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Vertex Operator Representation of Some Quantum Tori Lie Algebras

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Abstract. We are defining the trigonometric Lie subalgebras in $\bar{X}_\infty = \bar{A}_\infty(\bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty)$ which are the natural generalization of the well known Sin-Lie algebra. The embedding formulas into $\bar{X}_\infty$ are introduced. These algebras can be considered as some Lie algebras of quantum tori. An irreducible representation of A, B series of trigonometric Lie algebras is constructed. Special cases of the trigonometric Lie factor algebras, which can be considered as a quantum (preserving Lie algebra structure) deformation of the Kac-Moody algebras are considered.

1. Introduction

The trigonometric Sin-Lie algebra [1] is the one-dimensional extension of the quantum (Weyl-Moyal) [2]) deformation of the Poisson Lie algebra on the two-torus [3]. In the explicit realization [1] it is defined by the generators $T_{\bar{n}}$, the central element c and relations

$$[T_{\bar{n}}, T_{\bar{m}}] = 2i \sin \hbar_1 (\bar{n} \times \bar{m}) T_{\bar{n}+\bar{m}} + n_1 \delta_{\bar{n}+\bar{m}, \bar{0}} c, \tag{1}$$

where $\bar{n}$ and $\bar{m}$ are vectors belonging to a square integer lattice $\mathbf{Z}^2 \backslash (0,0)$, $\bar{n} \times \bar{m} = n_1 m_2 - m_1 n_2$ and $\hbar_1$ is an arbitrary real parameter.

The Lie algebra (1) is associated with an associative C^*-algebra, usually called irrational rotation algebra $A_{\hbar_1}$, which defines the noncommutative two-torus [4]. More precisely the C^*-algebra $A_{\hbar_1}$ is generated by two unitary operators U_1 and U_2 and the relation

$$U_2 U_1 = q^2 U_1 U_2, \qquad q = e^{i\hbar_1}.$$

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If we choose $T_{\bar{n}}=q^{n_1n_2}U_1^{n_1}U_2^{n_2}$, then the commutator $T_{\bar{n}}T_{\bar{m}}-T_{\bar{m}}T_{\bar{n}}$, which defines the Lie algebra structure on $A_{\hbar_1}$, coincides with the right-hand side of Eq. (1) with $c=0$.

There exist a one-dimensional extension of the Lie algebra $A_{\hbar_1}$, which is defined by the two-cocycle $\omega(T_{\bar{n}}, T_{\bar{m}})=\tau(\delta_1(T_{\bar{n}})T_{\bar{m}})$. (The most general two-cocycle corresponds to the $a\delta_1+b\delta_2$, where $\delta_2 U_2=U_2$, $\delta_2 U_1=0$, but we restrict ourselves to the special case $b=0, a=1$.) Here τ is an invariant trace operator on $A_{\hbar_1}$ defined by $\tau(\sum f_{n_1n_2}U_1^{n_1}U_2^{n_2})=f_{0,0}$ and δ_1 is one of the derivatives on $A_{\hbar_1}$ satisfying the condition $\delta_1 U_1=U_1$, $\delta_1 U_2=0$ [4]. The two-cocycle ω defines the one-dimensional extension of the Lie algebra $A_{\hbar_1}$, which coincides with (1).

Note, there is a natural connection between the quantum group $GL_q(2)$ and (1): the first one acts, naturally, on the generators $T_{\bar{n}}$: $T_{\bar{n}}\to T'_{\bar{n}}=q^{n_1n_2}U_1'^{n_1}U_2'^{n_2}$. Here U'_i are the images of the action of $GL_q(2)$ [5]. Due to one of the defining properties of $GL_q(2)$: $U'_2U'_1=q^2U'_1U'_2$, this action preserves the commutator relation (1).

In this paper we will start from the natural generalization of (1) which will be denoted by $\bar{A}_{\hbar}$, where $\hbar=(\hbar_1,\ldots,\hbar_k)$ and $\hbar_i$ are arbitrary real valued parameters. Let $\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbf{Z}^k$ be an integer valued multi-index and $(\hbar,\alpha)=\sum_{i=1}^{k}\hbar_i\alpha_i$ denotes the usual inner product[1]. The Lie algebra $\bar{A}_{\hbar}$ is defined by the generators $A_{\alpha,m}$, which are labeled by the multi-indices $(\alpha,m)\in\mathbf{Z}^k\times\mathbf{Z}\backslash(0,\ldots,0)$, the central element c and relations.

$$[A_{\alpha,m},A_{\beta,l}]=2i\sin[m(\hbar,\beta)-l(\hbar,\alpha)]A_{\alpha+\beta,m+l}+m\delta_{\alpha+\beta,\bar{0}}\delta_{m+n,0}c\,. \tag{2}$$

There is an explicit realization of (2) with $c=0$, which is a straightforward generalization of the corresponding realization for the Sin-Lie algebra [6, 7]. The Lie algebra (2) can be realized as a cross product of the algebra of the functions on k-torus, $T^k=\{t=(t_1,\ldots,t_k)\,|\,t_j \bmod 2\pi\}$ by the shift operator $U=e^{2(\hbar_1\partial/\partial t_1+\ldots+\hbar_k\partial/\partial t_k)}$.

Let $q=(q_1,\ldots,q_k)$, where $q_j=e^{i\hbar_j}$ and define $q^\alpha=\prod_{j=1}^{k}q_j^{\alpha_j}$, then the generators

$$A_{\alpha,m}=q^{m\alpha}e^{i(\alpha,t)}U^m\,, \tag{3}$$

satisfy the commutation relation (2) with $c=0$ where $m\alpha=m(\alpha_1,\ldots,\alpha_k)$. Therefore, the Lie algebra (2) can be associated with some subalgebra of the quantum $2k$-torus. We will denote the Lie algebra (2) with $c=0$ as $A_{\hbar}$. Then the bar over $A_{\hbar}$ means the one-dimensional central extension. [We will reserve the bar to denote a one-dimensional central extension of any Lie algebras, for example $\bar{A}_\infty$ will denote a one-dimensional extension of $A_\infty=gl(\infty)$ and so on.]

