Commun. Math. Phys. 146, 61–71 (1992)

Communications in
Mathematical
Physics

Quantum Ergodicity on the Sphere

Steven Zelditch

Department of Mathematics, The Johns Hopkins University, Baltimore, MD 21218, USA

Received March 23, 1990; in revised form October 10, 1991

Abstract. We prove that a random orthonormal basis of eigenfunctions on the standard sphere has quantum ergodic behavior.

As the title portends, this paper is about quantum ergodicity in the most completely integrable of examples: the Laplacian on S^2. The notion of quantum ergodicity we pursue here is the one which characterizes ergodicity of a Schrödinger (or Laplace) operator H in terms of the semi-classical behaviour of its eigenfunctions ([Sn, V, Be, Z.1–2, CdeV.2, HMR, ST]). Roughly, H is quantum ergodic if its orthonormal bases $\{\varphi_j\}$ of eigenfunctions have the following property: $(A\varphi_j, \varphi_j) \to \int_{S^*M} \sigma_A d\mu$ $(j \to \infty)$ for any 0^{th} order ΨDO A ($d\mu$ = Liouville measure, σ_A = principal symbol). This limit formula is a kind of quantum analogue of the Birkhoff ergodic theorem and is known to hold whenever the classical (e.g. geodesic) flow is ergodic. Otherwise it does not seem well understood: for example, it might (for all that is proved to date) even hold for a generic Laplacian. Our purpose here is, perversely, to investigate it on the sphere. Of course, the usual basis $\{Y^l_m\}$ of spherical harmonics does not have the ergodic property. But, due to the high degeneracy of eigenvalues, there is an infinite dimensional manifold of orthonormal bases of eigenfunctions. This manifold is actually a group and carries a unit mass Haar measure. Our main result is that, relative to this measure, almost all bases have the ergodic property.

To state the result more precisely we will need to introduce some terminology and background.

Throughout this paper we will be considering only the standard 2-sphere S^2, although our methods would work on many other spaces. We will usually omit explicit reference to the metric on S^2; all notation such as $L^2(S^2)$, Δ, etc., will refer to the standard metric.

We first recall that

$$L^2(S^2) = \bigoplus_l E_l \quad (\dim E_l = 2l + 1), \tag{1.1}$$

where E_l is the complex eigenspace of spherical harmonics of degree l. Equivalently, E_l is the eigenspace of the laplacian Δ of eigenvalue $l(l+1)$. We will let π_l denote the orthogonal projection onto E_l:

$$\pi_l: L^2(S^2) \to E_l. \tag{1.2}$$

Next, recall that a (hermitian) orthonormal basis for E_l is provided by the vectors $\{Y^l_m, m = -l, \dots, l\}$, determined (up to constants) by the conditions;

$$\begin{cases} \Delta Y^l_m = l(l+1) Y^l_m \\ (1/i)\partial/\partial\theta Y^l_m = m Y^l_m, \end{cases} \tag{1.3}$$

where $\partial/\partial\theta$ generates rotation around the Z-axis. See, for example, [T, 2.1] for the specific constants.

The basis $\{Y^l_m, l = 0, 1, 2, \dots; m = -l, \dots, l\}$ has some very special asymptotic properties as $l \to \infty$, reflecting the complete integrability of the geodesic flow on S^2. Indeed, for any fixed rational e, let $\mathscr{H}_e$ be the subspace of $L^2(S^2)$, spanned by $\{Y^l_m : m/l = e\}$. $\mathscr{H}_e$ is the quantum analogue of the invariant torus $T_e \subset S^*(S^2)$ for the geodesic flow, consisting of all great circles whose angle ϕ with the Z-axis satisfies: $\cos\phi = e$. The precise meaning of "analogue" may be stated in several ways:

(i) T_e is the microsupport of the quasi-mode $\mathscr{H}_e$ (more precisely, the cone through T_e; see [C-de V]);

(ii)
$$\lim_{\substack{l \to \infty \\ m/l = e}} (A Y^l_m, Y^l_m) = \int_{T_e} \sigma_A d\mu_e, \tag{1.4}$$

where A is a 0^{th} order pseudo-differential operator with symbol σ_A, and where $d\mu_e$ is the invariant probability measure on T_e for the torus action determined by the geodesic flow and by rotation about the z-axis (see [Z.1]);

(iii) $\mathscr{H}_e$ is the ladder subspace corresponding to the co-isotropic cone $\Phi^{-1}(\{r(m, l): m/l = e, r \in \mathbb{R}^+\})$, where Φ is the moment map for the above torus action ([G·S]).

Thus, the foliation of $S^*(S^2)$ by invariant torii T_e for the geodesic flow has for quantum analogue the decomposition of L^2 into ladders $\mathscr{H}_e$, invariant under Δ; sequences of eigen-functions in $\mathscr{H}_e$ concentrate, in the limit of high eigenvalues, on T_e; or, equivalently, functions in $\mathscr{H}_e$ can only have their wave front sets in the cone thru T_e.

