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The Extended Bethe Ansatz for Infinite $S = 1/2$ Quantum Spin Chains with Non-Nearest-Neighbor Interaction

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Abstract. It is shown that the description of the states of infinite $S = 1/2$ interacting spin systems with the Hamiltonian $H_s = -\frac{J}{2} \sum_{j \neq l,\, j,l \in \mathbf{Z}} a^2 \sinh^{-2} a(j-l) \frac{(\sigma_j \sigma_l - 1)}{2}$ can be performed by studying the hyperbolic Calogero-Sutherland eigenvalue problem. The construction of multimagnon wave functions in each N-magnon sector is based on solutions of the set of linear algebraic equations which determine also the structure of zonal spherical functions on symmetric spaces $X_N^- = SL(N, \mathbf{H})/\mathrm{Sp}(N)$ of negative curvature. The usual Bethe Ansatz for the XXX Heisenberg model corresponds to asymptotic forms of these wave functions at small values of a^{-1} or large distances between spins turned over the ferromagnetic ground state.

1. Introduction

Starting with the paper of Bethe [1], the investigation of one-dimensional exactly solvable models of interacting objects (spins, classical or quantum particles in the schemes of first and second quantization) has given a number of results both of physical and mathematical significance. One of the highlights in this branch of mathematical physics is the Yang-Baxter equation [2, 3] which serves as a source of continuous development in the study of various aspects of group theory and low-dimensional statistical mechanics. Most of the well-known statistical models both in one- and two-dimensional cases have solutions in the form of the Bethe Ansatz in its classical [1, 2, 4] or algebraic [5] versions with some more or less sophisticated modifications. On the other hand, there is a family of systems which were proved to be completely integrable [6–11], but the solutions were not included into the Bethe Ansatz [9–10, 16] or still remained unknown.

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In the continuous case these one-dimensional systems are described by the Hamiltonians

$$H_c = -\frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial}{\partial x_j}\right)^2 + \sum_{\alpha\in\Delta_+^N} V_\alpha(\alpha x), \tag{1}$$

where Δ_+^N is the variety of positive roots of classical Lie algebras realized as the vectors in $\mathbf{R}^N$ and the functions $V_\alpha(\xi)$ may depend on the length of the root vector but not on its direction [11]. The simplest but most interesting are examples with the roots of A_{N-1} algebra in which $V(\xi)=g^2\wp(\xi)$ or

$$V(\xi)=g^2a^2\sinh^{-2}(a\xi), \quad 0<a<\infty. \tag{2}$$

Periods of the Weierstrass $\wp$ function may be arbitrary and the hyperbolic case (2) is obtained from the general elliptic case if the real period is infinite.

The $S=\frac{1}{2}$ spin versions of these Hamiltonians,

$$H_s = -\frac{J}{2}\sum_{j\geqq l,\, j,l\in\mathbf{Z}} V(j-l)\frac{(\sigma_j\sigma_l-1)}{2}, \tag{3}$$

where $\{\sigma_j\}$ are usual Pauli matrices, have been proposed only recently. It is worth noting that, contrary to (1), the H_s is not singular and might be used in the models of ferro- and antiferromagnetism.

The degenerate case of $(j-l)^{-2}$ exchange has been treated in [17, 18] while in [19] I have shown that the isotropic Heisenberg model with nearest-neighbor spin interaction originally solved by Bethe [1] can be obtained as the limit of (3) with $V(\xi)=\wp(\xi)$ if the imaginary period of $\wp$ tends to zero with the inverse of the coupling J. Since in the case of nearest-neighbor interaction the eigenvectors of H_s have the form of the Bethe Ansatz, the following problem arises: how to find an appropriate extension of this Ansatz so as to get the solution of the eigenvalue problem for the Hamiltonian (3)?

In this paper I shall give the complete description of the H_s eigenvectors for infinite spin chains with the exchange interaction (2). In the simplest so-called ferromagnetic ground states (f.g.s.) $|0_\pm\rangle$ all spins have the same directions along some axis,

$$|0_\pm\rangle = \bigotimes_{j\in\mathbf{Z}} \chi_j^\pm, \quad \chi_j^+ = \begin{pmatrix}1\\0\end{pmatrix}_j, \quad \chi_j^- = \begin{pmatrix}0\\1\end{pmatrix}_j. \tag{4}$$

It is easy to see that f.g.s. (4) are eigenvectors of H_s, $H_s|0_\pm\rangle=0$. Due to the $SU(2)$ symmetry of (3), all other eigenvectors can be represented in the form

$$|\psi_N^\pm\rangle = \sum_{n\in\mathbf{Z}^N}\psi^{(N)}(n_1,\dots,n_N)\left(\prod_{\gamma=1}^{N} a_{n_\gamma}^+\right)|0_\pm\rangle, \quad N\in\mathbf{Z}_+,$$
$$\prod_{\beta>\gamma}^{N}(n_\beta-n_\gamma)\neq 0, \tag{5}$$

where a_j^+ is the operator which turns spin at j^{th} position to opposite direction $(a_j^+\chi_j^\pm=\chi_j^\mp)$ and the N-magnon wave function $\psi^{(N)}(n_1,\dots,n_N)$ is completely symmetric in its arguments. Explicit expressions for $\psi^{(N)}$ have been found in [19] for $N\leqq 2$ and in [20] for $N\leqq 4$ where one of the possible ways of $\psi^{(N)}$ construction for an arbitrary $N\in\mathbf{Z}_+$ has been also indicated.

As it will be shown later, a more simple solution of the problem consists in finding the remarkable correspondence between $\psi^{(N)}$ and zonal spherical functions (ZSF) on the symmetric space $X_N^- = SL(N,\mathbf{H})/\mathrm{Sp}(N)$ of negative curvature. As a consequence, the asymptotics of N-magnon scattering can be described in terms of the Harish-Chandra function [10, 12, 13] $c(\lambda)$, $\lambda \in \mathbf{R}^N$, the components of the vector λ being expressed through elliptic Weierstrass ζ function of magnon quasimomenta.

The usual Bethe Ansatz appears in two limiting situations: first, at small values of the imaginary period of (2) and second, at large distances between spins turned over the f.g.s.

The organization of the paper is as follows. The structure of ZSF on the symmetric spaces X_N^-, $N \in \mathbf{Z}_+$ is discussed in detail in Sect. 2. It is shown that the eigenfunctions of the Hamiltonian (1) in the case of A_{N-1} root system and $V(\xi)$ of the type (2) at $g^2=2$ can be constructed by solving the set of linear algebraic equations. The similar set appears also in the process of solving the eigenvalue problem for quantum spin Hamiltonian (3). It is just this fact that allows one to find the extension of the Bethe Ansatz for quantum spin chains with non-nearest-neighbor interaction. The proof is based on the representation of some infinite trigonometric sums through elliptic functions, as it is shown in Sect. 3. The last section contains a short summary and discussion of the results.

2. Zonal Spherical Functions on X_N^- and the Structure of Eigenvectors of H_c at $g^2=2$

Olshanetsky and Perelomov [10] were the first who noticed that the Hamiltonians of the type (1) at some values of the coupling g^2 can be obtained by singular transformation from radial parts of the second-order Laplace-Beltrami operators on symmetric spaces. So the eigenfunctions are related to ZSF and various results of group theory can be used. The most appropriate for our purposes are the ZSF $\Phi_k^{(N)}(x)$ $(k, x \in \mathbf{R}^N)$ on the symmetric space $X_N^- = SL(N,\mathbf{H})/\mathrm{Sp}(N)$. In [10] it was shown that $\Phi_k^{(N)}(x)$ can be written as

$$\Phi_k^{(N)}(x) = \left[\prod_{j>l}^{N} a \sinh^{-1} a(x_j - x_l)\right]^2 \psi_k^{(N)}(x), \tag{6}$$

where $\psi_k^{(N)}(x)$ are eigenfunctions of Calogero-Sutherland operator H^{CS} of the type (1, 2) at $g^2=2$,

$$H^{\mathrm{CS}} = -\frac{1}{2}\sum_{j=1}^{N}\left(\frac{\partial}{\partial x_j}\right)^2 + 2\sum_{j>l}^{N} a^2 \sinh^{-2} a(x_j - x_l). \tag{7}$$

