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On the Solvability of Painlevé II and IV

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Abstract. We introduce a rigorous methodology for studying the Riemann-Hilbert problems associated with certain integrable nonlinear ordinary differential equations. For concreteness we investigate the Painlevé II and Painlevé IV equations. We show that the Cauchy problems for these equations admit in general global, meromorphic in t solutions. Furthermore, for special relations among the monodromy data and for t on Stokes lines, these solutions are bounded for finite t.

1. Introduction

Flaschka and Newell [1] and Jimbo, Miwa, and Ueno [2] have introduced a powerful approach for studying the initial value problem of certain nonlinear ODE's: They have shown that solving such an initial value problem is essentially equivalent to solving an inverse problem for an associated isomonodromic linear equation. This inverse problem can be formulated in terms of monodromy data which can be calculated from initial data. Fokas and Ablowitz [3] have shown that the inverse problem can be formulated as a matrix, singular, discontinuous Riemann-Hilbert (RH) problem defined on a complicated contour. Hence techniques from RH theory can be employed to study the solvability of certain nonlinear ODE's. The above method, which is an extension of the inverse scattering transform method, is called inverse monodromic transform (IMT), and can be thought of as a nonlinear analogue of the Laplace's method for solving linear ODE's.

The six Painlevé transcendents, P I–P IV, are the most well known nonlinear ODE's that can be studied using the IMT method. P II has been studied in [1, 3], a special case of P III in [1], P IV and P V in [4]. We refer the interested reader to [4] for a historical perspective on the Painlevé equations. Here we only note that these equations: (a) have the Painlevé property, i.e. their solutions are free from movable critical points [5], (b) possess particular solutions which are either rational or can be expressed in terms of certain classical transcendental functions [6], (c) are intimately related to nonlinear PDE's solvable by the inverse scattering transform [7], (d) appear in a wide range of physical problems [8–17]. In particular it has been shown recently that Painlevé equations play an important role in two dimensional quantum gravity [18–20].

In this paper we introduce a rigorous methodology for studying the RH problems appearing in the IMT. For concreteness we investigate P II and P IV, however similar results can also be obtained for other nonlinear equations. We will show that the Cauchy problems for P II and P IV in general admit global meromorphic in t solutions. Furthermore, for special relations among the monodromy data and for t on Stokes lines, these solutions have no poles. It should be noted that although the inverse monodromy problem was formulated as a RH problem in [3, 4] the question of solvability of this RH problem was left open. In this paper, in addition to simplifying the formalism introduced in [3, 4], we are able to investigate the above question. Actually, the proof of the existence of meromorphic solutions (which is the first such proof using the isomonodromy approach) is quite more transparent than the original proof of Painlevé. Also, for the first time in this paper, a method is introduced using RH theory for finding those monodromy data (and hence those initial data) for which the solution is free from poles.

The RH problems associated with the IMT have several novelties. For example, in connection with P II one needs to reconstruct a sectionally holomorphic function $m(z;t)$ with the following properties: (a) $m(z;t)$ has certain jumps on the contour defined by $\mathrm{Re}\left(\frac{4iz^3}{3}+itz\right)=0$, (b) $m(z;t)$ tends to the identity for large z off the contour, but it oscillates on the contour, (c) $m(z;t)\sim\hat{m}_0(z;t)z^{\theta\sigma_3}$ as $z\to 0$, where $\hat{m}_0$ is analytic at $z=0$, $\sigma_3=\mathrm{diag}(1,-1)$ and θ is a constant parameter. The situation for P IV is similar but now the contour is defined by $\mathrm{Re}\left(\frac{z^2}{2}+tz\right)=0$ and $m(z;t)$, in addition to being singular at the origin, has also a singularity of the type $(1/z)^{\theta_\infty\sigma_3}$ on the contour. To study a RH problem of this type we first study a RH problem which is formulated on a new contour, obtained from the original one by: (a) inserting a circle around the origin, (b) performing a small clockwise rotation. The new RH problem is analytic both at the origin and at infinity and hence can be studied by standard methods, in particular is equivalent to a certain Fredholm integral equation. Having established the solution of the new RH problem it is straightforward to establish the solution of the original one.