The main object of the present paper is to contrast these asymptotic properties of the "completely integrable orthonormal basis" $\{Y^l_m\}$ with the properties of a "random" orthonormal basis of Laplace eigenfunctions. It turns out that the random basis behaves like an orthonormal basis of eigenfunctions on a Riemannian manifold with ergodic geodesic flow.

Let us recall this ergodicity of eigenfunctions ([Sn, Z.2, (CdV.2)]). To begin with, let (M, g) be any compact, Riemannian manifold, let G^t be the geodesic flow on S^*M, let Δ be the Laplacian and let $\{\varphi_j\}$ be an orthonormal basis of Laplace eigenfunctions: $\Delta\varphi_j = \lambda_j \varphi_j$. Let us say:

(1.5) Definition. *$\{\varphi_j\}$ is a Liouville-distributed basis if, for any* ΨDO *(pseudo-differential operator) A of order* 0,

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\sqrt{\lambda_j} \leq \lambda} |(A\varphi_j, \varphi_j) - \bar{\sigma}_A|^2 = 0. \tag{1.6}$$

Here, $N(\lambda)=\#\{j:\sqrt{\lambda_j}\leq\lambda\},(A\varphi_j,\varphi_j)$ is the matrix-coefficient of the ΨDO A, and $\bar{\sigma}_A=\frac{1}{\mathrm{vol}(S^*M)}\int_{S^*M}\sigma_A d\mu$, where $d\mu$ is the Liouville measure on S^*M.

To understand (1.6), we recall a standard lemma on bounded sequences $\{a_j\}$ in $\mathbb{C}$ ([Wa]).

This says:

$$\lim_N (1/N)\sum_{j\leq N}|a_j|^p=0 \quad (\text{any } p>0) \tag{1.7a}$$

iff

$$\lim_{k\to\infty} a_{j_k}=0, j_k\in\mathcal{S}\subset\mathbb{Z}^+, \text{ where } \mathcal{S}=\{j_k\} \text{ is a subsequence of } \mathbb{Z}^+ \text{ of}$$

$$\text{density } 1: 1/N\#\{j_k\in\mathcal{S}\}\cap\{1,2,\ldots,N\}\to 1. \tag{1.7b}$$

The limit formula (1.6) is thus equivalent to the existence of a subsequence $\mathcal{S}=\{j_k\}$ of spectral density one (in the obvious sense) for which

$$(A\varphi_{j_k},\varphi_{j_k})\to\bar{\sigma}_A. \tag{1.8}$$

The subsequence $\mathcal{S}$ a priori depends on A. However by a diagonalization argument one can show that (1.6) implies the existence of a subsequence $\mathcal{S}$ of density one for which (1.8) holds for all A (see [Z.2]).

The limit formulae (1.8) and (1.6) give a quantum analogue of ergodicity. For example if σ_A is the characteristic function of a nice subset $E\subset S^*M$, then $(A\varphi_j,\varphi_j)$ is interpreted as the probability that a free particle in state φ_j has its (position, momentum) in E. As the energy λ_j tends to infinity, this should tend to the probability that a free classical particle, i.e. a geodesic, goes through E. This probability, calculated according to (1.8), is just the Liouville measure of E.

This heuristic reasoning suggests that if the geodesic flow G^t is ergodic, then (1.8) should hold for any orthonormal basis of eigenfunctions. Indeed, this is the case ([Sn, Z.2, CdV]). Conversely, if (1.8) holds for *any* orthonormal basis of eigenfunctions, then the geodesic flow should be ergodic. This converse direction seems however to be quite difficult. Our main result in this paper implies that, if we replace the "any" by "some" or even "almost any," then in fact the converse is false. Indeed, we will show that the random orthonormal basis of eigenfunctions on S^2 is Liouville distributed.

It emerges that existence of a Liouville-distributed basis of eigenfunctions for a Laplacian Δ is not a good definition of the "quantum ergodicity " of Δ. It seems to us that a more reasonable criterion for quantum ergodicity is in terms of operator time-averages, where $\bar{A}=w\text{-}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-it\sqrt{\Delta}}Ae^{it\sqrt{\Delta}}dt$ is the time average of A. The criterion is: Δ (or $e^{it\sqrt{\Delta}}$) is ergodic if, for all ψDO's A or order 0, $\bar{A}=\bar{\sigma}_A(\mathrm{Id})+K$, where the Hilbert–Schmidt norm of $\pi_\lambda K\pi_\lambda$ is $o(N(\lambda))$ as $\lambda\to\infty$ (π_λ is the spectral projection for the interval $[0,\lambda]$ and $N(\lambda)=\mathrm{tr}\,\pi_\lambda$. In [Z1–Z2], we showed that Δ is ergodic in this sense if the geodesic flow is ergodic; such ergodicity also implies that all orthonormal bases are Liouville distributed. It is not yet clear to us if this ergodicity of Δ implies classical ergodicity.

