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A Matrix Integral Solution to two-dimensional W_p-Gravity

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Abstract. The p^{th} Gel'fand-Dickey equation and the string equation $[L, P] = 1$ have a common solution τ expressible in terms of an integral over $n \times n$ Hermitean matrices (for large n), the integrand being a perturbation of a Gaussian, generalizing Kontsevich's integral beyond the KdV-case; it is equivalent to showing that τ is a vacuum vector for a $\mathcal{W}_p^+$-algebra, generated from the coefficients of the vertex operator. This connection is established via a quadratic identity involving the wave function and the vertex operator, which is a disguised differential version of the Fay identity. The latter is also the key to the spectral theory for the two compatible symplectic structures of KdV in terms of the stress-energy tensor associated with the Virasoro algebra.

Given a differential operator

$$L = D^p + q_2(t)\, D^{p-2} + \cdots + q_p(t), \quad \text{with} \quad D = \frac{\partial}{dx}, \ t = (t_1, t_2, t_3, \dots), \ x \equiv t_1,$$

consider the deformation equations[1]

$$\frac{\partial L}{\partial t_n} = [(L^{n/p})_+, L] \qquad n = 1, 2, \dots, n \neq 0 \,(\text{mod}\, p) \tag{0.1}$$

(p-reduced KP-equation)

of L, for which there exists a differential operator P (possibly of infinite order) such that

$$[L, P] = 1 \quad \text{(string equation)}. \tag{0.2}$$

In this note, we give a complete solution to this problem. In section 1 we give a brief survey of useful facts about the I-function $\tau(t)$, the wave function $\Psi(t, z)$, solution of $\partial \Psi / \partial t_n = (L^{n/p})_x \Psi$ and $L^{1/p} \Psi = z \Psi$, and the corresponding infinite-dimensional plane V^0 of formal power series in z (for large z)

$$V^0 = \text{span}\{\Psi(t, z) \ \text{for all} \ t \in \mathbb{C}^\infty\}$$

[1] $\left(\sum_{-\infty}^{\infty} b_i D_i\right)_+ = \sum_0^{\infty} b_i D_i, \ (\sum b_i D^i)_- = \sum_{-\infty}^{-1} b_i D_i, \ (\sum b_i D^i)_j = b_j.$

in Sato's Grassmannian. The three theorems below form the core of the paper; their proof will be given in subseuqent sections, each of which lives on its own right.

Theorem 1. *After an appropriate time shift $t \to t + c$ (choice of time origin), the solution to $\partial L/\partial t_n = [(L^{n/p})_+, L]$ constrained to $[L, P] = 1$ with L and P differential operators is given by*[2]

$$L = S(t)\,D^p\,S(t)^{-1}, \quad S \equiv S(t) = \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)} D^{-n} \tag{0.3}$$

and, moduls a Taylor series in L with coefficients depending on $(t_2, t_3, \ldots)$,

$$P = \frac{1}{p} M L^{-\frac{p-1}{p}} + \sum_{i<1-p} c_i L^{i/p}, \quad t_p, t_{2p} = 0, \quad c_i \text{ constants}, \tag{0.4}$$

where τ satisfies the KP *hierarchy and*

$$M \equiv S\left(\sum_1^{\infty} k t_k D^{k-1}\right) S^{-1}. \tag{0.5}$$

After an appropriate rescaling $\tau(t) \curvearrowright \tau(t)\,e^{\Sigma t_i d_i}$, which alters S and M, but not L, we have

$$P = \frac{1}{p}\left(M L^{-\frac{p-1}{p}} - \frac{p-1}{2} L^{-1}\right), \tag{0.6}$$

with the requirement

$$\left(M L^{-\frac{p-1}{p}}\right)_- = \frac{p-1}{2} L^{-1}. \tag{0.7'}$$

In general we have

$$\begin{aligned}(M^j L^{k+j/p})_- &= \prod_{r=0}^{j-1}\left(\frac{p-1}{2} - r\right) L^{-1} && k = -1, \quad j = 1, 2, \ldots \\ &= 0 && k = 0, 1, 2, \ldots, \; j = 1, 2, \ldots.\end{aligned} \tag{0.7}$$

Corollary 1.1. [Kae-Schwarz], [Schw], [FKN2]. *The plane $V^0 \in Gr$ associated with the wave function $\Psi(t,z)$ of L (in Theorem 1) is invariant under the action of the differential operators L and P; they act on V^0 as z-operators, to wit*

$$L \to z^p \qquad P \to A_p = z^{\frac{p-1}{2}} \frac{d}{dz^p} z^{-\frac{p-1}{2}};$$

hence

$$z^p V^0 \subset V^0 \quad \textit{and} \quad A_p V^0 \subset V^0 \quad \textit{with} \quad [A_p, z^p] = 1$$

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[2] $\exp \sum_1^{\infty} t_i z^i = \sum_0^{\infty} z^n p_n(t)$, $p_n(-\tilde{\partial}) = p_n\left(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right)$.

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Corollary 1.2. *For L and P as above, the relation* $[L, P] = 1$ *is equivalent to*

$$-\frac{1}{p}\sum_{k \geqq p+1} k t_k \frac{\partial L}{\partial t_{k-p}} = 1. \tag{0.8}$$

In particular for $p = 2$ *(*KdV *equation), this is equivalent to*

$$\left(\sum_{k=3,5,\ldots} k t_k \frac{\partial}{\partial t_{k-2}} + \frac{t_1^2}{2}\right)\tau = 0. \tag{0.9}$$

So Theorem 1, inspired by work of Goeree [G], Krichever [K], T. Shiota [Sh] and Fukuma, Kawai and Nakayama [FKN 3], proves that if L and P are to satisfy (0.1) and (0.2), then L must satisfy [0.7′], which imposes strong constraints on τ, as will appear in Theorem 2.

Introduce the algebra $\mathscr{W}_{1+\infty}$, with generators $W_n^{(\nu)}$, defined by the vertex operator (as explained in Sect. 3 in the context of the Bäcklung transformation):

$$\begin{aligned} X(t,\lambda,\mu) &= e^{\sum_1^\infty t_i(\mu^i - \lambda^i)} e^{\sum_1^\infty (\lambda^{-i} - \mu^{-i}) \frac{1}{i}\frac{\partial}{\partial t_i}} \\ &= \sum_0^\infty \frac{(\mu-\lambda)^\nu}{\nu!} \frac{\partial^\nu}{\partial \mu^\nu} X(t,\lambda,\mu)\Big|_{\mu=\lambda} \\ &= \sum_0^\infty \frac{(\mu-\lambda)^\nu}{\nu!} \sum_{n=-\infty}^{\infty} \lambda^{-n-\nu} W_n^{(\nu)}; \end{aligned} \tag{0.10}$$

for explizit formulae, see (3.7) and the appendix. Also introduce the p-reduced algebra $\mathscr{W}_p$

$$\mathscr{W}_p = \{\text{algebra generated by } W_{jp}^{(\nu)},\ 1 \leqq \nu \leqq p,\ j \in \mathbb{Z}, \text{ with } t_p = t_{2p} = \ldots = 0\}$$

and the truncated sub-algebra

$$\mathscr{W}_p^+ = \left\{\begin{matrix}\text{closure underbracketing of } W_{jp}^{(\nu)},\ 1 \leqq \nu \leqq p,\ j = -1, 0, 1, \ldots \\ \text{with } t_p = t_{2p} = \ldots = 0\end{matrix}\right\} \tag{0.11}$$

note that $\mathscr{W}_{1+\infty}$ and $\mathscr{W}_p$ have a *central term*, whereas $\mathscr{W}_p^+$ does not; it implies that every element of $\mathscr{W}_p^+$ can be expressed as a bracket of two elements in $\mathscr{W}_p^+$ (see [FKN 2]).

Theorem 2. *Consider the differential operator*

$$L = D^p + \cdots + q_p(t) = S(t)\, D^p\, S(t)^{-1} \quad \textit{with} \quad S(t) = \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)} D^{-n};$$

then

$$\{\text{solutions } L \text{ of (0.1) and (0.2)}\} \Leftrightarrow \left\{\begin{matrix}\text{solutions } \tau \text{ of} \\ W\tau = 0 \text{ for all} \\ W \in W_p^+\end{matrix}\right\} \tag{0.12}$$

and the solution τ *is unique.*

The proof of this statement given in Sect. 4 hinges on the differential *Fay identity* (see Sect. 3), which plays an important role in this paper:

$$\Psi^*(t,\lambda)\,\Psi(t,\mu) = \frac{1}{\mu-\lambda} D \frac{X(t,\lambda,\mu)\,\tau(t)}{\tau(t)}$$

and so by Taylor's theorem and (0.10)

$$\nu\Psi^*(t,\lambda)\left(\frac{d}{d\lambda}\right)^{(\nu-1)}\Psi(t,\lambda)=D\left(\frac{1}{\tau}\sum_{n=-\infty}^{\infty}\lambda^{-n-\nu}W_n^{(\nu)}(\tau)\right).$$

In the context of the p-reduced KP equation (Felfand-Dickey hierarchy), it is natural to define so-called $\mathcal{W}_p$ stress-energy tensors (see Sect. 3 for more details); namely setting $y=\lambda^p$,

$$T_p^{(j)}(y)\equiv\sum_{n\in\mathbb{Z}}J_{np}^{(j)}y^{-n-j},\quad 1\leqq j\leqq p\quad\text{with}\quad t_{ip}=0\quad\text{all}\quad i\geqq 1,$$

for an appropriate choice of generators $J_{np}^{(j)}$ of $\mathcal{W}_p$. The p-reduced KP equation is known to have two (or more) symplectic structures and the $\mathcal{W}_p$ stress-energy tensors relate intimately to their spectral theory. For instance, $T_2^{(2)}(y)$ relates to the spectrum of the two symplectic structures D and $K\equiv(D^3+2(qD+Dq))/4$ in the following simple way (Proposition 3.4)

$$(K-yD)\,D\frac{T_2^{(2)}(y)\,\tau}{\tau}=-2.$$

We now state Theorem 3, which is proved and discussed in Sects. 5 and 6:

Theorem 3. *The unique solution to* (0.1) *and* (0.2) *is given by the limit (for large N) of*

$$\tau_p^{(N)}(t)=\frac{\tilde{A}_p^{(N)}(\Theta)}{\tilde{B}_p^{(N)}(\Theta)},\tag{0.13}$$

where $\tilde{A}_p^{(N)}$ and $\tilde{B}_p^{(N)}$ are the following integrals:

$$\tilde{A}_p(\Theta)=\int dZ\exp\operatorname{Tr}\left(\textit{non-linear terms in}\,\frac{(Z-\Theta)^{p+1}}{-(p+1)}\right)\tag{0.14}$$

and

$$\tilde{B}_p(\Theta)=\int dZ\exp\operatorname{Tr}\left(\textit{quadratic terms in}\,\frac{(Z-\Theta)^{p+1}}{-(p+1)}\right)\tag{0.15}$$

over the space of $N\times N$ skew-hermitian matrices, dZ being its invariant measure, $\Theta=\operatorname{diag}(\theta_1,\dots,\theta_N)$ and

$$t_i=\frac{1}{i}\sum_j\left(-\frac{1}{\theta_j}\right)^i\qquad i=1,2,\dots.$$

Corollary 3.1. *After a time shift $t_{p+1}\curvearrowright t_{p+1}+1$,*

$$\frac{\partial\tau_p}{\partial t_1}=\frac{p}{p+1}\frac{1}{\tilde{B}_p}\int dZ\,e^I\operatorname{tr}Z.$$

Ed. Witten [W1] conjectured that the partition function for 2d-gravity is a specific generating function for the intersection theory of moduli space and that its second derivative satisfies the string equation and the KdV equation. M. Kontsevich [K1] conjectured, also in the KdV-context, that the exponential of the same partition function has the matrix integral representation (0.13) for $p = 2$, based on the fundamental work on D. Bessis, Cl. Itzykson and J. B. Zuber [BIZ]; Kontsevich [K3] and Witten [W2] then showed that $2(\log \tau)''$ is a solution of KdV, using quite different methods: Kontsevich shows that the matrix integral representation is a τ-function, by a direct calculation, viewing τ as the determinant of a projection, whereas Witten shows that it is a vacuum vector for the Virasoro algebra (i.e. $L_i\tau = 0$ for $i = -1, 0, 1, 2, \ldots$); he then uses the independent observations of R. Dijkgraaf and E. and H. Verlinde [D-V-V] and M. Fukuma, H. Kawai and R. Nakayama [F-K-N1] that KdV and string equations are equivalent to being a vacuum vector for the Virasoro algebra. For general p, [D-V-V] and [F-K-N1] also conjectured the equivalence of the following sets

$$\{\tau \text{ a solution of the } p\text{-reduced KP and string equation}\}$$

and

$$\{\tau \text{ vacuum vector of a } \mathcal{W}_p\text{-algebra}\}$$

and Goeree [G] developed some of the mathematical machinery to show that this is true for $p = 3$ and indicated a possible approach in general.