Both $\Phi_k^{(N)}(x)$ and $\psi_k^{(N)}(x)$ are completely symmetric functions of $(x_1, \ldots, x_N)$. The eigenvalue of H^{CS} which corresponds to $\psi_k^{(N)}(x)$ equals

$$E(k) = \frac{1}{2}\sum_{j=1}^{N} k_j^2. \tag{8}$$

It follows from (6) and the Harish-Chandra series [12] for $\Phi_k^{(N)}(x)$ that $\psi_k^{(N)}(x)$ can be represented in the form

$$\psi_k^{(N)}(x) = \sum_{P \in \pi_N} \chi_k^{(N)}(Px).$$

Here π_N is the group of all permutations $\{P\}$ of the numbers from 1 to N, Px is the vector in $\mathbf{R}^N$ with the components $(x_{P1}, \dots, x_{PN})$ and $\chi_k^{(N)}(x)$ is the eigenvector of H^{CS} with the eigenvalue (8) which can be written as

$$\chi_k^{(N)}(x) = \exp\left(i \sum_{\mu=1}^{N} k_\mu x_\mu\right) \varphi_k^{(N)}(x), \tag{9}$$

where $\varphi_k^{(N)}(x)$ is periodic in each x_j,

$$\varphi_k^{(N)}(x) = \varphi_k^{(N)}(x_1, \dots, x_j + i\pi a^{-1}, \dots, x_N). \tag{10}$$

Moreover, it depends on k only through the combinations $\{a^{-1}(k_j - k_l)\}$.

The asymptotic behaviour of ZSF is completely determined by the Harish-Chandra c function [10]. In the limit $x_j - x_l \to +\infty$ $(j > l)$ one obtains

$$\lim \varphi_k^{(N)}(x) = c(a^{-1}k). \tag{11}$$

As it was shown in [13], for the X_N^- space $c(\lambda)$ can be written as

$$c(\lambda) = \prod_{\mu > \nu}^{N} \left\{ \frac{i}{2}(\lambda_\mu - \lambda_\nu) \left[1 + \frac{i}{2}(\lambda_\mu - \lambda_\nu) \right] \right\}^{-1} \prod_{\mu=1}^{N} (2\mu + 1)! \tag{12}$$

Unfortunately, no more detailed information about the properties of $\psi_k^{(N)}(x)$ can be obtained from the general theory of symmetric spaces which gives for ZSF only very cumbersome integral representations and multidimensional infinite series with coefficients being determined by a recurrent procedure. The simplified integral representation of ZSF on X_3^- has been constructed in [14]. The case of some other rank 2 symmetric spaces has been studied by the method of intertwining operators [15]. As for ZSF on X_N^- at $N \geqq 3$, the explicit construction of the differential operator which intertwines H^{CS} and the usual N-dimensional Laplacian $\sum_{j=1}^{N} \left(\frac{\partial}{\partial x_j} \right)^2$ has been proposed recently by Chalykh and Veselov [15]. The functions of type (11) have been represented in the form

$$\chi_k^{(N)}(x) = D_N \exp\left(i \sum_{\mu=1}^{N} k_\mu x_\mu\right), \qquad D_N = Q_N^{1 \dots N-1} D_{N-1}, \tag{13}$$

where

$$\begin{aligned} Q_n^{i_1 \dots i_l} &= Q_n^{i_1 \dots i_{l-1}} \left[\frac{\partial}{\partial x_{i_l}} - \frac{\partial}{\partial x_n} - 2a \coth a(x_{i_l} - x_n) \right] \\ &\quad + \sum_{s=1}^{l-1} 2a^2 \sinh^{-2}[a(x_{i_s} - x_{i_l})] Q_n^{i_1 \dots i_{s-1} i_{s+1} \dots i_{l-1}}, \qquad Q_n = 1, \\ &\quad 1 \leqq i_1, \dots, i_l < n, \qquad 1 \leqq n \leqq N. \end{aligned} \tag{14}$$

This double recurrence scheme of $\chi_k^{(N)}$ construction is also very cumbersome because of the presence of multiple differentiations. So, explicit calculations of $\chi_k^{(N)}$ have been performed only in the case $N = 3$. However, I'll show that the use of (14–15) allows one to reduce the construction of $\Phi_k^{(N)}(x)$ and $\chi_k^{(N)}(x)$ to a much simpler problem of solving the set of linear algebraic equations.

To start with, let us note that the function $\varphi_k^{(N)}(x)$ from (9) can be represented in the form

$$\varphi_k^{(N)} = R(\{\coth a(x_j - x_l)\}), \tag{15}$$

where R is a polynomial in the variables $\{\coth a(x_j-x_l)\}$, $1 \leqq j < l \leqq N$. It follows immediately from (13–14) and the representation of $\left(\frac{d}{dz}\right)^n [\coth z]^m$ as a polynomial in $\coth z$ of the degree $(m+n)$. As it can be seen from the structure of singularities of H^{CS} (7), the function $\varphi_k^{(N)}(x)$ has a simple pole of the type $[\sinh a(x_j-x_l)]^{-1}$ at each hyperplane $x_j-x_l=0$. As a consequence of (15), all the limits of $\varphi_k^{(N)}(x)$ as $x_j \to \pm\infty$, $1 \leqq j \leqq N$, must be finite. Combining these properties with the periodicity (10), one obtains that the eigenfunctions of H^{CS} can be written as

$$\chi_k^{(N)}(x) = \exp\left\{\sum_{\mu=1}^{N} [ik_\mu - a(N-1)]x_\mu\right\}\left[\prod_{\mu>\nu}^{N} \sinh a(x_\mu - x_\nu)\right]^{-1} S_k^{(N)}(\{y\}), \quad (16)$$

where $S_k^{(N)}(\{y\})$ is a polynomial in $y_\mu = \exp(2ax_\mu)$. The maximal power of each variable in $S_k^{(N)}(\{y\})$ cannot exceed $N-1$. Hence this polynomial can be represented in the form

$$S_k^{(N)}(\{y\}) = \sum_{m\in D^N} d_{m_1\dots m_N}(k) \prod_{\lambda=1}^{N} y_\lambda^{m_\lambda}, \quad (17)$$

where D^N is the hypercube in $\mathbf{Z}^N$,

$$m \in D^N \Leftrightarrow 0 \leqq m_\beta \leqq N-1, \quad \beta = 1, \dots, N, \quad (18)$$

and $d_{\{m\}}(k)$ is the set of N^N coefficients. It will be shown later that most of them vanish.

From the eigenvalue condition $[H^{\mathrm{CS}} - E(k)]\chi_k^{(N)}(x) = 0$ one obtains for $S_k^{(N)}(\{y\})$ the equation

$$\begin{aligned}&\sum_{\beta=1}^{N}\left[2y_\beta\frac{\partial}{\partial y_\beta}\left(y_\beta\frac{\partial}{\partial y_\beta} + ia^{-1}k_\beta - N + 1\right) - ia^{-1}k_\beta(N-1) + \frac{(N-1)(2N-1)}{3}\right]S_k^{(N)}\\ &-\sum_{\beta\neq\varrho}^{N}\frac{y_\beta + y_\varrho}{y_\beta - y_\varrho}\left[y_\beta\frac{\partial}{\partial y_\beta} - y_\varrho\frac{\partial}{\partial y_\varrho} + \frac{i}{2a}(k_\beta - k_\varrho)\right]S_k^{(N)} = 0. \quad (19)\end{aligned}$$

It can be satisfied only if for each $\beta, \varrho \leqq N$ the polynomial $\left[y_\beta\frac{\partial}{\partial y_\beta} - y_\varrho\frac{\partial}{\partial y_\varrho} + \frac{i}{2a}(k_\beta - k_\varrho)\right]S_k^{(N)}$ is divisible by $(y_\beta - y_\varrho)$. By the use of (17) this condition can be expressed in the form of $\frac{1}{2}(N-1)(2N-1)N^N$ linear equations for the coefficients $d_{\{m\}}(k)$,

$$\sum_{l\in\mathbf{Z}} d_{m_1\dots m_\beta+l\dots m_\varrho-l\dots m_N}(k)\left[m_\beta - m_\varrho + 2l + \frac{i}{2a}(k_\beta - k_\varrho)\right] = 0. \quad (20)$$