The motivations to introduce the Lie algebra (2) are as follows. First of all, an embedding formula (17) is the "maximal" generalization, among a certain class, of the corresponding formula from [7]. (The same result was discovered in [8] from a different point of view.) So that the Lie algebra (2) is the most general, among a certain (trigonometric) class, which can be embedded into an infinite-dimensional Lie algebra $\widehat{gl}(\infty)=\bar{A}_\infty$. Another series of trigonometric algebras can be introduced in a natural way. One can define $\bar{B}_{\hbar}$, $\bar{C}_{\hbar}$, and $\bar{D}_{\hbar}$ series of trigonometric Lie algebras as an intersection of $\bar{A}_{\hbar}$ with the $\bar{X}_\infty=\bar{B}_\infty$, $\bar{C}_\infty$ or $\bar{D}_\infty$ Lie algebras [9–11].

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[1] We will reserve Greek indices $\alpha,\beta,\gamma,\ldots$ to denote multi-indices and Latin indices $n,m,l,\ldots$ to denote the single ones

The following motivations were also very important for us. It is well known that Kac-Moody algebras of the A–D series can be embedded (periodically) into an infinite-dimensional matrix algebra $\bar{X}_\infty = \bar{A}_\infty$, $\bar{B}_\infty$, $\bar{C}_\infty$ or $\bar{D}_\infty$ and can be considered as a periodic reduction from $\bar{X}_\infty$ [10, 11]. There is an application of this fact to the theory of integrable equations. To any Lie algebras $\bar{X}_\infty$ one can associate an integrable hierarchy of nonlinear equations, the so-called generalized $\bar{X}_\infty$-Toda lattices [12]. Then A–D series of generalized Toda lattices [13, 14] can be obtained as a periodic reduction from the $\bar{X}_\infty$ ones [12].

The Sin-Lie algebra and its $\bar{A}_\hbar$–$\bar{D}_\hbar$ generalizations give a new example of non-periodic embedding into $\bar{X}_\infty$. On the other hand, there is a similar situation in the theory of integrable equations. As it was discovered recently [15], the general classes of some non-local integrable equations (the so-called ILW_n, $MILW_n$, and 2-dimensional non-local Toda lattice [16]) can be considered as a non-periodic reduction from $\bar{A}_\infty$ Toda lattice hierarchy. More definitely the reduction $\varphi_{l+N}(t_1+2i\hbar_1, t_2, t_3, \ldots) = \varphi_l(t_1, t_2, \ldots)$ in the $\bar{A}_\infty$ generalized Toda lattice also leads to integrable equations (see [15] for further details).

Since all times t_j are "equal" in the integrable hierarchies, the shift operator $U = \exp\left(\sum_{j=1}^{k} \hbar_j \partial/\partial t_j\right)$ in Eq. (3) reproduces this democracy between all the times. (Of course, one can put, formally, $k=\infty$ and work with an infinite set of times.)

Let us state the main results and describe the structure of this paper.

In Sect. 2 we apply the Kac-Kazhdan-Lepowsky-Wilson (KKLW) construction [17] to the case of $\bar{A}_\hbar$. We construct an irreducible highest weight representation of $\bar{A}_\hbar$ in terms of the vertex operators. We will follow the ideas of [8] and generalize the results of [7] to the case of $\bar{A}_\hbar$.

In Sect. 3 we derive the embedding formula into $\bar{A}_\infty$.

In Sect. 4 we introduce the $\bar{B}_\hbar$, $\bar{C}_\hbar$, and $\bar{D}_\hbar$ series of the trigonometric algebras and their embedding formulas into $\bar{X}_\infty = \bar{B}_\infty$, $\bar{C}_\infty$ or $\bar{D}_\infty$. Note, that in the case of $\bar{B}_\hbar$, when $\hbar = h_1$, we obtain the Weyl-Moyal quantization of the central extension of the so-called "area-preserving" algebra for the Klein bottle [18]. (Due to the fact that the Klein bottle is non-orientable, the term "area-preserving" is not well defined, but we will use it formally following [18]. The authors thank M. Olshanetsky for pointing this paper out to us.) As far as we know, the other series are novel.

In Sect. 5 we apply the KKLW approach to the case of $\bar{B}_\hbar$ Lie algebra and construct an irreducible highest weight representation of $\bar{B}_\hbar$ in terms of the vertex operators.

In Sect. 6 we investigate some special values of the parameter $\hbar = (\hbar_1, \ldots, \hbar_k)$. The most important value is $\hbar = \left(\frac{\pi}{N}, \hbar_1\right)$, when $\hbar_1 \notin \pi Q$ (here Q defines, as usual, the field of the rational numbers). We will show that in this case the Lie factor algebras $\bar{X}_{N,\hbar_1}$ (see Sects. 6.2, 6.3 for the definitions) are the Weyl-Moyal quantization of an appropriate Kac-Moody algebra. In particular, the vertex operator representation of $\bar{A}_{N,\hbar_1}$ or $\bar{B}_{N,\hbar_1}$, when $\hbar_1 = 0$, coincides with the basic representation of the $A_N^{(1)}$ or $D_{l+1}^{(2)}$ if $N=2l+2$ and $A_{2l}^{(2)}$ if $N=2l+1$ Kac-Moody algebras in the principal realization [17].

Finally, we would like to point out that our algebras are an example of the wide class of continuum Lie algebras introduced in [19]. We would also like to point out that similar objects appeared previously in the gauge theory of higher spins [20] and it will be interesting to clarify this connection.

Even though our main interest is to apply our algebras to integrable equations, our results, probably, may be useful in string theory (see [21] for motivation). We finally remark that some of the results of this paper were published in [22].

2. Vertex Operators Representation of $\bar{A}_\hbar$

In this section we will generalize the approach of [8] which is nothing but application of the Kac-Kazhdan-Lepowsky-Wilson construction [17] to the trigonometric-Lie algebras.

