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The Isomonodromy Approach to Matrix Models in 2D Quantum Gravity

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Abstract. We consider the double-scaling limit in the hermitian matrix model for 2D quantum gravity associated with the measure $\exp \sum_{j=1}^{N} t_j z^{2j}$, $N \geqq 3$. We show that after the appropriate modification of the contour of integration the Cross-Migdal-Douglas-Shenker limit to the Painlevé I equation (in the generic case of the pure gravity) is valid and calculate the nonperturbative parameters of the corresponding Painlevé function. Our approach is based on the WKB-analysis of the L-A pair corresponding to the discrete string equation in the framework of the Inverse Monodromy Method. Here we extend our results, which were obtained before for the particular cases $N=2, 3$. Our analysis complements the isomonodromy approach proposed by G. Moore to the general string equations that come from the matrix model in the continuous limit and differ in that we apply the isomonodromy technique to investigate the double scaling limit itself.

1. Introduction

We shall study the difference equation

$$n = w_n^{1/2} \sum_{j=1}^{N} j t_j (L^{2j-1})_{n,n-1}, \qquad L_{nm} \doteqdot \frac{1}{2} w_m^{1/2} \delta_{n+1,m} + \frac{1}{2} w_n^{1/2} \delta_{n-1,m}, \tag{1.1}$$

where $n \in \mathbb{Z}$, $t_j \in \mathbb{C}$, $1 \leqq j \leqq N$, $N \geqq 3$, are regarded given parameters, and L is the operator acting in the space $\psi = \{\psi_n\}_{n=-\infty}^{\infty}$ via $(L\psi)_n = \sum_{m=-\infty}^{\infty} L_{nm} \psi_m$. This nonlinear equation for the dependent variable $w_n \in \mathbb{C}$ has recently appeared in connection with a matrix model in 2D quantum gravity [1, 2] and for this reason we shall refer to it as the **discrete string equation**. We will outline, following [3], the

★ On leave of absence from Leningrad University, Leningrad, Russia physical derivation of (1.1) in Sect. 1.1. In this physical context one is interested in the initial value problem:

$$w_n=0 \quad \text{for} \quad n\leqq 0 \quad \text{and} \quad w_n=4\frac{h_n}{h_{n-1}}, \qquad n=1,2,\ldots,N-1, \tag{1.2}$$

where h_n are the normalized constants of orthogonal polynomials $P_n(z)$ with respect to the following measure,

$$h_n\delta_{nm}=\int_{-\infty}^{\infty} P_n(z)P_m(z)\exp\left(\sum_{j=1}^{N} t_j z^{2j}\right)dz, \qquad n=0,1,2,\ldots,t_N>0, \tag{1.3}$$

and P_n is a polynomial of degree n whose term z^n has coefficient equal to 1. Furthermore, in the context of matrix models (see again Sect. 1.1) one is interested in the asymptotic limit,

$$t_1=\frac{\beta}{2}, \quad t_k=\beta q_k, \quad 2\leqq k\leqq N, \quad \beta=C_1 h^{-5}, \quad \frac{n}{\beta}=C_2+C_1^{-1}h^4\xi,$$
$$w_n\sim\varrho(1-2h^2u(\xi)), \quad h\to 0. \tag{1.4}$$

It has been shown in [1, 2] that the limit (1.4) with an appropriate choice of the constants C_1, C_2, ϱ maps Eq. (1.1) into the Painlevé I (PI) equation for the function $u(\xi)$:

$$u_{\xi\xi}=6u^2+\xi. \tag{1.5}$$

Also, the authors of [1, 2] conjectured that the special solution of (1.1) characterized by the initial conditions (1.2) tends to a solution of (1.5).

A particular consequence of our analysis is that this conjecture is not true. Actually, this has already been known indirectly from [4]. However, a certain modification of this conjecture is indeed valid: Let in (1.3)

$$\int_{-\infty}^{\infty} dz\to s_1\int_{\Gamma_m^+} dz-s_2\int_{\Gamma_m^-} dz, \tag{1.6}$$

where $s_1, s_2\in\mathbb{C}$, $s_1 \neq s_2$, and the contours Γ_m^+, Γ_m^- are the **lines** corresponding to rays $\left\{\arg z=\frac{\pi}{N}(m+1),\ 0\leqq|z|<\infty\right\}$, $\left\{\arg z=-\frac{\pi}{N}(m+1),\ 0\leqq|z|<\infty\right\}$, respectively. Then, for each $m=0,1,\ldots,\left[\frac{N-3}{2}\right]$, there exists an open set of parameters q's for which it is possible to choose the constants C_1, C_2, and ϱ in (1.4) in such a way that the unique solution w_n of the discrete string equation (1.1) characterized by the initial condition (1.2) tends to a solution $u(\xi)$ of PI equation (1.5) (the details are given in Sect. 5).

Furthermore, this unique solution of PI is characterized by one of the following large ξ asymptotics:

$$u(\xi)=e^{\mp\frac{i\pi}{5}}\sqrt{\frac{|\xi|}{6}}+\alpha_{\pm}|\xi|^{-1/8}e^{-\frac{8i}{5}\left(\frac{3}{2}\right)^{1/4}|\xi|^{5/4}}+o(|\xi|^{-1/8}), \quad |\xi|\to\infty, \quad \arg\xi=\pi\mp\frac{2\pi}{5}, \tag{1.7}$$

$$u(\xi)=\left(\sqrt{\frac{-\xi}{6}}+O(\xi^{-2})\right)+\alpha_0(-\xi)^{-1/8}e^{-\frac{8}{5}\left(\frac{3}{2}\right)^{1/4}(-\xi)^{5/4}}(1+o(1)), \quad \xi\to-\infty, \tag{1.8}$$

where

$$\alpha_+ = -\frac{i}{\sqrt{8\pi}} e^{\frac{\pi i}{20}} \left(\frac{2}{3}\right)^{1/8} \frac{p}{1+p}, \quad \alpha_- = \frac{i}{\sqrt{8\pi}} e^{-\frac{\pi i}{20}} \left(\frac{2}{3}\right)^{1/8} \frac{1}{1+p},$$
$$\alpha_0 = \frac{i}{2} \frac{1}{\sqrt{8\pi}} \left(\frac{2}{3}\right)^{1/8} \frac{1-p}{1+p}, \quad p = -\frac{s_2}{s_1} \mp -1. \tag{1.9}$$

Note that in order for $u(\xi)$ to be real, one needs $|p|=1$. The so-called "triply truncated solution," which has been discussed intensively in connection with 2D gravity since the work [4], corresponds to $p=0$. This solution has infinitely many poles only in the sector $\frac{7}{5}\pi < \arg\xi < \frac{9}{5}\pi$, and has regular asymptotic behavior on the remaining Stokes rays, $\arg\xi = \pi \mp \frac{4}{5}\pi$ [see formulae (A.11) in Appendix A].

The distinguished feature of formulae (1.9) is that they don't depend on the concrete choice of the parameters q's, and the number $N \geqq 3$.

The particular cases $N=2,3$ have already been considered in the author's papers [5–8]. For the particular case $N=2$, $p=0$, the last of the equalities (1.9) was also obtained in [9].

Our analysis complements the scheme [10, 11] where the isomonodromy approach is used to study the continuous string equations (see Sect. 1.1).

To obtain the results listed above we made essential use of the asymptotic analysis of the PI equation developed in [12].

In order to investigate the general Cauchy problem of the string equation (1.1) we use the so-called isomonodromy method [30, 31]. This method, which is an extension of the inverse spectral method, relies on the association of a given nonlinear equation to a pair of linear equations known as the Lax pair. Actually, the string equation (1.1) is associated with **three** linear equations (see for example [3]),

$$L\psi = z\psi, \quad \partial_z \psi = 2\sum_{j=1}^{N} jt_j L_-^{2j-1}\psi, \quad \partial_{t_j}\psi = \left(L_-^{2j} + \frac{1}{2}L_0^{2j}\right)\psi, \tag{1.10}$$

where $(L_-)_{nm} = L_{nm}$ for $n>m$, $(L_-)_{nm}=0$ for $n \leqq m$ and $(L_0)_{nm} = L_{nm}\delta_{nm}$. This is a consequence of the fact that the string equation is a "similarity" reduction (or simply is compatible) with the Volterra hierarchy

$$\frac{\partial}{\partial t_j} \ln w_n = (L^{2j})_{n-1,n-1} - (L^{2j})_{n,n}. \tag{1.11}$$

Since, Eq. (1.11) is the compatibility condition of Eqs. (1.10a) and (1.10c) [13–15], Eq. (1.1) is the compatibility condition of Eqs. (1.10a) and (1.10b) [16, 3, 17, 5], and Eqs. (1.1) and (1.11) are compatible, it follows that Eq. (1.1) is compatible with all three linear equations (1.10).

It is more convenient to let $\Psi_n(z) = (\psi_n(z), \psi_{n-1}(z))^T$, and to write Eqs. (1.10) in matrix form. In Sect. 2 we shall show that the relevant matrix form of Eqs. (1.10) is given by

$$\Psi_{n+1}(z) = U_n(z)\Psi_n, \quad U_n(z) = \begin{pmatrix} 2zw_{n+1}^{-1/2} & -w_n^{1/2}w_{n+1}^{-1/2} \\ 1 & 0 \end{pmatrix}; \tag{1.12}$$

$$\frac{\partial \Psi_n(z)}{\partial z} = A_n(z)\Psi_n, \quad A_n(z) = \begin{pmatrix} a_n(z) & -\frac{w_n^{1/2}}{2z}(a_n(z)+a_{n+1}(z)) \\ \frac{w_n^{1/2}}{2z}(a_{n-1}(z)+a_n(z)) & -a_n(z) \end{pmatrix}, \tag{1.13}$$

where $a_n(z)$ is given by

$$a_n(z) = -\sum_{j=1}^{N} j t_j z^{2j-1} - w_n^{1/2} \sum_{j=1}^{N-1} z^{2j-1} \sum_{l=j+1}^{N} l t_l (L^{2l-2j-1})_{n,n-1}; \tag{1.14}$$

$$\frac{\partial \Psi_n(z)}{\partial t_j} = V_n(z)\Psi_n(z), \quad V_n(z) = \begin{pmatrix} v_n(z) & -\frac{w_n^{1/2}}{2z}(v_n(z) - r_{n+1}(z)) \\ \frac{w_n^{1/2}}{2z}(v_{n-1}(z) - r_n(z)) & r_n(z) \end{pmatrix}, \tag{1.15}$$

where $v_n(z)$ and $r_n(z)$ are given by

$$v_n(z) = -\frac{z^{2j}}{2} - \frac{1}{2} w_n^{1/2} \sum_{l=1}^{j-1} z^{2l} (L^{2j-2l-1})_{n,n-1} + \frac{w_{n+1}^{1/2}}{4} (L^{2j-1})_{n+1,n} - \frac{w_n^{1/2}}{4} (L^{2j-1})_{n,n-1}, \tag{1.16a}$$

$$r_n(z) = \frac{z^{2j}}{2} + \frac{1}{2} w_n^{1/2} \sum_{l=1}^{j-1} z^{2l} (L^{2j-2l-1})_{n,n-1} + \frac{w_n^{1/2}}{4} (L^{2j-1})_{n,n-1} - \frac{w_n^{1/2}}{4} (L^{2j-1})_{n-1,n-2}. \tag{1.16b}$$

For convenience of notation we have suppressed the t_j-dependence.

In Sect. 3 we study the general initial value problem of the discrete string equation (1.1), where w_n are given for $-(N-2) \leqq n \leqq N-1$. We show that this problem admits a global meromorphic in t_j solution. This solution can be obtained by solving a RH problem for the function $\Phi_n(z) = \Psi_n(z) \exp\left[\frac{1}{2} \sum_{j=1}^{N} t_j z^{2j}\right]$:

$$\Phi_n^-(z) = \Phi_n^+(z) e^{-\frac{1}{2}\sum_{j=1}^{N} t_j z^{2j}} S e^{\frac{1}{2}\sum_{j=1}^{N} t_j z^{2j}}, \tag{1.17}$$

$$\Phi_n(z) = \begin{pmatrix} (\beta_n^{(1)})^{-1/2} & 0 \\ 0 & (\gamma_n^{(1)})^{-1/2} \end{pmatrix} \left(I + \begin{pmatrix} \alpha_n^{(1)} & \beta_n^{(1)} \\ \gamma_n^{(1)} & \delta_n^{(1)} \end{pmatrix} \frac{1}{z} + O\left(\frac{1}{z^2}\right) \right) z^{n\sigma_3}, \quad z \to \infty; \tag{1.18}$$

this RH problem is defined on a contour which as $z \to \infty$ is asymptotic to the rays $\arg z = -\pi/2N + l\pi/2N$, $0 \leqq l \leqq 4N-1$. The jump matrix S depends on the monodromy data s_l, $0 \leqq l \leqq 4N-1$ which are defined on a $2N-1$-dimensional algebraic variety. Having obtained Φ_n, w_n follows from $w_n = 4\beta_n^{(1)}\gamma_n^{(1)}$. To prove this result we show that Φ_n can be obtained explicitly in terms of Φ_0 and then we use the rigorous results of [18] to establish the solvability of the RH problem for Φ_0. Also, in analogy with the results of [18], we find that if the monodromy data s_l satisfy certain constraints and if the t_j's are on certain rays, then Φ_0 is bounded for all finite t_j's (i.e. the existence of poles is excluded). An example is

$$\bar{s}_{l+1} = -s_{2N-l}, \quad 1 \leqq l \leqq N-1; \quad |s_0 - \bar{s}_1| < 2; \quad t_j \text{ imaginary}, \quad 1 \leqq j \leqq N. \tag{1.19}$$

The case of physical interest is the so-called triangular case, which corresponds to the special choice of the monodromy data $s_{2l+1} = 0$, $0 \leqq l \leqq 2N-1$, t_j real, $t_N > 0$.

In this case the above RH problem can be solved in closed form in terms of the orthogonal polynomials $P_n(z)$ [see Sect. 3, formula (3.21)].

In Sect. 4 we investigate the limit of the discrete string equation to PI equation. It turns out that it is more convenient to consider the limit of the associated Lax pair. We show that under the limit (1.4), where C_1, C_2, and ϱ are given by Eqs. (5.10) and (5.12), Eqs. (1.10a) and (1.10b) are mapped to the Lax pair for the PI equation. This Lax pair is expressed in terms of an eigenfunction $Y(k, \xi)$ (see Appendix A). The asymptotic relationship between Y and Ψ_n is

$$\Psi_n(z) = k^{-1/2}\begin{pmatrix} 1 & 1 \\ 1-kh & 1+kh \end{pmatrix}\sigma_3 Y(k,\xi)\sigma_3 + o(1), \qquad h \to 0. \tag{1.20}$$

In Sect. 5 using the methodology of [19] we investigate the limit of solutions of the discrete string equation under the ansatz (1.4). We show that under this limit only **certain** solutions of the discrete string equation tend to solutions of PI. We characterize the initial data of these solutions and also give a description of the corresponding solutions of PI. Our analysis involves the following steps.