Since the indices of $d_{\{m\}}(k)$ obey the restriction (18), the sum over l in (20) is finite and runs from $\max(-m_\beta, m_\varrho+1-N)$ to $\min(N-1-m_\beta, m_\varrho)$. Upon substituting (17) into (19) one also obtains the condition

$$\begin{aligned}&\sum_{m\in D^N}\left(\prod_{\lambda=1}^{N} y_\lambda^{m_\lambda}\right)d_{\{m\}}(k)\left\{\sum_{\beta=1}^{N}\left[2m_\beta^2 + \frac{2i}{a}k_\beta m_\beta - \left(2m_\beta + \frac{i}{a}k_\beta - \frac{2N-1}{3}\right)(N-1)\right]\right.\\ &\left.-\sum_{\beta\neq\varrho}^{N}\frac{y_\beta + y_\varrho}{y_\beta - y_\varrho}\left[m_\beta - m_\varrho + \frac{i}{2a}(k_\beta - k_\varrho)\right]\right\} = 0. \quad (21)\end{aligned}$$

As (21) must be satisfied for all $y\in\mathbf{R}^N$, one gets finally the second system of N^N linear equations for $d_{\{m\}}$ after performing the explicit division by $(y_\beta-y_\varrho)$ in the last term in (21) by the use of (20),

$$\begin{aligned}&d_{m_1\dots m_N}(k)\left\{\sum_{\beta=1}^N\left[2m_\beta\left(m_\beta+\frac{i}{a}k_\beta\right)-\left(2m_\beta+\frac{i}{a}k_\beta-\frac{2N-1}{3}\right)(N-1)\right]\right\}\\&\quad-\sum_{\beta\neq\varrho}^N\sum_{l\in\mathbf{Z}}\operatorname{sign}(l)\left[m_\beta-m_\varrho+2l+\frac{i}{2a}(k_\beta-k_\varrho)\right]\\&\quad\times d_{m_1\dots m_\beta+l\dots m_\varrho-l\dots m_N}(k)=0\,.\end{aligned}\tag{22}$$

Now the structure of $d_{\{m\}}(k)$ can be specified by the following four propositions.

Proposition 1. *$S_k^{(N)}(y)$ is a homogeneous polynomial of the degree $\frac{1}{2}N(N-1)$.*

The proof is based on the system (20). It is easy to see that $d_{\{m\}}(k)$ vanish if $m_\mu=m_\nu=0$ or $m_\mu=m_\nu=N-1$ for any pair (μ,ν).

Let us consider some $d_{m_1\dots m_N}(k)$ with $M=\sum_{\mu=1}^N m_\mu<\frac{N(N-1)}{2}$. Then at least two among the numbers $\{m_\mu\}$ must coincide. Let j be the minimal value of coinciding $\{m_\mu\}$. Choosing among these $\{m_\mu\}$ any pair $m_\varrho=m_\gamma=j$, let us express $d_{m_1\dots m_\varrho\dots m_\gamma\dots m_N}(k)$ through the coefficients $d_{m_1\dots m_\varrho+l\dots m_\gamma-l\dots m_N}(k)$ with the same value of M by using (20). Repeating this procedure for each of these coefficients, one finally obtains that the initial $d_{m_1\dots m_N}(k)$ is represented as a linear combination of those $d_{\{m\}}(k)$ which have at least two zero values of the indices $\{m\}$, i.e. vanish. The case of $M>\frac{N(N-1)}{2}$ can be treated in a similar way.

Proposition 2. *The $d_{\{m\}}(k)$ can be chosen as depending on k and a only through the combinations $a^{-1}(k_\mu-k_\nu)$.*

For the proof it is sufficient to show that the coefficients in the system (22) depend on k and a in this manner. It can be done by using the result of the preceding proposition and the simple relations

$$\sum_{\mu=1}^N k_\mu m_\mu=N^{-1}\left[\sum_{\mu=1}^N m_\mu\sum_{\nu=1}^N k_\nu+\frac{1}{2}\sum_{\mu\neq\nu}^N(k_\mu-k_\nu)(m_\mu-m_\nu)\right],$$

$$2\sum_{\mu=1}^N m_\mu^2=N^{-1}\left[\sum_{\mu\neq\nu}^N(m_\mu-m_\nu)^2+2\left(\sum_{\mu=1}^N m_\mu\right)^2\right].$$

So the system (22) can be rewritten in the form

$$\begin{aligned}&\sum_{\beta<\varrho}^N\left\{d_{m_1\dots m_N}(k)\left[\frac{m_\beta-m_\varrho}{N}\left(\frac{i}{2}(k_\beta-k_\varrho)+m_\beta-m_\varrho\right)+\frac{N+1}{6}\right]\right.\\&\quad\left.-\sum_{l\in\mathbf{Z}}\operatorname{sign}(l)\left[m_\beta-m_\varrho+\frac{i}{2a}(k_\beta-k_\varrho)+2l\right]d_{m_1\dots m_\beta+l\dots m_\varrho-l\dots m_N}(k)\right\}=0\,.\end{aligned}\tag{23}$$

Proposition 3. *Let $\{P\}$ be the following set of the numbers $\{m_\mu\}:m_\mu=P\mu-1$, where $P\in\pi_N$, $1\leqq\mu\leqq N$. The nonvanishing $d_{\{m\}}(k)$ with coinciding values of $\{m_\mu\}$ are expressed through $d_{\{P\}}^{(k)}$. The latter are determined by the system (20) up to some* *normalization constant* d_0,

$$d_{\{P\}}(k)=d_0\prod_{\lambda<\mu}^{N}\left[1+\frac{i}{2a}(k_{P^{-1}\lambda}-k_{P^{-1}\mu})\right]. \tag{24}$$

The first part of this statement is proved by the same scheme as was applied for the proof of Proposition 1. As for formula (24), let us note that in the case of noncoinciding $\{m_\mu\}$ the system (20) contains the subsystem of $\frac{N-1}{2}N!$ equations

$$\begin{aligned} &d_{m_1\ldots m_\mu\ldots m_\nu\ldots m_N}(k)\left[-1+\frac{i}{2a}(k_\mu-k_\nu)\right]\\ &\quad+d_{m_1\ldots m_\nu\ldots m_\mu\ldots m_N}(k)\left[1+\frac{i}{2a}(k_\mu-k_\nu)\right]=0, \qquad m_\nu=m_\mu+1. \end{aligned} \tag{25}$$

Let $P\in\pi_N$ be the permutation $(j\to m_j+1)$, $1\leqq j\leqq N$. If $R\in\pi_N$ permutes μ and ν leaving other numbers from 1 to N unchanged, then (25) can be written in the form

$$d_{\{P\}}(k)\left[1+\frac{i}{2a}(k_\nu-k_\mu)\right]=d_{\{PR\}}(k)\left[1+\frac{i}{2a}(k_\mu-k_\nu)\right]. \tag{26}$$

With the use of the condition $m_\nu=m_\mu+1$ one can represent the right-hand side of (24) as

$$\begin{aligned} d_{\{P\}}(k)=d_0&\left[1+\frac{i}{2a}(k_\mu-k_\nu)\right]\prod_{\lambda=1}^{m_\mu}\left[1+\frac{i}{2a}(k_{P^{-1}\lambda}-k_\mu)\right]\left[1+\frac{i}{2a}(k_{P^{-1}\lambda}-k_\nu)\right]\\ &\times\prod_{\varrho=m_\nu+2}^{N}\left[1+\frac{i}{2a}(k_\mu-k_{P^{-1}\varrho})\right]\left[1+\frac{i}{2a}(k_\nu-k_{P^{-1}\varrho})\right]\\ &\times\prod_{\lambda<\varrho,\ \lambda,\varrho\neq m_\mu+1,m_\nu+1}\left[1+\frac{i}{2a}(k_{P^{-1}\lambda}-k_{P^{-1}\mu})\right]. \end{aligned} \tag{27}$$

The corresponding expression for $d_{\{PR\}}(k)$ differs from (27) only by the change $k_\mu\leftrightarrow k_\nu$. So (26) is fulfilled for any $P\in\pi_N$.

