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Nonlinear Scattering with Nonlocal Interaction

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Abstract. We consider the scattering problem for the Hartree type equation in $\mathbb{R}^n$ with $n \geqq 2$:

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = (V * |u|^2)u,$$

where $V(x) = \sum_{j=1}^{2} \lambda_j |x|^{-\gamma_j}$, $(\lambda_1, \lambda_2) \neq (0,0)$, $\lambda_j \in \mathbb{R}$, $\gamma_j > 0$, and $*$ denotes the convolution in $\mathbb{R}^n$. We prove the existence of wave operators in $H^{0,k} = \{\psi \in L^2(\mathbb{R}^n); |x|^k \psi \in L^2(\mathbb{R}^n)\}$ for any positive integer k under the assumption $1 < \gamma_1, \gamma_2 < 2$. This is an optimal result in the sense that the existence of wave operators breaks down if $\min(\gamma_1, \gamma_2) \leqq 1$. The case where $1 < \gamma_1 < \gamma_2 = 2$ is also treated according to the sign of λ_2.

1. Introduction

We consider the scattering problem for the nonlinear Schrödinger equations of the form

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = (V * |u|^2)u, \tag{1.1}$$

where u is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ with $n \geqq 2$, Δ is the Laplacian in $\mathbb{R}^n$, V is real function on $\mathbb{R}^n$, and $*$ denotes the convolution in $\mathbb{R}^n$, The nonlocal interaction $(V * |u|^2)u$ is known as the Hartree type nonlinearity and the evolution equation (1.1) is derived from a multibody Schrödinger equation in the semi-classical limit or in the self-consistent field approximation for a quantum field of

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bosons. A typical form of V is given by the sum of two potentials

$$V = V_1 + V_2, \quad V_j(x) = \lambda_j |x|^{-\gamma_j}$$

with $\lambda_j \in \mathbb{R}$, $(\lambda_1, \lambda_2) \neq (0,0)$ and $\gamma_2 > \gamma_1 > 0$, which we treat in the sequel for simplicity.

Although there is a large literature on the Cauchy problem and on the asymptotic behavior of the solutions of (1.1) (see [1, 2, 4–13, 15, 20]), there still remains a gap between the lower bound of γ_1 which ensures the existence of wave operators for (1.1) and the upper bound of γ_2 for their nonexistence. Up to now the former is known as $\frac{4}{3}$ (limit excluded) [6, 7, 10, 13, 30] and the latter as 1 (limit included) [8, 13]. One of our purpose here is to fill this gap and prove the existence of the wave operators in the lowest possible case $\gamma_2 > \gamma_1 > 1$. To state our results more precisely, we introduce the following notations.

Notations. $\partial_t = \dfrac{\partial}{\partial t}$, $\partial_k = \dfrac{\partial}{\partial x_k}$, $U = U(t) = \exp\left(\dfrac{it}{2}\Delta\right)$, $M = M(t) = \exp\left(\dfrac{i|x|^2}{2t}\right)$, $J_k = J_k(t) = U(t)x_k U(-t) = x_k + it\partial_k = M(t)(it\partial_k)M(-t)$, $J = (J_1, \ldots, J_n)$, $\nabla = (\partial_1, \ldots, \partial_n)$; $(Uv)(t) = U(t)v(t)$, $(U^{-1}v)(t) = U(-t)v(t)$, $(J_k v)(t) = J_k(t)v(t)$, $J^\alpha = \prod_{k=1}^{n} J_k^{\alpha_k}$, $x^\alpha = \prod_{k=1}^{n} x_k^{\alpha_k}$ for multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$; L^p denotes the Lebesgue space $L^p(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n) \otimes \mathbb{C}^n$ with the norm denoted by $\|\cdot\|_p$; $H^{m,s}$ denotes the weighted Sobolev space defined by

$$H^{m,s} = \{\psi \in \mathscr{S}';\ \|\psi\|_{m,s} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}\psi\|_2 < \infty\}, \quad m, s \in \mathbb{R},$$

where $\mathscr{S}'$ denotes the space of tempered distribution on $\mathbb{R}^n$; $(\cdot,\cdot)$ denotes the scalar product in L^2 and various pairing of dual spaces of functions.

We now state our main results.

Theorem 1. *Let $1 < \gamma_1 < \gamma_2 \leqq 2$ and $n \geqq 2$. When $\gamma_2 = 2$, assume in addition that $\lambda_2 \geqq 0$ and $n \geqq 3$. Let $k \in \mathbb{N} = \{1, 2, \ldots\}$. Then:*

(1) *For any $\phi_+ \in H^{0,k}$ there exists a unique $\phi \in H^{0,k}$ such that*

$$\|U(-t)u(t) - \phi_+\|_{0,k} \to 0 \quad as \quad t \to \infty, \tag{1.3$_+$}$$

where u is the unique solution of

$$u(t) = U(t)\phi - i\int_0^t U(t-\tau)(V * |u|^2)u(\tau)\,d\tau, \quad t \in \mathbb{R}, \tag{1.4}$$

with $U^{-1}u \in C(\mathbb{R}; H^{0,k})$.

(2) *For any $\phi_- \in H^{0,k}$ there exists a unique $\phi \in H^{0,k}$ such that*

$$\|U(-t)u(t) - \phi_-\|_{0,k} \to 0 \quad as \quad t \to -\infty, \tag{1.3$_-$}$$

where u is the unique solution of (1.4).

Remark 1. (1) Theorem 1 extends the previous results (see [6, 7, 10, 13, 20]) and is optimal in the sense that if $\gamma_1 \leqq 1$, then for any nontrivial solution u, $U(-t)u(t)$ has no strong limit in L^2 as $t \to \pm\infty$ (see [8, 13]).

(2) By Theorem 1, the wave operators $W_\pm : \phi_\pm \mapsto \phi$ are well defined maps from $H^{0,k}$ into itself for any $k \in \mathbb{N}$.

Theorem 2. *Under the assumption of Theorem 1, the wave operators $W_\pm$ are continuous injection from $H^{0,k}$ to $H^{0,k}$. Moreover, $W_\pm$ are isometric in the sense that for any $\psi \in H^{0,1}$,*

$$\| W_\pm \psi \|_2 = \| \psi \|_2. \tag{1.5}$$

Remark 2. In the case of repulsive interactions $\lambda_j \geqq 0$, combining the result in [10] and Theorem 2, we see that the scattering operators $S: \phi_- \mapsto \phi_+$ is well defined as a map from $H^{0,1}$ into L^2 and isometric in the L^2 norm, i.e., $\| S\psi \|_2 = \| \psi \|_2$ for any $\psi \in H^{0,1}$.

In the case where $\gamma_2 = 2$ and $\lambda_2 < 0$, we construct scattering theory in $H^{0,k}$ with small data.