Fix one of an infinite number of infinite-dimensional Heisenberg subalgebra of $\bar{A}_\hbar$, namely, $s=\{A_{\bar{0},m} \mid m\in \mathbf{Z}\backslash 0\}$. The maximal commutative subalgebra of $\bar{A}_\hbar$ is $H=\{A_{\alpha,0} \mid \alpha\in\mathbf{Z}^k\backslash(0,\ldots,0)\}$. Define the generating functions $X_\alpha(z)$ to be

$$X_\alpha(z)=\sum_{\ell\in\mathbf{Z}} A_{\alpha,\ell} z^{-\ell},$$

where $\alpha\in\mathbf{Z}^k\backslash(0,\ldots,0)$ and z belongs to the complex plane. Note, that all generators of $\bar{A}_\hbar$ are either in s or are Laurent coefficients of the fields $X_\alpha(z)$.

It is easy to check that the following relations are satisfied:

$$[A_{\bar{0},m}, X_\alpha(z)]=2i\sin(m(\hbar,\alpha))z^m X_\alpha(z), \qquad m\in\mathbf{Z}\backslash 0, \qquad \alpha\in\mathbf{Z}^k\backslash(0,\ldots,0). \tag{4}$$

There is a standard irreducible representation of the Heisenberg subalgebra s in the space of polynomials in infinitely many variables $V=V[x_1,x_2,x_3,\ldots]$:

$$\pi_0(A_{\bar{0},m})=\partial/\partial x_m, \qquad \pi_0(A_{\bar{0},-m})=mx_m, \qquad \pi_0(c)=1, \qquad m>0. \tag{5}$$

If in Eq. (4) we substitute for $A_{\bar{0},\pm m}$, $m>0$ their representation above, we get a system of coupled differential equations for $X_\alpha(z)$.

This system has a unique (up to multiplication by arbitrary constants a_α) solution in the space of the differential operators on V [17, 11]:

$$\hat{X}_\alpha(z)=a_\alpha e^{2i\sum_{m\geq 1} z^m\sin(m(\hbar,\alpha))x_m}\, e^{2i\sum_{m\geq 1}\frac{z^{-m}}{m}\sin(m(\hbar,\alpha))\partial/\partial x_m}. \tag{6}$$

As we do not know at the moment that the ring of polynomials V coincides with an irreducible $\bar{A}_\hbar$-module or not, we cannot conclude that (5), (6) realize a representation of $\bar{A}_\hbar$. Nevertheless, in fact, it is true. More precisely, define

$$\hat{X}_{\alpha,\ell}=\frac{1}{2\pi i}\oint_\Gamma dz z^{\ell-1}\hat{X}_\alpha(z), \tag{7}$$

where the contour of integration Γ contains the point $0\in C$. Then

Theorem 1. (1) *The map*

$$\begin{aligned} \pi(A_{\alpha,\ell})&=\hat{X}_{\alpha,\ell}; \qquad \alpha\in\mathbf{Z}^k\backslash(0,\ldots,0); \\ \pi(A_{\bar{0},\ell})=\partial\backslash\partial x_\ell, \qquad \pi(A_{\bar{0},-\ell})&=\ell x_\ell, \qquad \pi(c)=1; \qquad \ell>0, \end{aligned} \tag{8}$$

gives an irreducible highest weight representation of the algebra (2) *in the space* $V=C[x_1,x_2,x_3,\ldots]$, *provided the constants* a_α *in* (6) *satisfy the equation:*

$$a_\alpha a_{\alpha'}=[q^{(\alpha+\alpha')}-q^{-(\alpha+\alpha')}]/[(q^\alpha-q^{-\alpha})\cdot(q^{\alpha'}-q^{-\alpha'})]a_{\alpha+\alpha'}. \tag{9}$$

(2) *The Eq.* (9) *has unique solution*

$$a_\alpha = \frac{q^\alpha}{q^\alpha - q^{-\alpha}} \tag{10}$$

up to the phase multiplier $e^{i(\alpha,\lambda)}$, $\lambda = (\lambda_1, \ldots, \lambda_k)$.
(3) *The vacuum state is given by* $|0\rangle = 1$, *and the highest weight* $\Lambda \in H^*$ *is defined by the equations* $\Lambda(A_{\alpha,0}) = a_\alpha$.

The proof of this theorem is based on a direct calculation of the commutation relations between the generators $\hat{X}_{\alpha,\ell}$. For this one needs to know the O.P.E. of two vertex operators (6) at different points z and ζ of the complex plane. It is simple to check that

$$\hat{X}_\alpha(z)\hat{X}_{\alpha'}(\zeta) = \frac{(z - q^{\alpha-\alpha'}\zeta)(z - q^{-(\alpha-\alpha')}\zeta)}{(z - q^{\alpha+\alpha'}\zeta)(z - q^{-(\alpha+\alpha')}\zeta)} :\hat{X}_\alpha(z)\hat{X}_{\alpha'}(\zeta):, \tag{11}$$

where $|z| > |\zeta|$, provided a_α satisfy Eq. (9). The normal ordering is defined so that all creation operators in $:\hat{X}_\alpha(z)\hat{X}_{\alpha'}(\zeta):$ are standing on the left.

Since the right-hand side of (11) is invariant under permutation $z \leftrightarrow \zeta$ and $\alpha \leftrightarrow \alpha'$ the calculation of the commutation relation $[\hat{X}_{\alpha,n}, \hat{X}_{\alpha',m}]$, where $\alpha + \alpha' \neq 0$ leads to the calculation of the contour integrals

$$\sum_{i=1}^{2} \frac{1}{(2\pi i)^2} \oint_\Gamma d\zeta \zeta^{m-1} \oint_{\Gamma_i} dz z^{n-1} F(z, \zeta),$$

where $F(z,\zeta)$ stands for the right-hand side of (11) applied to some state $|\phi\rangle$ and Γ_i are two small contours around the poles of (11); $z = q^{\alpha+\alpha'}\zeta$ and $z = q^{-(\alpha+\alpha')}\zeta$. (We would like to point out a phenomenon of shifting poles in (11) is very similar to those in the case of quantum Kac-Moody algebras [23]. We will be glad if it will be not an accident.) When $\alpha + \alpha' = 0$, the two simple poles combine into a second order pole and we get the usual situation.