Guided by Witten's computations in [W2] and by V. Kac and A. Schwarz's [K-S] observation that the wave functions (at some appropriate initial condition) is related to a generalization of the Airy function, we conjectured a matrix model for arbitrary p. This note contains a complete proof for $p \leqq 3$; a general proof hinges on the observation that a certain partial differential equation applied to the ratio (0.13) above produces at once the stress-energy tensor for W_p-gravity. It shows this algebra is naturally associated to these solutions and this should have a "physical" interpretation. Concurrently Kontsevich [K3] came up with the same model and the method, which he employs for $p = 2$, should work as well in general.

A link should also be made with the question discussed by J. J. Duistermaat and F. A. Grünbaum [D-G] to find an x-operator L and a λ-operator A such that $L\Psi(t,\lambda) = \lambda\Psi(t,\lambda)$ and $A\Psi((x,0,\ldots,0),\lambda) = f(x)\,\Psi((x,0,\ldots,0),\lambda)$, where $f(x)$ is a function of x. For second order L, there exists a solution L with unbounded potential $q(x)$, asymptotically linear, leading to the classical Airy equation.

1. Facts about τ

When the set of deformation equations

$$\frac{\partial Q}{\partial t_n} = [(Q^n)_+, Q] \qquad n = 1, 2, \ldots \tag{1.1}$$

for the pseudo-differential operator

$$Q = D + \sum_1^\infty a_j(t)\,D^{-j} \quad D = \frac{\partial}{\partial x},\ t = (x, t_2, \ldots)$$

has a solution, then Q conjugates to D, by means of $S(t) = 1 +$ pseudo-differential

$$Q = S(t)\,D\,S(t)^{-1}, \quad \text{with} \quad \frac{\partial S}{\partial t_n} = -(Q^n)_- S; \tag{1.2}$$

then $S(t)$ admits the representation

$$S(t) = \sum_{n=0}^{\infty} \frac{p_n(-\tilde{\partial})\,\tau(t)}{\tau(t)} D^{-n}$$

in terms of a tau-function τ satisfying the KP hierarchy.

Remark. The operator $S(t)$ is unique up to multiplication by S_0,

$$S(t) \curvearrowright S(t)\,S_0, \quad S_0 = 1 + \sum_1^{\infty} b_i D^{-i}, \quad b_i \text{ constants}, \tag{1.3}$$

since

$$Q \curvearrowright S(t)\,S_0\,D S_0^{-1}\,S(t)^{-1} = S(t)\,D S(t)^{-1} = Q.$$

Also a well-known fact is that the wave functions [3]

$$\Psi(t,z) = S e^{\sum_1^{\infty} t_i z^i} = e^{\sum_1^{\infty} t_i z^i} \frac{\tau(t - [z^{-1}])}{\tau(t)},$$

$$\Psi^*(t,z) = (S^T)^{-1} e^{-\sum_1^{\infty} t_i z^i} = e^{-\sum_1^{\infty} t_i z^i} \frac{\tau(t + [z^{-1}])}{\tau(t)}, \tag{1.4}$$

are solutions of

$$\frac{\partial \Psi}{\partial t_n} = (Q^n)_+ \Psi, \quad \frac{\partial \Psi^*}{\partial t_n} = -(Q^T)^n_+ \Psi^* \tag{1.5}$$

and

$$z\Psi = Q\Psi, \quad z\Psi^* = Q^T \Psi^*. \tag{1.6}$$

In view of the Heisenberg relation $[\partial/\partial z, z] = 1$, it is natural to compute, using (1.4)

$$\begin{aligned} \frac{\partial}{\partial z}\Psi &= \frac{\partial}{\partial z} S e^{\sum t_i z^i} \\ &= S \frac{d}{dz} e^{\sum_1^{\infty} t_i z^i} \\ &= S \sum_1^{\infty} k t_k D^{k-1} e^{\sum_1^{\infty} t_i z^i} \\ &= \left(S \sum_1^{\infty} k t_k D^{k-1} S^{-1} \right) \Psi \equiv M\Psi. \end{aligned} \tag{1.7}$$

Therefore, since $[\partial/\partial z, z] = 1$ and more generally

$$\left[\frac{1}{p} z^{-p+1} \frac{\partial}{\partial z}, z^p \right] = \left[\frac{\partial}{\partial z^p}, z^p \right] = 1, \quad \text{all } p \geqq 1 \tag{1.8}$$

[3] $[s] = \left(s, \frac{s^2}{2}, \frac{s^3}{3}, \ldots \right)$

we have

$$\left[Q^p, \frac{1}{p} M Q^{-p+1}\right] = 1, \quad \text{all } p \geqq 1. \tag{1.9}$$

We now prove the following identity, due to Goeree [G]

$$(M^n Q^{mp+n})_{-i-1} = \mathrm{Res}_z \left(z^{mp+n} \Psi^*(t,z)\, D^i \left(\frac{\partial}{\partial z}\right)^n \Psi(t,z) \right)$$
$$n = 0, 1, 2, \dots,$$
$$m = -1, 0, 1, \dots . \tag{1.10}$$

Proof. The proof is based on an identity of Date, Jimbo, Kashiwara, Miwa [DJKM] for general pseudo-differential operators $U(x, \partial/\partial x)$ and $V(x, \partial/\partial x)$, depending on x:

$$2\pi i \left(U\left(x, \frac{\partial}{\partial x}\right) V^T \left(x, \frac{\partial}{\partial x}\right)\right)_- \delta(x-y)$$
$$= \int U\left(x, \frac{\partial}{\partial x}\right) e^{xz}\, V\left(y, \frac{\partial}{\partial x}\right) e^{-yz}\, dz \quad H(x-y), \tag{1.11}$$

where $H(x) \equiv \left(\frac{d}{dx}\right)^{-1} \delta(x)$ is the Heavyside function; the integral can be evaluated by the residue theorem.

Setting

$$t = (t_1, t_2, \dots) \quad \text{and} \quad t' = (t_1', t_2, \dots)$$

we evaluate $(M^n Q^{mp+n}(t))_-$ in two different ways: on the one hand

$$(M^n Q^{mp+n}(t))_-\, \delta(t_1 - t_1') = \sum_1^\infty (M^n Q^{mp+n}(t))_{-i}\, D^{-i}\, \delta(t_1 - t_1')$$
$$= \sum_1^\infty (M^n Q^{mp+n}(t))_{-i} \frac{(t_1 - t_1')^{i-1}}{(i-1)!} H(t_1 - t_1'),$$

and on the other hand, using (1.11) in the third equality

$$(M^n Q^{mp+n}(t))_-\, \delta(t_1 - t_1')$$
$$= \left(S \left(\sum_\alpha \alpha t_\alpha D^{\alpha-1}\right)^n S^{-1} S D^{mp+n} S^{-1}\right)_- \delta(t_1 - t_1')$$
$$= \left(S(t) \left(\sum_\alpha \alpha t_\alpha D^{\alpha-1}\right)^n D^{mp+n} S^{-1}(t)\right)_- \delta(t_1 - t_1')$$
$$= \mathrm{Res}_z\, S\left(\sum \alpha t_\alpha D^{\alpha-1}\right)^n e^{\sum t_i z^i} (D^{mp+n} S^{-1})^T e^{-\sum t_i' z^i} H(t_1 - t_1')$$
$$= \mathrm{Res}_z \left(\frac{d}{dz}\right)^n \psi(t,z) \cdot z^{mp+n} \psi^*(t',z)\, H(t_1 - t_1'), \text{ using (1.7).}$$

Comparing these two expressions, when $t_1 > t_1'$, dividing by $H(t_1 - t_1')$, taking derivatives on both sides and letting $t_1 \searrow t_1'$, leads to (1.10).

When

$$L \equiv Q^p = D^p + q_2(t)\, D^{p-2} + \cdots + q_p(t) = S(t)\, D^p\, S(t)^{-1}$$

is a differential operator, then (1.1) becomes the p-reduced Gel'fand-Dickey hierarchy (p-reduced KP hierarchy)

$$\begin{aligned} \frac{\partial L}{\partial t_n} &= [(L^{n/p})_+, L] \qquad n = 1, 2, \ldots, \\ &= 0, \qquad\qquad\qquad n = p, 2p, 3p, \ldots. \end{aligned} \tag{1.12}$$

Conversely, if the differential operator L of order p satisfies (1.12), then $Q = L^{1/p}$ satisfies (1.1).

Incidentally, relation (1.9) amounts to

$$\left[L, \frac{1}{p} M L^{-1+\frac{1}{p}}\right] = 1, \tag{1.13}$$

where the second operator in the bracket is pseudo-differential.

The wave function Ψ leads naturally to the consideration of an infinite-dimensional plane V^0 in Gr, that is Sato's Grassmannian of linear spaces, containing *formal power series in z* ([Sa] or [SW]). It is defined as follows:

$$\begin{aligned} V^0 &= \operatorname{span}\left\{\Psi(t,z)|_{t=0}, \frac{\partial}{\partial x}\Psi(t,z)|_{t=0}, \frac{\partial}{\partial x^2}\Psi(t,z)|_{t=0}, \ldots\right\} \\ &= \operatorname{span}\{\Psi(t,z) \;\text{ all }\; t \in \mathbb{C}^\infty\}, \text{ using Taylor's theorem;} \end{aligned} \tag{1.14}$$

then it is well known that

$$V^t = \exp\left(-\sum_1^\infty t_i z^i\right) V^0.$$

Observe also that since V^0 is a linear space, it is closed under differentiation $\partial/\partial t_i$ up to any order.

2. Proof of Theorem 1

Since the flow must preserve $[L, P] = 1$, differentiating this relation with respect to t_n and using $\partial L/\partial t_n = [(L^{n/p})_+, L]$, we have

$$0 = \frac{\partial}{\partial t_n}[L, P] = \left[L, \frac{\partial P}{\partial t_n} - [(L^{n/p})_+, P]\right].$$

If $[L, P] = 1$ for some differential operator P, then L has the following property (see Shiota [Sh, Remark 3])

$$\{\text{differential } Q \text{ such that } [L, Q] = 0\} = \left\{\sum_{k=0}^\infty c_k L^k,\ c_k \in \mathbb{C}\right\}$$

and so

$$\frac{\partial P}{\partial t_n} - [(L^{n/p})_+, P] = \sum_{k=0}^\infty c_k^{(n)} L^k.$$

The most general solution for this equation in P has the form

$$P = \sum_{n=1}^{\infty} t_n \sum_{k=0}^{\infty} c_k^{(n)} L^k + \hat{P}$$

with

$$\frac{\partial \hat{P}}{\partial t_n} = [(L^{n/p})_+, \hat{P}],$$

and so, modulo a Taylor series in L, the operator P is a solution of

$$\begin{aligned} \frac{\partial P}{\partial t_n} &= [(L^{n/p})_+, P] \qquad n = 1, 2, \ldots, \\ &= 0 \qquad\qquad\qquad n = p, 2p, 3p, \ldots . \end{aligned} \tag{2.1}$$

Both L and P are independent of $t_p, t_{2p}, \ldots$, i.e. we may set $t_{rp} = 0$ $(r = 1, 2, \ldots)$ whenever it appears.