Theorem 3. *Let $1 < \gamma_1 < \gamma_2 = 2$, $\lambda_2 < 0$, and $n \geqq 3$. Let Q be a nontrivial solution of the elliptic equation*

$$\Delta Q - Q = 2(V_2 * |Q|^2)Q \tag{1.6}$$

such that

$$\begin{aligned} \| Q \|_2^2 &= \inf_{\substack{\psi \in H^1(\mathbb{R}^n) \\ \psi \not\equiv 0}} \{ \| \psi \|_2^2 ; \| \nabla \psi \|_2^2 + (V_2 * |\psi|^2, |\psi|^2) \leqq 0 \} \\ &= \inf_{\substack{\psi \in H^1(\mathbb{R}^n) \\ \psi \not\equiv 0}} \frac{\| \psi \|^2 \| \nabla \psi \|^2}{(V_2 * |\psi|^2, |\psi|^2)}. \end{aligned} \tag{1.7}$$

Let $k \in \mathbb{N}$. Then:

(1) *For any $\phi_\pm \in H^{0,k}$ with $\| \phi_\pm \|_2 < \| Q \|_2$ there exists a unique $\phi \in H^{0,k}$ satisfying $(1.3)_\pm$ with the unique solution of (1.4) and $\| \phi \|_2 = \| \phi_\pm \|_2$.*

(2) *Suppose in addition that $\lambda_1 \geqq 0$. Then for any $\phi \in H^{0,1}$ with $\| \phi \|_2 < \| Q \|_2$, there exist unique $\phi_\pm \in L^2$ such that $\| \phi \|_2 = \| \phi_\pm \|_2$ and*

$$\| U(-t)u(t) - \phi_\pm \|_2 \to 0 \quad as \quad t \to \pm \infty,$$

where u is the unique solution of (1.4).

(3) *Suppose in addition that $\lambda_1 = 0$. Then for any $\phi \in H^{0,k}$ with $\| \phi \|_2 < \| Q \|_2$, there exist unique $\phi_\pm \in H^{0,k}$ satisfying $(1.3)_\pm$ with the unique solution u of (1.4) and $\| \phi \|_2 = \| \phi_\pm \|_2$.*

Remark 3. (1) Equation (1.6) is a time-independent version of (1.1) and arises in various domain of physics. See [15–17] for the existence of positive solutions of (1.6) and for the associated minimization problems. The existence of nontrivial solutions of $H^{1,0}$ for (1.6) with (1.7) is proved by the same method as in [18]. The standard argument shows that $Q \in \mathscr{S}$.
(2) Theorem 3 clarifies the size of the ball where scattering theory for (1.1) is constructed in the critical case $\gamma_2 = 2$. In the case of Cauchy problem in the energy space $H^{1,0}$ for the nonlinear Schrödinger equation with the critical power nonlinearity, the use of stationary solution in the description of the size of data can be traced back to Weinstein [21]. See [15, 19, 22] for related results.
(3) The condition $\| \phi \|_2 < \| Q \|_2$ in part (3) is optimal. If $\phi = Q$, then $\phi \in \bigcap_{k \geqq 1} H^{0,k}$

and $u(t) = e^{it/2}\phi$ is the unique solution of (1.4), while $U(-t)u(t) = e^{it/2}U(-t)\phi$ has no limit in L^2 as $t \to \pm\infty$.
(4) Let $\lambda_1 = 0$. $B_k = \{\psi \in H^{0,k};\ \|\psi\|_2 < \|Q\|_2\}$. By Theorem 3, the wave operators $W_\pm : \phi_\pm \mapsto \phi$ and the scattering operator $S = W_+^{-1} \circ W_- : \phi_- \mapsto \phi_+$ are well defined maps from B_k into itself for any $k \in \mathbb{N}$.

Theorem 4. *Let $\lambda_1 = 0$, $\lambda_2 < 0$, $\gamma_2 = 2$ and $n \geqq 3$. Then the wave operators $W_\pm$ and scattering operators S are homeomorphisms from B_k to B_k. Moreover $W_\pm$ and S are isometric in the sense that for any $\psi \in B_1$,*

$$\|W_\pm \psi\|_2 = \|S\psi\|_2 = \|\psi\|_2. \tag{1.8}$$

We consider the small data scattering in L^2.

Theorem 5. *Let $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\gamma_2 = 2$ and $n \geqq 3$. Then there exists a constant $\varepsilon_0 > 0$ with the following properties. Let $B(\varepsilon_0) = \{\psi \in L^2;\ \|\psi\|_2 < \varepsilon_0\}$.*

(1) *For any $\phi \in B(\varepsilon_0)$ there exists a unique solution u of (1.4) such that $u \in C(\mathbb{R}; L^2) \cap L^{2+(4/n)}(\mathbb{R}; L^{2+(4/n)})$ and $\|u(t)\|_2 = \|\phi\|_2$. Moreover the map $\phi \mapsto u$ is continuous from $B(\varepsilon_0)$ to $L^\infty(\mathbb{R}; L^2) \cap L^{2+(4/n)}$).*

(2) *For any $\phi_+ \in B(\varepsilon_0)$ there exists a unique $\phi \in B(\varepsilon_0)$ such that*

$$\|U(-t)u(t) - \phi_+\|_2 \to 0 \quad as \quad t \to +\infty, \tag*{(1.9)$_+$}$$

where u is the unique solution of (1.4) given by part (1). *For any $\phi_- \in B(\varepsilon_0)$ there exists a unique $\phi \in B(\varepsilon_0)$ such that*

$$\|U(-t)u(t) - \phi_-\|_2 \to 0 \quad as \quad t \to -\infty, \tag*{(1.9)$_-$}$$

where u is the unique solution of (1.4) given by part (1).

(3) *For any $\phi \in B(\varepsilon_0)$ there exists unique $\phi_\pm \in B(\varepsilon_0)$ satisfying (1.9)$_\pm$, where u is the unique solution of (1.4) given by part* (1).

(4) *The wave operators $W_\pm : \phi_\pm \mapsto \phi$ and the scattering operator $S = W_+^{-1} \circ W_-$ are homeomorphisms from $B(\varepsilon_0)$ to $B(\varepsilon_0)$ and isometric in the L^2 norm.*

Remark 4. (1) The assumption on V is weakened as follows. It suffices to assume that V is a real function on $\mathbb{R}^n$ satisfying $|x|^2 V \in L^\infty$.
(2) If the initial datum ϕ takes the form $\phi = U(\pm s)\phi_{\pm s}$ for some $\psi_\pm \in L^2$ with $s > 0$ sufficiently large, then $\|U(\cdot)\phi\|_{L^\sigma(\mathbb{R}_\pm; L^\sigma)}$ are sufficiently small, where $\sigma = 2 + \dfrac{4}{n}$, so that the solution u of (1.4) exists on $\mathbb{R}_\pm$ (see [3] and the proof below).
(3) There are related results for small data scattering for (1.1) in the spaces strictly smaller than L^2, see [13, 20].

Remark 5. Theorems 3, 4 and 5 are optimal in the sense that the L^2 scattering theory for (1.1), even restricted in the small data, is impossible in the case where $\lambda_1 = 0$, $\lambda_2 < 0$ and $0 < \gamma_2 < 2$. Indeed, let Q satisfy

$$\Delta Q - Q = 2(V_2 * |Q|^2)Q$$

and let $\phi_\varepsilon(x) = \varepsilon^{1+(n-\gamma_2)/2}Q(\varepsilon x)$ for $\varepsilon > 0$. Then $u_\varepsilon(t) = e^{i\varepsilon^2 t/2}\phi_\varepsilon$ solves (1.1) and $\|u_\varepsilon(t)\|_2 = \varepsilon^{1-(\gamma_2/2)}\|Q\|_2 \to 0$ as $\varepsilon \to 0$, while $U(-t)u_\varepsilon(t) = e^{i\varepsilon^2 t/2}U(-t)\phi_\varepsilon$ has no limit in L^2 as $t \to \pm\infty$.