The final result is as follows:

$$[\hat{X}_{\alpha,m}, \hat{X}_{\alpha',\ell}] = 2i \sin[m(\hbar,\alpha') - \ell(\hbar,\alpha)]\hat{X}_{\alpha+\alpha',m+l}$$

when $\alpha + \alpha' \neq 0$,

$$[\hat{X}_{\alpha,m}, \hat{X}_{-\alpha,\ell}] = \begin{cases} -2i \sin[p(\hbar,\alpha)]\partial/\partial x_p, & p = m + \ell > 0, \\ 2ip \sin[p(\hbar,\alpha)] \cdot x_p, & p = -(m+\ell) > 0, \\ m \cdot 1, & m + \ell = 0. \end{cases} \tag{12}$$

Comparing (12), (8), and (2) one can see that the algebra of differential operators $\hat{X}_{\alpha,n}$, $\dfrac{\partial}{\partial x_m}$, x_m, 1, where $n \in Z$, $\alpha \in \mathbf{Z}^k \backslash (0, \ldots, 0)$ and $m > 0$ is closed under the operation of commutation (which is not evident without performing explicit calculations) and moreover, it realizes some representation of $\bar{A}_\hbar$. Since the restriction of the representation π on s is irreducible by construction, the representation π is also irreducible. Using the explicit form of the vertex operators $\hat{X}_\alpha(z)$ it is easy to check that $\hat{X}_{\alpha,0}|0\rangle = a_\alpha$ and $\hat{X}_{\alpha,n}|0\rangle = 0$, when $n > 0$. This means that we have constructed the highest weight representation of weight $\Lambda \in H^*$, where $\bar{A}_\hbar = n_+ + H + n_-$, n_+ (n_-) is generated by $A_{\alpha,\ell}$, $\ell > 0$ $(\ell < 0)$ and H is generated by $A_{\alpha,0}$ and coincides with the maximal commutative subalgebra of $\bar{A}_\hbar$.

The weight $\Lambda \in H^*$ is defined by its values on the bases of H from the equations $\Lambda(A_{\alpha,0}) = a_\alpha$.

3. Embedding into $\bar{A}_\infty = \widehat{gl}(\infty)$

By substituting $u = zq^\alpha$, $v = zq^{-\alpha}$, it is simple to check that the vertex operators $\hat{X}_\alpha(z)$ can be obtained from the standard vertex operator [10, 11]

$$Z(u,v) = \frac{u}{u-v}\left[e^{\sum_{m\geqq 1}(u^m - v^m)x_m} e^{-\sum_{m\geqq 1}(u^{-m} - v^{-m})\frac{1}{m}\frac{\partial}{\partial x_m}} - 1\right], \tag{13}$$

which realizes the basic representation of the $\bar{A}_\infty = \widehat{gl}(\infty)$ Lie algebra. Using the expansion $Z(u,v) = \sum_{i,j} Z_{ij} u^i v^{-j}$ and substituting the above substitution in, it is simple to check that

$$\hat{X}_{\alpha,m} = q^{m\alpha} \sum_{n\in\mathbf{Z}} q^{2n\alpha} \tilde{Z}_{n,n+m} + \delta_{m,0} a_\alpha \cdot c\,. \tag{14}$$

But the operators $\tilde{Z}_{n,n+m}$ realize the basic ($c=1$) representation of $\bar{A}_\infty$. This simple observation has a natural generalization.

Theorem 2. *Let $E_{i,j}$, $i,j \in \mathbf{Z}$ satisfy the commutation relation of $\widehat{gl}(\infty)$ with arbitrary central charge:*

$$[E_{i,j}, E_{k,\ell}] = E_{i,\ell}\delta_{j,k} - E_{k,j}\delta_{i,\ell} + \psi(E_{i,j}, E_{k,\ell}) \cdot c\,, \tag{15}$$

where the 2-cocycle ψ on A_∞ defines

$$\left.\begin{cases} \psi(E_{i,j}, E_{j,i}) = 1 = -\psi(E_{j,i}, E_{i,j}) & \text{if} \quad i \leqq 0, \quad j \geqq 1 \\ \psi(E_{i,j}, E_{k,\ell}) = 0 & \text{in all other cases} \end{cases}\right\}. \tag{16}$$

Then the generators $A_{\alpha,m}$ defined by the equation

$$A_{\alpha,m} = q^{m\alpha} \sum_{n\in\mathbf{Z}} q^{2n\alpha} E_{n,n+m} + \delta_{m,0} a_\alpha \cdot c\,, \tag{17}$$

where the constants a_α are defined by (10) *will satisfy* (2).

Note that Eqs. (16) are the natural generalization of the appropriate equations from [7, 8]. Equation (17) gives the explicit formula of the embedding $\bar{A}_\hbar$ into $\bar{A}_\infty$. If we define the standard basis vectors in the infinite-dimensional linear space C^∞ by e_k, $(e_k)_i = \delta_{k,i}$, then we can realize (17) with $c=0$ as the maps from C^∞ to C^∞ by

$$I(A_{\alpha,m})e_k = q^{-m\alpha} q^{2k\alpha} e_{k-m}\,. \tag{18}$$

4. $\bar{B}_\hbar, \bar{C}_\hbar, \bar{D}_\hbar$-Series of the Trigonometric Subalgebras in $\bar{X}_\infty = \bar{B}_\infty, \bar{C}_\infty, \bar{D}_\infty$

Since $A_\hbar$ is isomorphic to the subalgebra of $gl(\infty) = A_\infty$ defined by the explicit formula (18), one can define the Lie subalgebras $B_\hbar$, $C_\hbar$, and $D_\hbar$ in $A_\hbar$ in a very natural way.