The leading terms in asymptotic expansions of $\chi_k^{(N)}(x)$ are completely determined by the set $d_{\{P\}}(k)$ (24) as it follows from

Proposition 4. *Let* $(-1)^P$ *be the parity of the permutation* P. *If* $x_{P(\lambda+1)}-x_{P\lambda}\to+\infty$, $1\leqq\lambda\leqq N-1$, *then*

$$\lim\chi_k^{(N)}(x)\exp\left(-i\sum_{\lambda=1}^{N}k_\lambda x_\lambda\right)=(-1)^P 2^{\frac{N(N-1)}{2}}d_{\{P^{-1}\}}(k). \tag{28}$$

The scheme of the proof can be illustrated by the case $x_{\lambda+1}-x_\lambda\to+\infty$, $1\leqq\lambda\leqq N-1$. Then

$$\prod_{\lambda>\mu}^{N}\sinh a(x_\lambda-x_\mu)\sim 2^{\frac{-N(N-1)}{2}}\exp\left[-\sum_{\lambda=1}^{N}x_\lambda(N-2\lambda+1)\right]$$

and

$$\begin{aligned} &\chi_k^{(N)}(x)\exp\left[-i\sum_{\lambda=1}^{N}k_\lambda x_\lambda\right]\\ &\quad\sim 2^{\frac{N(N-1)}{2}}\sum_{m\in D^N}d_{\{m\}}(k)\exp\left[-2a\sum_{\lambda=1}^{N-1}(x_{\lambda+1}-x_\lambda)f_\lambda(\{m\})\right], \end{aligned}$$

where $f_\lambda(\{m\}) = \sum_{\tau=1}^{\lambda} (m_\tau - \tau + 1)$. Let $\{m\}$ be any set of numbers which corresponds to nonvanishing $d_{\{m\}}(k)$. Then $f_\lambda(\{m\}) \geqq 0$ and the equality holds for all λ from 1 to $N-1$ only if $m_\lambda = \lambda - 1$. The general case of an arbitrary $P \in \pi_N$ can be treated analogously.

Remark. If the constant d_0 in (24) is chosen in the form

$$d_0 = \prod_{q=1}^{N-1} (2q+1)! \prod_{\lambda>\mu}^{N} \left[\frac{i}{a}(k_\lambda - k_\mu)\right]^{-1} \prod_{\lambda \neq \mu}^{N} \left[1 + \frac{i}{2a}(k_\lambda - k_\mu)\right]^{-1},$$

then

$$d_{\{P\}}(k) = (-1)^P c(a^{-1}k_{P^{-1}}, \ldots, a^{-1}k_{P^{-1}}),$$

where the Harish-Chandra function $c(\lambda)$ is defined by (12). So the statement of Proposition 4 agrees with the asymptotic relation (11) of general ZSF theory.

Let us note also that, according to Proposition 3, the solutions of (20) must obey the system (23). Hence (23) can be treated as a consequence of (20). It would be interesting to find the direct algebraic proof of this fact.

To end this section, let us give some examples of explicit calculation of $d_{\{m\}}(k)$ for small N by Eqs. (20) and introduce the following notation. Let $[\lambda_1 \ldots \lambda_N]$ be the permutation $(1 \to \lambda_1, \ldots, N \to \lambda_N)$ and $r_{\lambda\mu} = i(2a)^{-1}(k_\lambda - k_\mu)$.

1. $N=3$. According to (24), the coefficients $d_{\{P\}}(k)$ can be written as

$$\begin{aligned} &d_{012}(k) = d_0(1+r_{12})(1+r_{13})(1+r_{23}), \quad d_{102}(k) = d_0(1+r_{21})(1+r_{23})(1+r_{13}),\\ &d_{210}(k) = d_0(1+r_{32})(1+r_{31})(1+r_{21}), \quad d_{021}(k) = d_0(1+r_{13})(1+r_{12})(1+r_{32}),\\ &d_{120}(k) = d_0(1+r_{31})(1+r_{32})(1+r_{12}), \quad d_{201}(k) = d_0(1+r_{23})(1+r_{21})(1+r_{31}). \end{aligned} \tag{29}$$

The last nonvanishing coefficient $d_{111}(k)$ is determined from (20) at $m_1 = 0$, $m_2 = 1$, $m_3 = 2$, $\beta = 1$, $\varrho = 3$,

$$d_{111}(k) = d_0(6 - r_{12}^2 - r_{13}^2 - r_{23}^2). \tag{30}$$

2. $N=4$. There are 24 coefficients of the $d_{\{P\}}(k)$ type. The other nonzero $d_{\{m\}}(k)$ with coinciding values of indices can be divided into three sets. The first two of them consist of coefficients with three coinciding indices. They can be calculated in the same way as $d_{111}(k)$ at $N=3$,

$$d_{1113}(k) = d_0(1+r_{14})(1+r_{24})(1+r_{34})(6 - r_{12}^2 - r_{13}^2 - r_{23}^2), \tag{31}$$

$$d_{2220}(k) = d_0(1+r_{41})(1+r_{42})(1+r_{43})(6 - r_{12}^2 - r_{13}^2 - r_{23}^2). \tag{32}$$

The other elements of these sets,

$$d_{1131}(k),\ d_{1311}(k),\ d_{3111}(k) \quad \text{and} \quad d_{2202}(k),\ d_{2022}(k),\ d_{0222}(k)$$

are obtained from (31) and (32) by the permutations [1243], [1342], [2341] of indices in $\{r_{\lambda\mu}\}$. The third set contains the coefficients

$$d_{1122}(k),\ d_{2211}(k),\ d_{2112}(k),\ d_{1221}(k),\ d_{1212}(k),\ d_{2121}(k)$$

with two pairs of coinciding indices. They are determined from Eqs. (20) with the use of the coefficients of the type (31) belonging to the first set. For example, taking in (20) $m_1=m_2=m_3=1$, $m_4=3$, $\beta=3$, $\varrho=4$ one obtains

$$d_{1113}(k)(-2+r_{34})+d_{1122}(k)r_{34}+d_{1131}(k)(2+r_{34})=0\,.$$

After simple calculations, $d_{1122}(k)$ can be written as

$$\begin{aligned}d_{1122}(k)=d_0[18+(r_{13}+r_{24})(14-2(r_{13}+r_{24})-r_{13}^2-r_{24}^2-3(r_{12}^2+r_{34}^2)\\ -r_{12}r_{34}+r_{12}^2r_{34}^2)-(r_{12}^2+r_{34}^2)(6+r_{13}r_{24}-r_{12}r_{34})-r_{13}r_{24}(r_{13}^2+r_{24}^2)]\,.\end{aligned} \tag{33}$$

The remaining five coefficients of these sets are given by (33) after the permutations [3412], [3214], [4123], [1324], [4123] of the indices in $\{r_{\lambda\mu}\}$.
3. For $N=5$ there are, besides $d_{\{P\}}(k)$, 171 nonzero coefficients $d_{\{m\}}(k)$ with coinciding values of indices. They are divided into 8 sets. For $N=6$ the number of these coefficients increases to 2112 and the number of various sets becomes equal to 21. In spite of this fast growth, the system (20) seems to be most convenient for the explicit calculations of the eigenfunctions of the Hamiltonian (7) and zonal spherical functions $\phi_k^{(N)}(x)$ on X_N^-.