In Sect. 3 we prove Theorems 1 and 2. The main point of the proof is to solve the integral equations

$$u(t) = U(t)\phi_{\pm} + i \int_t^{\pm\infty} U(t-\tau)(V * |u|^2)u(\tau)d\tau, \quad t \in \mathbb{R}, \tag{1.10}$$

for given data $\phi_{\pm}$ at infinity. We solve (1.10) near $t = \pm\infty$ by a contraction argument. For this purpose we define a suitable function space and a metric in order that the right-hand side of (1.10) defines a contraction map on that space. The choice of suitable function spaces depends on how the solutions of (1.10) should behave in the space and time, which is measured by the space-time integrability. We have found the best possible choice of the admissible pairs for indices [1] for the space-time integrability. With this choice, the space-time estimate of Strichartz type work well to play a crucial role. We then extend the solution of (1.10) to the whole time interval by the standard continuation procedure.

In Sect. 4 we prove Theorems 3 and 4. We make use of the pseudo-conformal identity to obtain decay estimates of solutions to (1.4). The assumption $\|\phi\|_2 < \|Q\|_2$ leads to a priori estimates for $\|Ju(t)\|_2$. The method here is comparable with that of Weinstein [21].

In Sect. 5 we prove Theorem 5. The nonlinearity in the assumption proves to admit a special function space where (1.10) is solvable globally in time by a simple contraction technique without any continuation argument.

2. Preliminaries

We collect here some preliminaries. Following [1, 3], we say that a pair (σ, ρ) of indices is *admissible* if

$$\frac{1}{2} - \frac{1}{n} < \frac{1}{\rho} \leqq \frac{1}{2}$$

and

$$\frac{2}{\sigma} = \frac{n}{2} - \frac{n}{q} \equiv \delta(\rho).$$

Lemma 2.1. (1) *For every $\phi \in L^2$ and for every admissible pair (σ, ρ), the function $t \mapsto U(t)\phi$ belongs to $C(\mathbb{R}; L^2) \cap L^\sigma(\mathbb{R}; L^\rho)$ and satisfies*

$$\|U(\cdot)\phi\|_{L^\sigma(\mathbb{R}; L^\rho)} \leqq C\|\phi\|_2,$$

where C is independent of $\phi \in L^2$.

(2) *Let I be an interval $I \subset \mathbb{R}$ and let $t_0 \in \bar{I}$. Let (κ, θ) be an admissible pair and let $v \in L^{\kappa'}(I; L^{\theta'})$, where $\frac{1}{\kappa'} + \frac{1}{\kappa} = \frac{1}{\theta'} + \frac{1}{\theta} = 1$. Then, for every admissible pair (σ, ρ), the function $t \mapsto \int_{t_0}^{t} U(t-\tau)v(\tau)d\tau \equiv (Gv)(t)$ belongs to $C(\bar{I}; L^2) \cap L^\sigma(I; L^\rho)$ and satisfies*

$$\|Gv\|_{L^\sigma(I; L^\rho)} \leqq C\|v\|_{L^{\kappa'}(I; L^{\theta'})},$$

where C is independent of $v \in L^{\kappa'}(I; L^{\theta'})$.

(3) *Let* $I=(t_0,\infty)$ *with* $t_0\in\mathbb{R}$. *Let* (κ,θ) *be an admissible pair and let* $v\in L^{\kappa'}(I;L^{\theta'})$. *Then, for every admissible pair* (σ,ρ), *the function* $t\mapsto\int_t^\infty U(t-\tau)v(\tau)d\tau\equiv(\tilde{G})(t)$ *belongs to* $C(\bar{I};L^2)\cap L^\sigma(I;L^\rho)$ *and satisfies*

$$\|\tilde{G}v\|_{L^\sigma(I;L^\rho)}\leqq C\|v\|_{L^{\kappa'}(I;L^{\theta'})},$$

where C *is independent of* $v\in L^{\kappa'}(I;L^{\theta'})$.

For Lemma 2.1, see [1, 3, 14, 23].

We now consider the Cauchy problem for (1.1) in the weighted L^2 spaces.

Proposition 2.1. (1) *Let* $1<\gamma_1<\gamma_2\leqq 2$ *and* $n\geqq 2$. *When* $\gamma_2=2$, *assume in addition that* $\lambda_2\geqq 0$ *and* $n\geqq 3$. *Let* $k\in\mathbb{N}=\{1,2,\dots\}$ *and* $t_0\in\mathbb{R}$. *Let* $\phi\in L^2$ *satisfy* $U(-t_0)\phi\in H^{0,k}$. *Then there exists a unique solution u of the integral equation*

$$U(-t)u(t)=U(-t_0)\phi-i\int_{t_0}^t U(-\tau)(V*|u|^2)u(\tau)d\tau \tag{2.1}$$

such that $u\in C(\mathbb{R};L^2)\cap L^\sigma_{\mathrm{loc}}(\mathbb{R};L^\rho)$ *for every admissible pair* (σ,ρ) *and that* $U^{-1}u\in C(\mathbb{R};H^{0,k})$. *Moreover, u satisfies*

$$\|u(t)\|_2=\|\phi\|_2,\quad t\in\mathbb{R}, \tag{2.2}$$

$$\begin{aligned}\|Ju(t)\|_2^2+t^2(V*|u(t)|^2,|u(t)|^2)&=\|Ju(s)\|_2^2+s^2(V*|u(s)|^2,|u(s)|^2)\\&\quad+4\int_s^t\tau(\tilde{V}*|u(\tau)|^2,|u(\tau)|^2)d\tau,\quad t,s\in\mathbb{R},\end{aligned} \tag{2.3}$$

where $\tilde{V}=V+\frac{1}{2}x\cdot\nabla V$. *Furthermore, the map* $\phi\mapsto u$ *is continuous from* $\mathscr{H}_k$ *to* $C(\mathbb{R};H^{0,k})$, *where* $\mathscr{H}_k=\{\phi\in L^2;U(-t_0)\phi\in H^{0,k}\}$ *with the norm* $|||\phi|||_k=\|U(-t_0)\phi\|_{0,k}$.

(2) *Let* $1<\gamma_1<\gamma_2=2$, $\lambda_2\leqq 0$ *and* $n\geqq 3$. *Let* Q *be as in Theorem 3. Let* $k\in\mathbb{N}$ *and* $t_0\in\mathbb{R}$. *Let* $\phi\in L^2$ *satisfy* $U(-t_0)\phi\in H^{0,k}$ *and* $\|\phi\|_2<\|Q\|_2$. *Then there exists a unique solution u of (2.1) such that* $u\in C(\mathbb{R};L^2)\cap L^\sigma_{\mathrm{loc}}(\mathbb{R};L^\rho)$ *for every admissible pair* (σ,ρ) *and that* $U^{-1}u\in C(\mathbb{R};H^{0,k})$. *Moreover, u satisfies (1.2) and*

$$\begin{aligned}\|Ju(t)\|_2^2+t^2(V*|u(t)|^2,|u(t)|^2)&=\|Ju(s)\|_2^2+s^2(V*|u(s)|^2,|u(s)|^2)\\&\quad+4\int_s^t\tau(\tilde{V}_1*|u(\tau)|^2,|u(\tau)|^2)d\tau,\quad t,s\in\mathbb{R},\end{aligned} \tag{2.4}$$

where $\tilde{V}_1=V_1+\frac{1}{2}x\cdot\nabla V_1$. *Furthermore, the map* $\phi\mapsto u$ *is continuous from* $\tilde{\mathscr{H}}_k$ to $C(\mathbb{R};H^{0,k})$, *where* $\tilde{\mathscr{H}}_k=\{\phi\in L^2;U(-t_0)\phi\in H^{0,k},\|\phi\|_2<\|Q\|_2\}$ *with the norm* $|||\phi|||_k=\|U(-t_0)\phi\|_{0,k}$.