Since $L = SD^pS^{-1}$, the constraint $[L, P] = 1$ amounts to

$$0 = [D^p, S^{-1}PS] - 1 = \left[D^p, S^{-1}PS - \frac{x}{p} D^{1-p}\right]$$

implying

$$S^{-1}PS - \frac{x}{p} D^{1-p} = \sum_{i=-\infty}^{\infty} c_i D^i, \quad c_i = c_i(t_2, t_3, \ldots) \tag{2.2}$$

we now specify the t-dependence of c_i; taking the derivative $\partial/\partial t_n$ for $n > 1$,

$$\begin{aligned} \sum_{i=-\infty}^{\infty} \frac{\partial c_i}{\partial t_n} D^i &= \frac{\partial}{\partial t_n} S^{-1}PS \\ &= -S^{-1}\frac{\partial S}{\partial t_n} S^{-1}PS + S^{-1}\frac{\partial P}{\partial t_n} S + S^{-1}P\frac{\partial S}{\partial t_n} \\ &= S^{-1}(L^{n/p})_- PS + S^{-1}[(L^{n/p})_+, P]S - S^{-1}P(L^{n/p})_- S, \text{ using (1.2)} \\ &= [S^{-1}L^{n/p}S, S^{-1}PS] \\ &= \left[D^n, \sum_{-\infty}^{\infty} c_i D^i + \frac{x}{p} D^{1-p}\right], \text{ using (2.2)} \\ &= \frac{1}{p}[D^n, x] D^{1-p}, \text{ since } c_i = c_i(t_2, t_3, \ldots) \\ &= \frac{n}{p} D^{n-1} D^{1-p} = \frac{n}{p} D^{n-p} \end{aligned}$$

leads to

$$\begin{aligned} \frac{\partial c_i}{\partial t_n} &= \frac{n}{p}\delta_{i, n-p} \qquad \text{for } n > 1,\ n \not\equiv 0 (\operatorname{mod} p) \\ &= 0 \qquad\qquad\quad \text{for } n = p, 2p, \ldots . \end{aligned}$$

Therefore

$$\begin{aligned} c_{n-p} &= \frac{n}{p} t_n + c_{n-p}(0) && \text{for } n > 1,\ n \not\equiv 0 (\operatorname{mod} p) \\ &= c_{n-p}(0) && \text{for } n = p, 2p, \ldots \\ &= c_{n-p}(0) && \text{for } n < 1 \end{aligned} \tag{2.3}$$

and

$$S^{-1} P S = \frac{1}{p} \sum_{\substack{n=2 \\ n \neq rp}}^{\infty} n t_n D^{n-p} + \frac{x}{p} D^{1-p} + \sum_{r=0}^{\infty} c_{rp} D^{rp} + \sum_{i < 1-p} c_i D^i,$$

with constants c_i. Since P is defined modulo $\mathbb{C}[L]$ and since $S D^{rp} S^{-1} = L^r$, we may remove, without harm, the terms $\sum c_{rp} D^{rp}$ from $S^{-1} P S$, leading to

$$\begin{aligned} S^{-1} P S &= \frac{1}{p} \sum_{n=2}^{\infty} n t_n D^{n-p} + \frac{x}{p} D^{1-p} + \sum_{i < 1-p} c_i D^i \\ &= \frac{1}{p} \sum_{n=1}^{\infty} n t_n D^{n-p} + \sum_{i < 1-p} c_i D^i, \end{aligned} \tag{2.4}$$

and thus, since $P = P_+$ and since $L^{i/p}$ $(i < 1-p)$ is strictly pseudo-differential,

$$\begin{aligned} P &= \frac{1}{p} S \sum_{1}^{\infty} n t_n D^{n-p} S^{-1} + \sum_{i < 1-p} c_i L^{i/p} \\ &= \frac{1}{p} S \sum_{1}^{\infty} n t_n D^{n-1} S^{-1} S D^{1-p} S^{-1} + \sum_{i < 1-p} c_i L^{i/p} \\ &= \frac{1}{p} M L^{\frac{1-p}{p}} + \sum_{i < 1-p} c_i L^{i/p}. \end{aligned} \tag{2.5}$$

As pointed out in (1.3), there remains the freedom to change $S(t) \curvearrowright S(t) S_0$ without modifying P_+ and L; in the expression (2.4), this will only affect the term $\frac{x}{p} D^{1-p}$. Indeed, setting $S_0 = 1 + \psi \equiv 1 + \sum_{1}^{\infty} b_i D^{-i}$ pseudo-differential, with constant coefficients, notice that

$$S_0^{-1} = 1 - \psi + \psi^2 + \ldots \quad \text{and} \quad S_0 \eta S_0^{-1} = \eta + [\psi, \eta](1 + \psi)^{-1},$$

and so

$$\begin{aligned} \frac{x}{p} D^{1-p} \curvearrowright\ & \frac{x}{p} D^{1-p} + \left[\psi, \frac{x}{p} D^{1-p}\right] (1 + \psi)^{-1} \\ &= \frac{x}{p} D^{1-p} + \sum_{1}^{\infty} b_i \left[D^{-i}, \frac{x}{p}\right] D^{1-p} (1 + \psi)^{-1} \\ &= \frac{x}{p} D^{1-p} - \sum_{1}^{\infty} \frac{i b_i}{p} D^{-i-1} D^{1-p} (1 + \psi)^{-1} \\ &= \frac{x}{p} D^{1-p} - \sum_{1}^{\infty} \frac{i b_i}{p} D^{-i-p} (1 + \psi)^{-1} \\ &= \frac{x}{p} D^{1-p} + \sum_{1}^{\infty} \left(- \frac{i b_i}{p} + F_i(b_1, b_2, \ldots, b_{i-1}) \right) D^{-i-p}, \end{aligned} \tag{2.6}$$

for some polynomial expression F_i. Therefore

$$
\begin{aligned}
S^{-1}PS \curvearrowright (SS_0)^{-1}PSS_0 &= \frac{1}{p}\sum_2^\infty nt_n D^{n-p} + \frac{x}{p}D^{1-p} + c_{-p}D^{-p} \\
&\quad + \sum_{i=1}^\infty \left(-\frac{ib_i}{p} + F_i(b_1,\dots,b_{i-1}) + c_{-i-p}\right)D^{-i-p} \\
&= \frac{1}{p}\sum_1^\infty nt_n D^{n-p} + c_{-p}D^{-p}
\end{aligned}
\tag{2.7}
$$

upon picking the b_i's such that

$$\frac{ib_i}{p} - F(b_1,\dots,b_{i-1}) = c_{-i-p}.$$

The map $S \curvearrowright SS_0$ has the following effect on Ψ and τ:

$$
\begin{aligned}
\Psi = Se^{\sum t_i z^i} \curvearrowright SS_0 e^{\sum t_i z^i} &= S\left(1 + \sum_1^\infty b_i z^{-i}\right)e^{\sum t_i z^i} \\
&= \left(1 + \sum_1^\infty b_i z^{-1}\right)\Psi
\end{aligned}
$$

$$\tau(t) \curvearrowright \tau(t)\, e^{\sum t_i d_i},$$

where $b_i = p_i\left(-d_1, -\frac{d_2}{2},\dots\right)$, $i = 1,2,\dots$. Finally it will be shown at the end of the proof of Corollary 1.1 that $c_{-p} = \frac{1-p}{2p}$; so by (2.5)

$$P = \frac{1}{p}\left(ML^{\frac{1-p}{p}} - \frac{p-1}{2}L^{-1}\right).$$

Therefore P is a differential operator if and only if

$$(ML^{-1+1/p})_- = \frac{p-1}{2}L^{-1}. \tag{2.8}$$

proving (0.7′) and thus (0.7) for $j = 1$ and $k = -1$.

To prove (0.7) in general we proceed by induction on j: assume that (0.7) holds up to j, then for $k = 0,1,2,\dots$

$$
\begin{aligned}
(M^j L^{k+j/p})_- &= (M^j L^{-1+j/p+(k+1)})_- \\
&= ((M^j L^{-1+j/p})L^{k+1})_- \\
&= ((M^j L^{-1+j/p})_- L^{k+1})_-, \text{since } L^{k+1} \text{ is a differential operator} \\
&= (cL^{-1}L^{k+1})_-, \text{using the inductive step} \\
&= c(L^k)_- = 0.
\end{aligned}
$$

From the commutation relation

$$\begin{aligned}[L^{n+j/p}, M] &= S\left[D^{pn+j}, \sum_1^\infty k t_k D^{k-1}\right] S^{-1} \\ &= S[D^{pn+j}, x] S^{-1} \\ &= (pn+j) S D^{pn+j-1} S^{-1} = (pn+j) L^{n+\frac{j-1}{p}},\end{aligned}$$

it follows that

$$\begin{aligned}(M^j L^{n+j/p})(M L^{m+1/p}) &= M^j (M L^{n+j/p} + [L^{n+j/p}, M]) L^{m+1/p} \\ &= M^{j+1} L^{m+n+\frac{j+1}{p}} + (pn+j) M^j L^{m+n+j/p}.\end{aligned}$$

Then, setting $m = -1$ and $n = 0$ into this relation, using the fact that $M^j L^{j/p}$ is a differential operator and the precise expression (2.8) for $M^j L^{-1+j/p}$ (both by the inductive step)

$$\begin{aligned}\left(M^{j+1} L^{-1+\frac{j+1}{p}}\right)_- &= ((M^j L^{j/p})(M L^{-1+1/p}))_- - j(M^j L^{-1+j/p})_- \\ &= ((M^j L^{j/p})(M L^{-1+1/p})_-)_- - j(M^j L^{-1+j/p})_- \\ &= \frac{p-1}{2}(M^j L^{j/p} L^{-1})_- - j(M^j L^{-1+j/p})_- \\ &= \left(\frac{p-1}{2} - j\right)(M^j L^{-1+j/p})_- \\ &= \left(\frac{p-1}{2} - j\right)\prod_{r=0}^{j-1}\left(\frac{p-1}{2} - r\right) L^{-1},\end{aligned}$$

concluding the proof of Theorem 1.

Proof of Corollary 1.1. This proof, inspired by Kac and Schwarz [K-S], seems more direct than theirs. Since the plane $V^0 = \text{span}\{\Psi(t,z), \text{ all } t \in \mathbb{C}^\infty\} \in Gr$ is closed under differentiation D^k and, in particular, under the action of the differential operators $L(t)$ and $P(t)$ (see (1.14)), we have

$$LV^0 \subset V^0 \quad \text{and} \quad PV^0 \subset V^0, \quad \text{with } [L, P] = 1. \tag{2.9}$$

Then

$$L(t)\,\Psi(t,z) = z^p\,\Psi(t,z) \in V^0, \quad \text{for all } t \in \mathbb{C}^\infty$$

and

$$\begin{aligned}P(t)\,\Psi(t,z) &= S\left(\sum_1^\infty \frac{n}{p} t_n D^{n-1} D^{1-p} + c_{-p} D^{-p}\right) e^{\sum_1^\infty t_i z^i} \\ &= S\left(z^{1-p}\sum_1^\infty \frac{n}{p} t_n D^{n-1} + c_{-p} z^{-p}\right) e^{\sum_1^\infty t_i z^i} \\ &= \frac{1}{p}\left(z^{1-p}\frac{\partial}{\partial z} + c_{-p} z^{-p}\right)\Psi \equiv A_p\,\Psi(t,\lambda) \in V^0\end{aligned} \tag{2.10}$$

for all t, using (1.7).