Recall [11] that there are subalgebras B_∞, C_∞, and D_∞ of A_∞ preserving the bilinear forms in $C^\infty \times C^\infty$:

(i) $$\langle e_i, e_j\rangle = (-1)^i \delta_{i,j} \quad \text{in the case of } B_\infty, \tag{19.1}$$

(ii) $$\langle e_i, e_j\rangle = (-1)^i \delta_{i,1-j} \quad \text{in the case of } C_\infty, \tag{19.2}$$

(iii) $$\langle e_i, e_j\rangle = \delta_{i,1-j} \quad \text{in the case of } D_\infty. \tag{19.3}$$

The one-dimensional central extensions of these subalgebras are defined by the 2-cocycle $r\cdot\psi$, where ψ is defined by (16) and $r=1/2$ for the B_∞ and D_∞ cases and $r=1$ for the case of C_∞.

Now, it is natural to look for those linear combinations $V_{\alpha,l}=\sum_\beta v_\alpha^\beta A_{\beta,l}$ ($V_{\alpha,l}=B_{\alpha,l}, C_{\alpha,l}, D_{\alpha,l}$), which preserve the bilinear forms (19.1)–(19.3). This will give the natural Lie subalgebras in $A_\hbar$.

Straightforward calculations give the following list of results.

4.1 The $\bar{B}_\hbar$ Case

Trigonometric basis $B_{\alpha,l}$ of $\bar{B}_\hbar$:

$$B_{\alpha,m} = A_{\alpha,m} - (-1)^m A_{-\alpha,m}. \tag{20}$$

The second order automorphism τ_B of $A_\hbar$ such that $\tau_B(B_{\alpha,m}) = B_{\alpha,m}$ is defined by:

$$\tau_B(A_{\alpha,m}) = (-1)^m A_{-\alpha,m}. \tag{21}$$

The commutation relations for $\bar{B}_\hbar$ in the trigonometric basis are:

$$\begin{aligned}[B_{\alpha,m}, B_{\beta,l}] &= 2i\sin(m(\hbar,\beta)-l(\hbar,\alpha))B_{\alpha+\beta,m+l} + (-1)^l 2i\sin(m(\hbar,\beta)+l(\hbar,\alpha))\\ &\quad\times B_{\alpha-\beta,m+l} + m(\delta_{\alpha+\beta,\bar{0}} - (-1)^m\delta_{\alpha-\beta,\bar{0}})\delta_{m+l,0}.\end{aligned} \tag{22}$$

The embedding of $\bar{B}_\hbar$ into $\bar{B}_\infty$ is given by:

$$B_{\alpha,m} = q^{m\alpha}\sum_{n\in\mathbf{Z}} q^{2n\alpha}(E_{n,n+m} - (-1)^m E_{-n-m,-n}) + \delta_{m,0} b_\alpha \cdot c, \tag{23}$$

where $b_\alpha = 1/2(q^\alpha + q^{-\alpha})/(q^\alpha - q^{-\alpha})$. In analogy with the $\bar{A}_\hbar$ case, it is possible to construct the vertex operator realization of the basic representation in the $\bar{B}_\hbar$ case. We will present these results in the next section.

In the special case, when $\hbar=\hbar_1$ and $\hbar_1 \notin \pi\mathbf{Q}$ the Lie algebra (22) coincides with the central extension of the Weyl-Moyal quantization of the "area-preserving" algebra for the Klein bottle [18]. As we were informed by M. Olshanetsky, some second-order automorphisms of $A_{\hbar_1}$ were also constructed in [25].

4.2. The $\bar{C}_\hbar$ Case

Trigonometric basis of $\bar{C}_\hbar$:

$$C_{\alpha,m} = A_{\alpha,m} - (-1)^m q^{2\alpha} A_{-\alpha,m}. \tag{24}$$

The second-order automorphism τ_C of $A_\hbar$ such that $\tau_C(C_{\alpha,m}) = C_{\alpha,m}$ is defined by:

$$\tau_C(A_{\alpha,m}) = (-1)^m q^{2\alpha} A_{-\alpha,m}. \tag{25}$$

The commutation relations for $\bar{C}_\hbar$ in the trigonometric basis are:

$$\begin{aligned}[C_{\alpha,m}, C_{\beta,l}] = & 2i\sin(m(\hbar,\beta)-l(\hbar,\alpha))C_{\alpha+\beta,m+l} \\ & +2i(-1)^l q^{2\beta}\sin(m(\hbar,\beta)+l(\hbar,\alpha))C_{\alpha-\beta,m+l} \\ & +m(2\delta_{\alpha+\beta,\bar{0}}-(-1)^m\cos(2\hbar,\alpha)\delta_{\alpha-\beta,\bar{0}})\delta_{m+l,0}\cdot c. \end{aligned} \tag{26}$$

The embedding of $\bar{C}_\hbar$ into $\bar{C}_\infty$ is given by:

$$C_{\alpha,m} = q^{m\alpha}\sum_{n\in\mathbf{Z}} q^{2n\alpha}(E_{n,n+m}-(-1)^m E_{1-n-m,1-n}) + 2\delta_{m,0}a_\alpha\cdot c. \tag{27}$$

4.3. The $\bar{D}_\hbar$ Case

Trigonometric basis of $\bar{D}_\hbar$:

$$D_{\alpha,m} = A_{\alpha,m} - q^{2\alpha}A_{-\alpha,m}. \tag{28}$$

The second-order automorphism τ_D of $A_\hbar$ such that $\tau_D(D_{\alpha,m}) = D_{\alpha,m}$ is defined by:

$$\tau_D(A_{\alpha,m}) = q^{2\alpha}A_{-\alpha,m}. \tag{29}$$

The commutation relations for $\bar{D}_\hbar$ in the trigonometric basis are:

$$\begin{aligned}[D_{\alpha,m}, D_{\beta,l}] = & 2i\sin(m(\hbar,\beta)-l(\hbar,\alpha))D_{\alpha+\beta,m+l} + 2iq^{2\beta}\sin(m(\hbar,\beta)+l(\hbar,\alpha)) \\ & \times D_{\alpha-\beta,m+l} + m(\delta_{\alpha+\beta,\bar{0}} - q^{2\alpha}\delta_{\alpha-\beta,\bar{0}})\delta_{m+l,0}\cdot c. \end{aligned} \tag{30}$$