Returning to our main result, we first note that the set $\mathcal{OB}$ of orthonormal basis of Laplace eigenfunctions of S^2 is a probability space. Indeed, any orthonormal basis $\{\varphi^\ell_m : m = -\ell, \ldots, \ell\}$ of E_ℓ corresponds to an element $\tau_\ell \in U(2\ell + 1)$ via:

$$\varphi^\ell_m = \tau_\ell Y^\ell_m. \tag{1.9}$$

It follows that the manifold of orthonormal bases of E_ℓ can be identified via (1.9) with the unitary group $U(2l+1)$. Hence, the infinite dimensional manifold of orthonormal bases of eigenfunctions can be identified with the product:

$$\mathcal{OB} \cong U(1) \times U(3) \times U(5) \times \cdots. \tag{1.10}$$

Now each $U(2\ell + 1)$ carries a Haar measure $d\mu_\ell$, normalized to have mass one. Hence $\mathcal{OB}$ carries the probability measure

$$d\mu_\infty = d\mu_1 \times d\mu_3 \times \cdots. \tag{1.11}$$

Let us denote by τ_∞ the sequence $(\tau_1, \tau_2, \tau_3, \cdots)$. Let us say, as in (1.6)–(1.8), that τ_∞ is Liouville distributed,

$$\tau_\infty \in \mathcal{LOB}, \tag{1.12}$$

if, for any ΨDO A of order 0,

$$\lim_{L\to\infty} (1/L) \sum_{\ell \leqq L} \left(\frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |(\pi_\ell A \pi_\ell \tau_\ell Y^\ell_m, \tau_\ell Y^\ell_m) - \bar{\sigma}_A|^2 \right) = 0. \tag{1.13}$$

Our main result is, then:

$$\mu_\infty(\mathcal{LOB}) = 1. \tag{1.14}$$

Thus, almost every orthonormal basis of Laplace eigenfunctions on S^2 behaves ergodically.

2. $\mu_\infty(\mathcal{LOB}) = 1$

Let A be a Hermitian ΨDO of order 0 on $L^2(S^2, \text{can})$. Associated to A is the sequence $\{\pi_\ell A \pi_\ell : \ell = 0, 1, 2, \ldots\}$ of finite rank Hermitian operators on the sequence $\{E_\ell\}$ of Hermitian vector spaces. Having fixed the basis $\{Y^\ell_m : m = -\ell, \ldots, \ell\}$ of E_ℓ, we can identify E_ℓ with $\mathbb{C}^{2\ell+1}$ and $\pi_\ell A \pi_\ell$ with a $(2\ell + 1) \times (2\ell + 1)$ Hermitian matrix. Thus, $i\pi_\ell A \pi_\ell$ may be considered an element of $\mathfrak{u}(2\ell + 1)$ (the Lie algebra of $U(2\ell + 1) \cong U(E_\ell)$. As such, it is conjugate to an element $\mu^\ell(A) = \begin{pmatrix} \mu^\ell_1 & & \\ & \ddots & \\ & & \mu^\ell_{2\ell+1} \end{pmatrix}$ of the Cartan subalgebra $\mathfrak{h}_{2\ell+1}$ of diagonal matrices in $U(2\ell + 1)$. $\mu^\ell = \mu^\ell(A)$ is uniquely determined if we require that $\mu^\ell_1 \geqq \mu^\ell_2 \geqq \cdots \geqq \mu^\ell_{2\ell+1}$, i.e. that μ^ℓ lies in the positive Weyl chamber $\mathfrak{h}_{2\ell+1,+}$ of $U(2\ell + 1)$.

Corresponding in turn to μ^ℓ are:

i) its adjoint orbit $\mathcal{O}_\mu$ in $\mathfrak{u}(2\ell + 1)$;

ii) the probability measure $dm_{\mu^\ell} = \dfrac{1}{2\ell + 1} \sum_{i=1}^{2\ell+1} \delta(\mu - \mu^\ell_i)$ on $\mathbb{R}$. (2.1)

The correspondences $\mu^\ell \mapsto \mathcal{O}_{\mu^\ell} \leftrightarrow dm_{\mu^\ell}$ are all 1–1. On the other hand, the correspondences $A \to \{\pi_\ell A \pi_\ell\} \to \{\mu^\ell(A)\}$ are not. Indeed, if $A^{\text{ave}} \overset{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} e^{-itP} A e^{itP} dt$, where $P|_{E_\ell} = \ell$, then $A^{\text{ave}} = \bigoplus_\ell \pi_\ell A \pi_\ell$; hence A and A^{ave} correspond to the same sequence $\{\pi_\ell A \pi_\ell\}$. A^{ave} is of course just the diagonal part of A, studied at some length in ([Wei, Wi, Gu, U]). Since the definition of $\mathcal{LOB}$ depends only on consideration of diagonal parts of ΨDO's, we will henceforth assume $A = A^{\text{ave}}$. In other words, we will always assume $[A, \Delta] = 0$. The bounded ΨDO's A commuting with Δ form a ring, denoted $\mathscr{C}$ in [U]. With no loss of generality, we assume henceforth that $A \in \mathscr{C}$.