Therefore, since $V^0 = \operatorname{span}\{\Psi(t,z) \text{ all } t \in \mathbb{C}^\infty\}$, the conditions (2.9) translate into t-independent conditions,

$$z^p V^0 \subset V^0 \quad \text{and} \quad A_p V^0 \subset V^0, \quad \text{with } [A_p, z^p] = 1\,. \tag{2.11}$$

We now prove a point, left open in the proof of Theorem 1, namely that $c_{-p} = (1-p)/2p$. The proof given below is based on calculations of [A] and [Schw], but is more straightforward. Consider the related pair of maps

$$\mathscr{A}_0\colon\ D^j e^{\sum_1^\infty t_i z^i} \curvearrowright D^j (S^{-1} P S)\, e^{\sum_1^\infty t_i z^i}$$

$$= D^j \left(\frac{1}{p} \sum_{\substack{2 \\ n \neq kp}}^{\infty} n t_n D^{n-p} + \frac{x}{p} D^{1-p} + c_{-p} D^{-p} \right) e^{\sum_1^\infty t_i z^i}$$

$$= \left(\frac{1}{p} \sum_{\substack{2 \\ n \neq kp}}^{\infty} n t_n D^{n-p+j} + \frac{x}{p} D^{1-p+j} + \frac{j}{p} D^{-p+j} + c_{-p} D^{-p+j} \right) e^{\sum_1^\infty t_i z^i}$$

using $D^j \cdot x = x D^j + j D^{j-1}$

$$= \left(\frac{j}{p} z^{-p} + c_{-p} z^{-p} \right) D^j e^{\sum_1^\infty t_i z^i} + \left\{ \begin{array}{l} \text{a linear combination of} \\ D^k e^{\sum_1^\infty t_i z^i},\ k \neq j \\ \text{with holomorphic coefficients in } t \end{array} \right\}$$

$$= z^{-p} \left(\frac{j}{p} + c_{-p} \right) D^j e^{\sum_1^\infty t_i z^i} + \left\{ \begin{array}{l} \text{a linear combination of} \\ D^k e^{\sum_1^\infty t_i z^i},\ 0 \leqq k \leqq p-1,\ k \neq j, \\ \text{with holomorphic coefficients in } t \\ \text{which are Laurent in } z^p \end{array} \right\}$$

and

$$\begin{aligned} \mathscr{A} = D^j \Psi \curvearrowright D^j P \Psi &= D^j A_p \Psi \\ &= A_p D^j \Psi \\ &= \left\{ \begin{array}{l} \text{a linear combination of} \\ \Psi,\ D\Psi,\ D^2\Psi, \ldots, \text{ with} \\ \text{holomorphic coefficients in } t \end{array} \right\}, \text{ since } A V^0 \subset V^0 \\ &= \left\{ \begin{array}{l} \text{a linear combination of} \\ \Psi,\ D\Psi,\ D^2\Psi, \ldots, D^{p-1}\Psi \\ z^p\Psi,\ z^p D\Psi, \ldots, D^{p-1}\Psi \\ z^{2p}\Psi, \ldots, \text{ with} \\ \text{holomorphic coefficients in } t \end{array} \right\}, \text{ since } z^p V^0 \subset V^0 \\ &= \left\{ \begin{array}{l} \text{a linear combination of} \\ \Psi,\ D\Psi, \ldots, D^{p-1}\Psi, \text{ with} \\ \text{coefficients polynomial in } z^p \\ \text{and holomorphic in } t \end{array} \right\}. \end{aligned}$$

Therefore $\mathscr{A}_0$ is represented by a matrix of the form

$$\begin{bmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{bmatrix} + \begin{bmatrix} c_{-p} & & 0 \\ & \frac{1}{p}+c_{-p} & \\ & & \ddots \\ 0 & & \frac{p-1}{p}+c_{-p} \end{bmatrix} z^{-p}$$

and $\mathscr{A}$ by a matrix holomorphic in t and polynomial in z^p. These two maps intertwine; the following diagram commutes:

$$\begin{array}{ccc} D^j e^{\sum_1^\infty t_i z^i} & \xrightarrow[U]{D^j S D^{-j}} & D^j S e^{\sum t_i z^i} = D^j \Psi \\ \Big\downarrow \mathscr{A}_0 & & \Big\downarrow \mathscr{A} \\ D^j (S^{-1} P S) e^{\sum_1^\infty t_i z^i} & \xrightarrow[U]{D^j S D^{-j}} & D^j P S e^{\sum_1^\infty t_i z^i} = A_p D^j \Psi \end{array}$$

and thus

$$\mathscr{A}_0 = U^{-1} \mathscr{A} U \quad \text{and} \quad Tr\, \mathscr{A}_0 = Tr\, \mathscr{A}\,.$$

Setting $y = z^p$, we have

$$\text{Res}_{y=\infty}\, Tr\, \mathscr{A}_0 = \frac{p(p-1)}{2p} + p c_{-p}$$

and

$$\text{Res}_{y=\infty}\, Tr\, \mathscr{A} = 0\,;$$

by the equality of the above traces, we have

$$\frac{p(p-1)}{2p} + p c_{-p} = 0\,,$$

confirming that $c_{-p} = (1-p)/2p$.

Proof of Corollary 1.2. To prove (0.8), compute

$$\begin{aligned} 1 = [L, P] &= \frac{1}{p}\left[L, \left(S\left(\sum_1^\infty k t_k D^{k-p}\right) S^{-1}\right)_+\right] \\ &= \frac{1}{p}\left[L, \left(S\left(\sum_{k \geqq p} k t_k D^{k-p}\right) S^{-1}\right)_+\right] \\ &= \frac{1}{p}\left[L, \sum_{k \geqq p} k t_k \left(L^{\frac{k-p}{p}}\right)_+\right] \\ &= \frac{1}{p} \sum_{k \geqq p+1} k t_k \left[L, \left(L^{\frac{k-p}{p}}\right)_+\right] \\ &= -\frac{1}{p} \sum_{k \geqq p+1} k t_k \frac{\partial L}{\partial t_{k-p}}\,. \end{aligned}$$

For $p = 2$, setting

$$L = S(t)\, D^2\, S(t)^{-1} = D^2 + 2(\log\tau)''$$

in the previous expression, one finds

$$-1 = \sum_{k=3,5,\dots} k t_k \frac{\partial}{\partial t_{k-2}} (\log\tau)''$$

$$= \left(\sum_{k=3,5,\dots} k t_k \frac{\partial \tau}{\partial t_{k-2}} \frac{1}{\tau} \right)''$$

leading to (0.9) upon integration.

3. Vertex Operators, the Fay Identity, $\mathscr{W}$-Algebras and the Spectral Theory for the Second Symplectic Structure

Given an arbitrary, but fixed parameter μ, the *Bäcklund-Darboux* transformation[4]

$$\Psi(t,z) = e^{\sum t_i z^i} \frac{\tau(t - [z^{-1}])}{\tau(t)} \curvearrowright \Psi_1(t,z) \equiv z^{-1} \frac{\{\Psi(t,z),\, \Psi(t,\mu)\}}{\Psi(t,\mu)}$$

$$= e^{\sum t_i z^i} \frac{\tau_1(t - [z^{-1}])}{\tau_1(t)}$$

transforms a wave function Ψ into a new wave function Ψ_1 and a τ-function into a new one

$$\tau(t) \curvearrowright X(t,\mu)\, \tau(t) = \tau_1(t) = e^{\sum_1^\infty t_i \mu^i} \tau(t - [\mu^{-1}]). \qquad (3.1)$$

In the Grassmannian picture (1.14), the transformation $\Psi \curvearrowright \Psi_1$ induces a transformation in Gr: (for precise statements and generalizations, see for instance [A-vM])

$$V^t \in \mathrm{Gr} \curvearrowright V_1^t \in \mathrm{Gr} \quad \text{such that} \quad z V_1^t \subset V^t. \qquad (3.2)$$

It is natural to consider the "inverse" $\tilde{X}(t,\lambda)$,

$$\tau_1 \curvearrowright \tilde{X}(t,\lambda)\, \tau_1 = e^{-\sum_1^\infty t_i \lambda^i} \tau_1(t + [\lambda^{-1}]); \qquad (3.3)$$

in the Grassmannian picture

$$V_1^t \in \mathrm{Gr} \curvearrowright \tilde{V}^t \in \mathrm{Gr} \quad \text{such that} \quad z V_1^t \subset \tilde{V}^t. \qquad (3.4)$$

It is not quite an inverse, since the following expression has a singularity, when $\lambda \to \mu$; indeed, using (0.10)

$$\tilde{X}(t,\lambda)\, X(t,\mu)\tau = \frac{\lambda}{\lambda - \mu} e^{\sum_1^\infty t_i(\mu^i - \lambda^i)} \tau(t + [\lambda^{-1}] - [\mu^{-1}])$$

$$\text{using } \exp\left(-\sum_1^\infty \frac{1}{i} \left(\frac{\mu}{\lambda}\right)^i \right) = 1 - \frac{\mu}{\lambda}$$

$$= \frac{\lambda}{\lambda - \mu} e^{\sum_1^\infty t_i(\mu^i - \lambda^i)} e^{\sum_1^\infty (\lambda^{-i} - \mu^{-i}) \frac{1}{i} \frac{\partial}{\partial t_i}} \tau(t)$$

$$\equiv \frac{\lambda}{\lambda - \mu} X(t,\lambda,\mu)\tau$$

$$= \frac{\lambda}{\lambda - \mu} \sum_{k=0}^\infty \frac{(\mu - \lambda)^k}{k!} \left(\sum_{l=-\infty}^\infty \lambda^{-l-k} W_l^{(k)}(\tau) \right), \qquad (3.5)$$

---

[4] $\{a,b\} = \frac{\partial a}{\partial x} b - a \frac{\partial b}{\partial x}$

where the expressions $W_n^{(\nu)}$ form the generators of a so-called $\mathcal{W}_{1+\infty}$-algebra, i.e. the commutators of two such generators is a (non-linear) polynomial of the generators. Here are a few generators:

$$W_n^{(1)} = J_n^{(1)} = \frac{\partial}{\partial t_n} + (-n)\,t_{-n}, \quad t_{-n} = 0 \quad \text{for } n > 0,$$
$$W_n^{(2)} = J_n^{(2)} - (n+1)\,J_n^{(1)},$$
$$W_n^{(3)} = J_n^{(3)} - \frac{3}{2}(n+2)\,J_n^{(2)} + (n+1)(n+2)\,J_n^{(1)},$$
$$W_n^{(4)} = J_n^{(4)} - 2(n+3)\,J_n^{(3)} + (2n^2+9n+11)\,J_n^{(2)} - (n+1)(n+2)(n+3)\,J_n^{(1)}, \ldots \tag{3.6}$$

with[5] (see also the appendix for explizit formulae)

$$J_n^{(2)} \equiv \sum_{i+j=n} :J_i^{(1)} J_j^{(1)}:, \quad J_n^{(3)} = \sum_{i+j+k=n} :J_i^{(1)} J_j^{(1)} J_k^{(1)}:$$
$$J_n^{(4)} = \sum_{i+j+k+l=n} :J_i^{(1)} J_j^{(1)} J_k^{(1)} J_l^{(1)}: - \sum_{i+j=n} :(iJ_i^{(1)})(jJ_j^{(1)}):, \text{ etc.} \ldots \tag{3.7}$$

In the Grassmannian picture, we have the following inclusions, using (3.2) and (3.4)

$$\begin{array}{lll} V^t \supset z V_1^t \subset & & \tilde{V}^t \\ \updownarrow \quad\quad \updownarrow & & \updownarrow \\ \tau(t) \curvearrowright \tau_1 = X(t,\lambda)\tau & \curvearrowright & \tilde{\tau} = X(t,\lambda,\mu)\tau \equiv e^{\sum_1^\infty t_i(\mu^i - \lambda^i)}\,\tau(t + [\lambda^{-1}] - [\mu^{-1}]). \end{array}$$

Consider now the generating functions (the stress-energy tensors)

$$W_\lambda^{(\nu)} = \sum_{n=-\infty}^{\infty} \lambda^{-n-\nu}\, W_n^{(\nu)} \quad \text{and} \quad J_\lambda^{(\nu)} = \sum_{n=-\infty}^{\infty} \lambda^{-n-\nu}\, J_n^{(\nu)}. \tag{3.8}$$

We now have the following relations, essentially a reformulation of the Fay identity.