The embedding of $\bar{D}_\hbar$ into $\bar{D}_\infty$ is given by:

$$D_{\alpha,m} = q^{m\alpha}\sum_{n\in\mathbf{Z}} q^{2n\alpha}(E_{n,n+m} - E_{1-n-m,1-n}) + \delta_{m,0}a_\alpha\cdot c. \tag{31}$$

Note that Eqs. (17), (20), (24), (28) give the construction of the representation of $A_\hbar$–$D_\hbar$ series in the space $C^\infty(S^1)$ of the complex valued functions on the circle $S^1 = \{t_1 \bmod 2\pi\}$ (compare with [6, 7]):

$$A_{\alpha,m} \to q^{-m\alpha}e^{-mt_1}e^{-2(\hbar,\alpha)\partial/\partial t_1}. \tag{32}$$

5. Vertex Operator Representation of $\bar{B}_\hbar$

In this section we will describe, briefly, the vertex operator construction for the case of $\bar{B}_\hbar$.

Fix an infinite-dimensional Heisenberg subalgebra of $\bar{B}_\hbar$, namely, $s = \left\{p_m = \frac{1}{2}B_{\bar{0},m},\ q = \frac{1}{m}B_{\bar{0},-m},\ m>0,\ m\ -\ odd\right\}$, and define the generating functions $B_{\alpha,l}(z)$ to be

$$B_\alpha(z) = \sum_{n\in\mathbf{Z}} B_{\alpha,n}z^{-n},$$

where $\alpha\in\mathbf{Z}^k\backslash(0,\ldots,0)$.

It is easy to check that

$$\begin{aligned} &[p_m, B_\alpha(z)] = 2i\sin[m(\hbar,\alpha)]z^m B_\alpha(z), \\ &[q_m, B_\alpha(z)] = -4i\sin[m(\hbar,\alpha)]z^{-m}B_\alpha(z). \end{aligned} \tag{33}$$

By the same arguments as in Sect. 2, Eq. (33) in which p_m and q_m ($m \geqq 1, m-odd$) are represented by the operators $\partial/\partial x_m$ and x_m have a unique, up to multiplication by an arbitrary constant, solution in the space of the differential operators on $V = C[x_1, x_3, x_5, \ldots]$:

$$\hat{B}_\alpha(z) = b_\alpha e^{2i \sum_{m \geqq 1;\, m-\text{odd}} z^m \sin[m(\hbar,\alpha)] x_m} e^{4i \sum_{m \geqq 1;\, m-\text{odd}} \frac{z^{-m}}{m} \sin[m(\hbar,\alpha)] \partial/\partial x_m} . \tag{34}$$

By similar arguments one can prove the following theorem

Theorem 3. *Let $\hat{B}_{\alpha,n} = \frac{1}{2\pi i} \oint_\Gamma dz \cdot z^{n-1} \hat{B}_\alpha(z)$, where the contour of integration Γ contains the point $0 \in \mathbf{C}$. Then*

(1) *The map*

$$\begin{aligned} &\pi(\tfrac{1}{2}B_{\bar{0},m}) = \partial/\partial x_m, \quad \pi(\tfrac{1}{2}B_{\bar{0},-m}) = x_m, \quad m \geqq 1, \quad m-\text{odd}, \\ &\pi(B_{\alpha,n}) = \hat{B}_{\alpha,n}, \quad \pi(c) = 1, \quad \alpha \in Z^k \backslash (0, \ldots, 0), \end{aligned} \tag{35}$$

gives an irreducible highest weight representation of the algebra (22) in the space $V = C[x_1, x_3, x_5, \ldots]$, provided the constants b_α in (34) satisfy the equation

$$b_\alpha b_{\alpha'} = \frac{1}{2} \frac{(q^\alpha + q^{-\alpha})(q^{\alpha'} + q^{-\alpha'})(q^{\alpha+\alpha'} - q^{-(\alpha+\alpha')})}{(q^\alpha - q^{-\alpha})(q^{\alpha'} - q^{-\alpha'})(q^{\alpha+\alpha'} + q^{-(\alpha+\alpha')})} b_{\alpha+\alpha'} . \tag{36}$$

(2) *The Eq. (36) has unique solution*

$$b_\alpha = \tfrac{1}{2}(q^\alpha + q^{-\alpha})/(q^\alpha - q^{-\alpha}) \tag{37}$$

up to the phase multiplier.
(3) *The vacuum state is given by $|0\rangle = 1$ and the highest weight $\Lambda \in H^*$ is defined by the equation $\Lambda(B_{\alpha,0}) = b_\alpha$, where $H = \{B_{\alpha,0}\}$ is the maximal commutative subalgebra of $\bar{B}_\hbar$.*

Note, that the vertex operators (34), (37) can be obtained from the vertex operator [10, 11]:

$$\Gamma_B(u,v) = \frac{1}{2} \frac{u-v}{u+v} e^{\sum_{m \geqq 1;\, m-\text{odd}} (u^m + v^m) x_m} e^{-2 \sum_{m \geqq 1;\, m-\text{odd}} \frac{1}{m} (u^{-m} + v^{-m}) \partial/\partial x_m} , \tag{38}$$

which realizes the basic representation of the $\bar{B}_\infty$ Lie algebra, by substituting $u = q^\alpha z$, $v = -q^{-\alpha} z$.

The embedding formula (23) is compatible with this substitution into (38) and (34), (37).

6. Realizations of the Lie Algebras $\bar{X}_\hbar$ for Special Values of the Parameter $\hbar$

In this section we consider some special values of the parameter $\hbar = (\hbar_1, \ldots, \hbar_k)$, which cover the cases of the Sin-Lie algebra and their generalization, Kac-Moody Lie algebras of the A–D types and Lie algebras which can be considered as $\hbar$-deformations of the corresponding Kac-Moody algebras.