3. The Solution of the Eigenvalue Problem for the Quantum Spin Hamiltonian H_s

As it was already mentioned in Sect. 1, the eigenvectors of the operator H_s (3) are classified with respect to the number N of spins turned over the ferromagnetic ground state. Let us fix $N\in\mathbf{Z}_+$ and consider the N-magnon eigenstates (5). The eigenvalue problem $H_s|\psi_N^{\pm}\rangle=\varepsilon_N|\psi_N^{\pm}\rangle$ can be represented in terms of the N-magnon wave function $\psi(n_1,\dots,n_N)$ as [20]

$$\begin{aligned}&\sum_{\beta=1}^{N}\sum_{s\in\mathbf{Z}_{[n]}} V(n_\beta-s)\psi(n_1,\dots,n_{\beta-1},s,n_{\beta+1},\dots,n_N)\\ &\quad=\psi(n_1,\dots,n_N)\left[\sum_{\beta\neq\gamma}^{N} V(n_\beta-n_\gamma)+J^{-1}\varepsilon_N-N\varepsilon_0\right],\\ &\qquad\qquad\prod_{\beta>\gamma}^{N}(n_\beta-n_\gamma)\neq 0\,,\end{aligned} \tag{34}$$

where $n\in\mathbf{Z}^N$ and the notation $\mathbf{Z}_{[n]}$ is used for the variety $\mathbf{Z}-(n_1,\dots,n_N)$. Motivated by the structure of $\psi(\{n\})$ at $N=2$ found in [19], I'll search the solutions to (34) in the form which is very similar to the ZSF on X_N^-. It is the properly symmetrized combination of the functions like $\chi_k^{(N)}(x)$ in the form (16–17) restricted to the integer values of arguments,

$$\begin{aligned}\psi(n_1,\dots,n_N)=&\prod_{\mu>\nu}^{N}[\sinh a(n_\mu-n_\nu)]^{-1}\sum_{P\in\pi_N}(-1)^P\exp\left(i\sum_{\lambda=1}^{N}k_{P\lambda}n_\lambda\right)\\ &\times\sum_{m\in D^N}\tilde{d}_{m_1\dots m_N}(k)\exp\left[a\sum_{\lambda=1}^{N}(2m_{P\lambda}-N+1)n_\lambda\right].\end{aligned} \tag{35}$$

Here $k_1,\dots,k_N\in\mathbf{R}(\mathrm{mod}\,2\pi)$ are the magnon quasimomenta for scattering states in which $\psi(\{n\})$ oscillates at infinity. Contrary to the eigenvectors of the Hamiltonian (7), there are, as it will be shown later, the bound states in which $\psi(\{n\})$ vanishes if $|n_\alpha-n_\beta|\to\infty$. All the other notations in (35) are the same as in the preceding section.

The eigenvalue problem reduces to finding the set $\tilde{a}_{m_1\dots m_N}(k)$ if the Ansatz (35) is correct. A crucial point is the calculation of the left-hand side of Eq. (34) with $\psi(\{n\})$ in the form (35),

$$\begin{aligned}\mathscr{L}(\{n\}) &= a^2 \sum_{\beta=1}^{N} \sum_{s\in \mathbf{Z}_{[n]}} [\sinh a(n_\beta - s)]^{-2}\psi(n_1,\dots,n_{\beta-1},s,n_{\beta+1},\dots,n_N)\\ &= \sum_{\beta=1}^{N}\sum_{P\in\pi_N}(-1)^P\left[\prod_{\mu>\nu;\,\mu,\nu\neq\beta}^{N}\sinh a(n_\mu-n_\nu)\right]^{-1}(-1)^{\beta-1}\\ &\quad\times\sum_{m\in D^N}\tilde{a}_{m_1\dots m_N}(k)\\ &\quad\times\exp\left\{\sum_{\gamma\neq\beta}[ik_{P\gamma}+a(2m_{P\gamma}-N+1)]n_\gamma\right\}W(k_{P\beta},m_{P\beta},\{n\}),\end{aligned} \tag{36}$$

where

$$\begin{aligned}W(k,m,\{n\}) &= \sum_{s\in\mathbf{Z}_{[n]}}\frac{a^2}{\sinh^2 a(s-n_\beta)}\left[\prod_{\lambda\neq\beta}^{N}\sinh a(n_\lambda - s)\right]^{-1}\\ &\quad\times\exp\{[ik+a(2m-N+1)]s\}.\end{aligned} \tag{37}$$

It is easy to see that the sum (37) converges for all m obeying (18) if $k\in\mathbf{C}$ is restricted to the strip $|\Im m k|<2a$. To calculate (37) explicitly, let us consider the function $W_q(x)$ of the variable $x\in\mathbf{C}$,

$$W_q(x)=\sum_{s\in\mathbf{Z}}\frac{a^2\exp(qs)}{\sinh^2 a(s-n_\beta+x)}\prod_{\lambda\neq\beta}^{N}[\sinh a(n_\lambda-s-x)]^{-1}, \tag{38}$$

$$q=ik+a(2m-N+1). \tag{39}$$

As it follows immediately from (38), $W_q(x)$ is double quasiperiodic,

$$W_q(x+i\pi a^{-1})=\exp[i\pi(N-1)]W_q(x),\; W_q(x+1)=\exp(-q)W_q(x). \tag{40}$$

Hence it can be treated on the torus $\mathbf{C}\backslash\Gamma$ obtained by the factorization of the complex plane by the lattice $\Gamma=l_1+i\pi a^{-1}l_2$ ($l_1,l_2\in\mathbf{Z}$). The only singularity of $W_q(x)$ on $\mathbf{C}\backslash\Gamma$ is the double pole at $x=0$. It arises from the terms with $s=n_1,\dots,n_N$ in (38). After simple calculations one obtains the first three terms of the Laurent decomposition of (38) near $x=0$,

$$W_q(x)=b_0x^{-2}+b_1x^{-1}+b_2+O(x), \tag{41}$$

where

$$b_0=\exp(qn_\beta)\prod_{\lambda\neq\beta}[\sinh a(n_\lambda-n_\beta)]^{-1}, \tag{42}$$

$$\begin{aligned}b_1 &= a\Bigg\{b_0\sum_{\gamma\neq\beta}^{N}\coth a(n_\gamma-n_\beta)-\sum_{\varrho\neq\beta}^{N}\exp(qn_\varrho)\\ &\quad\times\left[\sinh a(n_\beta-n_\varrho)\prod_{\lambda\neq\varrho}^{N}\sinh a(n_\lambda-n_\varrho)\right]^{-1}\Bigg\},\end{aligned} \tag{43}$$

$$\begin{aligned}b_2 &= a^2\Bigg\{b_0\Bigg[-\frac{1}{3}+\frac{N-1}{2}+\frac{1}{2}\sum_{\gamma\neq\delta\neq\beta}^{N}\coth a(n_\gamma-n_\beta)\coth a(n_\delta-n_\beta)\\ &\quad+\sum_{\gamma\neq\beta}^{N}\sinh^{-2}(n_\gamma-n_\beta)\Bigg]-\sum_{\varrho\neq\beta}^{N}\frac{\exp(qn_\varrho)}{\sinh a(n_\beta-n_\varrho)}\left[\prod_{\lambda\neq\varrho}^{N}\sinh a(n_\lambda-n_\varrho)\right]^{-1}\\ &\quad\times\left[\coth a(n_\beta-n_\varrho)+\sum_{\gamma\neq\varrho}^{N}\coth a(n_\gamma-n_\varrho)\right]\Bigg\}+W(k,m,\{n\}).\end{aligned} \tag{44}$$

Now let us construct the function $U_q(x)$ with the same quasiperiodicity conditions (40) and pole decomposition (41) by using the Weierstrass elliptic functions $\wp(x)$, $\zeta(x)$, and $\sigma(x)$ defined on the torus $\mathbf{C}\backslash\Gamma$. The proper form of $U_q(x)$ is

$$U_q(x) = -A\frac{\sigma(x+r)}{\sigma(x-r)}\exp(\delta x) \times \{\wp(x) - \wp(r) + \Delta[\zeta(x+r) - \zeta(x) - \zeta(2r) + \zeta(r)]\}, \tag{45}$$

where A, r, δ, and Δ are some constants. The term in braces is chosen so that it is double periodic and has a zero at $x=r$. So the only singularity of $U_q(x)$ on $\mathbf{C}\backslash\Gamma$ is at $x=0$ for all values of r and Δ.