Lemma 3.1 *(Fay identity). In the general KP-context, the wave function $\Psi(t,\lambda)$ and the adjoint wave function $\Psi^*(t,\mu)$ satisfy*

$$\Psi^*(t,\lambda)\,\psi(t,\mu) = \frac{1}{\mu-\lambda} D \frac{X(t,\lambda,\mu)\,\tau(t)}{\tau(t)}, \tag{3.9}$$

and thus

$$\nu\Psi^*(t,\lambda)\left(\frac{d}{d\lambda}\right)^{\nu-1}\psi(t,\lambda) = D\left(\frac{1}{\tau}\sum_{n=-\infty}^{\infty}\lambda^{-n-\nu}\; W_n^{\nu}(\tau)\right) = D\left(\frac{W_\lambda^{\nu}(\tau)}{\tau}\right). \tag{3.10}$$

Proof. Differentiating the Fay identity for τ-functions

$$\sum_{\substack{\text{cyclic}\\ \text{permutations of } 1,2,3}} (s_0 - s_1)(s_2 - s_3)\,\tau(t + [s_0] + [s_1])\,\tau(t + [s_2] + [s_3]) = 0$$

---
[5] $::$ means normal ordering, i.e., pull the differentiation to the right

with regard to s_0, then setting $s_0 = s_3 = 0$, dividing by $s_1 s_2$, and shifting t by $t \curvearrowright t - [s_2]$, lead to the differential Fay identity

$$\{\tau(t), \tau(t+[s_1]-[s_2]\} + (s_1^{-1} - s_2^{-1})(\tau(t+[s_1]-[s_2])\,\tau(t) - \tau(t+[s_1])\,\tau(t-[s_2])) = 0; \tag{3.11}$$

see Mumford [Mu] and [A-vM]. This relation (3.11) with $\lambda = s_1^{-1}$ and $\mu = s_2^{-1}$, multiplied with $\exp \sum_1^\infty t_i(\mu^i - \lambda^i)$ leads to equality $(*)$ below; we thus have

$$\begin{aligned}
\Psi^*(t,\lambda)\,\Psi(t,\mu) &= e^{-\sum t_i\lambda^i}\frac{\tau(t+[\lambda^{-1}])}{\tau(t)}\, e^{\sum t_i\mu^i}\frac{\tau(t-[\mu^{-1}])}{\tau(t)} \\
&\overset{*}{=} \frac{1}{\mu-\lambda} D\left(e^{\sum t_i(\mu^i-\lambda^i)}\frac{\tau(t+[\lambda^{-1}]-[\mu^{-1}])}{\tau(t)}\right) \\
&= \frac{1}{\mu-\lambda} D \frac{X(t,\lambda,\mu)\,\tau(t)}{\tau(t)} \\
&= \sum_{j=1}^{\infty}\frac{(\mu-\lambda)^{j-1}}{j!} D\left(\frac{1}{\tau}\sum_{n=-\infty}^{\infty}\lambda^{-n-j}W_n^{(j)}(\tau)\right) \\
&= \sum_{j=1}^{\infty}\frac{(\mu-\lambda)^{j-1}}{j!} D\frac{W_\lambda^{(j)}(\tau)}{\tau}.
\end{aligned}$$

Differentiating this relation with regard to μ and setting $\mu = \lambda$ leads to (3.10), ending the proof of Lemma 3.1.

Remark. It was pointed out to us by A. Radul that the Fay trisecant identity has already appeared in the context of quantum field theory; see for instance A. K. Raina [Rai].

Lemma 3.2. *For the p-reduced Gel'fand-Dickey equations*

$$(M^n L^{m+n/p})_{-i-1} = \operatorname{Res}_\lambda\left(\lambda^{mp+n}\Psi^*(t,\lambda)\,D^i\left(\frac{\partial}{\partial\lambda}\right)^n \Psi(t,\lambda)\right)$$
$$n = 0,1,2,3,\dots \quad m = -1,0,1,\dots \quad i = 0,1,2,\dots, \tag{3.13}$$

and in particular

$$\left(M^{\nu-1}L^{j+\frac{\nu-1}{p}}\right)_{-1} = \frac{1}{\nu} D\frac{W_{jp}^{(\nu)}(\tau)}{\tau},$$
$$\nu = 1,2,\dots, \quad j = -1,0,1,\dots. \tag{3.14}$$

Proof. Equation (1.10) applied to $Q = L^{1/p}$ leads to (3.13); in particular

$$\left(M^{\nu-1}L^{j+\frac{\nu-1}{p}}\right)_{-1} = \operatorname{Res}_\lambda\left(\lambda^{jp+\nu-1}\Psi^*(t,\lambda)\left(\frac{\partial}{\partial\lambda}\right)^{\nu-1}\Psi(t,\lambda)\right),$$

which by Lemma 3.1 leads to (3.14), ending the proof of Lemma 3.2.

For $p = 2$, the Gelfand-Dickey equations reduce to the KdV equation

$$\frac{\partial q}{\partial t_3} = Kq = \frac{1}{4}(q''' + 6qq' \quad (' = \partial(\partial x))$$

where

$$L = Q^2 = D^2 + q, \quad q = 2(\log\tau)''$$

$$K = \frac{1}{4}(D^3 + 2(qD + Dq)).$$

As is well-known, it has two compatible symplectic structures K and D (see [MM]). We now have

Lemma 3.3. *(Spectral theory for $K - z^2 D$). In the KdV case* ($p = 2$), the *wave functions $\Psi(t,z)$ and $\Psi^*(t,z)$ defined in* (1.4) *satisfy the following formulas*

(i) $\{\Psi^*, \Psi\} = -2z$,
(ii) $(K - z^2 D)\Psi^*\Psi = 0$,
(iii) $(K - z^2 D)\Psi^* \dfrac{\partial\Psi}{\partial z} = -z^2 + zD\Psi^*\Psi$.

Proof. Substituting

$$t \curvearrowright t - [s_1],\ s_1 \curvearrowright -z^{-1} \quad \text{and} \quad s_2 \curvearrowright z^{-1},$$

into the differential Fay identity (3.11) leads to (3.15)

$$\begin{aligned}&\{\tau(t - [-z^{-1}]), \tau(t - [z^{-1}])\}\\ &\quad - 2z(\tau(t - [z^{-1}])\tau(t\ [-z^{-1}]) - \tau(t)\tau(t - [-z^{-1}] - [z^{-1}])) = 0.\end{aligned}$$

Since in the KdV ($p = 2$) case $\tau(t) = \tau(t_1, t_3, t_5, \ldots)$ does not depend on $t_2, t_4, \ldots$, we have $\tau(t - [-z^{-1}] - [z^{-1}]) = \tau(t + [z^{-1}] - [z^{-1}]) = \tau(t)$ and $\tau(t - [-z^{-1}]) = \tau(t + [z^{-1}])$. Using $\{e^{-xz}a, e^{xz}b\} = \{a,b\} - 2zab$ and $\{a/e, b/e\} = \{a,b\}/e^2$, one computes

$$\begin{aligned}\{\Psi^*, \Psi\} &= \left\{e^{-xz}\frac{\tau(t + [z^{-1}])}{\tau(t)},\ e^{xz}\frac{\tau(t - [z^{-1}])}{t(t)}\right\}\\ &= \frac{1}{\tau(t)^2}(\{\tau(t + [z^{-1}]), \tau(t - [z^{-1}])\} - 2z\tau(t + [z^{-1}])\tau(t - [z^{-1}]))\\ &= -2z \quad \text{using (3.15)},\end{aligned}$$

which establishes (i).

Using the eigenrelations

$$(L - \lambda^2)\Psi^*(t,\lambda) = 0 \quad \text{and} \quad (L - \mu^2)\Psi(t,\mu) = 0$$

we compute

$$\begin{aligned}4K(\Psi^*(t,\lambda)\Psi(t,\mu)) &= (\lambda^2 + 3\mu^2)\Psi^*(t,\lambda)'\Psi(t,\mu)\\ &\quad + (\mu^2 + 3\lambda^2)\Psi^*(t,\lambda)\Psi(t,\mu)'.\end{aligned} \tag{3.16}$$

Setting $\lambda = \mu = z$ leads at once to (ii). Then taking the μ-derivative of (3.16) and setting $\lambda = \mu = z$ yield

$$
\begin{aligned}
(K - z^2 D)\,\Psi^*(t,z)\frac{\partial}{\partial z}\Psi(t,z) &= \\
&= \frac{3}{2} z(\Psi^*(t,z)\Psi(t,z))' - z(\Psi^*(t,z)\Psi(t,z)') \\
&= \frac{3}{2} z(\Psi^*(t,z)\Psi(t,z))' - z\left(z + \frac{1}{2}(\Psi^*(t,z)\Psi(t,z))'\right) \quad \text{using (i)} \\
&= -z^2 + z(\Psi^*(t,z)\Psi(t,z))',
\end{aligned}
$$

which establishes (iii), ending the proof of Lemma 3.3.

Having considered the generators of the $\mathcal{W}_{1+\infty}$-algebra, recall from the introduction the definition of

$$\mathcal{W}_p = \{W_{np}^{(j)},\ 1 \leqq j \leqq p,\ n \in \mathbb{Z},\ t_p = t_{2p} = \ldots = 0\}; \tag{3.17}$$

correspondingly define the $\mathcal{W}_p$-stress energy tensors (in terms of $y = z^p$)

$$T_p^{(j)}(y) = \sum_{n \in \mathbb{Z}} J_{np}^{(j)} y^{-n-j} \qquad 1 \leqq j \leqq p, \text{ with } t_{ip} = 0, \text{ all } i \geqq 1 \tag{3.18}$$

and the (truncated) $\mathcal{W}_p^{\,+}$-stress energy tensors (meromorphic part of $T_p^{(j)}(z)$)

$$\bar{T}_p^{(j)}(y) = \sum_{n \geqq -j+1} J_{np}^{(j)} y^{-n-j} \qquad 1 \leqq j \leqq p, \text{ with } t_{ip} = 0, \text{ all } i \geqq 1. \tag{3.19}$$

Then $T_p^{(j)}(y)$ can also be expressed in terms of so-called $p-1$ free bosons $\varphi_l^{(p)}$ $(l = 1, 2, \ldots, p-1)$, defined by

$$\frac{\partial \varphi_l^{(p)}}{\partial y} = \frac{1}{\sqrt{p}} \sum_{r=-\infty}^{\infty} J_{-l+rp}^{(1)} y^{-\frac{(-l+rp)}{p} - 1}, \tag{3.20}$$

as illustrated in the examples below.

Example 1. For each p, the operators

$$L_n = \frac{1}{2} J_{np}^{(2)} \qquad (\text{with } t_{ip} = 0,\ i \geqq 1) \tag{3.21}$$

are the generators of the Virasoro algebra, namely

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{1}{12}(n^3 - n)\,\delta_{n+m}. \tag{3.22}$$

In particular (see F-K-N1)

$$T_p^{(2)}(y) = p \sum_{i=1}^{p-1} : \frac{\partial \varphi_l^{(p)}}{\partial y} \frac{\partial \varphi_{p-l}^{(p)}}{\partial y} : + \frac{p^2 - 1}{6} \frac{1}{y^2} \tag{3.23}$$

and

$$T_p^{(3)}(y) = 6p^{3/2} \sum_{\substack{1 \leqq l_1, l_2, l_3 \leqq p-1 \\ l_1 + l_2 + l_3 = 0 \,(\mathrm{mod}\, p)}} : \frac{\partial \varphi_{l_1}}{\partial y} \frac{\partial \varphi_{l_2}}{\partial y} \frac{\partial \varphi_{l_3}}{\partial y} : . \tag{3.24}$$

Example 2. For $p = 3$, the $L_n = \frac{1}{2} J_{3n}^{(2)}$ and $W_n = J_{3n}^{(3)}$ are the generators of the $\mathscr{W}_3$-algebra with relations

$$\begin{aligned}[L_n, L_m] &= (n-m) L_{n+m} + \frac{1}{12}(n^3 - n)\,\delta_{n+m},\\ \left[\frac{1}{3} L_n, W_m\right] &= (2n-m)\, W_{n+m},\\ [W_n, W_m] &= \text{quadratic functions of } L_k \text{ and } W_k.\end{aligned} \tag{3.25}$$

As pointed out in the introduction, stress-energy tensors seem to also arise naturally in the context of the two (or more) compatible symplectic structures of the Gel'fand-Dickey equations, as we illustrate here for the KdV equation ($p = 2$), where

$$\frac{\partial q}{\partial t_3} = Kq = \frac{1}{4}(q''' \;\; 6qq'),$$

with two symplectic structures D and K, where

$$L = Q^2 = D^2 + q, \qquad q = 2(\log\tau)''$$

$$K \equiv \frac{1}{4}(D^3 + 2(qD + Dq)).$$

Proposition 3.4. *In the KdV case* ($p = 2$), *we have the following relations*

(i) $(K - z^2 D) D \sum_{k=-\infty}^{\infty} \frac{J_{2k-1}^{(1)}(\tau)}{\tau} z^{-2k} = 0.$

(ii) $(K - z^2 D) D \sum_{k=-\infty}^{\infty} \frac{J_{2k}^{(2)}(\tau)}{\tau} z^{-2k-2} = -2z^2$

or what is the same

$$(K - yD) D \frac{T_2^{(2)}(y)\tau}{\tau r} = -2 \qquad (y = z^2).$$

(iii) *recurrence relation*

(*a*) $KD \frac{J_{2n-1}^{(1)}(\tau)}{\tau} - D^2 \frac{J_{2n+1}^{(1)}(\tau)}{\tau} = 0 \qquad n = 0, 1, 2, \ldots$ (*Lenard relation*)

(*b*) $KD \frac{J_{2n-2}^{(2)}(\tau)}{\tau} - D^2 \frac{J_{2n}^{(2)}(\tau)}{\tau} = 0$ *for all* $n \in \mathbb{Z}$, $n \neq -1$

$= -2$ *for*[6] $n = -1$.