Even with this assumption, the correspondence $\{\pi_\ell A \pi_\ell\} \to \{\mu^\ell(A)\}$ is not 1–1. For example, if $A^g = T_g A T_g^*$, where $g \in SO(3)$ and T_g is the corresponding unitary translation on $L^2(S^2)$, then obviously $\mu^\ell(A^g) = \mu^\ell(A)$. (It is possible however that, conversely, $\mu^\ell(A_1) = \mu^\ell(A)(\forall \ell)$ implies $A_1 = A^g$ for some g.) Moreover, the correspondence is far from surjective. Indeed, the Szegö limit Theorems of Weinstein–Widom (loc. cit.) show that the sequences $\{\mu_\ell(A)\}$ have quite special asymptotic properties. The most important for this paper is that the measures $dm_{\mu^\ell}(A)$ have a weak limit dm_{μ_∞} as $\ell \to \infty$:

$$dm_\infty = \sigma_A^{\text{ave}} * d\mu, \tag{2.2}$$

where $d\mu$ is Liouville (i.e. Haar) measure on $S^*(S^2)$ (i.e. $SO(3)$), and where σ_A^{ave} is the averaged principal symbol of A:

$$\sigma_A^{\text{ave}}(x, \xi) = \frac{1}{2\pi} \int_0^{2\pi} \sigma_A((G^t(x, \xi))dt \quad (G^t = \text{geodesic flow}). \tag{2.3}$$

Actually (2.2) is only the first of a sequence of distributions associated to $\{\mu^\ell(A)\}$. Indeed, it is prove in [Wei] (see also [Gu, U]) that

$$\int_{\mathbb{R}} f\, dm_{\mu^\ell(A)} \sim \sum_{j=0}^{\infty} \beta_j(f) \ell^{-j} \tag{2.4}$$

for certain $\beta_j \in \mathscr{D}'(\mathbb{R})$ (depending on $A \in \mathscr{S}$), and with $\beta_0 = dm_\infty$. We will not use the higher β_j's in this paper. However, their existence constrains the sequence of orbits $\{\mathcal{O}_{\mu^\ell}\}$ associated to $A \in \mathscr{C}$.

We now reformulate our theorem in terms of the orbits $\{\mathcal{O}_{\mu^\ell}\}$.

First, we fix an Ad-invariant inner product $\langle \cdot \rangle$ on $u(2\ell + 1)$: for example, $B_0(X, Y) = \operatorname{Tr} XY$. Then let J_ℓ be the corresponding orthogonal projection: $u(2\ell + 1) \to h_\ell$. Thus,

$$J_\ell(a_{ij}) = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{2\ell+1, 2\ell+1} \end{pmatrix}. \tag{2.5}$$

Under the identifications $u \cong u^*, h \cong h^*$ induced by B_0, the orbits $\mathcal{O}_\mu$ go over to co-adjoint orbits $\mathcal{O}_\mu^*$. These are symplectic manifolds, and the maximal torus H_ℓ corresponding to h_ℓ acts on them in a Hamiltonian fashion (by conjugation). The moment map for this action can then be identified, as above, with J_ℓ.

Further, let $u(2\ell+1)\cong su(2\ell+1)\oplus\mathbb{R}$ be the decomposition in terms of trace-less and scalar parts: $a=a^0+\bar{a}$, where $\bar{a}=\left(\frac{1}{2\ell+1}\operatorname{tr}a\right)I_\ell$, with I_ℓ the $(2\ell+1)\times(2\ell+1)$ identity matrix, and where $a^0=a-a\in su(2\ell+1)$.

Then the condition: $\tau_\infty\in\mathscr{L}\mathcal{O}\mathscr{B}$ can be reformulated by:

$$\tau_\infty\in\mathscr{L}\mathcal{O}\mathscr{B} \quad \text{iff} \quad (\forall A\in\mathscr{C}): \quad \lim_{L\to\infty}(1/L)\sum_{\ell\leq L}\frac{1}{2\ell+1}|J_\ell(\tau_\ell^{-1}(\pi_\ell A\pi_\ell)\tau_\ell)-\bar{\sigma}_A I_\ell|^2=0. \tag{2.6}$$

In view of the Szegö limit formula,

$$\frac{1}{2\ell+1}\operatorname{tr}\pi_\ell A\pi_\ell=\bar{\sigma}_A+O_A(\ell^{-1}). \tag{2.7}$$

So, we can further reformulate this condition as:

$$\tau_\infty\in\mathscr{L}\mathcal{O}\mathscr{B}\Leftrightarrow(\forall A\in\mathscr{C}): \quad \lim_{L\to\infty}(1/L)\sum_{\ell\leq L}\frac{1}{2\ell+1}|J_\ell(\tau_\ell^{-1}(\pi_\ell A\pi_\ell)^0\tau_\ell)|^2=0. \tag{2.8}$$

To demonstrate the usefulness of this reformulation, let us first prove that, for a fixed $A\in\mathscr{C}$, there exists $\tau_\infty\in\mathcal{O}\mathscr{B}$ satisfying;

$$|(\tau_\ell^{-1}((\pi_\ell A\pi_\ell)\tau_\ell Y_m^\ell, Y_m^\ell)-\bar{\sigma}_A|=O_A(\ell^{-1})(\text{uniformly in } m=-\ell,\dots,\ell). \tag{2.9}$$

In fact, (2.9) follows from (2.7) and:

(2.10) Proposition. *Let $\mathscr{P}_\mu$ be the image $J_\ell(\mathcal{O}_\mu)$ of the orbit $\mathcal{O}_\mu$ under the moment map (a convex polytope, see [G–S]). Let dV_μ be the measure on $\mathscr{P}_\mu$ which is the pushforward under J_ℓ of the symplectic volume measure dv_μ on $\mathcal{O}_\mu$. Then the center of mass of $\mathscr{P}_\mu$, relative to dV_μ, is $\bar{\mu}$, where $\bar{\mu}=\left(\frac{1}{2\ell+1}\sum_{i=1}^{2\ell+1}\mu_i\right)I_\ell$.*

Proof. Let $\mu^0=\mu-\bar{\mu}$ be the traceless part of μ; $\mu^0\in h_\ell^0$ (the Cartan subalgebra of $su(2\ell+1)$. Obviously $\mathscr{P}_\mu=\mathscr{P}_{\mu^0}+\bar{\mu}$ (translation in $h_\ell=h_\ell^0\oplus\mathbb{R}$). The proposition is equivalent to: 0 is the center of mass of $\mathscr{P}_{\mu^0}$.

Thus, we claim:

$$\int_{\mathscr{P}_{\mu^0}} H\,dV_{\mu^0}(H)=0. \tag{2.11}$$

Since $\mathscr{P}_{\mu^0}=J_\ell(\mathcal{O}_{\mu^0})$ and $dV_{\mu^0}=J_{\ell_*}dv_{\mu^0}$, (2.11) is equivalent to:

$$\int_{\mathcal{O}_{\mu^0}} J_\ell(\xi)dv_{\mu^0}(\xi)=0, \tag{2.12i}$$

hence to

$$\int_{SU(2\ell+1)} J_\ell(\tau^{-1}\mu^0\tau)d\tau=0 \quad (d\tau=\text{Haar measure}). \tag{2.12ii}$$

But, clearly, $\int_{SU(2\ell+1)}\tau^{-1}\mu^0\tau d\tau=0.$ ■

We now prove (2.9). In view of (2.7), it follows from

$$(\forall A\in\mathscr{C})(\exists\tau_\infty\in\mathcal{O}\mathscr{B})(\forall\ell): J_\ell(\tau_\ell^{-1}(\pi_\ell A\pi_\ell)^0\tau_\ell)=0, \tag{2.13}$$

but this immediately follows from (2.10). ■

We now return to (2.8). Clearly this will involve the asymptotic dispersion from the mean of $\mathscr{P}_{\mu^\ell(A)}$ as $\ell\to\infty$. Among many possible formulations, the following seems quite convenient.

Fix $A\in\mathscr{C}$, and let $\{X_\ell^A\}$ denote the following sequence of positive random variables on $\mathcal{O}\mathscr{B}$:

$$X_\ell^A(\tau_\infty)=|J_\ell(\tau_\ell^{-1}(\pi_\ell A\pi_\ell)\tau_\ell)-\bar\sigma_A I_\ell|^2. \tag{2.14}$$

It is obvious that the X_ℓ^A are independent random variables (they involve different components of τ_∞). Further, the condition that $\tau_\infty\in\mathscr{L}\mathcal{O}\mathscr{B}$ is just that

$$(\forall A\in\mathscr{C})\lim_{L\to\infty}(1/L)\sum_{\ell\leq L}\frac{1}{2\ell+1}X_\ell^A(\tau_\infty)=0.$$

The statement of Theorem (1.14) is that this holds for almost all τ_∞. This can be reduced to Kolmogorov's strong law of large numbers ([I], p. 188) once the expected values and variances of the X_ℓ^A are calculated asymptotically.

Let us denote by:

i) $E(X)=\int_{\mathcal{O}\mathscr{B}}X(\tau_\infty)d\mu_\infty(\tau_\infty)$,

ii) $V(X)=E((X-EX)^2)$

the expected value, respectively the variance, of a random variable $X:\mathcal{O}\mathscr{B}\to\mathbb{R}$.

Our main lemma is:

(2.15) Lemma. *Fix $A\in\mathscr{C}$. Then:*

$$E(X_\ell^A)=\frac{1}{\operatorname{vol}(S^*S^2)}\int_{S^*S^2}|\sigma_A^{\mathrm{ave}}(\zeta)-\bar\sigma_A|^2d\mu(\zeta)+O_A(\ell^{-1}).$$

Proof. Evidently,

$$E(X_\ell^A)=\frac{1}{\operatorname{vol}U(2\ell+1)}\int_{U(2\ell+1)}|J_\ell(\tau^{-1}(\pi_\ell A\pi_\ell)\tau)-\bar\sigma_A I_\ell|^2d\tau\quad(d\tau=\text{Haar measure}). \tag{2.16}$$

We first note that, using (2.7), we may replace X_ℓ^A by Y_ℓ^A where:

$$Y_\ell^A(\tau_\infty)\overset{\text{def}}{=}|J_\ell(\tau_\ell^{-1}(\pi_\ell A\pi_\ell)\tau_\ell)-\overline{\pi_\ell A\pi_\ell}|^2. \tag{2.17}$$

Indeed, $X_\ell^A(\tau_\infty)-Y_\ell^A(\tau_\infty)=(2\ell+1)\left|\bar\sigma_A-\frac{1}{2\ell+1}\operatorname{tr}\pi_\ell A\pi_\ell\right|^2=O_A(1/\ell)$.