Proof. Part (1) can be proved in the same way as in the proof of Theorem 5 in [11] if we use the space-time estimate in the most general situation, as described in parts (1), (2) of Lemma 2.1. We shall prove part (2). By the standard method we obtain a unique local solution u of (1.1) such that $u\in C(I;L^2)\cap L^\sigma(I;L^\rho)$ for every admissible pair (σ,ρ) and $U^{-1}u\in C(I;L^2)$, where $I=[t_0-T,t_0+T]$ for some $T>0$ depending only on $|||\phi|||_k$. Moreover, we see that u satisfies (1.2) and (1.4) on I. In order to extend u to the whole time interval, we need *a priori* estimates for u. In a way similar to [6], we see that $\|U(-t)u(t)\|_{0,k}$ is bounded uniformly for $t\in I$ in terms of $|||\phi|||_k$

and $\sup_{t\in I}\| U(-t)u(t)\|_{0,1}$. Therefore we are reduced to obtaining from (2.2) and (2.4) on I the following *a priori* estimate

$$\sup_{t\in I}\| U(-t)u(t)\|_{0,1}\leqq C(|||\phi|||_1). \tag{2.5}$$

By (2.2) and (2.4) on I and (1.7), we have

$$\begin{aligned}
\| Ju(t)\|_2^2 &= \| xU(-t_0)\phi\|_2^2 + t_0^2(V*|u(t_0)|^2, |u(t_0)|^2)\\
&\quad - t^2(V*|M(-t)u(t)|^2, |M(-t)u(t)|^2) + 4\int_{t_0}^{t}\tau(\tilde V_1*|u(\tau)|^2, |u(\tau)|^2)d\tau\\
&\leqq |||\phi|||_1^2 + \frac{\|\phi\|_2^2}{\|Q\|_2^2}\| xU(-t_0)\phi\|_2^2 + C\|\phi\|_2^{4-\gamma_1}\| xU(-t_0)\phi\|_2^{\gamma_1}t_0^{2-\gamma_1}\\
&\quad + \frac{\|\phi\|_2^2}{\|Q\|_2^2}\| Ju(t)\|_2^2 + C\|\phi\|_2^{4-\gamma_1}\| Ju(t)\|_2^{\gamma_1}t_0^{2-\gamma_1}\\
&\quad + 4\int_{t_0}^{t}\tau(\tilde V_1*|u(\tau)|^2, |u(\tau)|^2)d\tau, \quad t\in I\backslash\{t_0\},
\end{aligned}$$

so that, for any $\varepsilon>0$

$$\begin{aligned}
\left(1-\frac{\|\phi\|_2^2}{\|Q\|_2^2}\right)\| Ju(t)\|_2^2 &\leqq \left(2+\frac{\|\phi\|_2^2}{\|Q\|_2^2}\right)|||\phi|||_1^2 + (C\|\phi\|_2^{4-\gamma_1}t_0^{2-\gamma_1})^{2/(2-\gamma_1)}\\
&\quad + C(\varepsilon)(\|\phi\|_2^{4-\gamma_1}t_0^{2-\gamma_1})^{2/(2-\gamma_1)} + \varepsilon\| Ju(t)\|_2^2\\
&\quad + 4\int_{t_0}^{t}\tau(\tilde V_1*|u(\tau)|^2, |u(\tau)|^2)d\tau, \quad t\in I\backslash\{t_0\}.
\end{aligned} \tag{2.6}$$

Thus the required estimate (2.5) can be obtained in the same way as in the proof of Lemma 3.5 in [11]. The remaining statement follows by the standard method.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We shall prove part (1) only. One can prove Part (2) analogously. We first consider the case $k=1$. Let ρ_j and θ_j satisfy

$$\frac{1}{\rho_j},\ \frac{1}{\theta_j}\in\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right), \tag{3.1}$$

$$1+\frac{2(1-\gamma_j)}{n}<\frac{1}{\rho_j}+\frac{1}{\theta_j}<1. \tag{3.2}$$

Let σ_j and κ_j satisfy

$$\frac{2}{\sigma_j}=\delta(\rho_j),\quad \frac{2}{\kappa_j}=\delta(\theta_j),$$

where $\delta(q)=\frac{n}{2}-\frac{n}{q}$. Then (σ_j,ρ_j) and (κ_j,θ_j) become admissible pairs.

For $R>0$ and $T \geqq 1$, we define

$$X=\left\{u \in C([T,\infty);L^2)\cap \bigcap_{j=1}^{2} L^{\sigma_j}(T,\infty;L^{\rho_j});\ Ju \in C_w([T,\infty);L^2)\cap \bigcap_{j=1}^{2} L^{\sigma_j}(T,\infty;L^{\rho_j}),\right.$$
$$\left.\|u\|_{L^\infty(T,\infty;L^2)}+\|Ju\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}(\|u\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}+\|Ju\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})})\leqq R\right\}.$$

Then X is a complete metric space with respect to the metric d given by

$$d(u,v)=\|u-v\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|u-v\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})},\quad u,v\in X. \tag{3.4}$$

For $\phi_+ \in H^{0,1}$ and $u \in X$ we define $\Phi(u)$ by

$$(\Phi(u))(t)=U(t)\phi_+ + i\int_t^\infty U(t-\tau)(V*|u|^2)u(\tau)d\tau,\quad t\geqq T. \tag{3.5}$$

Let p_j and q_j satisfy $\dfrac{2}{p_j}=2-\dfrac{\gamma_j}{n}-\left(\dfrac{1}{\theta_j}+\dfrac{1}{\rho_j}\right)$ and $\dfrac{1}{q_j}=1-\left(\dfrac{1}{\theta_j}+\dfrac{1}{\rho_j}\right)$. By the Hölder inequality and the Hardy–Littlewood–Sobolev inequality with $\dfrac{2}{p_j}=\dfrac{1}{q_j}+1-\dfrac{\gamma_j}{n}$,

$$\|(V_j*|u|^2)u\|_{\theta_j'}\leqq \|V_j*|u|^2\|_{q_j}\|u\|_{\rho_j}\leqq C\|u\|_{p_j}^2\|u\|_{\rho_j}. \tag{3.6}$$

We see from (3.2) that $\dfrac{1}{2}-\dfrac{1}{n}<\dfrac{1}{p_j}<\dfrac{1}{2}$. Therefore we use the Gagliardo–Nirenberg inequality to obtain

$$\begin{aligned}\|u\|_{p_j}&\leqq C\|\nabla M(-t)u\|_2^{\delta(p_j)}\|M(-t)u\|_2^{2-\delta(p_j)}\\&=Ct^{-\delta(p_j)}\|Ju\|_2^{\delta(p_j)}\|u\|_2^{1-\delta(p_j)}\\&\leqq CRt^{-\delta(p_j)}.\end{aligned} \tag{3.7}$$

By Lemma 2.1, (3.6), (3.7) and the Hölder inequality with $\dfrac{1}{\nu_j}=\dfrac{1}{\kappa_j'}-\dfrac{1}{\sigma_j}$,

$$\begin{aligned}&\|\Phi(u)\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|\Phi(u)\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\\&\leqq C\|\phi_+\|_2+C\sum_{j=1}^{2}\left(\int_T^\infty\|(V_j*|u|^2)u\|_{\theta_j'}^{\kappa_j'}dt\right)^{1/\kappa_j'}\\&\leqq C\|\phi_+\|_2+C\sum_{j=1}^{2}\left(\int_T^\infty\|u\|_{p_j}^{2\nu_j}dt\right)^{1/\nu_j}\left(\int_T^\infty\|u\|_{\rho_j}^{\sigma_j}dt\right)^{1/\sigma_j}\\&\leqq C\|\phi_+\|_2+C\sum_{j=1}^{2}T^{(1/\nu_j)-2\delta(p_j)}R^3,\end{aligned} \tag{3.8}$$

where we note that (3.2) implies $\dfrac{1}{\nu_j}>2-\dfrac{2}{\gamma_j}\geqq 0$ and $2\nu_j\delta(p_j)>1$. Putting $\varepsilon_j=$