By using the properties of sigma functions [21] one obtains

$$\frac{\sigma(x+r+1)}{\sigma(x-r+1)} = \exp(2\eta_1 r)\frac{\sigma(x+r)}{\sigma(x-r)}, \quad \frac{\sigma(x+r+i\pi a^{-1})}{\sigma(x-r+i\pi a^{-1})} = \exp(2\eta_2 r)\frac{\sigma(x+r)}{\sigma(x-r)}, \tag{46}$$

where $\eta_1 = 2\zeta\left(\frac{1}{2}\right)$ and $\eta_2 = 2\zeta\left(\frac{i\pi}{2a}\right)$. Comparing (46) with (40), one finds two equations for r and δ,

$$2\eta_1 r + \delta = -q, \quad 2\eta_2 r + i\pi a^{-1}\delta = i\pi(N-1).$$

Solving them with the use of the expression for q (39) and the Legendre relation

$$i\pi a^{-1}\eta_1 - \eta_2 = 2\pi i, \tag{47}$$

one finds both r and δ,

$$r = -\left(\frac{m}{2} + \frac{ik}{4a}\right), \quad \delta = a\left[N - 1 + \frac{4i}{\pi}r\zeta\left(\frac{i\pi}{2a}\right)\right]. \tag{48}$$

The Laurent decomposition of (45) at $x=0$ can be obtained by standard expansions of $\wp$, ζ, and σ [21],

$$U_q(x) = A[x^{-2} + (2\zeta(r) + \delta - \Delta)x^{-1} + \tfrac{1}{2}(2\zeta(r) + \delta - 2\Delta)(2\zeta(r) + \delta) + \Delta(2\zeta(r) - \zeta(2r)) - \wp(r)] + O(x). \tag{49}$$

The function $W_q - U_q$ is analytic on $\mathbf{C}\backslash\Gamma$ if A and δ obey the conditions

$$A = b_0, \quad A(2\zeta(r) + \delta - \Delta) = b_1. \tag{50}$$

But, according to the Liouville theorem, the only analytic function defined on a torus is a constant which must be zero because of quasiperiodicity of $W_q(x) - U_q(x)$ (40). Comparing now the decompositions (41) and (49) one obtains the expression of b_2 in terms of b_0, b_1, r, δ, and Δ,

$$b_2 = b_0[\tfrac{1}{2}(2\zeta(r) + \delta - 2\Delta)(2\zeta(r) + \delta) + \Delta(2\zeta(r) - \zeta(2r)) - \wp(r)]. \tag{51}$$

With the use of (42–44) and (51) the sum $W(k,m,\{n\})$ defined by (37) can be written as

$$
\begin{aligned}
W(k,m,\{n\})=a^2\Bigg\{&-\exp(qn_\beta)\left[\prod_{\lambda\neq\beta}^{N}\sinh a(n_\lambda-n_\beta)\right]^{-1}\\
&\times\left[\frac{(N-1)}{2}+\frac{1}{2}\sum_{\gamma\neq\mu\neq\beta}^{N}\coth a(n_\gamma-n_\beta)\coth a(n_\mu-n_\beta)\right.\\
&\left.+\sum_{\gamma\neq\beta}^{N}\sinh^{-2}a(n_\gamma-n_\beta)-a^{-1}\tilde{f}_a(r)\sum_{\gamma\neq\beta}^{N}\coth a(n_\gamma-n_\beta)+a^{-2}\tilde{\varepsilon}_a(r)\right]\\
&+\sum_{\varrho\neq\beta}^{N}\frac{\exp(qn_\varrho)}{\sinh a(n_\beta-n_\varrho)}\left[\prod_{\lambda\neq\varrho}^{N}\sinh a(n_\lambda-n_\varrho)\right]^{-1}\\
&\times\left[\coth a(n_\beta-n_\varrho)+\sum_{\gamma\neq\varrho}^{N}\coth a(n_\gamma-n_\varrho)-a^{-1}\tilde{f}_a(r)\right]\Bigg\},
\end{aligned}
\tag{52}
$$

where

$$\tilde{f}_a(r)=\zeta(2r)+\delta\,,\tag{53}$$

$$\tilde{\varepsilon}_a(r)=-\frac{a^2}{3}+\wp(r)+\frac{1}{2}(2\zeta(r)+\delta)(2\zeta(2r)-2\zeta(r)+\delta)\,.\tag{54}$$

Now let us show that $\tilde{\varepsilon}_a(r)$ and $\tilde{f}_a(r)$ are some polynomials in m. According to (44), r and δ can be written as

$$r=r_k-\frac{m}{2},\qquad \delta=a\left[N-1-\frac{2i}{\pi}m\zeta\left(\frac{i\pi}{2a}\right)\right]+\delta_k\,,$$

where

$$r_k=-\frac{ik}{4a},\qquad \delta_k=\frac{k}{\pi}\zeta\left(\frac{i\pi}{2a}\right).$$

By using the Legendre relation (47) and the quasiperiodicity of $\zeta(x)$

$$\zeta(x+l)=\zeta(x)+2l\zeta(\tfrac{1}{2}),\qquad l\in\mathbf{Z}\,,\tag{55}$$

one can represent (53) in the form

$$\tilde{f}_a(r)=f_a(k)-a(2m+1-N)\,,\tag{56}$$

where

$$f_a(k)=\zeta(2r_k)+\delta_k=\frac{k}{\pi}\zeta\left(\frac{i\pi}{2a}\right)-\zeta\left(\frac{ik}{2a}\right).\tag{57}$$

The calculation of explicit m dependence of $\tilde{\varepsilon}_a(r)$ is not so trivial. With the use of (55) one obtains

$$
\begin{aligned}
\tilde{\varepsilon}_a(r)=&-\frac{a^2}{3}+\wp\left(r_k-\frac{m}{2}\right)\\
&+\frac{1}{2}\left[2\zeta\left(r_k-\frac{m}{2}\right)+2m\zeta\left(\frac{1}{2}\right)+\delta_k-a(2m+1-N)\right]\\
&\times\left[2\zeta(2r_k)+\delta_k-a(2m+1-N)-2m\zeta\left(\frac{1}{2}\right)-2\zeta\left(r_k-\frac{m}{2}\right)\right].
\end{aligned}
$$

First, let us consider the case of even m. The periodicity of Weierstrass functions now can be used explicitly. One finds

$$\tilde{\varepsilon}_a(r)=\varepsilon_a(k)-a(2m+1-N)f_a(k)+\frac{a^2}{2}(2m+1-N)^2, \tag{58}$$

where

$$\varepsilon_a(k)=-\frac{a^2}{3}+\wp(r_k)+\frac{1}{2}(2\zeta(r_k)+\delta_k)(2\zeta(2r_k)-2\zeta(r_k)+\delta_k). \tag{59}$$

In the case of odd m one needs some more complex calculations. Note that $\tilde{\varepsilon}_a(k)$ in that case is given by

$$\begin{aligned}\tilde{\varepsilon}_a(k)=&-\frac{a^2}{3}+\wp\left(r_k-\frac{1}{2}\right)\\&+\frac{1}{2}\left[2\zeta\left(r_k-\frac{1}{2}\right)+2\zeta\left(\frac{1}{2}\right)+\delta_k-a(2m+1-N)\right]\\&\times\left[2\zeta(2r_k)+\delta_k-2\zeta\left(r_k-\frac{1}{2}\right)-2\zeta\left(\frac{1}{2}\right)-a(2m+1-N)\right].\end{aligned} \tag{60}$$

Now one can use the addition theorems for Weierstrass functions,

$$\wp\left(r_k-\frac{1}{2}\right)=-\wp(r_k)-\wp\left(\frac{1}{2}\right)+\frac{\omega^2}{4},\quad \zeta\left(r_k-\frac{1}{2}\right)+\zeta\left(\frac{1}{2}\right)=\zeta(r_k)+\frac{\omega}{2},$$
$$\omega=\wp'(r_k)\left[\wp(r_k)-\wp\left(\frac{1}{2}\right)\right]^{-1}.$$

The formula (60) is transformed to

$$\tilde{\varepsilon}_a(r)=\varepsilon_a(k)-a(2m+1-N)f_a(k)+\frac{a^2}{2}(2m+1-N)^2+\Phi(r_k),$$

where

$$\begin{aligned}\Phi(x)=&-2\wp(x)-\wp\left(\frac{1}{2}\right)\\&-\wp'(x)\left\{\frac{1}{4}\wp'(x)-[\zeta(2x)-2\zeta(x)]\left[\wp(x)-\wp\left(\frac{1}{2}\right)\right]\right\}\left[\wp(x)-\wp\left(\frac{1}{2}\right)\right]^{-2}.\end{aligned}$$

The final trick consists in the use of the formula $\zeta(2x)-2\zeta(x)=\frac{1}{2}\wp''(x)[\wp'(x)]^{-1}$ and the differential equation for $\wp(x)$. I find that $\Phi(x)$ vanishes and $\tilde{\varepsilon}_a(r)$ is given by (59) as in the case of even m.