(a) We use the WKB method to characterize the asymptotic behavior of the solution of $\Psi_{n_z} = A_n \Psi_n$ as $\beta \to \infty$. We denote by $\Psi_n^{\mathrm{WKB}}(z)$ the WKB-limit of this solutions. For large z the piecewise solution $\Psi_n(z)(\Psi_n^{(1)}, \ldots, \Psi_n^{(4N-1)})$ described in Sect. 3 can be expressed in terms of $\Psi_n^{\mathrm{WKB}}(z)$ by

$$\Psi_n(z) = \Psi_n^{\mathrm{WKB}}(z)\Lambda_n \varrho^{1/4} 2^{-1/2} \exp[\delta_\infty \beta\sigma_3], \quad \text{where} \quad \Lambda_n = \mathrm{diag}((\beta_n^{(1)})^{-1/2}, (\gamma_n^{(1)})^{-1/2}) \tag{1.21}$$

and δ_∞ is a certain function of ξ, n.

(b) The solution Ψ^{WKB} breaks down in the neighborhood of the turning points of the equation $\Psi_{n_z} = A_n \Psi_n$. Under certain assumption (given in Proposition 5.2) there exist $2N-4$ double turning points and 2 triple turning points. We denote by Ψ^{TTP} and Ψ_n^{DTP} the associate solutions of $\Psi_{n_z} = A_n \Psi_n$ at these turning points. The results of Sect. 4 indicate that Ψ_n^{TTP} is simply related to the eigenfunction Y associated with the isomonodromy analysis of PI [see Eq. (1.20)]. The dominant part of Ψ_n^{DTP} can be given in closed form, and does not contribute to the asymptotic analysis. At this point the WKB-analysis of (1.13) resembles the analysis of the z-equation corresponding to the continuous string equation (1.48) (see [10]).

(c) Using the results of (b) above and the fact that Ψ_n^{WKB} and Ψ_n^{TTP} can be related, we find a relationship between Ψ_n and Y. This, in turn, induces a relationship between $\{S\}$, the monodromy data associated with Ψ_n, and $\{G\}$, the monodromy data associated with Y.

(d) The case of physical interest corresponds to the triangular case. In this case the monodromy data $\{S\}$ are directly related to initial data for w_n. The relation between the monodromy data $\{G\}$ and the coefficients characterizing the large ξ asymptotic behavior of solutions of PI has already been obtained in [12]. Thus the result of (c) above provides a direct description of solutions of PI in terms of initial data of solutions of the discrete string equation [see Eq. (1.9)].

1.1. The Physical Model. The starting point of the theory of 2D quantum gravity is the partition function of the bosonic string which can be represented by the functional integral [20],

$$F = \sum_p \int Dg \int DX \exp\left\{-\lambda_1 \int_{\Sigma_p} \sqrt{g} - \frac{\lambda_2}{2\pi}\int_{\Sigma_p} R\sqrt{g} - \int_{\Sigma_p} \sqrt{g} g^{ab}\partial_a X^\mu \partial_b X^\mu\right\}. \tag{1.22}$$

The notation $\int Dg$ means the integration over all possible metrics on the 2-surface $\sum_p$ of genus p; $\int DX$ means the integration over all mappings $X:\sum_p \to \mathbb{R}^D$ (these mappings are the string fields). The entity R denotes the scalar curvature of the metric g. The constants λ_1 and λ_2 are the cosmological constant and the string coupling, respectively. The pure two-dimensional quantum gravity is associated with the partition function

$$F=\sum_p \int Dg \exp\left\{-\lambda_1 \int_{\Sigma_p} \sqrt{g}-\frac{\lambda_2}{2\pi}\int_{\Sigma_p} R\sqrt{g}\right\}. \tag{1.23}$$

The basic mathematical problem is to make sense of the formal expressions (1.22) and (1.23). One of the possible ways of achieving this is the following: Let $W(p; n_4, n_6, \dots, n_{2N})$ be the number of ways that $\sum_p$ can be covered with n_4 squares, n_6 hexagons, n_8 eight-gons etc. The basic idea is to approximate the functional integral

$$F_p(A)=\int Dg \int Dx\left\{-\lambda_1 \int_{\Sigma_p} \sqrt{g}-\frac{\lambda_2}{2\pi}\int_{\Sigma_p} R\sqrt{g}-\int_{\Sigma_p}\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X^\mu\right\}$$

as

$$F_p(A)\approx \frac{1}{\varepsilon} W_{\{x\}}(p;q)\, e^{-\lambda_1 A-\lambda_2(2-2p)}, \qquad \varepsilon\to 0, \tag{1.24}$$

where

$$W_{\{x\}}(p;q)=\sum_{n_4+n_6+\dots+n_{2N}=q} W(p;n_4,n_6,\dots,n_{2N})x_2^{n_4}x_3^{n_6}\dots x_N^{n_{2N}}, \qquad q=\frac{A}{\varepsilon}.$$

(The integral $\int Dg$ is over all metrics of total area A in $\sum_p$.) The variables x_i play only an auxiliary role as will be cleared below.

The derivation of (1.24) is based on the consideration of the triangulations of the 2-surfaces. (For the details and history of this question we refer the reader to the articles [21, 3].) It should be noted that Eq. (1.24) is consistent with the following argument. It is known [22] that

$$W_{\{x\}}(p,q)=e^{cq}q^{\gamma(2-2p)-1}b_p\left(1+O\left(\frac{1}{q}\right)\right), \qquad q\to\infty, \tag{1.25}$$

where

$$c,\gamma,b_p\equiv c\{x\},\gamma\{x\},b_p\{x\}.$$

The function $\gamma\{x\}$ for generic $\{x\}$ is given by

$$\gamma=-\tfrac{5}{4}; \tag{1.26}$$

on the other hand, for a special choice of $m-2$ of x's, it is possible to make

$$\gamma=-1-\frac{1}{2m}. \tag{1.27}$$

The quantity b_p depends on $\{x\}$ in such a way that,

$$\{x\}\to\{x'\} \Rightarrow b_p\to b_p b^{1-p}. \tag{1.28}$$

Substituting Eq. (1.25) into Eq. (1.24) it follows that

$$F_p(A)\approx\frac{1}{\varepsilon}e^{\frac{cA}{\varepsilon}}\left(\frac{A}{\varepsilon}\right)^{\gamma(2-2p)-1}e^{-\lambda_1 A-\lambda_2(2-2p)}, \qquad \varepsilon\to 0.$$

This implies the renormalization rule

$$\lambda_1=\frac{c}{\varepsilon}+\lambda_1^0, \qquad \lambda_2=-\gamma\ln\varepsilon+\lambda_2^0, \tag{1.29}$$

which in turn leads to the following power-like area dependence of $F_p(A)$:

$$F_p(A)\approx e^{-\lambda_1^0 A}A^{\gamma(2-2p)-1}e^{-\lambda_2^0(2-2p)}b_p. \tag{1.30}$$

On the other hand, from the scaling-gauge KPZ-theory [23–25, 21] one should have

$$F_p(A)\cong e^{-\lambda_1^0 A}A^{-1+(1-p)(\gamma_{\mathrm{str}}-2)}, \tag{1.31}$$

where

$$\gamma_{\mathrm{str}}=\frac{1}{12}[D-1-\sqrt{(D-1)(D-25)}].$$

Comparing (1.31) and (1.30) it follows that the approximation (1.24) is valid for the special dimensions

$$D=1-\frac{6}{m(m+1)}, \qquad m\geqq 2,$$

and that generic values of $\{x\}$ correspond to the pure gravity (1.23) ($m=2$, $D=0$).

Introducing the notation

$$\lambda=-e^{-\lambda_1\varepsilon}, \qquad n=e^{-\lambda_2}, \qquad \lambda_c=-e^{-c} \tag{1.32}$$

and assuming that

$$\int F_p(A)dA\approx\sum_q W_{\{x\}}(p;q)e^{-\lambda_1 q\varepsilon-\lambda_2(2-2p)}, \tag{1.33}$$

Eqs. (1.29), (1.33) yield the representation

$$\begin{gathered}F=\sum_{p=0}^{\infty}\int F_p(A)dA\cong\sum_{p=0}^{\infty}n^{2-2p}\sum_{q=0}^{\infty}(-\lambda)^q W_{\{x\}}(p;q),\\ \lambda\to\lambda_c, \qquad n\to\infty, \qquad n(\lambda-\lambda_c)^{-\gamma}=O(1).\end{gathered} \tag{1.34}$$

However, Eq. (1.34) cannot be accepted as the definition of the functional integral (1.22) because the series in the right-hand side has only asymptotic meaning. Actually, using Eq. (1.25) and the classical formula

$$\sum_{m=1}^{\infty}\frac{x^n}{m^s}\approx\Gamma(1-s)(1-x)^{s-1}, \qquad x\to 1, \qquad s<1,$$

Eq. (1.34) implies

$$F \equiv F(t) = \sum_{p=0}^{\infty} t^{2-2p} b_p \Gamma(\gamma(2-2p)) + \text{reg. terms}, \tag{1.35}$$

where

$$t = n\left(1 - \frac{\lambda}{\lambda_c}\right)^{-\gamma}. \tag{1.36}$$

Equations (1.29) and (1.32) suggest that the variable t has the meaning of the renormalized string coupling, and the asymptotic series (1.35) defines the perturbative theory for the partition function (1.22). Note that because of Eq. (1.28), the series in (1.35) does not depend on the individual value of x's (up to a redefinition of t), but only of the number m (m^{th} class of universality).

To obtain the nonperturbative definition of the partition function (1.22) one needs a well-defined generating function for the series (1.34). It follows from the results of [22] that a candidate for such a generating function can be taken in the form

$$\log Z_n\left(\frac{1}{2}, \frac{\lambda}{4n} x_2, \frac{\lambda^2}{6n^2} x_3, \ldots, \frac{\lambda^{N-1}}{2Nn^{N-1}} x_N\right), \tag{1.37}$$

where $Z_n(t_1, t_2, \ldots, t_N)$ is the partition function of the hermitian matrix model:

$$Z_n = \int D\Phi \exp\{-T_r U(\Phi)\}, \qquad U(z) = \sum_{j=1}^{N} t_j z^{2j}. \tag{1.38}$$

In (1.38), Φ is $n \times n$ hermitian matrix, and

$$D\Phi = \prod_i d\Phi_{ii} \prod_{i<j} d\Phi_{ij} d\bar{\Phi}_{ij}. \tag{1.39}$$

Accepting (1.37) as the generating function for (1.34) one can reduce the problem of calculating the functional integral (1.22) to the problem of calculating a special double-scaling limit of the well-defined finite-dimensional integral (1.38). [The problem is not trivial because the dimension of the integral (1.38) goes to infinity.] Letting

$$\Phi \to \beta^{1/2}\Phi, \qquad \frac{\lambda\beta}{n} = 1, \qquad \frac{x_j}{2j} = q_j,$$

the double limit can be formulated as follows:

Evaluate

$$\log Z_n\left(\frac{\beta}{2}, \beta q_2, \ldots, \beta q_N\right) \tag{1.40}$$

under the limit

$$\beta = C_1 h^{-4-\frac{2}{m}}, \qquad \frac{n}{\beta} = C_2 + C_1^{-1} h^4 \xi, \qquad h \to 0, \tag{1.41}$$

where

$$C_2 \equiv \lambda_c, \qquad \lambda - \lambda_c \equiv C_1^{-1} h^4 \xi, \qquad \xi \equiv -t^{\frac{2m}{2m+1}} (C_1 \lambda_c)^{\frac{1}{2m+1}}. \tag{1.42}$$

In order for the integral (1.40) to be a well-defined generating function for (1.34) the constants C_1, C_2 should be positive. Constant C_2 is a function of q's. Using the freedom in the choice of q's one can always reach the condition $C_2>0$ for sufficiently large N (as a matter of fact for $N\geqq 3$). Note also, that for generic values of q's $m=2$ in (1.41), (1.42).

To study the integral (1.38) it is natural to factor out the integration over the "angle" variables. Putting

$$\Phi=u^{-1}\Lambda u,$$

where u is a unitary matrix and Λ is a diagonal matrix,

$$\Lambda=\mathrm{diag}(z_1,\dots,z_n), \qquad z_i\in\mathbb{R},$$

we find

$$ud\Phi u^{-1}=d\Lambda+[\Lambda,duu^{-1}];$$

or introducing $d\tilde{\Phi}=ud\Phi u^{-1}$, $d\tilde{u}=duu^{-1}$,

$$d\tilde{\Phi}_{ii}=dz_i, \qquad d\tilde{\Phi}_{ij}=(z_i-z_j)d\tilde{u}_{ij}.$$

Thus

$$D\Phi=\prod_{i<j}(z_i-z_j)^2dz_1,\dots,dz_n\prod_{i<j}d\tilde{u}_{ij}d\bar{\tilde{u}}_{ij}. \tag{1.43}$$

Since the integrant in (1.38) does not depend on $\tilde{u}_{ij}$, Eq. (1.43) implies

$$Z_n(t_1,\dots,t_N)=\mathrm{const}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{n}dz_i\prod_{i<j}(z_i-z_j)^2\exp\left(-\sum_{i=1}^{n}U(z_i;t_1,\dots,t_N)\right). \tag{1.44}$$

Let (see Introduction) $P_n(z)$ be the orthogonal polynomials with respect to the measure $dze^{-U(z)}$ [see (1.3)]. Taking into account the equation

$$\det\{P_{j-1}(z_i)\}=\prod_{i<j}(z_i-z_j),$$

one can rewrite (1.44) as

$$\begin{aligned}Z_n&=\mathrm{const}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\prod_{i=1}^{n}dz_i\det{}^2\{P_{j-1}(z_i)\}e^{-\sum_{i=1}^{n}U(z_i)}\\&=\mathrm{const}\sum_{\sigma,\sigma'}(-1)^{\mathrm{sign}\,\sigma+\mathrm{sign}\,\sigma'}\prod_{i=1}^{n}\int_{-\infty}^{\infty}dz_ie^{-U(z_i)}P_{\sigma(i)-1}(z_i)P_{\sigma'(i)-1}(z_i)\\&=\mathrm{const}\,n!\prod_{i=1}^{n}h_{i-1}(t_1\dots t_N)\equiv\mathrm{const}\prod_{i=1}^{n}h_{i-1}(t_1\dots t_N).\end{aligned} \tag{1.45}$$

Equation (1.45) reduces the evaluation of $Z_n\left(\frac{\beta}{2},\beta q_2,\dots,\beta q_N\right)$, under the limit (1.41), to the evaluation of the normalized constants $h_n\left(\frac{\beta}{2},\beta q_2,\dots,\beta q_N\right)$, under the same limit. The latter in turn leads to the study of the discrete string equation (1.1), (1.2) under the limit (1.41), which coincides with the limit (1.4) for the case of $m=2$ (pure gravity).

Actually, letting

$$w_n = 4\frac{h_n}{h_{n-1}}, \qquad n = 1, 2, \dots,$$

one obtains Eq. (1.1) and the Volterra hierarchy (1.11). These equations are elementary consequences (see for example [7]) of the orthogonality condition (1.3) and of recurrence relation

$$zQ_n = \frac{1}{2}w_{n+1}Q_{n+1} + \frac{1}{2}w_nQ_{n-1}, \qquad Q_n = \frac{1}{\sqrt{h_n}}P_n. \tag{1.46}$$

The important recent achievement in the theory of the matrix model of the 2D quantum gravity is the discovery [1, 2] of the connection between the limit (1.41) in Eq. (1.1) and the theory of KdV-equation. Suppose that under the limit (1.41)

$$w_n \cong \varrho\left(1 - 2h^{\frac{4}{m}}u(\xi)\right). \tag{1.47}$$

Then, as it was shown in [1, 2], it is possible to determine ϱ in such a way that the function $u(\xi)$ will satisfy the ordinary differential equation

$$[H, A_m] = 1, \tag{1.48}$$

where $H = -d^2/d\xi^2 + u(\xi)$, and A_m is the A-operator associated with the m^{th} KdV equation. For the general case of pure gravity where $m = 2$, after an appropriate choice of the scaling constant C_1, one finds the first Painlevé equation

$$u_{\xi\xi} = 6u^2 + \xi. \tag{1.49}$$

It should be emphasized that the limiting string equation (1.48) depends only on m and not on the concrete choice of the parameters q_i (the property of universality).