$2\delta(p_j)-\frac{1}{\nu_j}>0$, one can rewrite (3.8) as

$$\|\Phi(u)\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|\Phi(u)\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\leqq C\|\phi_+\|_2+C\sum_{j=1}^{2}R^3T^{-\varepsilon_j}. \tag{3.9}$$

Next we have

$$\begin{aligned}J((V_j*|u|^2)u)&=M(t)(it\nabla)((V_j*|M(-t)u|^2(M(-t)u)\\&=2(V_j*(\mathfrak{I}(\bar{u}Ju)))u+(V_j*|u|^2)Ju,\end{aligned}$$

and therefore in the same way as in (3.6),

$$\begin{aligned}\|J((V_j*|u|^2)u)\|_{\theta_j'}&\leqq 2\|V_j*(\mathfrak{I}(\bar{u}Ju))\|_{1/(1/\theta_j'-1/p_j)}\|u\|_{p_j}+\|V_j*|u|^2\|_{q_j}\|Ju\|_{\rho_j}\\&\leqq C\|u\|_{p_j}^2\|Ju\|_{\rho_j},\end{aligned} \tag{3.10}$$

where we note that (3.1) implies $\frac{1}{\theta_j'}-\frac{1}{p_j}>\frac{\gamma_j-1}{2n}>0$ and that $\frac{1}{\theta_j'}-\frac{1}{p_j}=\left(\frac{1}{p_j}+\frac{1}{\rho_j}\right)-1+\frac{\gamma_j}{n}>0$. Letting J act on (3.5), we proceed as in (3.8) and obtain from (3.10)

$$\|J\Phi(u)\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|J\Phi(u)\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\leqq C\|x\phi_+\|_2+C\sum_{j=1}^{2}R^3T^{-\varepsilon_j}. \tag{3.11}$$

Let $u,v\in X$. As in (3.10), we have

$$\begin{aligned}&\|(V_j*|u|^2)u-(V_j*|v|^2)v\|_{\theta_j'}\\&\leqq\|V_j*(\bar{u}(u-v))\|_{1/(1/\theta_j'-1/p_j)}\|u\|_{p_j}+\|V_j*((\bar{u}-\bar{v})v)\|_{1/(1/\theta_j'-1/p_j)}\|u\|_{p_j}\\&\quad+\|V_j*|v|^2\|_{q_j}\|u-v\|_{\rho_j}\\&\leqq C(\|u\|_{p_j}^2+\|v\|_{p_j}^2)\|v-v\|_{\rho_j},\end{aligned} \tag{3.12}$$

so that in the same way as in (3.8),

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|\Phi(u)-\Phi(v)\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\\&\leqq C\sum_{j=1}^{2}\left(\left(\int_T^\infty\|u\|_{p_j}^{2\nu_j}dt\right)^{1/\nu_j}+\left(\int_T^\infty\|v\|_{p_j}^{2\nu_j}dt\right)^{1/\nu_j}\right)\|u-v\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\\&\leqq CR^3\sum_{j=1}^{2}T^{-\varepsilon_j}\|u-v\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}.\end{aligned}$$

We choose $C\|\phi_+\|_{0,1}\leqq\frac{R}{2}$ and $CR^2\sum_{j=1}^{2}T^{-\varepsilon_j}\leqq\frac{1}{2}$, we see that $\Phi:u\mapsto\Phi(u)$ is a contraction on X and hence Φ has a unique fixed point u. u satisfies

$$u(t)=U(t)\phi_++i\int_t^\infty U(t-\tau)(V*|u|^2)u(\tau)d\tau,\quad t\geqq T, \tag{3.14}$$

which implies as before

$$\begin{aligned}&\| U(-t)u(t)-\phi_+\|_2+\| x(U(-t)u(t)-\phi_+)\|_2\\&=\| u(t)-U(t)\phi_+\|_2+\| Ju(t)-U(t)x\phi_+\|_2\\&=\left\|\int_t^\infty U(t-\tau)(V*|u|^2)u(\tau)d\tau\right\|_2+\left\|\int_t^\infty U(t-\tau)J((V*|u|^2)u)(\tau)d\tau\right\|_2\\&\leqq CR^3\sum_{j=1}^2 t^{-\varepsilon_j}\to 0\quad\text{as}\quad t\to+\infty.\end{aligned}$$

Moreover, it follows from (3.14) that for all $t\geqq T$,

$$U(-t)u(t)=U(-T)u(T)-i\int_T^t U(-\tau)(V*|u|^2)u(\tau)d\tau. \tag{3.15}$$

Since $U(-T)u(T)\in H^{0,1}$, Proposition 2.1 proves that the solution u of (3.14) extends to all times and satisfies (3.15) for all $t\in\mathbb{R}$ with $U^{-1}u\in C(\mathbb{R};H^{0,1})$.

We now prove the uniqueness. It suffices to prove that if u satisfies (1.4), $U^{-1}u\in C(\mathbb{R};H^{0,1})$, and

$$\| U(-t)u(t)-\phi_+\|_{0,1}\to 0\quad\text{as}\quad t\to\infty, \tag{3.16}$$

then $u\in X$ for some $R, T>0$. By (3.16) we have $u, Ju\in L^\infty(T,\infty;L^2)$ for T sufficiently large. As in (3.7) we see that for any $t>T$,

$$\| u\|_{\rho_j}\leqq Ct^{-\delta(\rho_j)}C\| Ju\|_2^{\delta(\rho_j)}\| u\|_2^{1-\delta(\rho_j)}\leqq CRt^{-\delta(\rho_j)}, \tag{3.17}$$

which implies $u\in\bigcap_{j=1}^2 L^{\sigma_j}(T,\infty;L^{\rho_j})$. By Proposition 2.1, $Ju\in\bigcap_{j=1}^2 L^{\sigma_j}_{\mathrm{loc}}(\mathbb{R};L^{\rho_j})$ and for any $t>T$,

$$Ju(t)=U(t-T)(Ju)(T)-i\int_T^t U(t-\tau)J((V*|u|^2)u)(\tau)d\tau.$$

In the same way as before, we obtain from (3.17),

$$\sum_{j=1}^2\| Ju\|_{L^{\sigma_j}(T,t;L^{\rho_j})}\leqq C\|(Ju)(T)\|_2+C\sum_{j=1}^2 T^{-\varepsilon_j}\| Ju\|_{L^{\sigma_j}(T,t;L^{\rho_j})},$$

and therefore for T large enough

$$\sum_{j=1}^2\| Ju\|_{L^{\sigma_j}(T,t;L^{\rho_j})}\leqq C\| Ju(T)\|_2,$$

where C is independent of t. By the Fatou lemma, $Ju\in\bigcap_{j=1}^2 L^{\sigma_j}(T,\infty;L^{\rho_j})$. This proves $u\in X$ and hence the required uniqueness.