Corollary. *If* τ *satisfies the KdV equation and* $J_{-2}^{(2)}(\tau) = 0$ (i.e., $L_{-1}\tau = 0$), *then* $J_{2n}^{(2)}(\tau) = 0$ *for all* $n \geqq -1$ (i.e., $L_n \tau = 0$ *for all* $n \geqq -1$).

---

[6] for $n = -1$, it can also be written

$$KD \frac{J_{-4}^{(2)}(\tau)}{\tau} = D^2 \frac{(J_{-2}^{(2)} - x^2)\tau}{\tau}$$

Proof of Proposition 3.4. From Lemma 3.3 (ii), we have $(K - \lambda^2 D)\Psi^* \Psi = 0$ with

$$\Psi^* \Psi = \sum_{n=-\infty}^{\infty} \lambda^{-n-1} D \frac{W_n^{(1)}(\tau)}{\tau} = \sum_{n \text{ odd}} \lambda^{-n-1} D \frac{J_n^{(1)}(\tau)}{\tau}, \tag{3.26}$$

since τ is independent of $t_2, t_4, t_6, \ldots,$

leading to (i) and (iii, a) by identifying powers of λ. Then using again (3.10) for $v = 2$, relation (3.6), and the fact that $J_n^{(2)}(\tau)$ identically vanishes for odd n

$$2\Psi^* \frac{\partial \Psi}{\partial \lambda} = \sum_{n \text{ even}} \lambda^{+n-2} D \frac{J_n^{(2)}(\tau)}{\tau} - \sum_{n \text{ odd}} (n+1)\lambda^{-n-2} D \frac{J_n^{(1)}(\tau)}{\tau};$$

using (i), (iii, a) and (3.26), one computes

$$(K - \lambda^2 D) \sum_{n \text{ odd}} -(n+1)\lambda^{-n-2} D \frac{J_n^{(1)}(\tau)}{\tau} = 2 \sum_{n \text{ odd}} \lambda^{-n} D^2 \frac{J_n^{(1)}(\tau)}{\tau} = 2\lambda D \Psi^* \Psi.$$

Using this information, we have

$$\begin{aligned} -2\lambda^2 &= (K - \lambda^2 D) 2\Psi^* \frac{\partial \Psi}{\partial \lambda} - 2\lambda D \Psi^* \Psi \\ &= (K - \lambda^2 D) \sum_{n \text{ even}} \lambda^{-n-2} D \frac{J_n^{(2)}(\tau)}{\tau} \end{aligned}$$

establishing (ii) and thus also (iii, b).

Proof of Corollary. By relation (iii, a) for $n \geqq 0$, we have that $J_{-2}^{(2)}(\tau) = 0$ implies inductively $D^2 J_{2k}^{(2)}(\tau)/\tau = 0$ and so $J_{2k}^{(2)}(\tau) = 0$.

Remark 0. [DVV] have considered relations of the type (ii) for solutions τ of the KdV and string equations. Proposition 3.4 shows that such relations hold for general solutions of KdV, regardless of the string equation.

Remark 1. Recurrence relation (iii, a) is nothing but the by now classic Lenard relation

$$KD \frac{\partial \log \tau}{\partial t_{2n-1}} = D^2 \frac{\partial \log \tau}{\partial t_{2n+1}} \qquad (n \geqq 1).$$

Remark 2. Relations (iii, b) for $n \leqq -1$ turn out to be reducible to (iii, a). For instance for $n = -1$, relation (iii, b) can be written

$$\begin{aligned} KD &\frac{J_{-4}^{(2)}(\tau)}{\tau} - D^2 \frac{(J_{-2}^{(2)} - x^2)\tau}{\tau} \\ &= KD\left(2 \sum_{k=5,7,\ldots} k t_k \frac{\partial}{\partial t_{k-4}} \log \tau + 6 t_3\right) - D^2 \left(2 \sum_{k=3,5,\ldots} k t_k \frac{\partial}{\partial t_{k-2}} \log \tau\right) \\ &= \left(\sum_{k=5,7,\ldots} k t_k \frac{\partial}{\partial t_{k-2}} 2D^2 \log \tau + 3 t_3 \frac{\partial}{\partial t_1} 2D^2 \log \tau\right) \\ &\quad - \sum_{k=3,5,\ldots} k t_k \frac{\partial}{\partial t_{k-2}} 2D^2 \log \tau \\ &= 0, \end{aligned}$$

using $KDt_3 = q' t_3/2$ and (iii, a).

Remark 3. In Magri's theory (see [MM] and [McK]), integrability implies double eigenvalues for the Nyenhuis tensor $D^{-1}K$. How is the observation related to Proposition 3.4? Along a different vein, in a beautiful computation, Kirillov [Ki] has shown that changing variable x in $D^2 + q(x)$ by means of a diffeomorphism $x \curvearrowright s(x)$, leads to a new operator $D^2 + \tilde{q}(x)$ (after an appropriate "conjugation"), where $\tilde{q}(x)$ contains a Schwarzian derivative:

$$\tilde{q}(x) = s'(x)^2 q(s(x)) + \frac{1}{2}\left(\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2\right).$$

The infinitesimal deformation of this operation, thus belonging to the Virasoro algebra, leads at once to the second symplectic structure of KdV; this has been generalized for arbitrary p by [FIZ]. Another connection between $\mathcal{W}$-algebras and symplectic structures comes up as follows: the two symplectic structures yield two different Poisson brackets between the various functions $q_2(t), \ldots, q_p(t)$ of the differential operator L (fact first observed in the KdV case by Gervais [Ge]). Then expanding these functions into Fourier series and expressing the second Hamiltonian structure in terms of its Fourier coefficients lead to brackets between these Fourier coefficients; they exactly generate the $\mathcal{W}_p$-algebra. Consult for instance A. O. Radul [R]. The connection between these different points of view remains obscure.

4. Proof of Theorem 2

Step 1. If τ satisfies the p-reduced Gel'fand-Dickey and the string equations, then τ is a null-vector (vacuum-vector) for $\mathcal{W}_p{}^+$, which upon bracketing reads

$$\mathcal{W}_p{}^+ = \{J_{np}^{(\nu)} \quad 1 \leqq \nu \leqq p, \; n = -\nu + 1, \; -\nu + 2, \ldots\}. \tag{4.1}$$

Indeed if τ is a solution of $\partial L/\partial t_k = [(L^{k/p})_+, L]$ and $[L, P] = 1$, then according to Theorem 1 and Lemma 3.2 (in that order),

$$0 = \left(M^{\nu-1} L^{j+\frac{\nu-1}{p}}\right)_{-1} = \frac{1}{\nu} D \frac{W_{jp}^{(\nu)}(\tau)}{\tau} \quad \text{for } \nu = 1, 2, \ldots \text{ and } j = -1, 0, 1, \ldots,$$

implying

$$W_{jp}^{(\nu)}(\tau) = c\tau, \qquad c \in \mathbb{C}.$$

Since $\mathcal{W}_p{}^+$ has no central term, every element of $\mathcal{W}_p{}^+$ can be written as a commutator (see Lemma 4.2 of [FKN2]) of two elements of $\mathcal{W}_p{}^+$, implying the constant $c = 0$, and thus by (3.6),

$$J_{jp}^{(\nu)}(\tau) = 0 \qquad \text{for } \nu = 1, 2, \ldots, \; j = -1, 0, 1, \ldots,$$

which for $\nu = 1$, implies $\partial\tau/\partial t_{kp} = 0$; so we may set $t_{kp} = 0$ for $k = 1, 2, \ldots$.

That $\mathcal{W}_p{}^+$ is spanned by the generators in (4.1) is obtained by repeatedly bracketing $J_{-p}^{(\nu)}$ with $J_{-p}^{(2)}$, yielding $J_{(-\nu+1)p}^{(\nu)}$; for instance, from (3.23) we have

$$[\tfrac{1}{6} J_{-p}^{(2)}, J_{mp}^{(3)}] = (-2 - m) J_{(m-1)p}^{(3)},$$

and so $J_{-2p}^{(3)}$ can be generated from the higher ones but not $J_{-3p}^{(3)}$,

$$[\tfrac{1}{6} J_{-p}^{(2)}, J_{-p}^{(3)}] = -J_{-2p}^{(3)}, \quad \text{whereas} \quad [\tfrac{1}{6} J_{-p}^{(2)}, J_{-2p}^{(3)}] = 0.$$

This ends the proof of Step 1.

Step 2. The solution τ to the p-reduced Gel'fand-Dickey and the string equation $[L, P] = 1$ exists.

According to (2.9) and (2.10), the linear space $V^0 \in \mathrm{Gr}$ is invariant under the action of the operators $L(t)$ and $P(t)$, which act as (multiplication by) z^p and A_p respectively, with $[A_p, z^p] = 1$. By modifying the time-origin with the shift $t_{p+1} \curvearrowright t_{p+1} + 1$, the new operators $\bar{L}(t)$ and $\bar{P}(t)$ thus obtained still satisfy

$$\bar{L}\Psi = z^p \Psi$$

and

$$\bar{P}\Psi = \bar{A}_p \Psi,$$

where

$$\bar{A}_p = z^{\frac{p-1}{2}} \frac{d}{dz^p} z^{-\frac{p-1}{2}} + \frac{p+1}{p} z,$$

and $[\bar{A}_p, z^p] = 1$; indeed the shift $t_{p+1} \curvearrowright t_{p+1} + 1$ produces the linear term in $\bar{A}_p$, as appears from (2.10). Since $\bar{A}_p^k \Psi(0, z)$ blows up like z^k for $z \nearrow \infty$ and since in the big stratum, it is possible to find a basis whose functions blow up as z^k $(k = 0, 1, 2, \ldots)$, we have

$$V^0 = \mathrm{span}\{\Psi(0,z), \bar{A}_p \Psi(0,z), \bar{A}_p^2 \Psi(0,z), \bar{A}_p^3 \Psi(0,z), \ldots\};$$

but since $z^p V^0 \subset V^0$, the function $\Psi(0, z)$ must satisfy

$$z^p \Psi(0,z) \sum_{i=0}^{p} \alpha_i A_p^i \Psi(0,z), \qquad \alpha_p \neq 0, \tag{4.4}$$

for some constants α_i. Therefore the existence of a τ-function solution to p-reduced Gel'fand-Dickey and string reduces to the existence of a formal plane $V^0 \in \mathrm{Gr}$ containing a function $\Psi(0, z) = 1 + \sum_1^\infty c_i z^{-i}$ satisfying (4.4) for some constants α_i. The above differential Eq. (4.4) for $\Psi(0, z)$ with $\alpha_i = 0$ $(1 \leqq i \leqq p - 1)$ reduces by means of elementary transformations to an equation (in φ) for which a solution exists, namely the higher Airy function

$$\frac{d^p \varphi}{dy^p} = y\varphi, \quad \text{with} \quad \varphi(y) = \int \exp\left(-\frac{x^{p+1}}{p+1} + xy\right) dx. \tag{4.5}$$

This ends the proof of Step 2.

Step 3. The vacuum vector τ of $\mathscr{W}_p$ is unique.

