

Consider

$$K_t(x,\xi) := J\left(\frac{x}{t}\right)\left|\frac{x_a}{t} - \xi_a\right|$$

and $K(t) := K_t(x(t), \xi(t))$. Clearly, $K(t)$ is uniformly bounded. Now

$$\begin{aligned}\frac{d}{dt} K(t) = &-t^{-1} J\left(\frac{x(t)}{t}\right)\left|\frac{x_a(t)}{t} - \xi_a(t)\right| \\ &+ t^{-1}\left(\frac{x(t)}{t} - \xi(t)\right)\nabla J\left(\frac{x(t)}{t}\right)\left|\frac{x_a(t)}{t} - \xi_a(t)\right| \\ &+ J\left(\frac{x(t)}{t}\right)\left|\frac{x_a(t)}{t} - \xi_a(t)\right|^{-1}\left(\frac{x_a(t)}{t} - \xi_a(t)\right)\nabla_a I_a(x(t)). \qquad (5.4)\end{aligned}$$

Next note that we can find a family of continuous functions $\{j_b : b \subset a\}$ such that $0 \leq j_b \leq 1$, $\sum_{b \subset a} j_b = 1$ on $\operatorname{supp} \nabla J$, $\operatorname{supp} j_b \subset Y_b$ and J depends just on x_b on $\operatorname{supp} j_b$. Hence the second term on the right-hand side of (5.4) equals

$$\sum_{b \subset a} t^{-1}\left(\frac{x_b(t)}{t} - \xi_b(t)\right)\nabla J\left(\frac{x(t)}{t}\right) j_b\left(\frac{x(t)}{t}\right)\left|\frac{x_a(t)}{t} - \xi_a(t)\right|.$$

This is integrable by previous proposition. The third term is $O(t^{-1-\mu})$ and hence is integrable. Consequently

$$\int_1^\infty t^{-1} J\left(\frac{x(t)}{t}\right)\left|\frac{x_a(t)}{t} - \xi_a(t)\right| dt < \infty. \qquad \text{QED}$$

Proof of Theorem 3.1. Consider first $J \in C_0^\infty(X) \cap \mathscr{F}$. Then

$$\begin{aligned}\frac{d}{dt} J\left(\frac{x(t)}{t}\right) &= -t^{-1}\left(\frac{x(t)}{t} - \xi(t)\right)\nabla J\left(\frac{x(t)}{t}\right) \\ &= \sum_{a \in \mathscr{A}} t^{-1}\left(\frac{x_a(t)}{t} - \xi_a(t)\right)\nabla J\left(\frac{x(t)}{t}\right) j_a\left(\frac{x(t)}{t}\right),\end{aligned}$$

where $0 \leq j_a \leq 1$, $\sum_{a \in \mathscr{A}} j_a = 1$, $\operatorname{supp} j_a \subset Y_a$ and J depends just on x_a on $\operatorname{supp} j_a$. This is integrable by Proposition 5.3. Hence

$$\lim_{t \to \infty} J\left(\frac{x(t)}{t}\right) \qquad (5.5)$$

exists. If $J \in C_0(X)$ is arbitrary, it can be approximated by functions from $C_0^\infty(X) \cap \mathscr{F}$. Hence the limit (5.5) exists also for such functions.

Now choose $J \in C_0(X)$ equal to one on a large enough set so that $J\left(\frac{x(t)}{t}\right) = 1$ for $t > t_0$. Now

$$\frac{x(t)}{t} = J\left(\frac{x(t)}{t}\right)\frac{x(t)}{t} + \left(1 - J\left(\frac{x(t)}{t}\right)\right)\frac{x(t)}{t}.$$

The second part of the above expression is zero and the first is convergent by the above arguments. This proves the existence of the limit (3.2). The proof of the remaining statements of Theorem 3.1 is easy and is left to the reader. QED

6. Intracluster Motion

In this section we prove Theorem 3.2 which describes an upper bound on the growth of intracluster coordinates. Throughout this section we will use quite general assumptions on the potentials. Namely, we will suppose that

$$\int_0^\infty g(s)\,ds \leq \infty,$$

where

$$g(s) := \sup\{|\nabla V^b(x^b)| : b \in \mathscr{A}, |x^b| > s\}.$$

We set

$$G(s) := \int_s^\infty g(s_1)\,ds_1.$$

Note that there exists a unique solution $\mathbb{R}_+ \ni t \mapsto w(t) \in \mathbb{R}_+$ of the equation

$$\ddot{w}(t) = -g(w(t)) \tag{6.1}$$

such that $w(0) = 0$ and

$$\lim_{t\to\infty} \frac{w(t)}{t} = 0. \tag{6.2}$$

In fact, (6.1) is the e.m.g. by the Hamiltonian $\frac{1}{2}\dot{w}^2 - G(w)$ and condition (6.2) implies that this solution satisfies $\frac{1}{2}\dot{w}^2(t) - G(w(t)) = 0$. Let us give some examples.