Second, we observe that

$$\begin{aligned}E(Y_\ell^A)&=\frac{1}{\operatorname{vol}U(2\ell+1)}\int_{U(2\ell+1)}|J_\ell(\tau^{-1}\mu^\ell(A)\tau-\bar\mu^\ell|^2d\tau\\&=\frac{1}{\operatorname{vol}\mathcal{O}_{\mu^\ell(A)}}\int_{\mathcal{O}_{\mu^\ell(A)}}|J_\ell(\xi)-\bar\mu|^2dv_\mu(\xi).\end{aligned} \tag{2.18}$$

The identities in (2.18) suggest a way of calculating $E(Y_\ell^A)$ asymptotically.

Indeed, for any $\mu \in \mathscr{h}_\ell$, let us set:

$$D_2^2(\mu-\bar{\mu}) \stackrel{\text{def}}{=} \frac{1}{\operatorname{vol} U(2\ell+1)} \int_{U(2\ell+1)} |J_\ell(\tau^{-1}\mu\tau - \bar{\mu}|^2 d\tau$$

$$= \frac{1}{\operatorname{vol} \mathcal{O}_\mu} \int_{\mathcal{O}_\mu} |J_\ell(\xi) - \bar{\mu}|^2 d\nu_\mu(\xi). \tag{2.19}$$

(D_2 is the 2nd deviation of $\mathscr{P}_\mu$ from its mean).

Clearly, $D_2^2(\mu-\bar{\mu})$ is a symmetric, homogeneous polynomial of degree 2 in μ. So there exist constants a_ℓ, b_ℓ such that:

$$D_2^2(\mu-\bar{\mu}) = \frac{1}{2\ell+1}(a_\ell S_1^2(\mu) + b_\ell S_2(\mu)), \tag{2.20}$$

where $S_p(\mu) = \sum_{i=1}^{2\ell+1} \mu_i^p$ is the p^{th} power function. Also, clearly, $E(Y_\ell^A) = D_2^2(\mu^\ell(A) - \bar{\mu}^\ell(A))$. So we are reduced to calculating $\{a_\ell, b_\ell\}$ asymptotically. We claim:

$$\begin{aligned} a_\ell &= -\frac{1}{2\ell+1}[1 + O(1/\ell)] \\ b_\ell &= 1 + O(1/\ell) \end{aligned} , \tag{2.21}$$

where the O-symbols are independent of any parameters.

Proof of (2.21). First, we plug $\mu = (1, 1, \ldots, 1)$ into (2.20) and conclude:

$$(2\ell+1)a_\ell + b_\ell = 0. \tag{2.22}$$

Next, we introduce the Fourier transform of the orbit $\mathcal{O}_\mu(\mu \in \mathfrak{su}(2\ell+1))$:

$$\mathscr{F}_\mu(H) = \frac{1}{\operatorname{vol}(\mathcal{O}_\mu)} \int_{\mathcal{O}_\mu} e^{i<j_\ell(\xi),H>} d\nu_\mu(\xi) \quad (H \in \mathscr{h}_\ell^0). \tag{2.23}$$

Associated to $\langle \ , \ \rangle$ is a gradient ∇ and Laplacian Δ. They satisfy:

$$\begin{aligned} -\Delta e^{i\langle\mu,H\rangle} &= |\mu|^2 e^{i\langle\mu,H\rangle}, \\ (1/i)\nabla e^{i\langle\mu,H\rangle} &= \mu e^{i\langle\mu,H\rangle}. \end{aligned} \tag{2.24}$$

Clearly:

$$D_2^2(\mu-\bar{\mu}) = -\Delta \mathscr{F}_{\mu-\bar{\mu}}(H)|_{H=0}. \tag{2.25}$$

To make use of (2.23), we invoke a well-known formula [G–S, Sect. 33]:

$$\int_{\mathcal{O}_\mu} e^{i\langle j_\ell(\xi),H\rangle} \frac{d\nu_\mu(\xi)}{(2\pi)^n} = \sum_{w\in W} \frac{\varepsilon(w) e^{i\langle w\mu,H\rangle}}{\prod_{\alpha\in R_+} \langle \alpha, H\rangle}, \tag{2.26}$$

where $\mu \in \mathscr{h}_l^0, n = \dim \mathcal{O}_\mu, R_+$ is the set of positive roots for $\mathfrak{su}(2\ell+1)$, and W is the Weyl group of the pair $(G, G_\mu)(G = SU(2\ell+1), G_\mu = \text{stabilizer of } \mu)$.

We will use (2.26) with $\mu = \delta$, where $\delta = (1/2)\sum_{\alpha\in R_+} \alpha$.