6.1. The $\bar{B}_{\hbar_1}$, $\bar{C}_{\hbar_1}$, and $\bar{D}_{\hbar_1}$ Analogies of the Sin-Lie Algebra

Let us consider first the case when the vector $\hbar$ has only one component (i.e. $k=1$). Then $\hbar=\hbar_1$ and we require that $\hbar_1 \notin \pi Q$, so that the Lie algebra (2) coincides with (1). Let us rescale the generators $T_{\bar{n}}$ in a way equivalent to dividing the right-hand side of (1) by $-2i\hbar_1$. Then in the limit $\hbar_1 \to 0$ we obtain the area-preserving algebra on the two-torus $T^2=\{(t_1,t_2) \bmod 2\pi\}$ introduced in [3]:

$$[T_{\bar{n}}, T_{\bar{m}}]=(\bar{m}\times\bar{n})T_{\bar{n}+\bar{m}},$$

where $T_{\bar{n}}=e^{i(n_1t_1+n_2t_2)}$.

In the same limit, the $B_{\hbar_1}$ and $C_{\hbar_1}$ Lie algebras coincide with the "area-preserving" algebra for the Klein bottle [18], which is the Poisson Lie algebra of functions $f(t_1,t_2)$ on the torus T^2 satisfying an additional symmetry: $f(-t_1,t_2+\pi) = -f(t_1,t_2)$.

In the case of $D_{\hbar_1}$ in the same limit we obtain similarly the Poisson Lie algebra with additional condition: $f(-t_1,t_2)=-f(t_1,t_2)$.

Note that the Lie algebras $B_{\hbar_1}$, $C_{\hbar_1}$, and $D_{\hbar_1}$ do not possess the structure of a C^* algebra.

6.2. Quantum Two-Torus Algebra $\bar{A}_{N,\hbar_1}$

Let us take $\hbar=(\pi/N,\hbar_1)$, $\hbar_1\notin\pi Q$ and consider first the case of the $\bar{A}_{\hbar}$ series. Equation (17) shows that for this special value of the parameter $\hbar$ one can impose on the generators $A_{\alpha,m}$ ($\alpha=(n_1,n_2)$) and additional condition

$$A_{n_1+rN,n_2,m}=(-1)^{mr}A_{n_1,n_2,m}\,. \tag{39}$$

Let us denote by $\bar{A}_{N,\hbar_1}$ the factor algebra with respect to the relation (39). Let us show first how to realize the $\bar{A}_{N,\hbar_1}$ Lie algebra as the cross product of the $A_{N-1}^{(1)}\simeq \hat{sl}(N)$ Kac-Moody algebra by the shift operator $U_2=e^{2\hbar_1\partial/\partial t_1}$.

Let us choose a trigonometric basis in the $A_{N-1}^{(1)}$:

$$A_{n,m}=\omega^{\frac{nm}{2}}Q^nP^m\otimes U_1^m, \tag{40}$$

where

$$Q=\begin{pmatrix} \omega & & & & 0\\ & \omega^2 & & & \\ & & \ddots & & \\ & & & \omega^{N-1} & \\ 0 & & & & 1\end{pmatrix},\quad P=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix},$$

$$\omega=e^{\frac{2\pi i}{N}},\qquad U_1=e^{it_1}. \tag{41}$$

This is exactly the principal realization of $A_{N-1}^{(1)}$ [17, 11]. Fix the special bases in the cross product of $A_{N-1}^{(1)}$ by U_2:

$$A_{n_1,n_2,m}=\omega^{\frac{n_1n_2}{2}}q_1^{n_2m}Q^{n_1}P^m\otimes U_1^mU_2^{-n_2}. \tag{42}$$

The generators $A_{n_1,n_2,m}$ in Eq. (41) satisfy the commutation relations (2) (with $c=0$), for $\hbar=\left(\frac{\pi}{N},\hbar_1\right)$ and the additional condition (39).

Note that the above algebra is a very natural object in non-commutative geometry on the quantum two-torus [24].

The commutation relations (2) with $c\neq 0$ is also very natural. Let us define the one-dimensional central extension of $\mathrm{Mat}(N)\otimes A_{\hbar_1}$ by the two-cocycle

$$\omega(A(U_1,U_2),B(U_1,U_2))=\frac{1}{N}\tau\,\mathrm{tr}(\delta_1 A(U_1,U_2)B(U_1,U_2)), \tag{43}$$

where $A(U_1,U_2),B(U_1,U_2)\in\mathrm{Mat}(N)\otimes A_{\hbar_1}$ and the operators τ and δ_1 were defined in the Introduction. The generators (42) define a subalgebra in the central extension (43) of the Lie algebra $\mathrm{Mat}(N)\otimes A_{\hbar_1}$ which gives a realization of $\bar{A}_{N,\hbar_1}$. [That is the commutation relations (2) are satisfied.]

Another way to define $\bar{A}_{N,\hbar_1}$ is the following. Define a Lie subalgebra in the central extension (43) of $\mathrm{Mat}(N)\otimes A_{\hbar_1}$ by the condition $\tau\,\mathrm{tr}(A(U_1,U_2))=0$. Due to the property $\tau\,\mathrm{tr}[A(U_1,U_2),B(U_1,U_2)]=0$ this condition really defines some Lie subalgebra. Using the representation $U_1=e^{it_1}$, $U_2=e^{2\hbar_1\partial/\partial t_1}$ it is easy to show, that the above Lie algebra is isomorphic to $\bar{A}_{N,\hbar_1}$. (The proof is based on the isomorphism of the principal and basic realization of the $A_N^{(1)}$ Kac-Moody algebra [11].)

In the limit $\hbar_1\to 0$ $(U_2\to 1)$ one can obtain the Kac-Moody Lie algebra $A_{N-1}^{(1)}$, so that the $\bar{A}_{N,\hbar_1}$ can be considered as a quantum (Weyl-Moyal) deformation of $\hat{sl}(N)$. In general, the algebra $\bar{A}_\hbar$ for $\hbar=\left(\frac{\pi}{N},\hbar_1,\ldots,\hbar_k\right)$ can be considered as a k-parameter deformation of the $\hat{sl}(N)$ Lie algebra.