Coming back to the main object of interest, to the partition function

$$F(\xi) = \lim \log Z_n,$$

one obtains the relation

$$F_{\xi\xi} = -2u. \tag{1.50}$$

Indeed, the second difference of $\log Z_n$ satisfies the relation

$$\begin{aligned}\Delta^2 \log Z_n &\equiv \log Z_{n+1} - 2\log Z_n + \log Z_{n-1}\\ &= \log w_n + \text{explicit increasing (as } n\to\infty\text{) terms}.\end{aligned} \tag{1.51}$$

As it follows from (1.41), $n \to n \pm 1 \Rightarrow \xi \to \xi \pm h^{2/m}$. Because of this and (1.47), relation (1.51) implies (1.50) after a trivial additional regularization of Z_n.

In connection with these results, the analytical problem of the calculation of the parameters of the limiting solution $u(\xi)$ arises. It should be mentioned that some partial information about $u(\xi)$ has already been obtained. Indeed, the perturbative series (1.35) together with (1.50) show that

$$u(\xi) = \sum_{p=0}^{\infty} (-\xi)^{\frac{1}{m} - \frac{2m+1}{m}p} C_p \quad \text{as} \quad \xi \to -\infty. \tag{1.52}$$

For m odd the same type of behavior takes place at $\xi \to +\infty$ and, as it has been shown in [10], together with the reality condition determines $u(\xi)$ uniquely.

However, for m even the asymptotics (1.52) does not determine the solution in a unique way. For instance, in the case $m=2$ there is one-parameter family (see [12] and Appendix A) of solutions with the asymptotic (1.52):

$$u(\xi)=\sqrt{\frac{-\xi}{6}}+\sum_{p=1}^{\infty}(-\xi)^{\frac{1}{2}-\frac{5}{2}p}C_p+\alpha_0(-\xi)^{-1/8}e^{-\frac{8}{5}\left(\frac{3}{2}\right)^{1/4}(-\xi)^{5/4}}(1+o(1)),$$
$$\xi\to-\infty.$$

The problem is to determine the nonperturbative parameter α_0. The answer of this question for the general polynomial $U(z)$ is given in (1.9). Note, that α_0 does not depend on q's. This means the universality holds on the level of the limiting function $u(\xi)$ as well as on the level of the limit equations.

Our method for calculating α_0 has been outlined in the Introduction and is based on the WKB-analysis of the $L-A$ pair corresponding to (1.1). In accordance with our approach, the main parameters of the limit (1.41), (1.47), i.e. the constants C_2 and ϱ, are determined by the condition that the A-equation of the Lax pair has a triple turning point (see Sect. 4). This condition is the necessary condition for the limit (1.41), (1.47) to map the discrete string equation (1.1) into the Painlevé I equation (1.49) for the general case of $m=2$. The analogous condition for $m>2$, should be the existence of higher order turning points. This observation leads to the unexpected connection between the string equations (1.48) and the catastrophe theory (see [26]).

Remark 1. The theory of the general string equation (1.48) has been treated via the isomonodromy approach in [10, 11]. The nonperturbative parameter for $m=3$ has been calculated in [27]. The original approach to the string equations (1.48) based on methods of algebraic geometry was proposed in [28, 29]. The interesting idea of considering Eqs. (1.48) as the quantization of finite-gap potentials was put forward in [10] and [28].

Remark 2. In this article we consider the case of general position, $m=2$ (pure gravity). To extend our approach to the arbitrary even m, one needs the description of the solutions of (1.48) with $m=2k$ in terms of the corresponding monodromy data (the cases $m=2$, $m=3$, and $m=2k+1$ are studied in [12, 27, and 10], respectively).

2. The Lax Pair Formulation of the Discrete String Equation

In this section we start with the linear eigenvalue equation

$$\tfrac{1}{2}w_{n+1}^{1/2}\psi_{n+1}+\tfrac{1}{2}w_n^{1/2}\psi_{n-1}=z\psi_n\,. \tag{2.1}$$

In Eq. (2.1), ψ_n and z are the eigenfunction and eigenvalue, respectively, and w_n plays the role of the potential. We shall show that associated with (2.1) there exists: (i) A hierarchy of discrete nonlinear equations for w_n; (ii) a hierarchy of discrete nonlinear evolution equations for w_n. Both these hierarchies admit a Lax pair representation. In the case (ii) this is a well known fact [13–15]. We shall give the relevant matrix z-dependent Lax pairs explicitly.

Equation (2.1) can be written as

$$L\psi=z\psi\,,\qquad L\doteq\tfrac{1}{2}\Delta w^{1/2}+\tfrac{1}{2}w^{1/2}\Delta^{-1}\,, \tag{2.2}$$

where L acts in the space of sequences $\{\psi_n\}_{n=-\infty}^{n=\infty}$, and Δ is the shift operator. The coordinate form of L is given by

$$(L\psi)_n=\sum_{m=-\infty}^{\infty} L_{nm}\psi_m\,,\quad \text{i.e.}\quad L_{nm}=\frac{1}{2}w_m^{1/2}\delta_{n+1,m}+\frac{1}{2}w_n^{1/2}\delta_{n-1,m}\,. \tag{2.3}$$

In order to derive the associated nonlinear hierarchies, it is convenient to rewrite Eq. (2.1) into matrix form. Letting $\Psi_n=(\psi_n,\psi_{n-1})^T$, Eq. (2.1) becomes Eq. (1.12).

Proposition 2.1. *(The Matrix Lax pair of the discrete string equation.) The hierarchy of nonlinear discrete equations*

$$n+C=-\frac{1}{2}\sum_{j=1}^{N} C_{N-j}w_n^{1/2}(L^{2j-1})_{n,n-1}-\frac{1}{2}C_N\,, \tag{2.4}$$

where C and C_j, $j=0,\dots,N$, are arbitrary z-independent parameters, $n\in\mathbb{Z}$, $N\in\mathbb{Z}_+$, and $L_{n,m}$ is defined in Eq. (2.3), *admits the Lax pair formulation*

$$\Psi_{n+1}(z)=U_n(z)\Psi_n(z)\,,\qquad \frac{\partial\Psi_n(z)}{\partial z}=A_n(z)\Psi_n(z)\,, \tag{2.5}$$

where U_n is defined in (1.12) *and*

$$A_n(z)\doteqdot\begin{pmatrix} a_n(z) & -\frac{1}{2z}(a_n(z)+a_{n+1}(z)w_n^{1/2} \\ \frac{1}{2z}(a_n(z)+a_{n-1}(z))w_n^{1/2} & -a_n(z)\end{pmatrix}, \tag{2.6a}$$

$$a_n(z)=\frac{1}{2}\sum_{j=1}^{N} z^{2j-1}C_{N-j}+\frac{w_n^{1/2}}{2}\sum_{j=1}^{N-1} z^{2j-1}\sum_{l=j+1}^{N} C_{N-l}(L^{2l-2j-1})_{n,n-1}\,. \tag{2.6b}$$

Proof. The compatibility condition of Eqs. (2.5) yields $U_{n_z}=A_{n+1}U_n-U_nA_n$. Denoting the 11, 12, 21, and 22 entries of A_n by a_n, b_n, c_n, and d_n, respectively, and writing the compatibility condition into component form we find

$$z(a_{n+1}-a_n)+\tfrac{1}{2}w_{n+1}^{1/2}b_{n+1}+\tfrac{1}{2}w_n^{1/2}c_n=1\,,\qquad d_n-a_{n+1}-2zw_n^{-1/2}b_n=0\,, \tag{2.7}$$

$$b_n+c_{n+1}w_n^{1/2}w_{n+1}^{-1/2}=0\,,\qquad d_{n+1}-a_n+2zw_{n+1}^{-1/2}c_{n+1}=0\,. \tag{2.8}$$

Subtracting Eqs. (2.7b) and (2.8b), and using Eq. (2.8a) it follows that

$$a_{n+1}+d_{n+1}=a_n+d_n\,,\quad \text{or}\quad a_n(z)+d_n(z)=\gamma(z)\,.$$

The function $\gamma(z)$ can be taken zero without loss of generality, since it can be absorbed in Ψ_n via the transformation $\Psi_n\to\Psi_n\exp\left[-\frac{1}{2}\int^z\gamma(z')dz'\right]$; thus $d_n=-a_n$. Then Eqs. (2.7b) and (2.8b) imply that b_n and c_n are the expressions appearing in (2.6a), while Eq. (2.7a) becomes

$$1=z(a_{n+1}-a_n)+\frac{w_n}{4z}(a_n+a_{n-1})-\frac{w_{n+1}}{4z}(a_{n+1}+a_{n+2})\,. \tag{2.9}$$

In order to solve Eq. (2.9) we make the ansatz,

$$a_n(z)=\sum_{j=1}^{N}\alpha_n^{N-j}z^{2j-1}\,. \tag{2.10}$$

Substituting the above form of a_n in Eq. (2.9), and equating the terms with the same coefficients of z^j, we find the equation

$$1=\frac{w_n}{4}(\alpha_n^{N-1}+\alpha_{n-1}^{N-1})-\frac{w_{n+1}}{4}(\alpha_{n+1}^{N-1}+\alpha_{n+2}^{N-1}), \tag{2.11}$$

as well as the recurrence relations

$$\begin{aligned}&\alpha_{n+1}^0=\alpha_n^0;\\ &\alpha_{n+1}^k-\alpha_n^k=-\frac{w_n}{4}(\alpha_n^{k-1}+\alpha_{n-1}^{k-1})+\frac{w_{n+1}}{4}(\alpha_{n+1}^{k-1}+\alpha_{n+2}^{k-1}), \quad k=1,\dots,N-1.\end{aligned} \tag{2.12}$$

Equations (2.12) determine α_n^{N-1} in terms of w_n, and then Eq. (2.11) yields a nonlinear discrete equation for w_n.

We shall show that the solution of Eqs. (2.12) is given by

$$\alpha_n^k=\sum_{j=0}^{k}C_{k-j}\hat{\alpha}_n^j;\quad \hat{\alpha}_n^0=\frac{1}{2},\quad \hat{\alpha}_n^j=\frac{1}{2}w_n^{1/2}(L^{2j-1})_{n,n-1},\quad j=1,2,\dots. \tag{2.13}$$

Because of the linearity of Eqs. (2.12) it is sufficient to prove that $\hat{\alpha}_n^k$ is a particular solution of Eqs. (2.12): Using that $L_{mn}=L_{nm}$, we find

$$\begin{aligned}\hat{\alpha}_n^{k-1}+\hat{\alpha}_{n-1}^{k-1}&=\frac{1}{2}w_n^{1/2}(L^{2k-3})_{n,n-1}+\frac{1}{2}w_{n-1}^{1/2}(L^{2k-3})_{n-2,n-1}\\ &=\sum_{l=-\infty}^{\infty}\left(\frac{1}{2}w_l^{1/2}\delta_{n,l}+\frac{1}{2}w_{n-1}^{1/2}\delta_{n-2,l}\right)(L^{2k-3})_{l,n-1}\\ &=\sum_{l=-\infty}^{\infty}L_{n-1,l}(L^{2k-3})_{l,n-1}=(L^{2k-2})_{n-1,n-1}.\end{aligned}$$

Using this expression, it follows that the right-hand side of Eq. (2.12b) becomes

$$\begin{aligned}&\frac{w_{n+1}}{4}(L^{2k-2})_{n+1,n+1}-\frac{w_n}{4}(L^{2k-2})_{n-1,n-1}\\ &=\frac{w_{n+1}^{1/2}}{2}\left[\frac{1}{2}w_{n+1}^{1/2}(L^{2k-2})_{n+1,n+1}+\frac{1}{2}w_n^{1/2}(L^{2k-2})_{n-1,n+1}\right]\\ &\quad-\frac{1}{2}w_n^{1/2}\left[\frac{1}{2}w_n^{1/2}(L^{2k-2})_{n-1,n-1}+\frac{1}{2}w_{n+1}^{1/2}(L^{2k-2})_{n-1,n+1}\right]\\ &=\frac{w_{n+1}^{1/2}}{2}(L^{2k-1})_{n,n+1}-\frac{w_n^{1/2}}{2}(L^{2k-1})_{n-1,n},\end{aligned}$$

which equals the left-hand side of Eq. (2.12b). Equation (2.12b) for $k=1$ is satisfied with $\hat{\alpha}_n^0=\frac{1}{2}$.

Using Eqs. (2.13) into (2.10) we find

$$\begin{aligned}a_n(z)&=\sum_{j=1}^{N}z^{2j-1}\sum_{l=0}^{N-j}C_{N-j-l}\hat{\alpha}_n^l\\ &=\frac{1}{2}\sum_{j=1}^{N}z^{2j-1}C_{N-j}+\frac{1}{2}w_n^{1/2}\sum_{j=1}^{N-1}z^{2j-1}\sum_{l=1}^{N-j}C_{N-j-l}(L^{2l-1})_{n,n-1}.\end{aligned}$$

Letting $l \to l-j$, this equation becomes Eq. (2.6b).

The right-hand side of Eq. (2.11) is of the same form as the right-hand side of Eq. (2.12b), hence Eq. (2.11) can be written as $1=\alpha_n^N-\alpha_{n+1}^N$. Therefore, $\alpha_n^N=-n-C$, which is Eq. (2.4).

If we allow w_n to depend on t_j, $j=1,2,\ldots$, it can be shown, following a similar analysis, that the linear eigenvalue equation (2.1) can also be associated with a hierarchy of nonlinear evolution (with respect to t_j) equations.

Proposition 2.2. *(The Matrix Lax pair of the Volterra hierarchy.) The Volterra hierarchy,*

$$\frac{\partial}{\partial t_j}\ln w_n=(L^{2j})_{n-1,n-1}-(L^{2j})_{n,n}, \tag{2.14}$$

admits the Lax pair formulation

$$\Psi_{n+1}(z)=U_n(z)\Psi_n(z), \qquad \frac{\partial\Psi_n(z)}{\partial t_j}=V_n(z)\Psi_n(z), \tag{2.15}$$

where U_n is defined in Eq. (1.12), *and V_n is given by Eq.* (1.15). *For convenience of notation we have suppressed the t_j-dependence. This is the matrix z-dependent representation of the known scalar pair* [13–15].