1) $g(w) = w^{-1-\mu}$. Then

$$w(t) = \left(\frac{(2+\mu)^2}{2(1+\mu)}\right)^{\frac{1}{\mu}} t^{2(2+\mu)^{-1}}.$$

2) $g(w) = e^{-\theta w}$. Then

$$w(t) = \frac{2}{\theta}\left(\ln\left(t + \sqrt{\frac{2}{\theta}}\right) - \ln\sqrt{\frac{2}{\theta}}\right).$$

3) Suppose that $\operatorname{supp} g = [0, s_0]$. Then there exists t_0 such that for $t \geq t_0$ we have $w(t) = s_0$.

The following theorem is a kind of generalization of Theorem 3.2.

Theorem 6.1. *Suppose that $x(t)$ is a solution of the* e.m.g. by H *such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} \in Z_a .$$

Then there exists c such that

$$|x^a(t)| \leq c(w(t)+1) . \tag{6.3}$$

Proof. Replace X with X^a throughout Sect. 4. Construct a function $r(x^a)$. By a scaling argument we can always assume that $r(x^a)$ depends just on x^b on $\{x^a : |x^b| < 1\}$ for $b \subset a$.

We will consider the function $r(t) := r_t(x^a(t))$ where $r_t(x^a) := w(t) r\left(\frac{x^a}{w(t)}\right)$ and $w(t)$ is the solution of (6.1) described at the beginning of this section. Now

$$\left|\nabla V_a(x^a) \nabla r\left(\frac{x^a}{w(t)}\right)\right| \leq c_2 g(w(t)) .$$

Moreover, by Theorem 3.1

$$|\nabla I_a(x(t))| \leq c_3 g(c_0 t) .$$

By (4.11) and (4.12) we obtain:

$$\frac{d^2}{dt^2} r(t) \geq -c_1 |\ddot{w}(t)| - c_2 g(w(t)) - c_3' g(c_0 t) . \tag{6.4}$$

We know that $\ddot{w}(t) \leq 0$ and $g(w(t)) = -\ddot{w}(t)$. Moreover, for large enough time $w(t) \leq c_0 t$. Hence for $t > t_0$

$$\frac{d^2}{dt^2} (r(t) - (c_1 + c_2 + c_3') w(t)) \geq 0 . \tag{6.5}$$

We also know that

$$\lim_{t\to\infty} t^{-1} (r(t) - (c_1 + c_2 + c_3') w(t)) = 0 . \tag{6.6}$$

Now (6.5) and (6.6) imply

$$\frac{d}{dt} (r(t) - (c_1 + c_2 + c_3') w(t)) \leq 0 . \tag{6.7}$$

Hence

$$r(t) - (c_1 + c_2 + c_3') w(t) \leq c_4 . \tag{6.8}$$

This clearly implies (6.3). QED

7. Asymptotics of Intercluster Motion

In this section we prove Theorem 3.3 which describes the asymptotics of the "sub-a" coordinate of an a-solution of the e.m.g. by H.

We start with the proof of a) which describes the most rough asymptotics valid for all μ. Let $0 < t < T$. Then the equation of motion satisfied by $x(t)$ and $y_a(t)$ imply the following integral equations:

$$x_a(t) = x_a(0) + \dot{x}_a(T)t + \left(\int_0^t s + t \int_t^T \right) \nabla_a I_a(x(s))\, ds \tag{7.1}$$

and

$$y_a(t) = y_a(0) + \dot{y}_a(T)t + \left(\int_0^t s + t \int_t^T \right) \nabla_a I_a(y_a(s))\, ds\,. \tag{7.2}$$

We subtract these two equations and let $T \to \infty$. We obtain

$$\begin{aligned} y_a(t) - x_a(t) = y_a(0) - x_a(0) + &\left(\int_0^t s + t \int_t^\infty \right) \\ &\times (\nabla_a I_a(y_a(s)) - \nabla_a I_a(x(s)))\, ds\,. \end{aligned} \tag{7.3}$$

We set $z_a(t) := y_a(t) - x_a(t)$. Note that

$$\begin{aligned} &|\nabla_a I_a(y_a(s)) - \nabla_a I_a(x(s))| \\ &\quad \le |\nabla_a I_a(x_a(s)) - \nabla_a I_a(x(s))| + |\nabla_a I_a(y_a(s)) - \nabla_a I_a(x_a(s))| \\ &\quad \le c\langle s\rangle^{-2-\mu}(\langle s\rangle^{2(2+\mu)^{-1}} + |z(s)|)\,. \end{aligned} \tag{7.4}$$