6.3. *Quantum Two-Torus Algebras of the $\bar{B}_\hbar$, $\bar{C}_\hbar$, and $\bar{D}_\hbar$ Series* $\left(\hbar=\left(\frac{\pi}{N},\hbar_1\right)\right)$

The generators of the $\bar{B}_\hbar$, $\bar{C}_\hbar$, and $\bar{D}_\hbar$ series when $\hbar=\left(\frac{\pi}{N},\hbar_1\right)$ are labeled by the triplet of indices (n_1,n_2,m). Analogously to the case of $\bar{A}_{N,\hbar_1}$ one can require that [cf. Eq. (39)]

$$V_{n_1+rN,n_2,m}=(-1)^{rm}V_{n_1,n_2,m}, \tag{44}$$

where $V=B$, C or D. Let us define the algebras $\bar{B}_{N,\hbar_1}$, $\bar{C}_{N,\hbar_1}$, and $\bar{D}_{N,\hbar_1}$ as the Lie algebras which are obtained from $\bar{B}_\hbar$, $\bar{C}_\hbar$ or $\bar{D}_\hbar$ putting $\hbar=\left(\frac{\pi}{N},\hbar_1\right)$ in (22), (26) or (30) and factorizing by the relations (44). Substituting (42) into (20), (24) or (28) one can obtain an explicit realization of the Lie algebras $\bar{B}_{N,\hbar_1}$, $\bar{C}_{N,\hbar_1}$, and $\bar{D}_{N,\hbar_1}$.

Note that since the Lie algebras constructed above do not possess the structure of C^*-algebras, there is no realization of them as cross products of the corresponding Kac-Moody algebras by the shift operator.

The following theorem shows the importance of the above algebras.

Theorem 4. *In the limit $\hbar_1 \to 0$,*

1. $\bar{B}_{N,\hbar_1}$ *goes into* $D^{(2)}_{l+1}$ *if* $N=2l+2$ *and into* $A^{(2)}_{2l}$ *if* $N=2l+1$.
2. $\bar{C}_{N,\hbar_1}$ *goes into* $C^{(1)}_{l}$ *if* $N=2l$ *and into* $A^{(2)}_{2l}$ *if* $N=2l+1$.
3. $\bar{D}_{N,\hbar_1}$ *goes into* $D^{(1)}_{l}$ *if* $N=2l$ *and into* $B^{(1)}_{l}$ *if* $N=2l+1$.

Proof. Following [1] we introduce the trigonometric bases in the classical Lie algebras of B, C, D types. Fix the trigonometric basis in $sl(N)$:

$$A_{n,m} = \omega^{\frac{nm}{2}} Q^n P^m ,$$

where $P, Q \in sl(N)$ are defined in (41). Then the trigonometric basis in B_l can be chosen as:

$$B_{n,m} = A_{n,m} - \omega^n A_{-n,m} , \qquad 0 < n \leqq l, \qquad 0 \leqq m \leqq 2l,$$

where $A_{n,m} \in sl(2l+1)$. It is easy to check the relation $SB^tS = -B$, where $S = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & 1 & \vdots \\ 1 & \dots & 0 \end{pmatrix}$ and thus $B_{n,m} \in B_l$.

There is a trigonometric basis in C_l of the type:

$$C_{n,m} = A_{n,m} - (-1)^m \omega^n A_{-n,m} , \qquad 0 \leqq n \leqq l, \qquad 0 \leqq m \leqq 2l-1 .$$

The Lie algebra generated by $\{C_{n,m}\}$ consists of $(2l \times 2l)$ complex matrices which are antisymmetric with respect to the bilinear form $(\tilde{S}x, y)$, where

$$\tilde{S} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 & \dots & (-1)^{l-1} \\ & -1 \,\cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \dots & 0 \end{pmatrix}$$

and hence coincides with the C_l.

There are two ways to fix the trigonometric basis in D_l:

(i) $D_{n,m} = A_{n,m} - \omega^n A_{-n,m}$, $0 < n \leqq l$, $0 \leqq m \leqq 2l-1$,
(ii) $D'_{n,m} = A_{n,m} - (-1)^m A_{-n,m}$, $0 \leqq n \leqq l-1$, $0 \leqq m \leqq 2l-1$,

where $A_{n,m} \in sl(2l)$. The matrices $D_{n,m}$ are antisymmetric with respect to the skew diagonal. Matrices $D'_{n,m}$ preserve the nondegenerate, symmetric bilinear form with the matrix

$$\tilde{\tilde{S}} = \begin{pmatrix} 0 & 0 & \dots & & -1 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 1 & \dots & & 0 & 0 \\ -1 & 0 & \dots & & 0 & 0 \\ 0 & 0 & \dots & & 0 & 1 \end{pmatrix} .$$

In all cases the second index m defines the $\mathbf{Z}/N\mathbf{Z}$ gradation. Therefore, there exist the following bases in Kac-Moody Lie algebras of the types B, C, D:

$$\begin{aligned} &C_{n,m}\otimes U_1^m \quad \text{in } C_l^{(1)},\\ &B_{n,m}\otimes U_1^m \quad \text{in } B_l^{(1)}, \end{aligned} \tag{45}$$

$D_{n,m}\otimes U_1^m$ in $D_l^{(1)}$ or $D'_{n,m}\otimes U_1^m$ in $D_l^{(2)}$, $X_{n,m}\otimes U_1^m$ and $Y_{n,m}\otimes U_1^m$ in $A_{2l}^{(2)}$ $[X_{n,m}=C_{n,m},\ Y_{n,m}=D'_{n,m},\ 0\leqq n\leqq l,\ 0\leqq m\leqq 4l+1$ and $A_{n,m}\in sl(2l+1)]$.

Now, we can return to the algebras $\bar{B}_\hbar$, $\bar{C}_\hbar$, and $\bar{D}_\hbar$, with $\hbar=\left(\frac{\pi}{N}, \hbar_1\right)$ and relation (44).

Comparing (42), (20), (24), and (28) with (45) one can easily check the statements of the theorem.

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