The generators $J_m^{(l)}$ of

$$\mathscr{W}_p = \{J_{(-v+r)p}^{v+1}, \ 0 \leqq v \leqq p-1, \ r = 0, 1, 2, \ldots\}$$

have the form

$$\begin{aligned} J_m^{(l)} = & \sum_{i_1 + \ldots + i_l = m} : J_{i_1}^{(1)} J_{i_2}^{(1)} \ldots J_{i_l}^{(1)} : \\ & + \sum_{k<l} \sum_{i_1 + \ldots + i_k = m} c_{i_1 \ldots i_k} : J_{i_1}^{(1)} J_{i_2}^{(1)} \ldots J_{i_k}^{(1)} : \end{aligned} \tag{4.6}$$

for some constants $c_{i_1 \dots i_k}$. Making the substitution $t_{p+1} \curvearrowright t_{p+1} + 1$,

$$\sum_{i_1+\dots+i_l=m} :J_{i_1}^{(1)} \dots J_{i_l}^{(1)}: = l(1+t_{p+1})^{l-1} \frac{\partial}{\partial t_{m+(l-1)(p+1)}} + \dots$$
$$= l \frac{\partial}{\partial t_{m+(l-1)(p+1)}}$$
$$+ \left(\begin{matrix} \text{non-linear terms} \\ \text{of the form } t_{\alpha_1}, \dots, t_{\alpha_m} \dfrac{\partial^s}{\partial t_{\beta_1} \dots \partial t_{\beta_s}} \end{matrix} \right)$$
$$+ \left(\begin{matrix} \text{higher order linear} \\ \text{differential operators} \end{matrix} \right),$$

and similarly for the second half of (4.6). Hence

$$J_{(-v+r)p}^{v+1} = (v+1) \frac{\partial}{\partial t_{v+rp}} + \sum_{v'<v} c_{v'} \frac{\partial}{\partial t_{v+rp-(v-v')p}} + \left(\begin{matrix} \text{non-linear terms} \\ \text{as above} \end{matrix} \right)$$
$$+ \left(\begin{matrix} \text{higher order linear} \\ \text{differential operators} \end{matrix} \right). \tag{4.7}$$

Thus possibly after taking linear combinations we find new generators of $\mathscr{W}_p^+$ of the form

$$H_i = \frac{\partial}{\partial t_i} + (\text{non-linear terms}) + \left(\begin{matrix} \text{higher order linear} \\ \text{differential operators} \end{matrix} \right)$$
$$i = 1, 2, \dots \quad \text{and} \quad \neq p, 2p, \dots$$

To prove uniqueness we must show $\tau(0) = 0$ implies $\tau \equiv 0$; that is, by Taylor, all partial derivatives of τ vanish at $t = 0$. Indeed, one shows inductively that all derivatives of τ with respect to $t_1, t_2, \dots, t_k$ (at $t = 0$) vanish as a consequence of $H_i \tau = 0$ for $1 \leqq i \leqq k$.

Step 4. To prove Theorem 2, we now proceed as follows: letting I and II be the two sets in (0.12). Step 1 implies at once the inclusion $I \subseteqq II$ in (0.12). According to Step 2, the space I of solutions is non-empty and according to Step 3, the space II contains exactly one function. Therefore $I = II$, ending the proof of Theorem 2.

5. An Explicit Solution of Gel'fand-Dickey and String (Theorem 3)

In showing $\tau_p^{(N)}(t)$ of (0.13) is a $\mathscr{W}_p$-vacuum vector, *a first step* consists of making the following substitution $X = Z - \Theta$ and $\Lambda = (-\Theta)^p$, yielding (remember $it_i = \operatorname{Tr}(-\Theta)^{-i} = \operatorname{Tr} \Lambda^{-i/p}$),

$$
\begin{aligned}
\tau_p^{(N)}(t) &= \frac{\tilde{A}_p^{(N)}(\Theta)}{\tilde{B}_p^{(N)}(\Theta)} \\
&= \frac{\int dZ \exp \operatorname{Tr}\left(\text{non-linear terms in } -\frac{(Z-\Theta)^{p+1}}{p+1}\right)}{\int dZ \exp \operatorname{Tr}\left(\text{quadratic terms in } -\frac{(Z-\Theta)^{p+1}}{p+1}\right)} \\
&= \frac{\int dZ \exp \operatorname{Tr} -\frac{1}{p+1}((Z-\Theta)^{p+1} + (-1)^{p+1}((p+1)\, Z\Theta^p - \Theta^{p+1}))}{\int dZ \exp\left(-\sum_{i,j} \frac{Z_{ij}Z_{ji}(\theta_i^p - \theta_j^p)}{2(\theta_i - \theta_j)}\right)} \\
&= \frac{\int dX \exp \operatorname{Tr} -\frac{1}{p+1}(X^{p+1} + (-1)^{p+1}((p+1)\,(X+\Theta)\,\Theta^p - \Theta^{p+1}))}{\text{constant} \prod_{i,j}^{N} \left(\frac{\theta_i^p - \theta_j^p}{\theta_i - \theta_j}\right)^{-1/2}} \\
&= \frac{\int dX \exp \operatorname{Tr}\left(-\frac{X^{p+1}}{p+1} + (-\Theta)^p X\right)}{\text{constant}\left(\prod_{ij=1}^{N} \frac{\theta_i^p - \theta_j^p}{\theta_i - \theta_j}\right)^{-1/2} \exp \operatorname{Tr} \frac{p}{p+1}(-\Theta)^{p+1}} \\
&= \text{constant} \frac{\int dX \exp \operatorname{Tr}\left(-\frac{X^{p+1}}{p+1} + X\Lambda\right)}{\left(\prod_{1 \leq i,j \leq N} \frac{\lambda_i^{1/p} - \lambda_l^{1/p}}{\lambda_i - \lambda_j}\right)^{1/2} \prod_{i=1}^{N} \exp \frac{p}{p+1} \lambda_i^{\frac{p+1}{p}}} \\
&\equiv \text{constant} \frac{A_p^{(N)}(\Lambda)}{B_p^{(N)}(\Lambda)}\,.
\end{aligned} \tag{5.1}
$$

In a *second step*, we exhibit a PDE for $A_p(\Lambda)$. To do this consider first

$$A = A_p(Y) \quad \text{with all entries of } Y = Y^\dagger \text{ non-zero}\,^7.$$

Then, since by integration by parts

$$\int dX \frac{\partial}{\partial X_{ij}} \exp \operatorname{Tr}\left(-\frac{X^{p+1}}{p+1} + XY\right) = 0$$

we have

$$\int dX(-(X^p)_{ji} + Y_{ji}) \exp \operatorname{Tr}\left(-\frac{X^{p+1}}{p+1} + XY\right) = 0,$$

and thus

$$Y_{ji}A - \sum_{r_2,\ldots,r_p} \frac{\partial^p A}{\partial Y_{ir_2} \partial Y_{r_2 r_3} \ldots \partial Y_{r_p j}} = 0. \tag{5.2}$$

[7] $Y^\dagger \equiv \bar{Y}^T$

But since $A(Y)$ is invariant under conjugation of Y, we have

$$A(Y) = A(UYU^\dagger) = A(\lambda),$$

where

$$Y = U^\dagger \lambda U, \quad U^\dagger U = I, \quad \lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N).$$

Then, differentiating the latter by Y_{ij}, leads to

$$\frac{\partial \lambda_\alpha}{\partial Y_{ij}} = U^\dagger_{j\alpha} U_{\alpha_i}. \tag{5.3}$$

We shall need quantities like

$$F_1(\alpha,\beta) = \sum_{i,j} U^\dagger_{i\beta} \frac{\partial \lambda_\alpha}{\partial Y_{ij}} U_{\beta j} = \delta_{\alpha\beta},$$

$$F_2(\alpha,\beta) = \sum_{i,j,k} U^\dagger_{i\beta} \frac{\partial^2 \lambda_\alpha}{\partial Y_{ij} \partial Y_{jk}} U_{\beta k} = \frac{1}{\lambda_\alpha - \lambda_\beta} \quad \text{if } \beta \neq \alpha$$

$$= -\sum_{\gamma \neq \beta} \frac{1}{\lambda_\gamma - \lambda_\beta} \quad \text{if } \beta = \alpha,$$

$$F_3(\alpha,\beta) = \sum_{i,j,k,l} U^\dagger_{i\beta} \frac{\partial^3 \lambda_\alpha}{\partial Y_{ij} \partial Y_{jk} \partial Y_{kl}} U_{\beta l} = -F_2(\alpha,\beta)^2 + 2F_2(\alpha,\alpha) F_2(\alpha,\beta) \text{ if } \beta \neq \alpha$$

$$= -\sum_{\gamma \neq \beta} F_3(\gamma,\beta) \quad \text{if } \beta = \alpha$$

$$F_2(\alpha,(\overset{\beta}{\gamma})) = \sum_{i,j,k,l} U^\dagger_{i\beta} U^\dagger_{k\gamma} \frac{\partial^2 \lambda_\alpha}{\partial Y_{ij} \partial Y_{jk}} U_{\gamma j} U_{\beta l} = F_2(\alpha,\beta) \quad \text{if } \alpha = \gamma,\ \beta \neq \alpha$$

$$= F_2(\alpha,\gamma) \quad \text{if } \alpha = \beta,\ \gamma \neq \alpha$$

$$= 0 \quad \text{otherwise}. \tag{5.4}$$

Then multiplying (5.2) to the left and to the right by $U^\dagger_{il}$ and U_{lj}, summing over i, j and using the chain rule

$$\frac{\partial A}{\partial Y_{ij}} = \sum_\alpha \frac{\partial A}{\partial \lambda_\alpha} \frac{\partial \lambda_\alpha}{\partial Y_{ij}},$$

$$\frac{\partial^2 A}{\partial Y_{ij} \partial Y_{jk}} = \sum_{\alpha,\beta} \frac{\partial \lambda_\alpha}{\partial Y_{ij}} \frac{\partial \lambda_\beta}{\partial Y_{jk}} \frac{\partial^2 A}{\partial \lambda_\alpha \partial \lambda_\beta} + \sum_\alpha \frac{\partial A}{\partial \lambda_\alpha} \frac{\partial^2 \lambda_\alpha}{\partial Y_{ij} \partial Y_{jk}}, \quad \text{etc.} \dots,$$

one finds the partial differential equations

$$\mathscr{P}_l^{(2)}(A) = \frac{\partial^2 A}{\partial \lambda_l^2} + \sum_{\alpha \neq l} F_2(\alpha,l) \left(\frac{\partial A}{\partial \lambda_\alpha} - \frac{\partial A}{\partial \lambda_l} \right) + \lambda_l A = 0, \qquad (l = 1, \dots, N) \tag{5.5}$$

$$\mathscr{P}_l^{(3)}(A) = \frac{\partial^3 A}{\partial \lambda_l^3} + \sum_{\alpha \neq l} F_2(\alpha,l) \left(\frac{\partial}{\partial \lambda_\alpha} - \frac{\partial}{\partial \lambda_l} \right) \left(\frac{\partial}{\partial \lambda_\alpha} + 2 \frac{\partial}{\partial \lambda_l} \right) A$$

$$+ \sum_{\alpha \neq l} F_3(\alpha,l) \left(\frac{\partial}{\partial \lambda_\alpha} - \frac{\partial}{\partial \lambda_l} \right) A + \lambda_l A = 0, \quad (l = 1, \dots, N) \tag{5.6}$$

etc. ..., with $F_2(\alpha, l)$, $F_3(\alpha, l)$, ... given by (5.4).

We now define

$$t_i \equiv \frac{1}{i} \sum_{j=1}^{N} \lambda_j^{-i/p} \qquad i = 1, 2, \ldots$$

which become independent time-variables when $N \nearrow \infty$. In Sect. 6 we show that $\tau_p^N(t)$ is indeed a function of t only. Now set $A_p(\lambda) = \tau_p^{(N)}(t)\, B_p(\lambda)$ in the partial differential equations above (5.5) and (5.6) and take the following derivatives (set $' = \partial/\partial\lambda_\alpha$):

$$\frac{A'}{B} = \tau' + \tau(\log B)', \quad \frac{A''}{B} = \tau'' + 2\tau'(\log B)' + \tau((\log B)'' - (\log B)'^2), \ldots,$$

and, using a symmetrization procedure,

$$\tau' = \sum_\alpha \frac{\partial \tau}{\partial t_\alpha} t'_\alpha, \quad \tau'' = \sum_{\alpha,\beta} \frac{\partial^2 \tau}{\partial t_\alpha \partial t_\beta} t'_\alpha t'_\beta + \sum_\alpha \frac{\partial \tau}{\partial t_\alpha} t'',$$

$$\tau''' = \sum_{\alpha,\beta,\gamma} \frac{\partial^3 \tau}{\partial t_\alpha \partial t_\beta \partial t_\gamma} t'_\alpha t'_\beta t'_\gamma + \frac{3}{2} \sum_{\alpha,\beta} \frac{\partial^2 \tau}{\partial t_\alpha \partial t_\beta} (t'_\alpha t'_\beta)' + \ldots .$$

Letting $N \nearrow \infty$, we find by means of a *not* straightforward calculation that

$$\frac{1}{B_2} \mathscr{P}_l^{(2)}(A_2) = \frac{1}{B_2} \mathscr{P}_l^{(2)}(B_2 \tau) = \bar{T}_2^{(2)}(y)\tau_2 |_{y=-\lambda_l}, \quad \text{for } p = 2, \tag{5.7}$$

$$\frac{1}{B_3} \mathscr{P}_l^{(3)}(A_3) = -\frac{1}{27} \bar{T}_3^{(3)}(y)\tau_3 - \frac{\sqrt{3}}{18}\left(\frac{\partial \varphi_1^{(3)}}{\partial y} + \frac{\partial \varphi_2^{(3)}}{\partial y}\right) \bar{T}_3^{(2)}(y)\tau_3$$
$$+ R^+(y)\tau_3 |_{y=\lambda_l}, \quad \text{for } p = 3, \tag{5.8}$$

where

$$R^+(y) = \frac{1}{9} \sum_{j \geqq 0} y^j \left(\sum_{n \geqq -1} t_{3(n+j+3)} J_{3n}^{(2)} \right)\Bigg|_{y=\lambda_l}, \qquad (p = 3),$$

where $\bar{T}_p^{(j)}(y) = \sum_{n \geqq -j+1} J_{np}^{(j)} y^{-n-j} (t_{ip} = 0,$ all $i \geqq 1)$ is the truncated stress-energy tensor associated with $\mathscr{W}_p^{\,+}$ and introduced in (3.18) and $\varphi_l^{(p)}$ the bosons introduced in (3.20).