Proof. Actually Eq. (2.4) is associated with a larger than (2.14) class of integrable equations. To derive these equations we consider the compatibility condition of Eqs. (2.15), which yields $U_{n_t}=V_{n+1}U_n-U_nV_n$. Denoting the 11, 12, 21, and 22 entries of V_n by v_n, q_n, p_n, and r_n, respectively, and writing the compatibility condition into component form we find for q_n and p_n the expressions given in (1.15), as well as

$$r_n=-v_n-\tfrac{1}{2}(\ln w_n)_{t_j}, \tag{2.16a}$$

and

$$\begin{aligned}-\frac{z}{2}(\ln w_{n+1})_{t_j}=z(v_{n+1}-v_n)&+\frac{w_n}{4z}(v_{n-1}+v_n)-\frac{w_{n+1}}{4z}(v_{n+1}+v_{n+2})\\&-\frac{w_{n+1}}{8z}(\ln w_{n+2})_{t_j}+\frac{w_n}{8z}(\ln w_n)_{t_j}.\end{aligned} \tag{2.16b}$$

[In Eqs. (2.16), subscripts t_j denote partial derivatives with respect to t_j.] In order to solve Eq. (2.16b) we make the ansatz

$$v_n(z)=\sum_{k=0}^{j}\beta_n^{j-k}z^{2k}. \tag{2.17}$$

Substituting this form of v_n in Eq. (2.16b), and equating the terms with the same coefficient z^k, we find

$$-\frac{1}{2}\partial_t\ln w_{n+1}=\beta_{n+1}^j-\beta_n^j+F_n^{j-1}, \qquad F_n^k \doteq \frac{w_n}{4}(\beta_{n-1}^k+\beta_n^k)-\frac{w_{n+1}}{4}(\beta_{n+1}^k+\beta_{n+2}^k), \tag{2.18}$$

as well as the recurrence relations

$$\beta_{n+1}^0=\beta_n^0\,;\qquad \beta_{n+1}^k-\beta_n^k=-F_n^{k-1}\,,\qquad k=1,2,\ldots,j-1\,;$$
$$w_{n+1}\beta_{n+1}^j-w_n\beta_{n-1}^j=w_{n+1}\frac{F_{n+1}^{j-1}}{2}-w_n\frac{F_{n-1}^{j-1}}{2}\,. \tag{2.19}$$

Equations (2.19) determine β_n^k, $k=0,\ldots,j$, and then Eq. (2.18) yields a nonlinear evolution equation for w_n. Furthermore, Eqs. (2.16a) and (2.17) imply the associated form of $V_n(z)$. We note that Eqs. (2.19a) and (2.19b) are identical to Eqs. (2.12), thus

$$\beta_n^0=\frac{C_0}{2}\,;\qquad \beta_n^l=\frac{C_l}{2}+\frac{w_n^{1/2}}{2}\sum_{r=1}^{l}C_{l-r}(L^{2r-1})_{n,n-1}\,,\qquad l=1,\ldots,j-1\,. \tag{2.20}$$

Using this general form of β_n^l we obtain nonlinear evolution equations which are linear combinations of the Volterra hierarchy. To obtain Eq. (2.14) we restrict ourselves to the choice

$$\beta_n^0=-\frac{1}{2}\,,\qquad \beta_n^k=-\frac{w_n^{1/2}}{2}(L^{2k-1})_{n,n-1}\,,\qquad k=1,\ldots,j-1\,. \tag{2.21}$$

Equation (2.19c) yields $\beta_n^j=\frac{F_n^{j-1}}{2}$, which [using (2.21)] was calculated in the derivation of Proposition 2.1, and gives

$$\beta_n^j=-\frac{w_n^{1/2}}{4}(L^{2j-1})_{n,n-1}+\frac{w_{n+1}^{1/2}}{4}(L^{2j-1})_{n+1,n}\,. \tag{2.22}$$

The right-hand side of Eq. (2.18a) reduces to $\beta_{n+1}^j+\beta_n^j$ which equals

$$\tfrac{1}{4}\{w_{n+1}^{1/2}(L^{2j-1})_{n+1,n}+w_{n+2}^{1/2}(L^{2j-1})_{n+2,n+1}\}$$
$$-\tfrac{1}{4}\{w_n^{1/2}(L^{2j-1})_{n,n-1}+w_{n+1}^{1/2}(L^{2j-1})_{n+1,n}\}\,;$$

the first bracket, which was also calculated in Proposition 2.1, equals $\frac{1}{2}(L^{2j})_{n+1,n+1}$, and the second one equals $\frac{1}{2}(L^{2j})_{n,n}$, hence Eq. (2.18a) is Eq. (2.14). Equations (1.16) follow from the substitution of Eqs. (2.21) in Eqs. (2.16a) and (2.17).

It is possible to allow the C_j's appearing in Eq. (2.6) to evolve in t_j in such a way that the Lax pairs (2.5) and (2.15) are compatible. The equations $\Psi_{n_{t_j}}=V_n\Psi_n$ and $\Psi_{n_z}=A_n\Psi_n$ are compatible iff

$$\frac{\partial}{\partial z}V_n=\frac{\partial}{\partial t_j}A_n+[A_n,V_n]\,. \tag{2.23}$$

Differentiating the compatibility condition of Eqs. (2.5) with respect to t_j, and the compatibility condition of Eqs. (2.15) with respect to z we obtain

$$\frac{\partial}{\partial z}V_n=\frac{\partial}{\partial t_j}A_n+[A_n,V_n]+F_n\,,\quad\text{where}\quad F_{n+1}U_n-U_nF_n=0\,. \tag{2.24}$$

The solution of Eq. (2.24b) is precisely of the form (2.6), thus if $(F_n)_{11}$ is zero then $F_n=0$. It can be shown that

$$C_{N-j}=-2jt_j \tag{2.25}$$

is a sufficient condition for $(F_n)_{11}=0$.

3. The RH Formulation of the Discrete String Equation

In this section we use the isomonodromy approach to solve Eq. (1.1), which as it was shown in Sect. 2, is the compatibility condition of Eqs. (1.12), (1.13), (1.15). Equation (1.13) plays a fundamental role in the subsequent analysis, while Eqs. (1.12) and (1.15) play only auxiliary roles.

3.1. The Direct Problem. The basic idea of the isomonodromy method is to use Eq. (1.13) to formulate an inverse problem for $\Psi_n(z)$ in terms of appropriate monodromy data. This can be achieved by determining the analytic structure of solutions of Eq. (1.13) with respect to $z\in\mathbb{C}$. Since Eq. (1.13) is a linear ODE in z, the analytic structure of Ψ_n depends only on $A_n(z)$. Actually, Eq. (1.13) has only one singularity, namely an irregular singular point at $z=\infty$. A formal solution at $z=\infty$ has the form,

$$\Psi_n\sim\Psi_n^{(\infty)},\qquad \Psi_n^{(\infty)}=\hat{\Psi}_n^{(\infty)}\exp\left[\left(-\frac{1}{2}\sum_{j=1}^{N}t_jz^{2j}+n\ln z\right)\sigma_3\right],\qquad z\to\infty\,, \tag{3.1}$$

where $\sigma_3=\operatorname{diag}(1,-1)$, and $\hat{\Psi}_n^{(\infty)}$ is a formal power series in $\frac{1}{z}$. However, the actual asymptotic behavior of Ψ_n changes form in certain sectors of the complex z-plane (Stoke's phenomenon). These sectors are determined by $\Re\sum_{j=1}^{N}t_jz^{2j}=0$; thus for large z the boundaries of the sectors, which we call $\sum_l$, are asymptotic to the rays $\arg z=\frac{-\pi}{2N}+l\frac{\pi}{N}$, $0\leqq l\leqq 4N-1$ (we have assumed that t_N is imaginary). Let Ω_l be the sector containing the boundary $\sum_l$, i.e. $z\in\Omega_0$, $-\frac{\pi}{2N}\leqq\arg z<0$, etc. Then if $\Psi_n\sim\Psi_n^{(\infty)}$ as $z\to\infty$ in Ω_0, it turns out that $\Psi_n\sim\Psi_n^{(\infty)}S_1S_2\ldots S_l$, as $z\to\infty$ in Ω_{l+1}, $0\leqq l\leqq 4N-1$. The matrices S_l, $0\leqq l\leqq 4N-1$, are triangular and are called Stokes matrices. Alternatively, it is more convenient to introduce different solutions $\Psi_n^{(l)}$, $0\leqq l\leqq 4N$ such that $\Psi_n^{(l)}$ is asymptotic to $\Psi_n^{(\infty)}$ in Ω_l. Then $\Psi_n^{(l+1)}=\Psi_n^{(l)}S_l$, $0\leqq l\leqq 4N-1$; also it can be shown that $\Psi_n^{(0)}(z)=\Psi_n^{(4N)}(ze^{2i\pi})e^{2i\pi n\sigma_3}=\Psi_n^{(4N)}(ze^{2i\pi})$. Therefore,

$$\Psi_n^{(l+1)}(z)=\Psi_n^{(l)}(z)S_l\,,\qquad 0\leqq l\leqq 4N-2\,,\qquad \Psi_n^{(0)}(z)=\Psi_n^{(4N-1)}(ze^{2i\pi})S_{4N-1}\,, \tag{3.2}$$

$$\Psi_n^{(l)}(z)\sim\hat{\Psi}_n^{(\infty)}e^{\left(-\frac{1}{2}\sum_{j=1}^{N}t_jz^{2j}+n\ln z\right)\sigma_3}\,,\quad\text{as}\quad z\to\infty\quad\text{in }\Omega_l\,. \tag{3.3}$$

The Stokes matrices have the form

$$S_{2l}=\begin{pmatrix}1 & s_{2l}\\ 0 & 1\end{pmatrix},\qquad S_{2l+1}=\begin{pmatrix}1 & 0\\ s_{2l+1} & 1\end{pmatrix},\qquad l=0,\ldots,2N-1\,. \tag{3.4}$$

The case $N=4$ is illustrated in Fig. 3.1.

There exists a symmetry relationship for A_n, which in turn implies a symmetry relationship for Ψ_n:

$$A_n(-z)=-\sigma_3A_n(z)\sigma_3\;\Rightarrow\;\Psi_n^{(l)}(-z)=(-1)^{n+1}\sigma_3\Psi_n^{(l+2N)}(z)\sigma_3\,. \tag{3.5}$$

Equation (3.5b) implies that the Stokes matrices satisfy the constraint

$$S_{l+2N}=\sigma_3S_l\sigma_3=S_l^{-1}\,. \tag{3.6a}$$

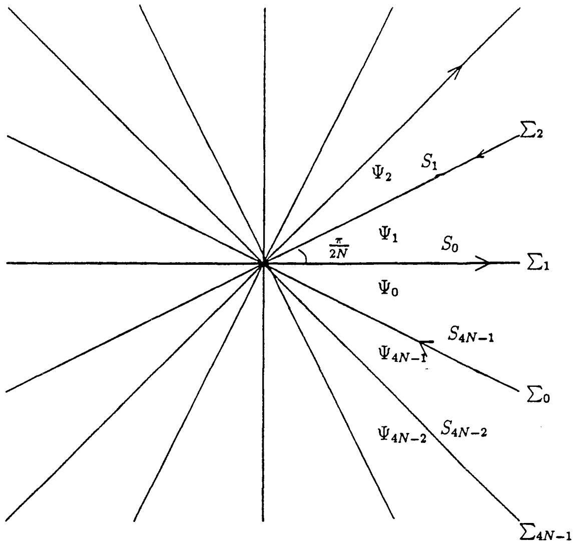

Fig. 3.1 $(N=4)$

Also Eqs. (3.2) imply the consistency condition

$$S_0 S_1 \dots S_{4N-1} = I. \tag{3.6b}$$

The constraints (3.6) identify the set of the monodromy data as a $2N-1$-dimensional algebraic variety. Given this set, Eqs. (3.2) and (3.3) define a RH problem for the function Ψ_N. The quantity w_n can then be reconstructed via

$$w_n = 4\beta_n^{(1)}\gamma_n^{(1)}, \tag{3.7}$$

where $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ are appropriate asymptotic coefficients in the expression

$$\hat{\Psi}_n^{\infty} = \Lambda_n\left(I + \begin{pmatrix} \alpha_n^{(1)} & \beta_n^{(1)} \\ \gamma_n^{(1)} & \delta_n^{(1)} \end{pmatrix}\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right) \tag{3.8}$$

and Λ_n is a diagonal matrix. Equation (3.7) implies that w_n depends only on the orbits of the action

$$S_l \mapsto \exp(\delta\sigma_3) S_l \exp(-\delta\sigma_3), \qquad \delta \in \mathbb{C}. \tag{3.9}$$

This action is well defined on the algebraic variety specified by Eqs. (3.6).

Since Λ_n depends on n and on t, it follows that the monodromy data will also depend in general on n and t. However, it is possible to normalize Ψ_n in such a way that, if w_n satisfies Eqs. (1.1) and (1.11), then the monodromy data are n and t independent (this is a usual situation in the isomonodromy method [31]). The correct normalization is achieved by choosing Λ_n so that the formal solution $\Psi_n^{(\infty)}$ defined in Eqs. (3.1), (3.8), is also a formal solution of Eqs. (1.12) and (1.15). This is the case if

$$\Lambda_n = \operatorname{diag}((\beta_n^{(1)})^{-1/2}, (\gamma_n^{(1)})^{-1/2}). \tag{3.10}$$

3.2. The Inverse Problem

Theorem 3.1. *The Cauchy problem for the discrete string equation* (1.1) *always admits a global meromorphic in* t_j *solution. This solution can be obtained by solving the RH problem defined with respect to the orientation shown in Fig. 3.1:*

$$\Phi_n^-(z)=\Phi_n^+(z)e^{-\frac{\sigma_3}{2}\sum_{j=1}^{N} t_j z^{2j}} S e^{\frac{\sigma_3}{2}\sum_{j=1}^{N} t_j z^{2j}}, \tag{3.11 a}$$

$$\Phi_n=\begin{pmatrix}(\beta_n^{(1)})^{-1/2} & 0\\ 0 & (\gamma_n^{(1)})^{-1/2}\end{pmatrix}\left(I+\begin{pmatrix}\alpha_n^{(1)} & \beta_n^{(1)}\\ \gamma_n^{(1)} & \delta_n^{(1)}\end{pmatrix}\frac{1}{z}+O\left(\frac{1}{z^2}\right)\right)z^{n\sigma_3}, \qquad z\to\infty, \tag{3.11 b}$$

where $S=S_{2l})^{-1}$ *on* $\sum_{2l+1}$, *and* $S=S_{2l+1}$ *on* $\sum_{2l+2}$, $0\leqq l\leqq 2N-1$. *This RH problem is uniquely defined in terms of the monodromy data* S_l, $0\leqq l\leqq 4N-1$, *defined on the* $2N-1$*-dimensional algebraic variety given by Eqs.* (3.6). *Having obtained* Φ_n, w_n *follows from Eq.* (3.7).

Proof. We first note that Eqs. (3.11) are a consequence of Eqs. (3.2), (3.3), (3.8) and of the change of variables $\Psi_n=\Phi_n\exp\left[-\frac{1}{2}\sum_{j=1}^{N} t_j z^{2j}\right]\sigma_3$; the specific form of the jump S follows from the orientation chosen in Fig. 3.1 ($\Psi_1^+=\Psi_0^- S_0$, $\Psi_2^- =\Psi_1^+ S_1$, etc.).

The solvability of the RH (3.11) for $n=0$ follows from the general results of [32] as extended in [18]. In particular, the difficulty of the existence of oscillations (as opposed to decay) on the contour, can be handled as in [18] by performing a small clockwise rotation. Also the existence of meromorphic in t_j solutions is a consequence of the explicit analytic dependence of the jump matrices on t_j.