According to (52, 56–59) the left-hand side (36) of the eigenvalue equation (34) can be represented as follows,

$$\mathscr{L}(\{n\})=\mathscr{L}_1(\{n\})+\mathscr{L}_2(\{n\})+\mathscr{L}_3(\{n\}),$$

where

$$\mathscr{L}_1(\{n\})=\psi(n_1,\ldots,n_N)\left[\sum_{\beta=1}^{N}\varepsilon_a(k_\beta)-\sum_{\beta\neq\gamma}^{N}\frac{a^2}{\sinh^2(n_\beta-n_\gamma)}\right], \tag{61}$$

$$\mathcal{L}_2(\{n\}) = -a^2 \prod_{\mu>\nu}^{N} [\sinh a(n_\mu - n_\nu)]^{-1} \sum_{P\in\pi_N} (-1)^P \sum_{m\in D^N} \tilde{d}_{m_1,\ldots,m_N}(k)$$
$$\times \sum_{\beta\neq\varrho}^{N} \exp\left\{ \sum_{\gamma\neq\beta,\varrho}^{N} [ik_{P\gamma} + a(2m_{P\gamma} - N + 1)]n_\gamma \right\}$$
$$\times \exp\{[i(k_{P\beta} + k_{P\varrho}) + 2a(m_{P\beta} + m_{P\varrho} - N + 1)]n_\varrho\}$$
$$\times [\sinh a(n_\beta - n_\varrho)]^{-1}$$
$$\times \left[\coth a(n_\beta - n_\varrho) + \sum_{\gamma\neq\varrho}^{N} \coth a(n_\gamma - n_\varrho) - a^{-1} f_a(k_{P\beta}) + 2m_{P\beta} - N + 1 \right]$$
$$\times \prod_{\lambda\neq\beta,\varrho}^{N} \frac{\sinh a(n_\lambda - n_\beta)}{\sinh a(n_\lambda - n_\varrho)}, \tag{62}$$

$$\mathcal{L}_3(\{n\}) = -a^2 \prod_{\mu>\nu}^{N} [\sinh a(n_\mu - n_\nu)]^{-1} \sum_{P\in\pi_N} (-1)^P \sum_{m\in D^N} \tilde{d}_{m_1,\ldots,m_N}(k)$$
$$\times \exp\left\{ \sum_{\gamma=1}^{N} [ik_{P\gamma} + a(2m_{P\gamma} - N + 1)]n_\gamma \right\}$$
$$\times \left\{ \sum_{\beta=1}^{N} \left[\frac{N-1}{2} - a^{-1}(2m_{P\beta} - N + 1) f_a(k_{P\beta}) + \frac{(N-1-2m_{P\beta})^2}{2} \right] \right.$$
$$- \sum_{\beta\neq\gamma}^{N} [a^{-1} f_a(k_{P\beta}) + N - 1 - 2m_{P\beta}] \coth a(n_\gamma - n_\beta)$$
$$\left. + \sum_{\beta\neq\gamma\neq\nu}^{N} \coth a(n_\gamma - n_\beta) \coth a(n_\nu - n_\beta) \right\}. \tag{63}$$

It is easy to see that if one chooses the N-magnon energy as [20]

$$\varepsilon_N = -J \sum_{\beta=1}^{N} [\varepsilon_a(k_\beta) - \varepsilon_0] = -J \sum_{\beta=1}^{N} \left\{ \wp\left(\frac{ik_\beta}{4a}\right) + \frac{1}{2}\left[\frac{k_\beta}{\pi} \zeta\left(\frac{i\pi}{2a}\right) - 2\zeta\left(\frac{ik_\beta}{4a}\right) \right] \right.$$
$$\left. \times \left[\frac{k_\beta}{\pi} \zeta\left(\frac{i\pi}{2a}\right) + 2\zeta\left(\frac{ik_\beta}{4a}\right) - 2\zeta\left(\frac{ik_\beta}{2a}\right) \right] - 2ia\zeta\left(\frac{i\pi}{2a}\right) \right\}, \tag{64}$$

then $\mathcal{L}_1(\{n\})$ exactly coincides with the right-hand side of Eq. (35). Now the problem consists in finding the conditions under which $\mathcal{L}_2(\{n\})$ and $\mathcal{L}_3(\{n\})$ vanish. Let us consider first the expression (62) for $\mathcal{L}_2(\{n\})$.

Let Q be the permutation $(\beta \leftrightarrow \varrho)$ which doesn't change all other indices from 1 to N. The sum over permutations in (62) can be transformed as

$$\mathcal{L}_2(\{n\}) = -a^2 \prod_{\mu>\nu}^{N} [\sinh a(n_\mu - n_\nu)]^{-1} \sum_{m\in D^N} \tilde{d}_{m_1,\ldots,m_N}(k)$$
$$\times \sum_{P\in\pi_N} (-1)^P \sum_{\beta\neq\varrho}^{N} [F(P) - F(PQ)],$$

where

$$F(P) = \exp\left\{ \sum_{\gamma\neq\beta,\varrho}^{N} [ik_{P\gamma} + a(2m_{P\gamma} - N + 1)]n_\gamma \right\}$$
$$\times \exp\{[i(k_{P\beta} + k_{P\varrho}) + 2a(m_{P\beta} + m_{P\varrho} - N + 1)]n_\varrho\}$$
$$\times \sinh^{-1} a(n_\beta - n_\varrho) \prod_{\lambda\neq\beta,\varrho}^{N} \frac{\sinh a(n_\lambda - n_\beta)}{\sinh a(n_\lambda - n_\varrho)}$$
$$\times \frac{1}{2}\left\{ 2m_{P\beta} - a^{-1} f_a(k_{P\beta}) + \coth a(n_\beta - n_\varrho) + \sum_{\gamma\neq\varrho}^{N} \coth a(n_\gamma - n_\varrho) - N + 1 \right\}.$$

Since $F(PQ)$ differs from $F(P)$ only by the first two terms in the last braces, one can represent $\mathscr{L}_2(\{n\})$ in the form

$$\mathscr{L}_2(\{n\}) = -a^2 \prod_{\mu>\nu}^{N} [\sinh a(n_\mu - n_\nu)]^{-1} \sum_{P\in\pi_N} (-1)^P$$
$$\times \sum_{\beta\neq\varrho}^{N} \exp\left[\sum_{\gamma\neq\beta,\varrho}^{N} [ik_{P\gamma} + a(2m_{P\gamma} - N + 1)] n_\gamma\right]$$
$$\times \sinh^{-1} a(n_\beta - n_\varrho) \prod_{\lambda\neq\beta,\varrho}^{N} \frac{\sinh a(n_\lambda - n_\beta)}{\sinh a(n_\lambda - n_\varrho)}$$
$$\times \sum_{\{m_\lambda\}\in D^N,\ \lambda\neq P\beta, P\varrho} \sum_{s=0}^{2(N-1)} \exp\{[i(k_{P\beta} + k_{P\varrho}) + 2a(s - N + 1)] n_\varrho\}$$
$$\times [N - |s - N + 1|]^{-1} \sum_{m_{P\beta} + m_{P\varrho} = s} \sum_{l\in\mathbf{Z}} \tilde{d}_{m_1 \dots m_{P\beta}+l \dots m_{P\varrho}-l \dots m_N}$$
$$\times \left[m_{P\beta} - m_{P\varrho} - \frac{a^{-1}}{2} f_a(k_{P\beta}) + \frac{a^{-1}}{2} f_a(k_{P\varrho}) + 2l\right]. \tag{65}$$

Comparing the last sum in (65) with Eqs. (20), I conclude that $\mathscr{L}_2(\{n\})$ vanishes if

$$\tilde{d}_{m_1 \dots m_N}(k) = d_{m_1 \dots m_N}(if_a(k)), \tag{66}$$

where $d_{\{m\}}(if_a(k))$ is an arbitrary solution of the system (20) with k_μ replaced by $if_a(k_\mu)$, $1 \leqq \mu \leqq N$.