First, let us denote by $n(\mu, H)$, respectively $d(H)$, the numerator, respectively the denominator in (2.26):

i) $$n(\mu, H) = \sum_{w \in W} \varepsilon(w) e^{i\langle w\cdot\mu, H\rangle},$$

ii) $$d(H) = \underset{\alpha\in R_+}{\pi} \langle \alpha, H\rangle. \tag{2.27}$$

Let us also denote by $D(H)$ the Weyl denominator:

iii) $$D(H) = \prod_{\alpha\in R_+} (e^{\langle \alpha/2, H\rangle} - e^{-\langle \alpha/2, H\rangle}).$$

Evidently $d(H)$ is the term of order $|R_+|$ in the Taylor expansion of $D(H)$. Further, by Weyl's denominator formula

$$D(H) = \sum_{w\in W} \varepsilon(w) e^{\langle w\delta, H\rangle}. \tag{2.28}$$

Thus:

$$n(\delta, H) = D(H). \tag{2.29}$$

Since $\bar{\delta} = 0$, the thing to calculate is $-\Delta\left(\frac{D(H)}{d(H)}\right)\Big|_{H=0}$.

But

$$\begin{aligned}
-\Delta\left(\frac{D(H)}{d(H)}\right)\Big|_{H=0} &= (-\Delta)\prod_{\alpha\in R_+} \frac{2\sin\langle \alpha/2, H\rangle}{\langle \alpha, H\rangle}\Big|_{H=0}\\
&= (-\Delta)\prod_{\alpha\in R_+}\left(1 - \frac{1}{3!}\langle \alpha/2, H\rangle^2 + O(|H|^3)\right)\Big|_{H=0}\\
&= \frac{1}{4\cdot 3!}\left(\Delta \sum_{\alpha\in R_+} \langle \alpha, H\rangle^2|_{H=0}\right)\\
&= \frac{2}{3!}|R_+| \quad (\text{as } |\alpha|^2 = 4 \text{ for } \alpha\in R_+).
\end{aligned} \tag{2.30}$$

For $SU(2\ell+1), R_+ = \{e_i - e_j : i > j : e_i(\mu_1, \ldots, \mu_{2\ell+1}) = \mu_i\}$. So $|R_+| = \ell(2\ell+1)$ and $D_2^2(\delta) = (1/6)\ell(2\ell+1) = \frac{1}{2\ell+1}(a_\ell S_1^2(\delta) + b_\ell S_2(\delta))$. But $\delta = \ell e_1 + (\ell-1)e_2 + \cdots + e_\ell - e_{\ell+2} - \cdots - \ell e_{2\ell+1}$, so $S_2(\delta) = 2[1^2 + 2^2 + \cdots + \ell^2] = \frac{\ell(\ell+1)(2\ell+1)}{3}$; and, $S_1(\delta) = 0$. Hence $D_2^2(\delta) = \frac{\ell(\ell+1)}{3} b_\ell = \frac{\ell(2\ell+1)}{6}$. Thus, $b_\ell = 1 + 0(1/\ell)$. Combining with (2.22) we get (2.21). ■

This concludes our discussion of (2.21). Returning to (2.15)–(2.18), we see:

$$\begin{aligned}
E(Y_\ell^A) &= \frac{1}{2\ell+1}(1 + O(1/\ell))S_2(\mu^\ell(A)) - \left(\frac{1}{2\ell+1}\right)^2 (1 + O(1/\ell))S_1(\mu^\ell(A))^2\\
&= \frac{1}{2\ell+1} S_2(\mu^\ell(A)) - \left(\frac{1}{2\ell+1}\right)^2 S_1(\mu^\ell(A))^2 + O(\|\mu^\ell(A)\|_\infty 1/\ell)^2.
\end{aligned} \tag{2.31}$$

Obviously, $\|\mu^\ell(A)\|_\infty < \|A\|$ (operator norm); while by the Szegö Theorem,

$\frac{1}{2\ell+1} S_2(\mu^\ell(A)) = \frac{1}{\operatorname{vol}(S^*(S^2))} \int_{S^*(S^2)} |\sigma_A^{\text{ave}}|^2 + O_A(1/\ell)$, and $\frac{1}{2\ell+1} S_1(\mu^\ell(A)) = \bar{\sigma}_A + O_A(1/\ell)$. Lemma (2.15) follows. ■

We next recall Kolmogorov's strong law of large numbers. Let $\{f_n\} \subset L^2$ be an independent sequence, and let $S_n = \sum_{k=1}^{n} f_k$. Then:

$$\text{(KSL)} \quad \{V(f_n)\} \text{ bounded implies } (1/n)[S_n - E(S_n)] \to 0 \quad \text{(a.s.)} \tag{2.32}$$

(almost surely). (See [I, Theorem 4.5.3]).

We set:

$$f_\ell \overset{\text{def}}{=} \frac{1}{2\ell+1} X_\ell^A, \tag{2.33}$$

So:

i)
$$E(f_\ell) = \frac{1}{2\ell+1} D_2^2(\sigma_A^{\text{ave}} - \bar{\sigma}_A) + O_A(1/\ell^2) \tag{2.34}$$

(in an obvious notation);

ii)
$$f_\ell(\tau_\infty) \leqq \frac{1}{2\ell+1} |J_\ell(\tau_\ell^{-1}(\pi_\ell A \pi_\ell)\tau_\ell) - \bar{\sigma}_A I_\ell|^2 = O_A(1).$$

Clearly $V(f_\ell)$ is bounded.