However, the above RH problem possesses two novelties: (a) Because of the boundary condition (3.11 b), Φ_n involves a polynomial P_n of degree n and the question arises of how to determine this polynomial. (b) In order to prove that the function w_n defined by Eq. (3.7) solves the discrete string equation (1.1), it is necessary to prove that the solution of the RH problem (3.11) satisfies Eqs. (1.12), (1.13), and (1.15). (This is sometimes referred to, in the literature, as proving that the inverse problem solves the direct problem.) This step presents a technical difficulty for Eqs. (1.13) and (1.15) because z enters in a polynomial of degree $2N-1$ and N is arbitrary. We will solve these problems as follows:

(a) We will derive the solution of Φ_n in terms of Φ_0; in this process the form of P_n will be determined.
(b) We shall prove that the solution of (3.11) solves Eq. (1.12). This is rather simple since z enters only linearly in Eq. (1.12). This proof also will clarify the reason for choosing A_n in the form (3.10). We shall also prove that $\Psi_{n_z}=A_n\Psi_n$ and $\Psi_{n_{t_j}}=V_n\Psi_n$, where A_n and V_n are polynomials of z or degree $2N-1$ and $2N$, respectively. Then it follows from the results of Sect. 2 that the RH problem (3.11) also solves Eqs. (1.13), (1.15). Indeed, if $\Psi_{n+1}=U_n\Psi_n$ and $\Psi_{nz}=A_n\Psi_n$, where U_n is given and A_n is a polynomial in z of degree $2N-1$, it was shown in Sect. 2 that A_n must be of the form (1.13). Similarly for V_n.

(a) Φ_n *in Terms of* Φ_0. Since the jump matrices are independent of n, the RH problems for both Φ_n and Φ_0 can be denoted by $\Phi_n^- = \Phi_n^+ J$, $\Phi_0^- = \Phi_0^+ J$, or

$$\Phi_n^-(\Phi_0^-)^{-1} = \Phi_n^+(\Phi_0^+)^{-1};$$

$$\Phi_n \sim \Lambda_n \begin{pmatrix} P_n + O\left(\frac{1}{z}\right) & O\left(\frac{1}{z^{n+1}}\right) \\ Q_{n-1} + O\left(\frac{1}{z}\right) & \frac{1}{z^n} + O\left(\frac{1}{z^{n+1}}\right) \end{pmatrix}, \qquad \Phi_0 \sim \Lambda_0 \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix},$$

where P_n is a polynomial of degree n whose z^n term has coefficient 1, Q_{n-1} is a polynomial of degree $n-1$, $\alpha_0 = 1 + O\left(\frac{1}{z}\right)$, $\delta_0 = 1 + O\left(\frac{1}{z}\right)$, $\beta_0 = O\left(\frac{1}{z}\right)$, and $\gamma_0 = O\left(\frac{1}{z}\right)$. Thus

$$\Lambda_n^{-1}\Phi_n = \begin{pmatrix} (P_n\delta_0)_+ & -(P_n\beta_0)_+ \\ (Q_{n-1}\delta_0)_+ & -(Q_{n-1}\beta_0)_+ \end{pmatrix} (\Lambda_0)^{-1}\Phi_0, \tag{3.12}$$

where $(P_n\delta_0)_+$ means multiplying P_n by δ_0 and keeping only the non-negative powers of z. We assume that Φ_0 is known, therefore, $\alpha_0, \beta_0, \gamma_0$, and δ_0 are known to any desired order. The matrix appearing in Eq. (3.12) depends on the $2n$ coefficients of P_n $(p_{n-1}, p_{n-2}, \ldots, p_0)$ and of Q_{n-1} $(q_{n-1}, q_{n-2}, \ldots, q_0)$. These $2n$ parameters can be determined as follows. The large z asymptotics of Φ_n indicates that the coefficients of the terms z^j, $-n \leqq j \leqq n-1$ of the 12 entry of the right-hand side of Eq. (3.12) must be zero. Similarly, the coefficients of the terms z^j, $-(n-1) \leqq j \leqq n-2$ of the 22 entry of the right-hand side of Eq. (3.12) must be zero, while the coefficient of the z^{-n} term of this entry must be one. The coefficients of the non-negative powers of z^j are zero by construction; the rest of these requirements imply precisely $2n$ equations for the $2n$ unknown parameters. It is easily seen that the relevant equations have a triangular structure thus they are always solvable. As an example we shall consider below $n=2$.

In both parts (a) and (b) we shall make use of the symmetry relationship (3.6a) of the monodromy data. It is easy to show that this symmetry induces a symmetry for $\Phi_n(z)$:

$$\Phi_n(-z) = (-1)^n \sigma_3 \Phi_n(z) \sigma_3 . \tag{3.13}$$

Equation (3.13) implies

$$\begin{gathered} P_n = z^n + p_{n-2}z^{n-2} + \ldots, \qquad Q_{n-1} = q_{n-1}z^{n-1} + q_{n-3}z^{n-3} + \ldots, \\ \alpha_0 = 1 + \frac{\alpha_0^{(2)}}{z^2} + O\left(\frac{1}{z^4}\right), \qquad \delta_0 = 1 + \frac{\delta_0}{z^2} + O\left(\frac{1}{z^4}\right), \\ \beta_0 = \frac{\beta_0^{(1)}}{z} + O\left(\frac{1}{z^3}\right), \qquad \gamma_0 = \frac{\gamma_0^{(1)}}{z} + O\left(\frac{1}{z^3}\right). \end{gathered} \tag{3.14}$$

Using these equations, Eq. (3.12) in the case $n=2$ becomes

$$\Lambda_n^{-1}\Phi_2 = \begin{pmatrix} z^2 + p_0 + \delta_0^{(2)} & -\beta_0^{(1)}z \\ -q_1 z & q_1\beta_0^{(1)} \end{pmatrix} \Lambda_0^{-1}\Phi_0 .$$

Demanding that the coefficients of $\frac{1}{z}$ and of $\frac{1}{z^2}$ in the 12 and 22 terms are 0 and 1, respectively, we find $\beta_0^{(1)}p_0+\beta_0^{(3)}=0$ and $q_1(\beta_0^{(1)}\delta_0^{(2)}+\beta_0^{(3)})=1$. These equations determine p_0 and q_1 in terms of the asymptotics of Φ_0.

It should be mentioned that the transition from Φ_0 to Φ_n described above is the particular case of the general Schlesinger transformation [31].

(b) *The RH Problem* (3.11) *Solves Eq.* (1.12). Using the relationship between Ψ_n and Φ_n it follows that $\Psi_{n+1}\Psi_n^{-1}=\Phi_{n+1}\Phi_n^{-1}$. But since Ψ_{n+1} and Ψ_n have the same jumps we deduce that $\Psi_{n+1}\Psi_n^{-1}$ is a polynomial, thus it equals $\lim_{z\to\infty}(\hat{\Phi}_{n+1}z^{\sigma_3}\hat{\Phi}_n^{-1})$. Using Eq. (3.11b) to compute this limit [and taking into consideration the symmetry condition (3.13)] we find

$$\Psi_{n+1}\Psi_n^{-1}=\begin{pmatrix}\frac{\lambda_{n+1}}{\lambda_n}z & -\frac{\lambda_{n+1}}{\mu_n}\beta_n^{(1)}\\ \frac{\mu_{n+1}\gamma_{n+1}^{(1)}}{\lambda_n} & 0\end{pmatrix},\qquad \Lambda_n\doteq\mathrm{diag}(\lambda_n,\mu_n). \tag{3.15}$$

The $\frac{1}{z}$ term of the 22 entry of the above equation implies $\beta_n^{(1)}\gamma_{n+1}^{(1)}=1$. Using $\lambda_n=(\beta_n^{(1)})^{-1/2}$, $\mu_n=(\gamma_n^{(1)})^{-1/2}=(\beta_{n-1}^{(1)})^{1/2}$, Eq. (3.15) becomes

$$\Psi_{n+1}=\begin{pmatrix}z\left(\frac{\beta_{n+1}^{(1)}}{\beta_n^{(1)}}\right)^{-1/2} & -\left(\frac{\beta_{n+1}^{(1)}}{\beta_n^{(1)}}\right)^{-1/2}\left(\frac{\beta_n^{(1)}}{\beta_{n-1}^{(1)}}\right)^{1/2}\\ 1 & 0\end{pmatrix}.$$

The definition $w_n=4\beta_n^{(1)}\gamma_n^{(1)}=4\beta_n^{(1)}/\beta_{n-1}^{(1)}$ reduces this equation to Eq. (1.12).

3.3. A Vanishing Lemma. For certain constraints of the monodromy data and for the t_j's on certain rays, the RH problem for the function Φ_0 is uniquely solvable, which in turn implies that Φ_0 cannot have poles for finite t_j's.

We denote by $f^\dagger(z)\doteq(f(\bar{z}))^*$ the Schwartz reflection of a matrix function f (* denotes transposition and complex conjugate). Consider the RH problem $\Phi_0^-=\Phi_0^+J$ on the contour $\sum$ containing the real axis. Then it is easy to prove [18] that if $\sum$ and J are Schwartz reflection invariant, then a sufficient condition for the solvability of this RH problem is $\mathrm{Re}J>0$ on the real axis. A direct application of this result to the RH for the function Φ_0 fails. However, it is possible to use analytic continuation of the original RH problem and then apply the above result. The situation is precisely analogous to the one studied in [18]. For brevity of presentation we give the result for N even only.

Lemma 3.1. *Assume that N is even, that t_j, $1\leqq j\leqq N$, are imaginary, and that the monodromy data s_l, in addition to Eqs.* (3.6), *also satisfy*

$$\bar{s}_{l+1}=-s_{2N-l},\qquad 1\leqq l\leqq N-1,\qquad |s_0-\bar{s}_1|<2. \tag{3.16}$$

Then the RH problem (3.11) *with $n=0$ is uniquely solvable.*

Proof.

Since $\bar{t}_j=-t_j$, $\exp\left[-\frac{1}{2}\sum_1^N t_jz^{2j}\sigma_3\right]^\dagger=\exp\left[\frac{1}{2}\sum_1^N t_jz^{2j}\sigma_3\right]$. Also using analytic continuation, we find $\Psi_2=\Psi_1S_1=\Psi_0S_0S_1$, or $\Psi_0^-=\Psi_2^+(S_0S_1)^{-1}$, etc. The re-

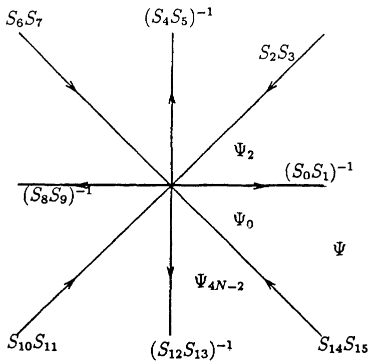

Fig. 3.2 $(N=4)$

quirement of the invariance under the Schwartz reflection implies $(S_2S_3)^* = S_{4N-2}S_{4N-1} = S_{2N-2}^{-1}S_{2N-1}^{-1}, \ldots, (S_{2N-2}S_{2N-1})^* = S_{2N+2}S_{2N+3} = S_2^{-1}S_3^{-1}$, where we have used the symmetry (3.6a). These equations imply (3.16a). The requirement that $\mathrm{Re}J>0$ on the real axis implies $\mathrm{Re}S_0S_1>0$ and $\mathrm{Re}S_{2N}S_{2N+1} = \mathrm{Re}S_0^{-1}S_1^{-1}>0$. Demanding that both the trace and the determinant of the matrix $S_0S_1+(S_0S_1)^*$ are positive we find (3.16b).

3.4. The Triangular Case. If the monodromy data s_l have a special form, then the RH problem (3.11) can be solved in closed form. This is, for example, the case when

$$\begin{aligned} &S_{2l+1}=I\,, \qquad l=0,\ldots,2N-1\,, \\ &t_j\in\mathbb{R}\,, \qquad j=1,\ldots,N\,, \qquad t_N>0\,. \end{aligned} \tag{3.17}$$

We denote by Γ_k the contours asymptotic to the rays at angles $-\frac{\pi}{2N}+(2k-1)\frac{\pi}{2N}$, $k=1,\ldots,2N$. Using the orientation of Fig. 3.3, the relevant RH becomes $\Psi_n^- = \Psi_n^+ S$, where S is either S_{2l}^{-1} or S_{2l}. In the case for example of $N=4$, S on $\Gamma_1,\ldots,\Gamma_8$ is given by S_0^{-1}, S_2, S_4^{-1}, S_6, S_0, S_2^{-1}, S_4, S_6^{-1}.

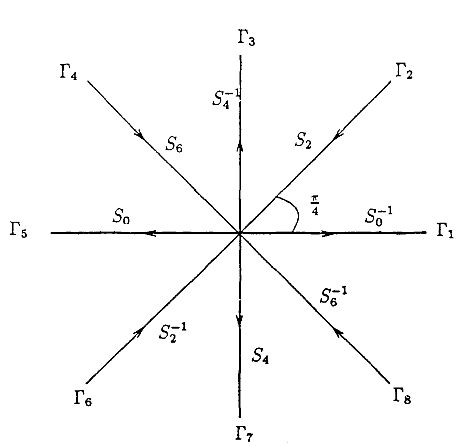

Fig. 3.3

We define ϕ_n by

$$\Lambda_n \phi_n e^{-\frac{1}{2}\sum_{j=1}^{N} t_j z^j \sigma_3} = \Psi_n, \tag{3.18}$$

then ϕ_n satisfies the RH problem

$$\begin{gathered}\phi_n^-(z) = \phi_n^+(z)\begin{pmatrix} 1 & se^{-U(z)} \\ 0 & 1 \end{pmatrix}, \\ U(z) = \sum_1^N t_j z^{2j}, \quad \phi_n(z) \sim \begin{pmatrix} z^n + O(z^{n-1}) & O(z^{-n-1}) \\ O(z^{n-1}) & z^{-n} + O(z^{-n-1}) \end{pmatrix}, \quad z \to \infty,\end{gathered} \tag{3.19}$$

where s is either s_{2k-2} or $-s_{2k-2}$. This RH problem is triangular and hence it can be solved in closed form: The 11 and 21 components yield $(\phi_n^+)_{11} = (\phi_n^-)_{11}$ and $(\phi_n^+)_{21} = (\phi_n^-)_{21}$. Using these equations and (3.19b) we find

$$(\phi_n^+)_{11} = P_n(z), \quad (\phi_n^+)_{21} = Q_{n-1}(z), \tag{3.20}$$

where P_n and Q_{n-1} are arbitrary polynomial of degree n with the only restriction that the coefficient of z^n in P_n equals 1. Using Eqs. (3.20) in the 12 and 22 entry of Eq. (3.19a) we can find $(\phi_n)_{12}$ and $(\phi_n)_{22}$:

$$\phi_n = \begin{pmatrix} P_n & \frac{1}{2i\pi} \int \frac{e^{-U(\mu)} P_n(\mu) d\mu}{\mu - z} \\ Q_{n-1} & \frac{1}{2i\pi} \int \frac{e^{-U(\mu)} Q_{n-1}(\mu) d\mu}{\mu - z} \end{pmatrix}, \tag{3.21}$$

where the integral $\int$ is defined along the **lines** corresponding to the rays Γ_k, $k = 1, \ldots, N$,

$$\int = \sum_{k=1}^{N} s_{2k-2} \int_{\hat{\Gamma}_k} \tag{3.22}$$

and the orientation of $\int_{\hat{\Gamma}_k}$ is indicated in Fig. 3.4.