We insert (7.4) into (7.3) and obtain

$$|z_a(t)| \le x\langle t\rangle^{\max(0,-\mu+2(2+\mu)^{-1})} + \left(\int_0^t s + t \int_t^\infty \right) \langle s\rangle^{-2-\mu}|z(s)|\, ds\,. \tag{7.5}$$

We know a priori that

$$|z_a(t)| \le c\langle t\rangle\,.$$

We insert this into (7.5) and obtain

$$|z_a(t)| \le c\langle t\rangle^{\max(0,1-\mu)}\,.$$

After a sufficient number of iterations we get

$$|z_a(t)| \le c\langle t\rangle^{\max(0,-\mu+2(2+\mu)^{-1})}\,.$$

Now let us prove b). For simplicity we shall consider only the case $\mu > \sqrt{3} - 1$. Other cases are similar and simpler. Let $y_a(t)$ be any solution of the e.m.g. by h_a such that $\lim_{t\to\infty} \dot{y}_a(t) = p_a^+$ (as in a)). If $0 < t < T$ then the following identities are true:

$$\dot{x}_a(t) = \dot{x}_a(T) + \int_t^T \nabla_a I_a(x(s))\, ds$$

and

$$\dot{y}_a(t) = \dot{y}_a(T) + \int_t^T \nabla_a I_a(y_a(s))\, ds\,.$$

We subtract one from the other and let $T \to \infty$. Thus

$$\dot{y}_a(t) - \dot{x}_a(t) = \int_t^\infty (\nabla_a I_a(y_a(s)) - \nabla_a I_a(x(s)) ds . \tag{7.6}$$

Equation 7.4 and the boundedness of $|x_a(t) - y_a(t)|$ obtained in a) implies:

$$|\dot{y}_a(t) - \dot{x}_a(t)| \le c \int_t^\infty \langle s\rangle^{-2-\mu+2(2+\mu)^{-1}} ds \le c\langle t\rangle^{1-\mu+2(2+\mu)^{-1}} . \tag{7.7}$$

Now if $\mu > \sqrt{3} - 1$ then $-\mu + 2(2+\mu)^{-1} < 0$. Hence in this case the right-hand side of (7.7) is integrable. Consequently there exists

$$\lim_{t\to\infty} (y_a(t) - x_a(t)) =: y_a^+ .$$

Arguments that belong to the standard 2-body classical long range theory show that there exists another unique solution $\tilde{y}_a(t)$ of the e.m.g. by h_a such that

$$\lim_{t\to\infty} (y_a(t) - \tilde{y}_a(t)) = y_a^+$$

and

$$\lim_{t\to\infty} \frac{d}{dt} y_a(t) = \lim_{t\to\infty} \frac{d}{dt} \tilde{y}_a(t) .$$

$\tilde{y}_a(t)$ is the solution we have been looking for. This proves b).

Standard 2-body classical short range scattering theory says that if $\mu > 1$ then there exists a unique $\tilde{y}_a^+ \in X_a$ such that

$$\lim_{t\to\infty} (\tilde{y}_a(t) - tp_a^+ - \tilde{y}_a^+) = 0 .$$

This proves c). QED.

8. Asymptotic Completeness

In this section we prove Theorem 3.4. One can, somewhat loosely, describe it as a theorem about the existence of "classical wave operators" and their "asymptotic completeness." Unfortunately, the conditions that we have to impose on potentials to prove this theorem are very restrictive, namely, we have to assume that all the potentials decay faster than any exponential.

Note that the proof of Theorem 8.1 has actually little to do with the structure of N-body systems. The most important ingredients of this proof are Theorem 3.1 and the following general fact about the stability of solutions of Newton's equation perturbed with a force that decays exponentially in time.

Theorem 8.1. *Suppose that $\theta > 0$ and $\theta^2 > \kappa > 0$. Let*

$$\mathbb{R}_+ \ni t \mapsto G(t) \in X$$

and

$$\mathbb{R}_+ \times X \ni (t,z) \mapsto F(t,z) \in X$$

satisfy

$$|G(t)| \le \sigma\, e^{-\theta t}, \qquad F(t,0) = 0 \quad \textit{and} \quad |\nabla_z F(t,z)| \le \kappa .$$

Then there exists a unique solution $z(t)$ of the equation

$$\ddot{z}(t) = G(t) + F(t, z(t)) \tag{8.1}$$

such that

$$\lim_{t\to\infty} e^{\theta t} z(t) = 0 \tag{8.2}$$

and

$$\lim_{t\to\infty} t\dot{z}(t) = 0\,. \tag{8.3}$$

Proof. Let $0 < t < T$. Then

$$z(t) = z(T) - \dot{z}(T)(T-t) + \int_t^T (s-t)(G(s) + F(s, z(s)))\,ds\,. \tag{8.4}$$