(KSL) therefore implies:

$$\frac{1}{L} \sum_{\ell \leqq L} \left[\frac{1}{2\ell+1} X_\ell^A(\tau_\infty) - \frac{1}{2\ell+1} D_2^2(\sigma_A^{\text{ave}} - \bar{\sigma}_A) \right] \to 0 \quad \text{(a.s.)}. \tag{2.35}$$

Hence,

$$\frac{1}{L} \sum_{\ell \leqq L} \frac{1}{2\ell+1} X_\ell^A(\tau_\infty) \to 0 \quad \text{(a.s.)}, \tag{2.36}$$

or equivalently,

$$\frac{1}{N(L)} \sum_{\ell \leqq L} X_\ell^A(\tau_\infty) \to 0 \quad \text{(a.s.)}, \tag{2.37}$$

($N(L) = \#$ eigenvalues $\leqq L^2$). (2.36)–(2.37) is the criteria that τ_∞ be Liouville distributed, at least as regards $A \in \mathscr{C}$. To get rid of the dependence on A, we first set:

$$\mathscr{LOB}_A = \left\{ \tau_\infty : \frac{1}{N(L)} \sum_{\ell \leqq L} X_\ell^A(\tau_\infty) \to 0 \right\}. \tag{2.38}$$

We have just seen that $\mu_\infty(\mathscr{LOB}_A) = 1$ for any fixed A.

But $\mathscr{LOB}_A$ only depends on the principal symbol σ_A. Further, choose an orthonormal basis $\{\varphi_n\}$ of $L^2(S^*(S^2), d\mu)$, so that linear combinations $\sum a_n \varphi_n$, with rapidly decaying a_n, are dense in $C(S^*(S^2))$. It is straightforward to show that:

$$\mathscr{LOB} = \bigcap_n \mathscr{LOB}_{\operatorname{Op}(\varphi_n)}, \tag{2.39}$$

where $\mathrm{Op}(\varphi_n)$ is any ΨDO with principal symbol φ_n. Since each member at right has measure 1, we conclude our main theorem:

$$\mu_\infty(\mathscr{L}\mathscr{O}\mathscr{B}) = 1. \tag{2.40}$$

Acknowledgements. This paper was written while the author was visiting at the Institut Fourier in Grenoble. Thanks are due to Y. Colin de Verdiere for his support, and to C. Anne, M. Brion, G. Courtois and T. Kappeler for many helpful comments.

References

[B] Berry, M.V.: Regular and irregular semi-classical wavefunctions. J. Phys. A: Math. Gen. **10**, 2083–2091 (1977)
[CdeV.1] Colin de Verdière, Y.: Quasi-modes sur les variétés riemannienes compactes. Invent. Math. **43**, 15–52
[CdeV.2] Colin de Verdière, Y.: Ergodicité et fonctions propres du Laplacien. Sem. Bony-Sjostrand-Meyer, 1984–1985, exposé no. XIII (1985)
[Gu] Guillemin, V.: Band asymptotics in two dimensions. Adv. Math. **42**, 48–282 (1981)
[G-S] Guillemin, V., Sternberg, S.: Symplectic techniques in physics. Cambridge: Camb. U. Press, 1984
[HMR] Helffer, B., Martinez, A., Robert, D.: Ergodicité et limite semi-classique. Commun. Math. Phys. **109**, 313–326 (1986)
[I] Ito, K.: Introduction to Probability Theory. Cambridge: Cambridge Univ. Press 1984
[ST] Schrader, R., Taylor, M.: Semi-classical asymptotics, gauge fields, and quantum chaos. J. Funct. Anal. **83**, 258–316 (1989)
[Sn] Snirelman, A.I.: Ergodic properties of eigenfunctions. Usp. Math. Nank **29**, 181–182 (1974)
[T] Terras, A.: Harmonic analysis on symmetric spaces and applications I. Berlin, Heidelberg, New York: Springer 1985
[U] Uribe, A.: A Symbol calculus for a class of Pseudodifferential Operator on S^n and band asymptotics. J. Funct. Anal. **59**, 535–556 (1984)
[V] Voros, A.: Semi-classical ergodicity of quantum eigenstates in the Wigner representation. In: Stochastic Behaviour in Classical and Quantum Hamiltonian Systems. SLNP **93**, 326–333 (1979)
[Wa] Walters, P.: An introduction to ergodic theory. Berlin, Heidelberg New York: Springer 1982
[Wei] Weinstein, A.: Asymptotics of eigenvalue clusters for the Laplacian plus a potential. Duke Math. J. **44**, 883–892 (1977)
[Wi] Widom, H.: Eigenvalue distribution theorems for certain homogeneous spaces. J. Funct. Anal. **32** (1979)
[Z.1] Zelditch, S.: Quantum transition amplitudes for ergodic and for completely integrable systems. J. Fun. Anal. **94**, 415–436 (1990)
[Z.2] Zelditch, S.: Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke Math. J. **55**, 919–941 (1987)

Communicated by Ya.G. Sinai