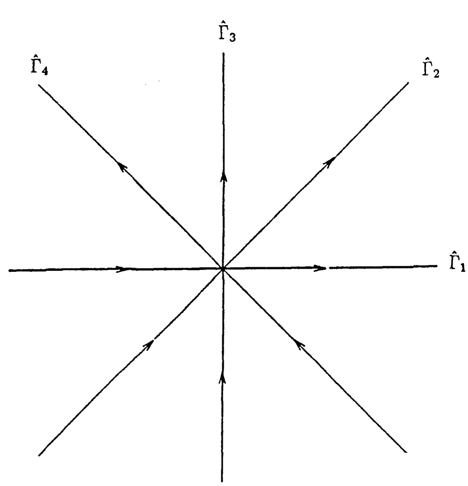

Fig. 3.4 ($N = 4$)

The function ϕ_n satisfies the boundary condition (3.19b) iff

$$\int \mu^l e^{-U(\mu)} P_n(\mu) = 0\,, \qquad l = 0, 1, \dots, n-1$$

and (3.23)

$$-\frac{1}{2i\pi}\int \mu^l e^{-U(\mu)} Q_{n-1}(\mu) = \delta_{l,n-1}\,, \qquad l = 0, 1, \dots, n-1\,.$$

These equations imply that P_n and Q_n are simply related and that P_n are orthogonal polynomials with respect to the measure $\int e^{-U(\mu)}$:

$$h_n\delta_{nl} = \int P_n(z)P_l(z)e^{-U(z)}dz\,, \quad P_n(z) = z^n + \dots\,, \quad Q_{n-1}(z) = -\frac{2\pi i}{h_{n-1}}P_{n-1}(z)\,. \tag{3.24}$$

Using the explicit formula (3.21) one obtains that in the case under consideration

$$w_n = 4\frac{h_n}{h_{n-1}}\,.$$

This means that the discrete string equation (1.1) with the initial data (1.2) corresponds to the special triangular form (3.17) of the RH problem (3.11). The monodromy data s_{2k-2}, $k = 1, \dots, N$ appear explicitly in the initial data (1.2) through the redefinition of the integration in (1.3),

$$\int_{-\infty}^{\infty} dz \to \sum_{k=1}^{N} s_{2k-2} \int_{\Gamma_k} dz \tag{3.25}$$

(the basic case corresponds $s_{2k-2} = 0$, $k = 2, \dots, N$).

Remark. Formulae (1.12)–(1.16) for the matrices $U_n(z)$, $A_n(z)$, and $V_n(z)$ can be derived in the case under consideration directly from the explicit formula (3.21) (without use of the general theorem 3.1).

4. The Continuous Limit of the Discrete String Equation to Painlevé I

We consider the Lax pair (2.5), which, recalling that $\Psi_n = (\psi_n, \psi_{n-1})^T$, can be written as

$$z\psi_n = \tfrac{1}{2}w_{n+1}^{1/2}\psi_{n+1} + \tfrac{1}{2}w_n^{1/2}\psi_{n-1}\,, \tag{4.1}$$

$$\psi_{n_z} = a_n\psi_n + b_n\psi_{n-1}\,, \qquad b_n \div \frac{-1}{2z}(a_n + a_{n+1})w_n^{1/2}\,, \tag{4.2}$$

where a_n [see Eq. (1.14)], after interchanging the summations $\sum_{j=1}^{N-1}$ and $\sum_{l=j+1}^{N}$, is given by

$$a_n(z) = -\sum_{j=1}^{N} jt_jz^{2j-1} - w_n^{1/2}\sum_{l=2}^{N} lt_lz^{2l-1}\sum_{m=1}^{l-1} z^{-2m}(L^{2m-1})_{n,n-1}\,. \tag{4.3}$$

We shall show that under a certain continuous limit, the Lax pair (4.1) and (4.2) reduces to the Lax pair of Painlevé I equation.

Proposition 4.1. *Consider the transformations*

$$w_n \sim \varrho(1-2h^2u(\xi)), \quad \psi_n(z) \sim \psi(k,\xi), \quad w_{n\pm 1} \sim \varrho(1-2h^2u(\xi \pm h)),$$
$$\psi_{n\pm 1}(z) \sim \psi(k, \xi \pm h), \quad z = \varrho^{1/2}\left(1+\frac{k^2h^2}{2}\right), \quad h \to 0, \tag{4.4}$$

where ϱ satisfies

$$t_1\varrho + \sum_{l=2}^{N} \frac{l^2 t_l}{2^{2l-1}} \varrho^l C_{2l}^l = 0, \quad C_\beta^\alpha = \binom{\beta}{\alpha}. \tag{4.5}$$

(i) *If $\psi_n(z)$ satisfies the Lax pair given by Eqs.* (4.1) *and* (4.2), *then $\psi(k,\xi)$ satisfies the Lax pair of the Painlevé I equation,*

$$\psi_{\xi\xi} = (2u+k^2)\psi, \tag{4.6}$$

$$\psi_k = -\frac{k}{2}\varrho^{1/2}h^5 R[u_\xi\psi + 2(k^2-u)\psi_\xi], \quad R \doteq \frac{1}{3}\sum_{l=2}^{N}\frac{l^2 t_l}{2^{2l-2}}\varrho^{l-\frac{1}{2}}(l-1)C_{2l}^l. \tag{4.7}$$

(ii) *If A_n is defined by Eq.* (1.13b), *then the determinant of A_n has a third order zero as $h \to 0$.*

Proof. We first derive (ii). Equations (4.4a, c) implies $L \sim \frac{\varrho^{1/2}}{2}(\Delta + \Delta^{-1})$, thus

$$L^{2m-1} \sim \left(\frac{\varrho^{1/2}}{2}\right)^{2m-1} \sum C_{2m-1}^l \Delta^{-l}\Delta^{2m-1-l},$$

hence (4.8)

$$(L^{2m-1})_{n-1,n} \sim \left(\frac{\varrho^{1/2}}{2}\right)^{2m-1} C_{2m-1}^m.$$

Equations (4.4a), (1.13b) implies that

$$-\det A_n \sim \frac{a^2}{z^2}(z^2-\varrho), \tag{4.9}$$

where a is the limit of a_n. Hence if $a(z)|_{z=\varrho^{1/2}}=0$, the determinant of A_n will have a third order zero. Using Eq. (4.8b) in Eq. (4.3) it follows that

$$a(\varrho^{1/2}) \sim -\sum_{j=1}^{N} jt_j\varrho^{j-\frac{1}{2}} - \sum_{l=2}^{N} lt_l\varrho^{l-\frac{1}{2}} \sum_{m=1}^{l-1} \frac{C_{2m}^m}{2^{2m}},$$

where we have used $2C_{2m-1}^m = C_{2m}^m$. It is easily shown by induction that

$$\sum_{m=1}^{l-1} \frac{C_{2m}^m}{2^{2m}} = l\frac{C_{2l}^l}{2^{2l-1}} - 1. \tag{4.10}$$

Using this equation to simplify $a(\varrho^{1/2})=0$ we find Eq. (4.5).

We now derive (i). The limit of Eq. (4.1) to Eq. (4.6) is straightforward. In order to derive the limit of Eq. (4.2) to (4.7) we need $a_n(z)$ and $b_n(z)$ correct to $O(h^4)$. For this purpose we must know $(L^{2m-1})_{n,n-1}$ correct to $O(h^4)$. Because the operators w

and Δ do commute up to $O(h^3)$, it follows that

$$(L^{2m-1})_{n,n-1}=\frac{\varrho^{m-\frac{1}{2}}}{2^{2m}}C_{2m}^{m}[1-(2m-1)h^2u]+O(h^3)\,. \tag{4.11}$$

More detailed analysis shows that the $O(h^3)$ in (4.11) can be replaced by $O(h^4)$. Substituting (4.11) and (4.4a, e) in Eqs. (4.3) and (4.2b) and taking into account (4.5) we find

$$\begin{aligned}a_n(z)&=-\frac{k^2h^2}{2}\left\{\sum_{j=1}^{N}2j^2t_j\varrho^{j-\frac{1}{2}}+\sum_{l=2}^{N}lt_l\varrho^{l-\frac{1}{2}}\sum_{m=1}^{l-1}\frac{C_{2m}^{m}}{2^{2m}}(2l-2m)\right\}\\&\quad+h^2u\sum_{l=2}^{N}lt_l\varrho^{l-\frac{1}{2}}\sum_{m=1}^{l-1}2m\frac{C_{2m}^{m}}{2^{2m}}+O(h^4)\,,\\b_n(z)&=\frac{k^2h^2}{2}\left\{\sum_{j=1}^{N}2j^2t_j\varrho^{j-\frac{1}{2}}+\sum_{l=2}^{N}lt_l\varrho^{l-\frac{1}{2}}\sum_{m=1}^{l-1}\frac{C_{2m}^{m}}{2^{2m}}(2l-2m)\right\}\\&\quad-2h^2u\sum_{l=2}^{N}lt_l\varrho^{l-\frac{1}{2}}\sum_{m=1}^{l-1}m\frac{C_{2m}^{m}}{2^{2m}}\\&\quad-h^3u_\xi\sum_{l=2}^{N}lt_l\varrho^{l-\frac{1}{2}}\sum_{m=1}^{l-1}m\frac{C_{2m}^{m}}{2^{2m}}+O(h^4)\,.\end{aligned} \tag{4.12}$$

It is easily shown by induction that

$$\sum_{m=1}^{l-1}m\frac{C_{2m}^{m}}{2^{2m}}=\frac{l(l-1)}{2^{2l-1}}\frac{C_{2l}^{l}}{3}\,. \tag{4.13}$$

Using Eqs. (4.5), (4.10), and (4.13), the expressions of $a_n(z)$ and $b_n(z)$ can be simplified,

$$a_n(z)=(-k^2+u)h^2R+O(h^4)\,,\quad b_n(z)=(k^2-u)h^2R-\frac{h^3u_\xi}{2}R+O(h^4)\,, \tag{4.14}$$

where R is defined by Eq. (4.7b).

After obtaining the limits of $a_n(z)$ and $b_n(z)$, the limit of Eq. (4.2) to Eq. (4.7) is straightforward:

$$k^{-1}h^{-2}\varrho^{-1/2}\psi_k=(a_n+b_n)\psi-hb_n\psi_\xi\,.$$

Substituting the expression for a_n and b_n in this equation we find Eq. (4.7a).

Proposition 4.1 suggests the proper relationship between β and h, in order for the discrete string equation to tend to PI. Letting $t_l=\beta q_l$, $1\leqq l\leqq N$, $q_1=\frac{1}{2}$, Eq. (4.7b) yields $R=\beta J$, $J=\frac{1}{3}\sum_{l=2}^{N}\frac{l^2q_l}{2^{2l-2}}\varrho^{l-1/2}(l-1)C_{2l}^{l}$ then in order for ψ_k in Eq. (4.7a) to be $O(1)$ it follows that $\beta h^5=C_1$, which is chosen as $C_1=-4\varrho^{-1/2}J^{-1}$, in order for (4.7a) to be the Lax operator for PI equation (1.5).

Also, in order for (4.4c) and (4.4d) to be the consequences of (4.4a) and (4.4b), respectively, we need

$$\frac{n}{\beta}=C_2+C_1^{-1}h^4\xi\,,$$

where

$$C_2 = \sum_{j=1}^{N} j q_j \frac{\varrho^j}{2^{2j}} C_{2j}^j$$

because of (1.1). This completes the description of the parameters of the limit (1.4) that maps the discrete equation (1.1) into the Painlevé I equation (1.5). Letting $\Psi = (\psi, \psi_\xi)^T$ we obtain a matrix Lax pair for Painlevé I equations. The transformation

$$\Psi(\xi, k) = k^{-1/2} \begin{pmatrix} 1 & 1 \\ k & -k \end{pmatrix} \sigma_3 Y(\xi, k) \sigma_3 , \tag{4.14}$$

maps this Lax pair to the Lax pair of PI studied in [12]

$$\begin{aligned} Y_\xi &= \left[\left(k + \frac{u}{k} \right) \sigma_3 - \frac{iu}{k} \sigma_2 \right] Y , \\ Y_k &= \left[(4k^4 + 2u^2 + \xi) \sigma_3 - i(4uk^2 + 2u^2 + \xi) \sigma_2 - \left(2ku_\xi + \frac{1}{2k} \right) \sigma_1 \right] Y , \end{aligned} \tag{4.15}$$

where σ_j, $j = 1, 2, 3$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relationship $\psi_{n \pm 1} \sim \psi(\xi \pm h)$ implies $\Psi_n(z) \sim M \Psi(\xi, k)$, $h \to 0$, where $M_{11} = M_{21} = 1$, $M_{12} = 0$, $M_{22} = -h$. Then Eq. (4.14) yields

$$\Psi_n(z) \sim k^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 - hk & 1 + hk \end{pmatrix} \sigma_3 Y(\xi, k) \sigma_3 .$$

5. The WKB Analysis

In this section we perform the analysis of the double-scaling limit (1.4), which has been outlined in points (a)–(d) of the Introduction.

5.1. The WKB-Solution.

Proposition 5.1 (The WKB-Solution). *Consider the equation*

$$\begin{gathered} \frac{d\Psi_n}{dz} = \beta \hat{A}_n(z) \Psi_n , \\ \hat{A}_n \textit{ is } A_n \textit{ of Eq.} (1.13\text{b}) \textit{ with } t_j \to q_j , \quad 1 \leqq j \leqq N , \quad q_1 \doteqdot \tfrac{1}{2} . \end{gathered} \tag{5.1}$$

Let

$$\begin{gathered} \beta = C_1 h^{-5} , \quad \frac{n}{\beta} = C_2 + C_1^{-1} h^4 \xi , \\ w_n \sim \varrho(1 - 2h^2 u(\xi)) , \quad h \to 0 . \end{gathered} \tag{5.2}$$

The WKB-solution of Eq. (5.1) *under the limit* (5.2) *is given by*

$$\Psi_n^{\mathrm{WKB}}(z)=(z^2-\varrho)^{-1/4}\left[\begin{pmatrix} m^{1/2}(z) & m^{-1/2}(z) \\ m^{-1/2}(z) & m^{1/2}(z)\end{pmatrix}(1+O(h^2))\right.$$
$$\left.+O\left(\frac{h^2}{z^2-\varrho}\right)+O\left(\frac{h^3z^{2N-1}}{a(z)(z^2-\varrho)^{1/2}}\right)\right]e^{\beta\sigma_3\int_{z_0}^{z}\mu(z')dz'}, \qquad (5.3)$$

where

$$m(z)=\frac{z+(z^2-\varrho)^{1/2}}{\varrho^{1/2}},$$
$$\mu(z)=(-\det\hat{A}_n(z))^{1/2}=A(z)\left(1-\frac{w_n}{z^2}\right)^{1/2}+O\left(\frac{h^4z^{2N-2}}{(z^2-\varrho)^{1/2}}\right), \qquad (5.4)$$

$$A(z)=a(z)+h^2u(\xi)\sum_{j=1}^{N-1}z^{2j-1}\sum_{l=j+1}^{N}lq_l\frac{\varrho^{l-j}}{2^{2l-2j-1}}(l-j)C_{2l-2j}^{l-j}, \qquad (5.5\text{a})$$

$$a(z)=-\sum_{j=1}^{N}jq_jz^{2j-1}-\varrho^{1/2}\sum_{j=1}^{N-1}z^{2j-1}\sum_{l=j+1}^{N}lq_l\frac{\varrho^{l-j-\frac{1}{2}}}{2^{2l-2j}}C_{2l-2j}^{l-j}, \qquad (5.5\text{b})$$

and z_0 is any of the zeros of the function $a(z)$. The asymptotic representation (5.3) *is valid along the Stokes lines given by*

$$\operatorname{Re}\int_{z_0}^{z}\mu(z')dz'=0, \qquad (5.6)$$

and away from $z=z_0$ and from $z=\varrho^{1/2}$. More precisely, we assume that $h^2/(z^2-\varrho)$, $h^3/(z^2-\varrho)^{3/2}$, $h^3/(z-z_l)$ where $z_l\neq\varrho^{1/2}$ and $a(z_l)=0$, are small.