Case. $p = 2$. When $N \nearrow \infty$, the λ_l move independently for fixed t_i and are thus indeterminates; therefore

$$\bar{T}_2^{(2)}(-\lambda_l)\tau_2 = \sum_{n \geqq -1} (-\lambda_l)^{-n-2} J_{2n}^{(2)}(\tau_2) = 0 \quad \text{implies} \quad J_{2n}^{(2)}(\tau_2) = 0$$

and so, τ_2 is a vacuum vector for the truncated Virasoro algebra ($p = 2$). This is a reinterpretation of an argument of Kontsevich [K2].

Case. $p = 3$. As before, for large N, λ_l plays the role of an indeterminate and all the coefficient of the various power in (5.8) must vanish. Since $R^+(y)\tau_3$ contains the only positive y-powers of (5.8), we have

$$\sum_{n \geqq -1} t_{3(n+j+3)} J_{3n}^{(2)}(\tau_3) = 0 \qquad \text{for } j \geqq 0.$$

Since t_{3k} does not appear in $J_{3n}^{(2)}(\tau_3)$, they are also indeterminates, and so all $J_{3n}^{(2)}(\tau_3) = 0$ for $n \geqq -1$, i.e. $\bar{T}_3^{(2)}(y)\tau_3 = 0$. Therefore again from (5.8)

$$0 = \bar{T}_3^{(3)}(y)\tau_3 = \sum_{n \geqq -2} y^{-n-3} J_{3n}^{(3)}(\tau_3)$$

yielding $J_{3n}^{(3)}(\tau_3) = 0$ for $n \geqq -2$. This shows τ_3 is a vacuum vector for the truncated algebra $\mathcal{W}_3^{+}$. Therefore also from Theorem 2, the function τ_3 is a solution of the Boussinesq and string equations. The proof for general p proceeds along similar lines.

Proof of Corollary 3.1. Defining with Witten [W2] the operator

$$\Delta_n = \sum_i \theta_i^n \frac{\partial}{\partial \theta_i},$$

one checks that

$$\Delta_{1+rp} t_k = (-1)^{rp}(k - rp)\, t_{k-rp} \qquad (k = 1, 2, \ldots)$$

and, using the explicit expression (5.1) for $\tilde{B}_p$, that

$$\Delta_{1-p}\tilde{B}_p = \frac{(-1)^{p-1}}{2} \tilde{B}_p \sum_{\substack{i+j=p \\ i,j \geqq 1}} i t_i j t_j .$$

On the one hand, we have using the two formulas above

$$\Delta_{1-p}(\tau_p \tilde{B}_p) = \frac{(-1)^{p-1}}{2} \tilde{B}_p \left(\sum_{-i-j=-p} i t_i j t_j + 2 \sum_{-i+j=-p} i t_i \frac{\partial}{\partial t_j} \right) \tau, \tag{5.9}$$

and on the other hand, using the explicit representation (5.1) for $\tilde{B}_p$ in terms of the integral (letting $\tilde{A}_p = \int dZ\, e^I$)

$$\begin{aligned} \Delta_{1-p}\tilde{A}_p &= \int dZ\, e^I \operatorname{Tr}\left(\Theta^{1-p} \frac{\partial I}{\partial \Theta} \right) \\ &\overset{*}{=} \int dZ\, e^I \operatorname{Tr}\left(\Theta^{1-p} \left(\frac{\partial I}{\partial \Theta} + \frac{\partial I}{\partial Z} \right) \right) \\ &= \int dZ\, e^I \operatorname{Tr} \Theta^{1-p} \frac{\partial}{\partial Z} \left(\frac{p(p+1)}{2} \frac{Z^2(-\Theta)^{p-1}}{p+1} \right) \\ &= (-1)^{p-1} p \int dZ\, e^I \operatorname{Tr} Z. \end{aligned} \tag{5.10}$$

Equality (*) follows from the observation that by integration by parts

$$\int dZ \sum_{ij} \frac{\partial}{\partial Z_{ij}} (M_{ij} e^I) = 0.$$

Since $\tau_p \tilde{B}_p = \tilde{A}_p$, comparing (5.9) and (5.10) leads to

$$\tilde{B}_p J_{-p}^{(2)} \tau_p = 2p \int dZ\, e^I \operatorname{Tr} Z.$$

By means of the (often used) time shift $t_{p+1} \curvearrowright t_{p+1} + 1$ (see for instance Sect. 4, Step 2),

$$J^{(2)}_{-p} \curvearrowright J^{(2)}_{-p} + 2(p+1)\frac{\partial}{\partial t_1};$$

then, since $J^{(2)}_{-p}\tau_p = 0$ by Theorem 3, the result of Corollary 3.1 follows.

6. An Explicit Evaluation of $\tau_p(t)$

We shall evaluate $\tau_p(t) = A_p(\lambda)/B_p(\lambda)$, the ratio of determinants, in the style of the classical formula for Schur polynomials, using an integration formula of Mehta [Me], following Kontsevich [K3] in the KdV case. This will immediately prove $\tau_p(t)$ is a formal sum in the variables $t_i = \frac{1}{i}\sum_j \lambda_j^{-i/p}$, a fact taken for granted in Sect. 5. Indeed, Mehta observed if Φ is a conjugacy invariant function on the space of hermitian $N \times N$ matrices, then for any diagonal hermitian matrix Y

$$\int_{\text{(Hermitian matrices)}} \Phi(X)\, e^{-\sqrt{-1}\,\mathrm{tr}\,XY}\, dX$$
$$= (-2\pi\sqrt{-1})^{N(N-1)/2}(V(Y))^{-1} \int_{\text{(diagonal matrices)}} \Phi(D)\, e^{-\sqrt{-1}\,\mathrm{tr}\,DY}\, V(D)\, dD$$

with

$$V(\mathrm{diag}(X_1, X_2, \ldots, X_N)) \equiv \prod_{i<j}(X_j - X_i) = \det[X_i^{j-1}]_{1 \leqq i,j \leqq N}.$$

From this it follows that (c is a constant)

$$A_p(\lambda) = c\,\frac{\det\left(\frac{\partial^{j-1}}{\partial y} a_p(\lambda_i)\right)_{1 \leqq i,j \leqq N}}{\det(\lambda_i^{j-1})_{1 \leqq i,j \leqq N}} \tag{6.1}$$

with

$$a_p(y) = \int e^{\left(-\frac{x^{p+1}}{p+1} + xy\right)} dx; \tag{6.2}$$

here we have made use of

$$\int x^{j-1} e^{\left(-\frac{x^{p+1}}{p+1} + xy\right)} dx = \left(\frac{\partial}{\partial y}\right)^{j-1} a_p(y).$$

Substituting into (6.1) the specific expression (6.2) of $a_p(y)$ with the following asymptotic expansion for large y (see [K-S]):

$$a_p(y) = y^{-\frac{p-1}{2p}} \exp\left(\frac{p}{p+1} y^{\frac{p+1}{p}}\right) \sum_0^\infty a_n y^{-\frac{p+1}{p}n},$$

and using

$$\left(\frac{\partial}{\partial y}\right)^{j-1} a_p(y) = c' y^{-\frac{p+1}{2p}} \exp\left(\frac{p}{p+1} y^{\frac{p+1}{p}}\right) g_j\left(y^{-\frac{1}{p}}\right),$$

where

$$g_j(s) = s^{-j}(1 + a_1^{(j)} s + a_2^{(j)} s^2 + \ldots) \equiv s^{-j} h_j(s), \quad s \text{ small};$$

yields

$$A_p(\lambda) = c'' \prod_k \lambda_k^{-\frac{p+1}{2p}} \exp\left(\frac{p}{p+1} \lambda_k^{\frac{p+1}{p}}\right) \frac{\det\left(g_j\left(\lambda_i^{-\frac{1}{p}}\right)\right)_{1 \leqq i, j \leqq N}}{\prod_{i<j} (\lambda_i - \lambda_j)}.$$

Thus (see (5.1) for $B_p(\lambda)$)

$$\begin{aligned}
\tau_p(t) &= \frac{A_p(\lambda)}{B_p(\lambda)} = \prod_i \lambda_i^{-\frac{p+1}{2p}} \cdot \left\{\prod_i \lambda_i^{\frac{p+1}{p}} \prod_{i<j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i^{1/p} - \lambda_j^{1/p}}\right)\right\} \frac{\det\left(g_j\left(\lambda_i^{-\frac{1}{p}}\right)\right)}{\prod_{i<j} (\lambda_i - \lambda_j)} \\
&= \Big(\prod_i \lambda_i^{1/p} \cdot \prod_{i<j} (\lambda_i^{1/p} - \lambda_j^{1/p})\Big)^{-1} \det g_j(\lambda_i^{-1/p})) \\
&= \frac{\det g_j\left(\lambda_i^{-\frac{1}{p}}\right)_{1 \leqq i, j \leqq N}}{\det(\lambda_i^{j/p})_{1 \leqq i, j \leqq N}} \\
&= \frac{\det[\lambda_i^{j/p} h_j(\lambda_i^{-1/p})]_{1 \leqq i, j \leqq N}}{\det[\lambda_i^{j/p}]_{1 \leqq i, j \leqq N}} \\
&= \frac{\det[(\lambda_i^{-1/p})^{N-j} h_j(\lambda_i^{-1/p})]_{1 \leqq i, j \leqq N}}{\det[(\lambda_i^{-1/p})^{N-j}]_{1 \leqq i, j \leqq N}}, \quad \text{after multiplying the } i^{\text{th}} \text{ row of both matrices by } \lambda_i^{-N/p} \\
&= \frac{\det(H_j(\mu_i))_{1 \leqq i, j \leqq N}}{\prod_{i<j} (\mu_i - \mu_i)} \equiv \frac{H(\mu_1, \mu_2, \ldots, \mu_N)}{\prod_{i<j} (\mu_i - \mu_j)}
\end{aligned}$$

with

$$\mu_i \equiv \lambda_i^{-1/p}, \quad H_j(s) \equiv s^{N-j} h_j(s) = s^{N-j}\left(1 + \sum_1^{\infty} a_i^{(j)} s^i\right).$$

Therefore $H(\mu_1, \ldots, \mu_N)$ is a formal *power series* in the μ_i, skew-symmetric in its arguments, and so divisible in the ring of formal power series by $\prod_{i<j} (\mu_i - \mu_j)$. Then, the ratio $H(\mu)/\prod_{i<j} (\mu_i - \mu_j)$ is a symmetric function in the μ_i, and hence (as in the polynomial case) a formal series in the elementary symmetric variables $\pi_j = \sum \mu_i^j$, $j = 1, 2, \ldots$; therefore $\tau_p(t)$ is a formal series in the $t_j = \pi_j / j$, $j = 1, 2, \ldots$, as claimed.

7. Appendix

$$J_n^{(1)} = \frac{\partial}{\partial t_n} - n t_{-n} \quad \text{with} \quad t_n = 0 \text{ if } n < 0,$$

$$J_n^{(2)} = \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{-i+j=n} i t_i \frac{\partial}{\partial t_j} + \sum_{-i-j=n} (i t_i)(j t_j),$$

$$J_n^{(3)} = \sum_{i+j+k=n} \frac{\partial^3}{\partial t_i \partial t_j \partial t_n} + 3 \sum_{-i+j+k=n} i t_i \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_k}$$

$$+ 3 \sum_{-i-j+k=n} (i t_i)(j t_j) \frac{\partial}{\partial t_k} + \sum_{-i-j-k=n} (i t_i)(j t_j)(k t_k).$$

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