We next consider the case $k\geqq 2$. For $R>0$ and $T\geqq 1$, we define

$$\begin{aligned}X_k=\Big\{&u\in C([T,\infty);L^2)\cap\bigcap_{j=1}^2 L^{\sigma_j}(T,\infty;L^{\rho_j});\\&J^\alpha u\in C_w([T,\infty);L^2)\cap\bigcap_{j=1}^2 L^{\sigma_j}(T,\infty;L^{\rho_j}),|\alpha|\leqq k,\end{aligned}$$

$$\sum_{|\alpha|\leqq k}\left(\|J^\alpha u\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|J^\alpha u\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\right)\leqq R\Bigg\}.$$

Then X_k is complete with respect to the metric d given by (3.4). For $\phi_+\in H^{0,k}$ and $u\in X_k$ we consider $\Phi(u)$ given by (3.5). We prove that $\Phi: u\mapsto\Phi(u)$ is a contraction on X_k for some R and T. Let α satisfy $|\alpha|=l\leqq k$. We have

$$J^\alpha((V*|u|^2)u)(\tau)=(V*|u|^2)J^\alpha u+(V*(\bar u J^\alpha u))u+(-1)^k(V*(u\overline{J^\alpha u})u$$
$$+\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \alpha_1,\alpha_2,\alpha_3\neq\alpha}}\frac{(-1)^{|\alpha_1|}\alpha!}{\prod_{j=1}^{3}\alpha_j!}(V*(\overline{J^{\alpha_1}u}J^{\alpha_2}u))J^{\alpha_3}u,$$

so that in the same way as in (3.10),

$$\|J^\alpha((V_j*|u|^2)u)\|_{\theta_j'}\leqq C\|u\|_{p_j}^2\|J^\alpha u\|_{\rho_j}+C\sum_{\substack{\alpha_1+\alpha_2+\alpha_3=\alpha\\ \alpha_1,\alpha_2,\alpha_3\neq\alpha}}\|J^{\alpha_1}u\|_{p_j}\|J^{\alpha_2}u\|_{p_j}\|J^{\alpha_3}u\|_{\rho_j}. \tag{3.18}$$

By the Gagliardo–Nirenberg inequalities, for β with $|\beta|\leqq l-1$,

$$\|J^\beta u\|_{p_j}\leqq C\|u\|_{p_j}^{1-a(\beta)}\left(\sum_{|\gamma|=l}\|J^\gamma u\|_{\rho_j}\right)^{a(\beta)}t^{|\beta|-la(\beta)}, \tag{3.19}$$

$$\|J^\beta u\|_{\rho_j}\leqq C\|u\|_{p_j}^{1-b(\beta)}\left(\sum_{|\gamma|=l}\|J^\gamma u\|_{\rho_j}\right)^{b(\beta)}t^{|\beta|-lb(\beta)}, \tag{3.20}$$

where

$$a(\beta)=\frac{\dfrac{|\beta|}{n}}{\dfrac{l}{n}+1-\dfrac{\gamma_j}{2n}-\dfrac{3}{2\rho_j}-\dfrac{1}{2\theta_j}}$$

and

$$b(\beta)=\frac{\dfrac{|\beta|}{n}+1-\dfrac{\gamma_j}{2n}-\dfrac{3}{2\rho_j}-\dfrac{1}{2\theta_j}}{\dfrac{l}{n}+1-\dfrac{\gamma_j}{2n}-\dfrac{3}{2\rho_j}-\dfrac{1}{2\theta_j}}.$$

We note here that $a(\beta)$, $b(\beta)\in[0,1)$ if $|\beta|\leqq l-1$ and that $a(\alpha_1)+a(\alpha_2)+b(\alpha_3)=1$ if $|\alpha_1+\alpha_2+\alpha_3|=l$ and $|\alpha_1|,|\alpha_2|,|\alpha_3|\leqq l-1$. Collecting (3.7), (3.18), (3.19) and (3.20), we have

$$\sum_{|\alpha|=l}\|J^\alpha((V_j*|u|^2)u)\|_{\theta_j'}\leqq C\|u\|_{p_j}^2\sum_{|\gamma|=l}\|J^\gamma u\|_{\rho_j}\leqq CR^2t^{-2\delta(p_j)}\sum_{|\gamma|=l}\|J^\gamma u\|_{\rho_j}. \tag{3.21}$$

This leads to

$$\sum_{|\alpha|\leqq k}\left(\|J^\alpha\Phi(u)\|_{L^\infty(T,\infty;L^2)}+\sum_{j=1}^{2}\|J^\alpha\Phi(u)\|_{L^{\sigma_j}(T,\infty;L^{\rho_j})}\right)$$

$$\leqq C\|\phi_+\|_{0,k} + CR^2 \sum_{j=1}^{2} \sum_{|\alpha|\leqq k} \left(\int_T^\infty t^{-2\nu_j \delta(p_j)} dt \right)^{1/\nu_j} \left(\int_T^\infty \| J^\alpha u \|_{p_j}^{\sigma_j} dt \right)^{1/\sigma_j}$$

$$\leqq C\|\phi_+\|_{0,k} + CR^3 \sum_{j=1}^{2} T^{-\varepsilon_j}.$$

Therefore Φ maps X_k into itself and is a contraction in the metric d provided $C\|\phi_+\|_{0,k} \leqq \frac{R}{2}$ and $CR^2 \sum_{j=1}^{2} T^{-\varepsilon_j} \leqq \frac{1}{2}$. This shows that Φ has a unique fixed point u in X_k. Exactly as in the preceding estimate, we have

$$\| U(-t)u(t) - \phi_+ \|_{0,k} \leqq C \sum_{j=1}^{2} t^{-\varepsilon_j} \to 0 \quad \text{as} \quad t \to \infty.$$

The remaining statement of the theorem follow in the same way as in the case $k = 1$.

Proof of Theorem 2. We consider the + case only, since the other case can be treated similarly. We first prove that W_+ is injective. Let $\phi_+, \psi_+ \in H^{0,k}$ satisfy $W_+(\phi_+) = W_+(\psi_+) = \phi$. Let u be the solution of (1.4). Then

$$\begin{aligned} \|\phi_+ - \psi_+\|_{0,k} &\leqq \| U(-t)u(t) - \phi_+ \|_{0,k} \\ &\quad + \| U(-t)u(t) - \psi_+ \|_{0,k} \to 0 \quad \text{as} \quad t \to \infty. \end{aligned}$$

This proves $\phi_+ = \psi_+$ as required. We next prove that W_+ is continuous from $H^{0,k}$ into itself. Let $\phi_+, \psi_+ \in H^{0,k}$ and let $\phi = W_+\phi_+$, $\psi = W_+\psi_+$. Let u and v be the corresponding solutions of (1.4) with initial data ϕ and ψ, respectively. By the argument in the proof of Theorem 1, there exist $T \geqq 1$ and $R > 0$ such that $u, v \in X_k$,

$$J^\alpha u(t) = U(t)x^\alpha \phi_+ + i \int_t^\infty U(t-\tau) J^\alpha((V * |u|^2)u)(\tau) d\tau, \quad |\alpha| \leqq k,$$

$$J^\alpha v(t) = U(t)x^\alpha \psi_+ + i \int_t^\infty U(t-\tau) J^\alpha((V * |v|^2)u)(\tau) d\tau, \quad |\alpha| \leqq k,$$