If we let $T \to \infty$, use (8.2) and (8.3), then we obtain

$$z(t) = \int_t^\infty (s-t)(G(s) + F(s, z(s)))\,ds\,. \tag{8.5}$$

Introduce the Banach space

$$Z := \left\{ z \in C(\mathbb{R}_+, X) \colon \lim_{t\to\infty} e^{\theta t} z(t) = 0 \right\},$$

equipped with the norm $\|z\| := \sup_{t>0} |e^{\theta t} z(t)|$. We denote

$$Z_\gamma := \{ z \in Z \colon \|z\| \le \gamma \}\,.$$

Equation (8.5) can be rewritten as

$$z = Pz\,, \tag{8.6}$$

where

$$Pz(t) := \int_t^\infty (s-t)(G(s) + F(s, z(s)))\,ds\,. \tag{8.7}$$

Our theorem will follow immediately from the following lemma.

Lemma 8.2. *Let $\gamma \ge \dfrac{\sigma}{\theta^2 - \kappa}$. Then P maps and Z_γ into itself and is a contradiction.*

Proof. Note that $|F(t, z(t))| \le \kappa |z(t)|$ and

$$\int_t^\infty (s-t) e^{-\theta s} ds = \theta^{-2} e^{-\theta t}\,.$$

Thus

$$|Pz(t)| \le (\sigma + \kappa\gamma)\theta^{-2} e^{-\theta t} \le (\gamma(\theta^2 - \kappa) + \gamma\kappa)\theta^{-2} e^{-\theta t} = \gamma e^{-\theta t}\,.$$

This shows that P maps Z_γ into itself.

Now

$$|Pz_1(t) - Pz_2(t)| \le \int_t^\infty (s-t)\,(\sup|\nabla F(s,z)|)\,|z_1(s) - z_2(s)|\,ds$$

$$\le \int_t^\infty (s-t)\,\kappa e^{-\theta s}\|z_1 - z_2\|\,ds = \kappa\theta^{-2}e^{-\theta s}\|z_1 - z_2\|\,.$$

Hence

$$\|Pz_1 - Pz_2\| \le \kappa\theta^{-2}\|z_1 - z_2\|\,.$$

This shows that P is a contraction. QED

The following corollary of Theorem 8.1 describes how one can compare solutions of two Newton's equations.

Corollary 8.3. *Suppose that we are given functions*

$$\mathbb{R}_+ \ni t \mapsto x_1(t) \in X\,, \qquad X \ni x \mapsto F_1(x) \in X\,,$$

and

$$X \ni x \mapsto F_2(x) \in X\,.$$

Suppose that $\theta > 0$, $\theta^2 > \kappa$,

$$|\nabla F_2(x)| \le \kappa\,, \qquad |F_2(x_1(t)) - F_1(x_1(t))| \le \sigma e^{-\theta t}$$

and

$$\ddot{x}_1(t) = F_1(x_1(t))\,.$$

Then there exists a unique solution $\mathbb{R}_+ \mapsto x_2(t)$ *of the equation*

$$\ddot{x}_2(t) = F_2(x_2(t))$$

such that

$$\lim_{t\to\infty} e^{\theta t}(x_1(t) - x_2(t)) = 0$$

and

$$\lim_{t\to\infty} t(\dot{x}_1(t) - \dot{x}_2(t)) = 0\,.$$

Proof. We set $z(t) := x_2(t) - x_1(t)$ and obtain the following equation:

$$\ddot{z}(t) = G(t) + F(t, z(t))\,,$$

where

$$G(t) := F_2(x_1(t)) - F_1(x_1(t))$$

and

$$F(t,z) := f_2(x_1(t) + z) - F_2(x_1(t))\,.$$

Then we apply Theorem 8.1. QED

Proof of Theorem 3.4. Equation (3.11) implies that

$$|y^b(t)| \ge \varepsilon t - c \tag{8.8}$$

for any $b \not\subset a$ and some $\varepsilon > 0$. Hence it is clear that for any $\theta > 0$ there exists σ such that

$$|\nabla I_a(y(t))| \le \sigma e^{-\theta t}\,.$$

Analogously, (3.14) implies

$$|x^b(t)| \geq \varepsilon t - c \tag{8.9}$$

for any $b \not\subset a$ and some $\varepsilon > 0$. Consequently, also $\nabla I_a(x(t))$ decays faster than any exponential.

We will apply Corollary 8.3 twice. First we set $x_1(t) := y(t)$, $x_2(t) := x(t)$, $F_1(x) := -\nabla V_a(x)$ and $F_2(x) := -\nabla V(x)$. Note that $F_1(x) - F_2(x) = \nabla I_a(x)$. Corollary 8.3 implies a).

Next we interchange 1 and 2. Another application of Corollary 8.3 yields b). QED

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