Proof. The WKB-solution of Eq. (5.1) can be represented as [33]

$$\Psi_n^{\mathrm{WKB}}(z)\sim T_n(z)\exp\left\{\beta\sigma_3\int^{z}\mu(z')dz'-\int^{z}\operatorname{diag}\left(T_n^{-1}(z')\frac{d}{dz'}T_n(z')dz'\right)\right\},$$

where

$$\mu(z)=(-\det\hat{A}_n(z))^{1/2},$$

and $T_n(z)$ is the matrix diagonalizing $\hat{A}_n(z)$, i.e. in our case $T_n^{-1}\hat{A}_nT_n=\mu\sigma_3$. We choose T_n in the form $(T_n)_{11}=(T_n)_{12}=1$, $(T_n)_{21}=(\mu-\hat{a}_n)/\hat{b}_n$, $(T_n)_{22}=-(\mu+\hat{a}_n)/\hat{b}_n$, where $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ are the 11, 12, and 21 entries of $\hat{A}_n$. Using Eq. (4.11) we find

$$\hat{a}_n(z)=A(z)+O(h^4),$$
$$\hat{b}_n=-\frac{w_n^{1/2}}{z}A(z)+O(h^3z^{2N-1}),\qquad \hat{c}_n=\frac{w_n^{1/2}}{z}A(z)+O(h^3z^{2N-1}), \qquad (5.7)$$

where $A(z)$ is defined in (5.5). These equations imply for $\mu_n=(\hat{a}_n^2+\hat{b}_n\hat{c}_n)^{1/2}$ the estimate given in Eq. (5.4b). It should be mentioned that the terms of order $O(h^3)$ in $\hat{b}_n$ and $\hat{c}_n$ are the same. Using this estimate and Eqs. (5.7) we find

$$T_n(z)=\begin{pmatrix}1 & 1\\ m_n^-(z) & m_n^+(z)\end{pmatrix}+O\left(\frac{h^3z^{2N-2}}{a(z)}\right);\qquad m_n^{\pm}(z)\doteqdot\frac{z\pm(z^2-w_n)^{1/2}}{w_n^{1/2}}. \qquad (5.8)$$

This equation implies

$$\operatorname{diag}(T_n^{-1}T_{n_z}) = \frac{1}{m_n^+ - m_n^-}\operatorname{diag}(-m_{n_z}^-, m_{n_z}^+) + O\left(\frac{h^3 z^{2N-2}}{a^2(z)(z^2-\varrho)^{1/2}}\right), \quad (5.9\text{a})$$

and therefore,

$$\int^z \operatorname{diag} T_n^{-1} T_{n_{z'}} dz' = -\frac{1}{2}\ln\left(\frac{z+(z^2-w_n)^{1/2}}{w_n}\right)\sigma_3 + \frac{1}{4}\ln(z^2-w_n)\sigma_0 + O\left(\frac{h^3 z^{2N-2}}{a(z)(z^2-\varrho)^{1/2}}\right). \quad (5.9\text{b})$$

The usual analysis of the corresponding integral equation shows that the error in Ψ_n^{WKB} is of order $\frac{h^3 z^{2N-1}}{a(z)(z^2-\varrho)^{1/2}}$, thus taking into consideration the estimates (5.8) and (5.9) we find

$$\Psi_n^{\mathrm{WKB}}(z) = (z^2-w_n)^{-1/4}\left[\begin{pmatrix}(m_n^+)^{1/2} & (m_n^+)^{-1/2}\\ (m_n^+)^{-1/2} & (m_n^+)^{1/2}\end{pmatrix} + O\left(\frac{h^3 z^{2N-1}}{a(z)(z^2-\varrho)^{1/2}}\right)\right] e^{\beta\sigma_3\int^z \mu(z')dz'}.$$

This equation immediately implies Eq. (5.3).

Proposition 5.2 (The Solution Near the Turning Points). *Assume that ϱ is a positive solution of the equation*

$$\frac{1}{2}\varrho + \sum_{l=2}^{N}\frac{l^2 q_l}{2^{2l-1}} C_{2l}^l \varrho^l = 0, \quad (5.10)$$

so that $\pm\varrho^{1/2}$ are the zeros of the function $a(z)/z$, where $a(z)$ is defined in Eq. (5.5b). *Also assume that this function has exactly $2N-4$ zeros z_l, $1 \leqq l \leqq 2N-4$, $z_l \neq z_j$, $z_l \neq \pm\varrho^{1/2}$. Then*

(i) *the points $z = z_l$, $1 \leqq l \leqq 2N-4$ are double turning points of Eq.* (5.1), *while the points $\pm\varrho^{1/2}$ are triple turning points.*
(ii) *Let $\Psi_{n_+}^{\mathrm{TTP}}$ and $\Psi_{n_-}^{\mathrm{TTP}}$ denote the solutions of* (5.1) *near the triple turning points $z = \varrho^{1/2}$ and $z = -\varrho^{1/2}$, respectively. Then*

$$\Psi_{n_\pm}^{\mathrm{TTP}}(z) \sim k^{-1/2}\begin{pmatrix}1 & 1\\ 1-hk & 1+hk\end{pmatrix}\sigma_3 Y(\xi,k)\sigma_3, \quad z = \pm\varrho^{1/2}\left(1+\frac{k^2}{2}h^2\right), \quad (5.11)$$

if the parameters in the limit (5.2) *are chosen as*

$$C_1 = -4\varrho^{-1/2}J^{-1}, \quad C_2 = \sum_{j=1}^{N}\frac{jq_j}{2^{2j}}C_{2j}^j \varrho^j,$$
$$J = \frac{1}{3}\sum_{l=2}^{N}\frac{l^2 q_l}{z^{2l-2}}\varrho^{l-\frac{1}{2}}(l-1)C_{2l}^l. \quad (5.12)$$

Here, $Y(\xi,k)$ is the eigenfunction associated with the isomonodromy solution of PI (see Appendix A). Furthermore,

$$\Psi_{n_\pm}^{\mathrm{TTP}}(z) = \Psi_n^{\mathrm{WKB}}(z)(\pm\varrho^{1/2}h)^{1/2} e^{-\delta_\pm\sigma_3}(I+o(1)), \quad h\to 0, \quad (5.13)$$

where $\delta_\pm$ are certain n-independent functions of ξ.

(iii) *Let* $\Psi_{n_l}^{\mathrm{DTP}}$ *denote the solution of* (5.1) *near* $z=z_l$. *Then*

$$\Psi_{n_l}^{\mathrm{DTP}}(z)=\begin{pmatrix} M_l^{1/2} & M_l^{-1/2} \\ M_l^{-1/2} & M_l^{1/2}\end{pmatrix} e^{w_l\left(1-\frac{\varrho}{z_l^2}\right)^{1/2}\frac{\lambda^2}{2}\sigma_3}+o(1), \qquad h\to 0;$$
$$M_l \doteq \frac{z_l+(z_l^2-\varrho)^{1/2}}{\varrho^{1/2}}, \qquad w_l=A_z(z_l)z_l^2C_1, \qquad z=z_l(1+h^{5/2}\lambda). \tag{5.14}$$

Furthermore,

$$\Psi_{n_l}^{\mathrm{DTP}}(z)=\Psi_n^{\mathrm{WKB}}(z)(z_l^2-\varrho)^{1/4}(I+o(1)), \qquad h\to 0. \tag{5.15}$$

Proof. (i) The turning points of Eq. (5.1) are determined by the zeros of $\det\hat{A}_n$, thus asymptotically they are determined by $\left(\frac{a(z)}{z}\right)^2(z^2-\varrho)=0$. It was shown (see Sect. 4) that under the condition (5.10), $a(\varrho^{1/2})=0$, which implies the existence of two triple turning points.
(ii) Equations (5.11) and (5.12) were derived in Sect. 4. To derive Eq. (5.13), we investigate Ψ_n^{WKB} and $\Psi_{n_\pm}^{\mathrm{TTP}}$ near $z=\pm\varrho^{1/2}$. We give the relevant formulae for z near $\varrho^{1/2}$: Near $z=\varrho^{1/2}$ we introduce the variable k by $z=\varrho^{1/2}\left(1+\frac{k^2}{2}h^2\right)$. Using $a_n=h^2J(u-k^2)+o(h^4)$ we find

$$-\det\hat{A}_n=J^2h^6\left(k^6+\frac{k^2}{2}\xi+r\right)+o(h^6), \qquad r\doteq\frac{1}{4}u_\xi^2-u^3-\frac{1}{2}\xi u, \tag{5.16}$$

thus

$$\beta\int_{\varrho^{1/2}}^{z}\mu(z')dz'\sim\beta\varrho^{1/2}Jh^5\int_0^k\sqrt{k^6+\frac{k^2}{2}\xi+r}\,dk\sim\frac{4}{5}k^5+k\xi+\delta_+, \tag{5.17}$$

where we have used (5.12a). In the matching domain $k^2\sim h^{-\varepsilon}$, $\varepsilon>0$, the WKB-solution Ψ_n^{WKB} can be represented as

$$\Psi_n^{\mathrm{WKB}}\sim(\varrho^{1/2}h)^{-1/2}k^{-1/2}\begin{pmatrix}1+\frac{h}{2}k & 1-\frac{h}{2}k \\ 1-\frac{h}{2}k & 1+\frac{h}{2}k\end{pmatrix}e^{\left(\frac{4}{5}k^5+k\xi+\delta_+\right)\sigma_3}.$$

Comparing this equation and Eq. (5.11) and, taking into account the known behavior of $Y(\xi,k)$ as $k\to\infty$ (see Appendix A) we find (5.13).
(iii) Near $z=z_l$ we introduce $z=z_l(1+h^{5/2}\lambda)$. Using Eq. (5.7) and expanding the matrix $\hat{A}_n(z)$ in a Taylor series at $z=z_l$ we find that near $z=z_l$ Eq. (5.1) becomes

$$\frac{d}{d\lambda}\Psi_n(\lambda)=\left[\lambda w_l\begin{pmatrix}1 & -\frac{\varrho^{1/2}}{z_l} \\ \frac{\varrho^{1/2}}{z_l} & -1\end{pmatrix}+O(h^{1/2})\right]\Psi_n(\lambda).$$

The $O(1)$ term of this equation can be solved exactly and is given by Eq. (5.14). Expanding Ψ_n^{WKB} near $z=z_l$ and comparing with $\Psi_{n_l}^{\mathrm{DTP}}$ we find Eq. (5.15).

5.2. Calculation of the Parameters of the PI Equation. The WKB-solution presented in Proposition 5.1 specifies the large β behavior of the eigenfunction $\Psi_n(z)$ characterized in Theorem 3.1. The solutions $\Psi_n(z)$ and $\Psi_n^{\mathrm{WKB}}(z)$ have different normalizations; comparing their large z behavior it follows that

$$\Psi_n(z) = \Psi_n^{\mathrm{WKB}} \frac{A_n \varrho^{1/4}}{\sqrt{2}} e^{\beta\delta_\infty \sigma_3}, \quad \delta_\infty = \lim_{z\to\infty}\left[-\int_{z_0}^{z} \mu(z')dz' - \frac{1}{2}\sum_{j=1}^{N} q_j z^{2j} + \frac{n}{\beta}\ln z \right]. \tag{5.18}$$

It was shown in Proposition 5.2 that near the triple turning points $z = \pm\varrho^{1/2}$, $\Psi_n(z)$ can be approximated by $\Psi_{n_\pm}^{\mathrm{TTP}}$ which are proportional to Y [see Eq. (5.11)]. Since $\Psi_{n_\pm}^{\mathrm{TTP}}$ and Ψ_n^{WKB} are related [see Eq. (5.13)] we obtain a relationship between $\Psi_n(z)$ and Y:

$$\begin{gathered} \Psi_n(z) = C\,Y(\xi, k)\sigma_3 A_n e^{(\beta\delta_\infty + \delta_\pm)\sigma_3}(I + o(1)), \\ C \doteqdot (\pm 2hk)^{-1/2}\begin{pmatrix} 1 & 1 \\ 1-hk & 1+hk \end{pmatrix}\sigma_3, \quad h \to 0. \end{gathered} \tag{5.19}$$

Recall that $z = \pm\varrho^{1/2}\left(1 + \dfrac{h^2k^2}{2}\right)$, and Y is the representation of the piecewise solution $(Y^{-1}, Y^0, Y^1, Y^2, Y^3, Y^8)$. Therefore, in the z-complex plane Y has the piecewise representation $(Y^{-1}, Y^0, Y^1, Y^2, Y^3)$. Since Stokes lines connect turning points to turning points or to infinity, and since they cannot cross, it follows that one of these Y's (Y^3) connects with the other turning point, while the other four Y's (Y^{-1}, Y^0, Y^1, Y^2) connect with some of the Ψ_n's.

Equation (5.19) and the independence of the monodromy data $\{S\}$ of n and β imply that for the Ψ_n's which are connected to Y's the following relationship between monodromy data is valid

$$S = M_\pm^{-1} G M_\pm, \qquad M_\pm \doteqdot \lim_{h\to 0} \sigma_3 A_n e^{(\beta\delta_\infty + \delta_\pm)\sigma_3}. \tag{5.20}$$

It should be mentioned that because of (3.9) we do not need to calculate the diagonal matrix M_+ explicitly. At the same time, formula (5.20b) gives us the characterization of the asymptotic behavior of the quantities $\beta_n^{(1)}$, $\gamma_n^{(1)}$ from (3.10) (cp. with [7]).

We recall that near the double turning points z_l, $\Psi_n(z)$ can be approximated by $\Psi_{n_l}^{\mathrm{DTP}}$. Then, using Eqs. (5.14), (5.15), and (5.18), we get a relationship between $\Psi_n(z)$ and the $O(1)$ approximation of $\Psi_{n_l}^{\mathrm{DTP}}$ [see the right-hand side of Eq. (5.14)]. This equation is analogous to Eq. (5.19). The $O(1)$ approximation of $\Psi_{n_l}^{\mathrm{DTP}}$ is represented by four functions (because of the occurrence of λ^2). However, the associated Stokes multipliers are equal to the identity matrix (cp. with the "continuous" calculation in [10]). Thus for the Ψ_n's which are connected with the $O(1)$ approximation of the solution around the double turning points, we find

$$S = I, \tag{5.21}$$

instead of Eq. (5.20).

Equations (5.20) and (5.21) allow us, for a given distribution of the location of the triple and double turning points, to decide for which monodromy data of the discrete string equation (and hence which initial data), the limit (1.4) exists. An exhaustive investigation of all possibilities will be given elsewhere. Here we discuss some generic cases and we assume that there exist $2N-4$ distinct, real double points.