Subtracting the equations above and estimating the resulting equation in X_k in the same way as in the proof of Theorem 1, we obtain for any $t \geqq T$,

$$\begin{aligned} &\sum_{|\alpha|\leqq k} \left(\| J^\alpha(u-v) \|_{L^\infty(t,\infty;L^2)} + \sum_{j=1}^{2} \| J^\alpha(u-v) \|_{L^{\sigma_j}(t,\infty;L^{\rho_j})} \right) \\ &\quad \leqq C\|\phi_+ - \psi_+\|_{0,k} + CR^2 \sum_{|\alpha|\leqq k} \sum_{j=1}^{2} t^{-\varepsilon_j} \| J^\alpha(u-v) \|_{L^{\sigma_j}(t,\infty;L^{\rho_j})}. \end{aligned} \tag{3.22}$$

By (3.22) there exists $t_0 \geqq T$ such that

$$\begin{aligned} &\sum_{|\alpha|\leqq k} \left(\| J^\alpha(u-v) \|_{L^\infty(t_0,\infty;L^2)} + \sum_{j=1}^{2} \| J^\alpha(u-v) \|_{L^{\sigma_j}(t_0,\infty;L^{\rho_j})} \right) \\ &\quad \leqq 2C\|\phi_+ - \psi_+\|_{0,k}. \end{aligned} \tag{3.23}$$

Proposition 2.1 and (3.23) imply that W_+ is continuous from $H^{0,k}$ into itself. We now prove (1.5). Let $\psi \in H^{0,1}$, let $\phi = W_+\psi$ and let u be the solution of (1.4). Then

$\| u(t) \|_2 = \| \phi \|_2$ for any $t \in \mathbb{R}$. By (1.3),

$$| \| \phi \|_2 - \| \psi \|_2 | = | \| U(-t)u(t) \|_2 - \| \psi \|_2 | \leqq \| U(-t)u(t) - \psi \|_2 \to 0 \quad \text{as} \quad t \to \infty,$$

which proves (1.5).

4. Proof of Theorems 3 and 4

In view of Proposition 2.1, part (1) of Theorem 3 follows in the same way as in the proof of Theorem 1 and part (2) follows in the same way as in [10]. We shall prove part (3) of Theorem 3 and then Theorem 4 in the + case. The other case can be proved analogously. Let $\phi \in H^{0,k}$ satisfy $\| \phi \|_2 < \| Q \|_2$ and let u be the solution of (1.4). We shall show that $J^\alpha u \in L^\infty(\mathbb{R}_+; L^2) \cap L^\sigma(\mathbb{R}_+; L^\sigma)$ for all α with $|\alpha| \leqq k$, where

$$\sigma = 2 + \frac{4}{n}.$$

We note here that

$$\frac{2}{\sigma} = \delta(\sigma).$$

By Proposition 2.1, u satisfies

$$\| u(t) \|_2 = \| \phi \|_2, \qquad t \in \mathbb{R}. \tag{4.1}$$

$$\| Ju(t) \|_2^2 + t^2 (V_2 * |u(t)|^2, |u(t)|^2) = \| x\phi \|_2^2, \quad t \in \mathbb{R}. \tag{4.2}$$

By (1.7), (4.1) and (4.2), we obtain

$$\| Ju(t) \|_2^2 = \| x\phi \|_2^2 + t^2 (V_2 * |M(-t)u(t)|^2, |M(-t)u(t)|^2) \leqq \| x\phi \|_2^2 + \frac{\| \phi \|_2^2}{\| Q \|_2^2} \| Ju(t) \|_2^2, \quad t \neq 0,$$

and therefore

$$\| Ju(t) \|_2^2 \leqq \left(1 - \frac{\| \phi \|_2^2}{\| Q \|_2^2} \right)^{-1} \| x\phi \|_2^2, \quad t \in \mathbb{R}. \tag{4.3}$$

As in (3.7), for any $q \in \left(2, \frac{2n}{n-2} \right]$ we have by (4.1) and (4.3),

$$\| u(t) \|_q \leqq C t^{-\delta(q)}, \quad t > 0. \tag{4.4}$$

Since $\frac{1}{2} - \frac{1}{n} < \frac{1}{\sigma} < 2$, as in (3.21), we have

$$\sum_{|\alpha| \leqq k} \| J^\alpha ((V_2 * |u|^2) u) \|_{\sigma'} \leqq C \| u \|_p^2 \sum_{|\alpha| \leqq k} \| J^\alpha u \|_\sigma, \tag{4.5}$$

where $\frac{1}{p} = \frac{1}{2} - \frac{1}{n} + \frac{1}{n+2}$. By Lemma 2.1, (4.4) and (4.5), we estimate

$$J^\alpha u(\tau) = U(\tau - T)(J^\alpha u(T)) - i \int_T^\tau U(\tau - s) J^\alpha ((V_2 * |u|^2) u)(s) ds$$

in $L^\infty(T,t;L^2)\cap L^\sigma(T,t;L^\sigma)$ for $t>T>0$ to obtain

$$\begin{aligned}
&\sum_{|\alpha|\leqq k}(\|J^\alpha u\|_{L^\infty(T,t;L^2)}+\|J^\alpha u\|_{L^\sigma(T,t;L^\sigma)})\\
&\leqq C\sum_{|\alpha|\leqq k}\left(\|J^\alpha u(T)\|_2+\left(\int_T^t s^{-(n+2)\delta(\sigma)}ds\right)^{2/(n+2)}\|J^\alpha u\|_{L^\sigma(T,t;L^\sigma)}\right)\\
&\leqq C\sum_{|\alpha|\leqq k}(\|J^\alpha u(T)\|_2+T^{-2(n-1)/(n+2)}\|J^\alpha u\|_{L^\sigma(T,t;L^\sigma)}).
\end{aligned}\tag{4.6}$$

Choosing $T>0$ large enough, we have

$$\sum_{|\alpha|\leqq k}(\|J^\alpha u\|_{L^\infty(T,t;L^2)}+\|J^\alpha u\|_{L^\sigma(T,t;L^\sigma)})\leqq C\sum_{|\alpha|\leqq k}\|J^\alpha u(T)\|_2.\tag{4.7}$$

Since the right-hand side of (4.7) is independent of $t>T$, the Fatou lemma proves that $J^\alpha u\in L^\infty(T,\infty;L^2)\cap L^\sigma(T,\infty;L^\sigma)$ for all α with $|\alpha|\leqq k$ and therefore our claim follows. In the same way as in the proof of Theorem 2, it follows that for any α with $|\alpha|\leqq k$ the map $\phi\mapsto J^\alpha u$ is continuous from $H^{0,k}$ into $L^\infty(\mathbb{R}_+;L^2)\cap L^\sigma(\mathbb{R}_+;L^\sigma)$. By (4.6), we have for $\tau>T>0$,

$$\begin{aligned}
&\|U(-\tau)u(\tau)-U(-T)u(T)\|_{0,k}\\
&\leqq C\sum_{|\alpha|\leqq k}\|x^\alpha(U(-\tau)u(\tau)-U(-T)u(T))\|_2\\
&\leqq C\sum_{|\alpha|\leqq k}\left\|\int_T^\tau U(\tau-s)J^\alpha((V_2*|u|^2)u)(s)ds\right\|_2\\
&\leqq CT^{-2(n-1)/(n+2)}\sum_{|\alpha|\leqq k}\|J^\alpha u\|_{L^\sigma(T,\infty;L^\sigma)}\to 0\quad\text{as}\quad T\to\infty.
\end{aligned}\tag{4.8}$$

This implies that there exists a unique $\phi_+\in H^{0,k}$ such that

$$\|U(-t)u(t)-\phi_+\|_{0,k}\to 0\quad\text{as}\quad t\to+\infty,\tag{4.9}$$

which in turn shows that $\mathrm{Range}(W_+)=H^{0,k}$. Moreover,

$$\phi_+=U(-t)u(t)-i\int_t^\infty U(-\tau)(V_2*|u|^2)u(\tau)d\tau.\tag{4.10}$$

Since $\phi\mapsto J^\alpha u$ is continuous from $H^{0,k}$ into $L^\infty(\mathbb{R}_+;L^2)\cap L^\sigma(\mathbb{R}_+;L^\sigma)$ for any α with $|\alpha|\leqq k$, we see from (4.10) that $\phi\mapsto\phi_+$ is continuous from $H^{0,k}$ into itself. This proves that W_+ is a homeomorphism from $H^{0,k}$ into itself. Therefore (1.8) follows from (1.5).