The final step is the transformation of $\mathscr{L}_3(\{n\})$. Upon symmetrizing the sums over $\beta, \gamma\nu$ in (63) one finds

$$\sum_{\beta\neq\gamma\neq\nu}^{N} \coth a(n_\gamma - n_\beta) \coth a(n_\nu - n_\beta) = \frac{1}{3} N(N-1)(N-2),$$

$$\sum_{\beta\neq\gamma}^{N} [a^{-1} f_a(k_{P\beta}) - 2m_{P\beta} + N - 1] \coth a(n_\gamma - n_\beta)$$
$$= \sum_{\beta\neq\gamma}^{N} [m_\beta - m_\gamma - (2a)^{-1}(f_a(k_\beta) - f_a(k_\gamma))] \coth a(n_{P^{-1}\beta} - n_{P^{-1}\gamma}).$$

With the use of these relations the formula (63) reduces to

$$\mathscr{L}_3(\{n\}) = -a^2 \prod_{\mu>\nu}^{N} [\sinh a(n_\mu - n_\nu)]^{-1}$$
$$\times \sum_{P\in\pi_N} (-1)^P \exp\left\{\sum_{\gamma=1}^{N} [ik_\gamma - a(N-1)] n_{P^{-1}\gamma}\right\} R(P, \{n\}),$$

where

$$R(P, \{n\}) = \sum_{m\in D^N} \tilde{d}_{m_1 \dots m_N}(k) \exp\left(2a \sum_{\lambda=1}^{N} n_{P^{-1}\lambda} m_\lambda\right)$$
$$\times \left\{\sum_{\beta=1}^{N} \left[\frac{1}{2}(N - 1 - 2m_\beta)^2 + \frac{N^2-1}{6} - a^{-1} f_a(k_\beta)(2m_\beta - N + 1)\right]\right.$$
$$\left. - \sum_{\beta\neq\gamma}^{N} [m_\beta - m_\gamma - 2a^{-1}(f_a(k_\beta) - f_a(k_\gamma))] \coth a(n_{P^{-1}\beta} - n_{P^{-1}\gamma})\right\}.$$

Introducing the notation $\exp(2an_{P^{-1}\lambda})=y_\lambda$, $1\leqq\lambda\leqq N$ at fixed P, one finds

$$R(P,\{n\})=\sum_{m\in D^N}\tilde{d}_{m_1\dots m_N}(k)\left(\prod_{\lambda=1}^{N}y_\lambda^{m_\lambda}\right)$$
$$\times\left\{\sum_{\beta=1}^{N}\left[2m_\beta^2-2m_\beta a^{-1}f_a(k_\beta)-\left(2m_\beta-a^{-1}f_a(k_\beta)-\frac{2N-1}{3}\right)(N-1)\right]\right.$$
$$\left.-\sum_{\beta\neq\gamma}^{N}\frac{y_\beta+y_\gamma}{y_\beta-y_\gamma}[m_\beta-m_\gamma-2a^{-1}(f_a(k_\beta)-f_a(k_\gamma))]\right\}. \tag{67}$$

Now it is easy to see that after replacing $\tilde{d}_{m_1\dots m_N}(k)\to d_{m_1\dots m_N}(k)$, $if_a(k_\mu)\to k_\mu$ the right-hand side of (67) exactly coincides with the left-hand side of (21). As it was shown in Sect. 2, it must vanish for all $y\in\mathbf{R}^N$ if $\{d_{m_1\dots m_N}(k)\}$ is an arbitrary solution of the system (20). Hence both $\mathscr{L}_2(\{n\})$ and $\mathscr{L}_3(\{n\})$ vanish under the conditions (66) and the Ansatz (35) satisfies the eigenvalue problem (34) with the N-magnon energy ε_N given by (64).

The asymptotic behaviour of the N-magnon wave functions $\psi(n_1,\dots,n_N)$ (36) as $a\to\infty$ or $|n_\mu-n_\nu|\to\infty$ can be found on the base of Proposition 4 in Sect. 2 (or, equivalently, from the general theory of ZSF on X_N^-). In the former case one obtains the usual Bethe Ansatz [1, 4] as a consequence of (29) and the relation

$$\lim_{a\to\infty}a^{-1}[f_a(k_1)-f_a(k_2)]=i\left(\cot\frac{k_1}{2}-\cot\frac{k_2}{2}\right).$$

This result is quite natural since the terms with the interaction of non-nearest-neighbor spins disappear in (3) in this limit.

In the case of finite a one obtains the following asymptotic form of the N-magnon wave functions. If the distances between the positions of turned spins tend to infinity so that $n_{P(\lambda+1)}-n_{P\lambda}\to+\infty$, $1\leqq\lambda\leqq N-1$, then

$$\psi(n_1,\dots,n_N)\sim\sum_{Q\in\pi_N}(-1)^{QP}\exp\left(i\sum_{\lambda=1}^{N}k_{Q\lambda}n_\lambda\right)$$
$$\times\prod_{\mu<\nu}^{N}\left\{1-\frac{1}{2a}[f_a(k_{QP\mu})-f_a(k_{QP\nu})]\right\}, \tag{68}$$

where $f_a(k)$ is expressed through the Weierstrass ζ function according to (57). Hence the multimagnon scattering matrix is factorized as it would be for integrable models. Note that $\psi(n_1,\dots,n_N)$ doesn't vanish at infinity as the magnon quasimomenta $\{k_\mu\}$ are restricted to $\mathbf{R}(\mathrm{mod}\,2\pi)$.

There are also various kinds of bound complexes in N-magnon states for which some terms in the asymptotic expansion (68) vanish. These configurations are determined by the roots of various systems of transcendental equations like $1-2a^{-1}[f_a(k_\mu)-f_a(k_\nu)]=0$ with all $\{k_\mu\}$ lying in the rectangle $|\mathfrak{Re}k|<\pi$, $|\mathfrak{Im}k|<a$, and $K=\sum_{\mu=1}^{N}k_\mu\in\mathbf{R}$. Evidently, these states have no analogs in the variety of eigenvectors of the Hamiltonian (7). It would be interesting to prove that the minimum of the N-magnon energy ε_N (64) in the ferromagnetic case $J>0$ at given K and $N\geqq3$ is reached on the N-magnon bound state determined by the system of equations

$$\sum_{\mu=1}^{N}k_\mu=K,\quad 1-(2a)^{-1}[f_a(k_\mu)-f_a(k_{\mu+1})]=0,\quad 1\leqq\mu\leqq N-1.$$

4. Discussion

As it follows from the results of the preceding section, the eigenvectors of the Hamiltonian H_s of infinite quantum spin chains with the exchange interaction (2) are tightly related to ZSF on X_N^-. The integral representations of the general ZSF theory [11] or the recurrently constructed intertwining operators [16] can be used for the investigation of their global properties. But, as for computational schemes, I haven't found any more simple way than solving the linear system (20). The explicit expressions for multimagnon wave functions at $N \leqq 4$ following from (29–33) can be obtained much more easily than it has been done in [20] by rather complicated calculations.

The connection of quantum spin chains having non-nearest-neighbor exchange with the Yang-Baxter equation and the corresponding algebraic structures of the quantum inverse scattering method is not clear up to this time. The validity of the triangle Yang-Baxter relation can be proved for the S matrix of the fermionic version of these models but it doesn't serve as a guide to the construction of a full set of eigenvectors, especially in the case of periodic chains with the exchange given by the elliptic Weierstrass $\wp$ function. Another question under study concerns the correspondence between the radial parts of high order Laplace-Beltrami operators on X_N^- and the operators commuting with H_s which have been found in [19]. The investigation of that aspect of the above-mentioned analogy between the objects of the theory of symmetric spaces and quantum spin chains with the Hamiltonians (2, 3) seems to be one of the possible ways of understanding the algebraic nature of their complete integrability.

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