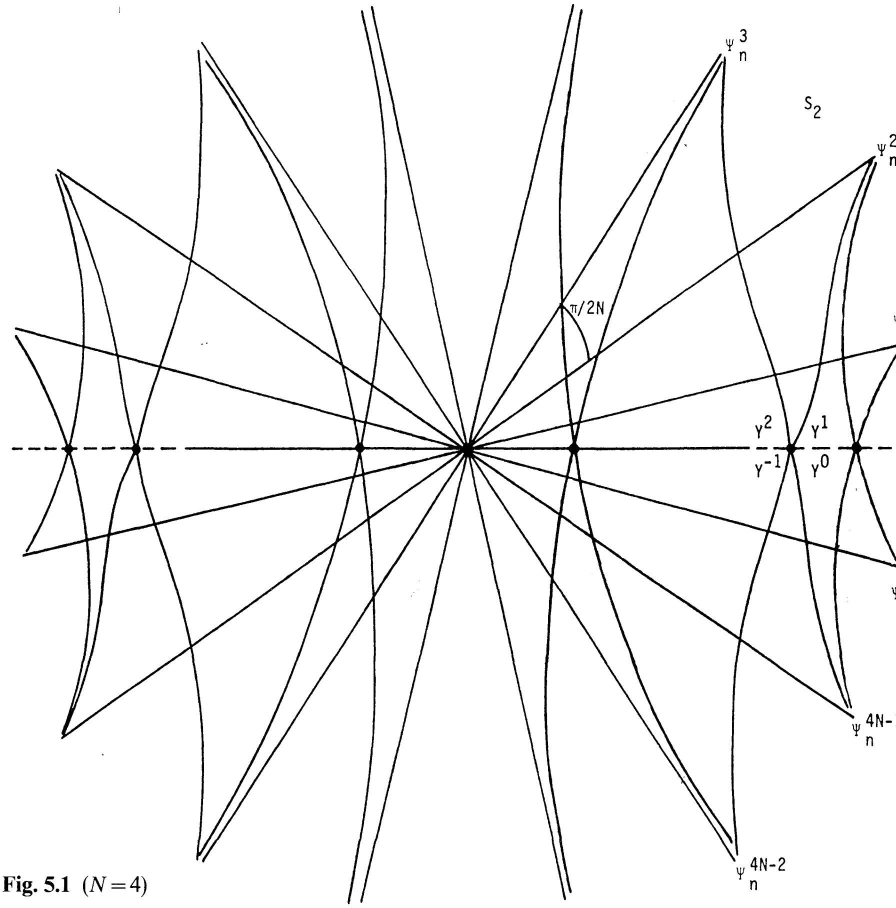

Fig. 5.1 $(N=4)$

Example 1. Only one z_l is to the right of $\varrho^{1/2}$.

This situation is illustrated in Fig. 5.1. The relationships $Y^0 = Y^{-1}G_{-1}$ and $\Psi_n^{4N-1} = \Psi_n^{4N-2}S_{4N-2}$ imply a relationship between G_{-1} and S_{4N-2}, the relations $Y^2 = Y^1 G_1$ and $\Psi_n^3 = \Psi_n^2 S_2$, imply a relationship between G_1 and S_2; the relations $Y^1 = Y^0 G_0$ and $\Psi_n^2 = \Psi_n^{4N-1}S_{4N-1}S_0S_1$ imply a relationship between G_0 and $S_{4N-1}S_0S_1$:

$$S_{4N-2} = M_+^{-1}G_{-1}M_+\,, \qquad S_2 = M_+^{-1}G_1 M_+\,, \qquad S_{4N-1}S_0S_1 = M_+^{-1}G_0M_+\,. \tag{5.22}$$

Using the triple turning point $-\varrho^{1/2}$ we find similar relations for S_{2N-2}, S_{2+2N}, S_{2N-1}, S_{2N}, S_{1+2N}. Using the double turning points it follows that all S's except S_{4N-2}, S_2, S_{2N-2}, S_{2+2N} are equal to identity. In particular, Eq. (5.22c) implies

$$G_0 = \sigma_1 G_5 \sigma_1 = I\,. \tag{5.23}$$

Thus we are dealing with the one-parameter family of solutions of PI characterized by (A.5). Equations (5.22a) and (5.22b) give (M_+ is diagonal!)

$$\frac{g_1}{g_4} = -\frac{s_2}{s_{2N-2}}\,,$$

which together with $g_1+g_4=i$ imply

$$g_1=\frac{i}{1+p}, \quad g_4=\frac{ip}{1+p}, \quad p=-\frac{s_{2N-2}}{s_2}. \tag{5.24}$$

Example 2. No z_l is to the right of $\varrho^{1/2}$.

This situation is illustrated in Fig. 5.2. In this case we get the same relations as in (5.22a, b) but for the matrices S_1, S_{4N-1}. In the triangular case (3.7), which corresponds to the matrix model, these matrices must be trivial. This implies the equalities

$$g_1=g_4=0 \tag{5.25}$$

for the monodromy parameters of the Pi function $u(\xi)$. Equations (5.25) contradict to the cyclic relations (A.4); the Stokes multipliers g_1, g_4 couldn't be zero simultaneously. This means that in this case the asymptotic behavior (1.4) for the solutions w_n corresponding to the triangular monodromy data (3.7) is not valid.

The two examples considered above are typical. It is obvious, that Eqs. (5.25) will always arise in the triangular case if the number of the points z_l on the right of $\varrho^{1/2}$ are even. On the other hand, if this number is odd, we get a situation similar to Example 1. More precisely, if the number of the points z_l on the right of $\varrho^{1/2}$ is $2m+1$, $m=0,1,\ldots,\left[\frac{N-3}{2}\right]$, then all S's except $S_{4N-2m-2}$, $S_{2N-2m-2}$, S_{2m+2}, $S_{2N+2m+2}$ are equal to identity, and the formula (5.24) should be rewritten as

$$g_1=\frac{i}{1+p}, \quad g_4=\frac{ip}{1+p}, \quad p=-\frac{s_{2N-2m-2}}{s_{2m+2}}. \tag{5.26}$$

Obviously, all of the above conclusions will still be true if the double turning points z_l are in a small neighborhood of the real axis.

For every possible location of the double points z_l one can conclude that in the basic triangular case (all S's are trivial except S_0, S_{2N}) the limit (1.4) does not exist. Indeed, the only possibility to get $S_0 \neq I$ is to have the situation depicted in Fig. 5.2, which leads to the contradictory equality (5.25).

In the matrix model of 2D gravity the parameters q_j, $j=2,\ldots,N$, play an auxiliary role. The above analysis shows, that the zeros of the polynomial $a(z)/z$ (i.e. the turning point z_l and $\varrho^{1/2}$) provide a more convenient set of independent parameters for the double scaling limit (1.4). Also, this indicates the correspondence between the hierarchy of the classes of universality (see Sect. 1.1) and the hierarchy of the types of the turning points of system (5.1) (for more details see [8, 26]).

Summarizing the above considerations and taking into account the results of the PI equation obtained in [12] and presented in Appendix A, we come to the description of the PI solution $u(\xi)$ given in the Introduction.

Remark 5.1. The result of the last two sections can easily be made rigorous. Indeed, consider for example the case corresponding to Fig. 5.1. Putting in the coefficients of Eq. (5.1) the asymptotic ansatz (1.4) where PI-function $u(\xi)$ has been chosen in accordance with (5.24), we get the system

$$\frac{d\Psi_n}{dz}=\beta\hat{A}_n^{as}(z)\Psi_n. \tag{5.27}$$

The calculations carried out above show that the Stokes matrices $S^{sa}(h)$ of the system (5.27) have the initial Stokes matrices S as the limit when $h\to 0$. Using the general properties of the RH-problem (3.11) established in Sect. 3, one can conclude that the initial matrix $\hat{A}_n(z)$ has matrix $\hat{A}_n^{as}(z)$ as its limit at $h \to 0$.

Remark 5.2. Let's write down the diagonal matrix M_+ as

$$M_+ = \beta e^{\alpha\sigma_3}.$$

Then, formulae (5.22) provide us with the explicit expression for the constant α,

$$\alpha = -\frac{1}{2}\log\frac{is_2^2}{s_{2N-2}-s_2}$$

or

$$\alpha = -\frac{1}{2}\log\frac{is_{2m+2}^2}{s_{2N-2m-2}-s_{2m+2}}$$

in the general "solvable" triangular case.

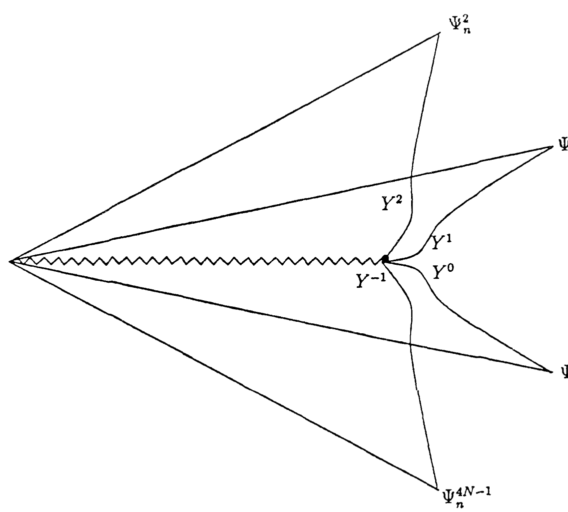

Fig. 5.2

Appendix A

According to the isomonodromy method the main role in the investigation of the PI-equation (1.5) is played by the second equation in (4.15). This equation has two singular points; a regular singularity at $\eta = 0$ and an irregular at $\eta = \infty$. Following [12] we shall introduce the monodromy data for the second equation in (4.12) as the set of Stokes matrices G_j, $j \in \mathbb{Z}$, defined by the equations

$$G_j = [Y^j(\kappa)]^{-1} Y^{j+1}(\kappa), \tag{A.1}$$

Here $Y^j(\kappa)$, $j \in \mathbb{Z}$, are the canonical solutions determined by the asymptotics

$$\begin{aligned} &Y^j(\kappa) = \left(I + O\left(\frac{1}{\kappa}\right)\right)\exp\left\{\sigma_3\left(\frac{4}{5}\kappa^5 + \xi\kappa\right)\right\}, \\ &k \to \infty \quad \text{in} \quad \frac{\pi}{5}\left(j - \frac{1}{2}\right) \leqq \arg\kappa < \frac{\pi}{5}\left(j + \frac{1}{2}\right). \end{aligned} \tag{A.2}$$

The Stokes matrices G_j have the usual triangular structure

$$G_{2l+1}=\begin{pmatrix}1 & g_{2l+1}\\ 0 & 1\end{pmatrix}, \qquad G_{2l}=\begin{pmatrix}1 & 0\\ g_{2l} & 1\end{pmatrix}$$

and they satisfy the relations

$$G_{j+5}=\sigma_1 G_j \sigma_1\,, \qquad j\in\mathbb{Z}; \qquad G_1G_2G_3G_4G_5=i\sigma_1\,. \tag{A.3}$$

This implies that the monodromy data for the second equation in (4.15) can be parametrized by the Stokes multipliers $\{g_j\}_{j=1}^{5}$ connected by the relations

$$g_5=i(1+g_2g_3), \qquad g_3+g_1(1+g_2g_3)=i, \qquad g_4=i(1+g_1g_2). \tag{A.4}$$

The monodromy data $\{g_j\}_1^5$ provide a parametrization of the solutions of the PI-equation (1.5). An alternative parametrization is provided by the asymptotic characteristics of the solution $u(\xi)$ on one of the "nonlinear Stokes ray," given by

$$\arg\xi=\pi, \qquad \pi\pm\frac{2\pi}{5}, \qquad \pi\pm\frac{4\pi}{5}.$$

The main result of [12] is the calculation of the explicit form of the connection between these two parametrizations. In particular, for the special case

$$g_5=0 \tag{A.5}$$

and, as a consequence,

$$g_3=i, \qquad g_2=i, \qquad g_4+g_1=i, \tag{A.6}$$

the following asymptotic behavior for $u(\xi)$ has been obtained:

$$\begin{aligned}\arg\xi=\pi-\frac{2\pi}{5}:\quad u(\xi)=e^{-\frac{i\pi}{5}}\sqrt{\frac{|\xi|}{6}}-\frac{e^{\frac{i\pi}{20}}}{\sqrt{8\pi}}\left(\frac{2}{3}\right)^{1/8} g_4|\xi|^{-1/8}\\ \times\exp\left\{\frac{i8}{5}\left(\frac{3}{2}\right)^{1/4}|\xi|^{5/4}\right\}+o(|\xi|^{-1/8}),\end{aligned} \tag{A.7}$$

$$\begin{aligned}\arg\xi=\pi+\frac{2\pi}{5}:\quad u(\xi)=e^{\frac{i\pi}{5}}\sqrt{\frac{|\xi|}{6}}+\frac{e^{\frac{-i\pi}{20}}}{\sqrt{8\pi}}\left(\frac{2}{3}\right)^{1/8} g_1|\xi|^{-1/8}\\ \times\exp\left\{\frac{-8i}{5}\left(\frac{3}{2}\right)^{1/4}|\xi|^{5/4}\right\}+o(|\xi|^{-1/8}).\end{aligned} \tag{A.8}$$

Moreover, in [12] using ideas based on the analytical continuation of the asymptotics (A.7), (A.8), the following description of the asymptotics of the solution $u(\xi)$ on the ray $\arg\xi=\pi$ is proposed:

$$\begin{aligned}u(\xi)=\sqrt{\frac{-\xi}{6}}+\ldots+\frac{1}{\sqrt{8\pi}}\left(\frac{2}{3}\right)^{1/8}\frac{g_1-g_4}{2}|\xi|^{-1/8}\\ \times\exp\left\{-\frac{8}{5}\left(\frac{3}{2}\right)^{1/4}(-\xi)^{5/4}\right\}(1+o(1)).\end{aligned} \tag{A.9}$$

The behavior of the function $u(\xi)$ on the rays $\arg\xi=\pi\pm\frac{4\pi}{5}$ is more complicated. It depends on the combinations

$$1+ig_1\left(\text{ray } \arg\xi=\pi-\frac{4\pi}{5}\right) \quad\text{and}\quad 1+ig_4\left(\text{ray } \arg\xi=\pi+\frac{4\pi}{5}\right). \tag{A.10}$$

For example, if $1+ig_1=0$ (i.e. $g_1=i$, $g_4=0$), the asymptotics of $u(\xi)$ on the rays $\arg\xi=\pi-\frac{4\pi}{5}$ and $\arg\xi=\pi+\frac{4\pi}{5}$ are given by Eq. (A.11) and (A.12), respectively:

$$\begin{aligned} u(\xi)=e^{-\frac{2i\pi}{5}}\sqrt{\frac{|\xi|}{6}}+\ldots+\frac{e^{-\frac{9i\pi}{10}}}{\sqrt{8\pi}}\left(\frac{2}{3}\right)^{1/8}\frac{1}{2}|\xi|^{-1/8} \\ \times\exp\left\{-\frac{8}{5}\left(\frac{3}{2}\right)^{1/4}|\xi|^{5/4}\right\}(1+o(1)), \end{aligned} \tag{A.11}$$

$$u(\xi)=-e^{\frac{2i\pi}{5}}\sqrt{\frac{|\xi|}{6}}+\frac{e^{-\frac{7i\pi}{20}}}{\sqrt{8\pi}}\left(\frac{2}{3}\right)^{1/8}|\xi|^{-1/8}\exp\left\{\frac{i8}{5}\left(\frac{3}{2}\right)^{1/4}|\xi|^{5/4}\right\}+o(|\xi|^{-1/8}). \tag{A.12}$$

For details and explicit formulae for the cases $1+ig_1>0$ and $1+ig_4\geqq 0$ we refer to [12].

As it follows from the asymptotic formulae (A.7–A.11), in the case $g_5=0=g_4$, $g_1=g_2=g_3=i$ we obtain the so-called "triply truncated solution," the solution having infinitely many poles only in the sector $\frac{7}{5}\pi<\arg\xi<\frac{9}{5}\pi$.

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