5. Proof of Theorems 5

Let $\sigma=2+\dfrac{4}{n}$ as in the preceding section. For $R_0\geqq R>0$ we define

$$Y=\{u\in C(\mathbb{R};L^2)\cap L^\sigma(\mathbb{R};L^\sigma);\ \|u\|_{L^\infty(\mathbb{R};L^2)}\leqq R_0,\ \|u\|_{L^\sigma(\mathbb{R};L^\sigma)}\leqq R\}.$$

Then Y is a complete metric space under the metric $d(u,v)=\|u-v\|_{L^\sigma(\mathbb{R};L^\sigma)}$. For

$\phi \in L^2$ and $u \in Y$, we consider $\Phi(u)$ given by

$$(\Phi(u))(t) = U(t)\phi - i\int_0^t U(t-\tau)(V_2 * |u|^2)u(\tau)d\tau. \tag{5.1}$$

As in (3.6), we have

$$\begin{aligned}\|(V_2 * |u|^2)u\|_{\sigma'} &\leqq C\|u\|_p^2\|u\|_\sigma \\ &\leqq C\|u\|_2^{2-(4/n)}\|u\|_\sigma^{(4/n)+1} \leqq CR_0^{2-(4/n)}\|u\|_\sigma^{(4/n)+1},\end{aligned} \tag{5.2}$$

where $\frac{1}{p} = \frac{1}{2} - \frac{1}{n} + \frac{1}{n+2}$, where is decomposed as $\frac{1}{p} = \frac{2}{n\sigma} + \frac{1-2/n}{2}$. By (5.1) and (5.2), we have

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(\mathbb{R};L^2)} &\leqq \|\phi\|_2 + C\|(V_2 * |u|^2)u\|_{L^{\sigma'}(\mathbb{R};L^{\sigma'})} \\ &\leqq \|\phi\|_2 + CR_0^{2-(4/n)}\|u\|_{L^\sigma(\mathbb{R};L^\sigma)}^{1+(4/n)} \\ &\leqq \|\phi\|_2 + CR_0^{2-(4/n)}R^{1+(4/n)}\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\|\Phi(u)\|_{L^\sigma(\mathbb{R};L^\sigma)} &\leqq \|U(\cdot)\phi\|_{L^\sigma(\mathbb{R};L^\sigma)} + C\|(V_2 * |u|^2)u\|_{L^{\sigma'}(\mathbb{R};L^{\sigma'})} \\ &\leqq \|U(\cdot)\phi\|_{L^\sigma(\mathbb{R};L^\sigma)} + CR_0^{2-(4/n)}R^{1+(4/n)}.\end{aligned} \tag{5.4}$$

For $u, v \in Y$, as in (3.2) and (5.2), we have

$$\begin{aligned}\|(V_2 * |u|^2)u - (V_2 * |v|^2)v\|_{\sigma'} &= C(\|u\|_p^2 + \|v\|_p^2)\|u-v\|_\sigma \\ &\leqq CR_0^{2-(4/n)}(\|u\|_\sigma^{4/n} + \|v\|_\sigma^{4/n})\|u-v\|_\sigma.\end{aligned} \tag{5.5}$$

By (5.5),

$$\begin{aligned}&\|\Phi(u) - \Phi(v)\|_{L^\sigma(\mathbb{R};L^\sigma)} \\ &\leqq \|(V_2 * |u|^2)u - (V_2 * |v|^2)v\|_{L^{\sigma'}(\mathbb{R};L^{\sigma'})} \\ &\leqq CR_0^{2-(4/n)}(\|u\|_{L^\sigma(\mathbb{R};L^\sigma)}^{4/n} + \|v\|_{L^\sigma(\mathbb{R};L^\sigma)}^{4/n})\|u-v\|_{L^\sigma(\mathbb{R};L^\sigma)} \\ &\leqq CR_0^{2-(4/n)}R^{4/n}\|u-v\|_{L^\sigma(\mathbb{R};L^\sigma)}.\end{aligned} \tag{5.6}$$

We now choose $\|\phi\|_2 \leqq \frac{R_0}{2}$ and $CR_0^{2-(4/n)}R^{4/n} \leqq \frac{1}{2}$ with $R \in (0, R_0)$ sufficiently small. If $\|U(\cdot)\phi\|_{L^\sigma(\mathbb{R};L^\sigma)} \leqq \frac{R}{2}$, it follows from (5.3), (5.4) and (5.6) that $\Phi: u \mapsto \Phi(u)$ maps Y into itself and is a contraction in the metric d. By Lemma 2.1, the condition $\|U(\cdot)\phi\|_{L^\sigma(\mathbb{R};L^\sigma)} \leqq \frac{R}{2}$ is always accomplished by choosing $\|\phi\|_2$ to be small enough. We have thus proved that for any $\phi \in L^2$ with $\|\phi\|_2$ sufficiently small, Φ has a unique fixed point u in Y. Similarly, we see from (5.6) that $\phi \mapsto u$ is continuous from a small L^2 ball centered at the origin into $L^\infty(\mathbb{R}; L^2) \cap L^\sigma(\mathbb{R}; L^\sigma)$. By approximating ϕ by sequence in H^1 in the same way as in [1, 14], we conclude from the continuous dependence above and from the L^2 conservation for the corresponding H^1 solutions that $\|u(t)\|_2 = \|\phi\|_2$. This proves part (1). We next prove part (3). Let ϕ and u be as above. Fort $t>s$, we have

$$U(-t)u(t)=U(-s)u(s)-i\int_s^t U(-\tau)(V_2*|u|^2)u(\tau)d\tau, \tag{5.7}$$

and therefore by (5.2),

$$\|U(-t)u(t)-U(-s)u(s)\|_2=\left\|\int_s^t U(t-\tau)(V_2*|u|^2)u(s)ds\right\|_2$$
$$\leqq C\|\phi\|_2^{2-(4/n)}\left(\int_s^t \|u(\tau)\|_\sigma^\sigma d\tau\right)^{1-\sigma}\to 0 \quad \text{as} \quad t>s\to\infty. \tag{5.8}$$

This proves part (3). We proceed to part (2). We treat the + case only. The other case can be proved similarly. For $\phi_+\in L^2$ and $u\in Y$ we consider

$$(\Phi(u))(t)=U(t)\phi_+ + i\int_t^\infty U(t-\tau)(V_2*|u|^2)u(\tau)d\tau. \tag{5.9}$$

In the same way as in the proof of part (1), we find that Φ has a unique fixed point in Y if $\|\Phi_+\|_2$ is small enough. As in (5.8), that solution u satisfies (5.7) and the wave operator W_+ is given by

$$\phi=W_+\phi_+=\phi_+ + i\int_0^\infty U(-\tau)(V_2*|u|^2)u(\tau)d\tau. \tag{5.10}$$

This proves part (2). Part (4) follows in the same way as in the proof of Theorems 2 and 4.

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