The matrices defining the jumps on the relevant contours have an explicit analytic dependence on t. This has important implications: (i) The associated Fredholm integral equation depends analytically on t, and since it is solvable at $t=0$ (which follows from the direct problem), it has solutions meromorphic in t [21]. (ii) It is possible to introduce a quadratic from of the type $mhm^\dagger$, where h is piecewise constant, as opposed to the standard form $mm^\dagger$. Here † denotes Schwarz reflection, i.e. $f^\dagger(z)=f(\bar{z})^*$, where $*$ denotes complex conjugation and transposition. Using this form it follows that for certain constraints on the monodromy data and for t on Stokes lines the homogeneous RH problem has only the zero solution (i.e. there exist a vanishing lemma). Alternatively, it is possible to define an equivalent RH problem on fewer contours and then use the standard quadratic form $mm^\dagger$. In the case of P II,

$$y_{tt}=2y^3+ty+\theta, \tag{1.1}$$

for real t, one finds the constraints

$$b=\bar{b} \quad \text{and} \quad |a-\bar{c}|<2, \qquad |\mathrm{Re}\,\theta|<\tfrac{1}{2}, \tag{1.2}$$

where the monodromy data a, b, c are related via [1]

$$a+b+c+abc=-2i\sin\theta\pi. \tag{1.3}$$

The special cases

$$a=\bar{c}, \qquad \theta=0 \tag{1.4}$$

and

$$a=c=i\alpha, \qquad |\alpha|<1, \qquad \theta=0 \tag{1.5}$$

were suggested by Its and Novokshenov [22] who have studied the large t behavior of Painlevé equations. These authors have formally shown that if the monodromy data satisfy (1.4) or (1.5) then the solution of P II for real and large t has no poles. They have also noted that one of the two cases corresponds to y being purely imaginary (see also [26, 27]).

In proving some of the above results we choose the parameters of the given Painlevé equation in such a way that the singularities at zero and at infinity are integrable. However, this is without loss of generality since there exist Schlessinger transformations [4] which shift by an integer or by a half integer all the parameters of a given Painlevé equation.

In Sect. 2 we review the direct problem (see [1, 4]) and we also show that it is not necessary to consider the connection matrix as part of the monodromy data. In Sect. 3 we prove the uniqueness and existence of global solutions meromorphic in t. (Such a result does not exist for integrable PDE's). Furthermore, we note that for t in a bounded region in $\mathbb{C}$, if the absolute value of the monodromy data is sufficiently small then the relevant operators are always invertible. In order to give sharp estimates one would need to calculate certain operator norms on a complicated contour. This is beyond the scope of this paper; however it appears that the technique of decomposing the complicated RH problem to a series of simple RH problems [3] could be useful in this respect. In Sect. 4 we derive the vanishing lemmas for P II and P IV, i.e. we find certain constraints on the monodromy data which guarantee the solvability of the associated RH problem without the small norm assumption. These constraints imply, in principle, certain constraints on the initial data. For example, we show that if y, the solution of P II, is purely imaginary these constraints are always satisfied.

2. The Direct Problem for P II and P IV

2.1. P II

The P II equation (1.1) can be written as the compatibility condition of the following system of equations:

$$Y_z=AY, \tag{2.1a}$$

$$Y_t=BY, \tag{2.1b}$$

where

$$A=-i(4z^2+t+2y^2)\sigma_3+\left(4zy-\frac{\theta}{z}\right)\sigma_1-2y_t\sigma_2, \qquad B=-iz\sigma_3+y\sigma_1, \tag{2.2}$$

the Pauli matrices σ_j, $j=1,2,3$ are defined by

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}0&0\\1&-1\end{pmatrix},\tag{2.3}$$

and $Y(z,t)$ is a 2×2 matrix valued function in $\mathbb{C}\times\mathbb{C}$.

The essence of the direct problem is to establish the analytic structure of Y with respect to z, in the entire complex z-plane. It should be pointed out that, in contrast to the analogous problem in the inverse scattering transform (see the discussion in [3]), this task here is straightforward: Equation (2.1a) is a linear ODE in z, therefore its analytic structure is completely determined by its singular points. Equation (2.1a) has a regular singular point at the origin (if $\theta\neq 0$) and an irregular singular point at infinity.

(i) *Analysis near* $z=0$. It is well known that if the coefficient matrix of a linear ODE has a regular singular point at $z=0$, then the solution in the neighbourhood of $z=0$ can be obtained via a convergent power series. In this particular case

$$Y_0(z)=\hat{Y}_0(z)z^{\theta\sigma_3},\quad \text{as}\quad z\to 0,\qquad \theta\neq\frac{2n+1}{2},\qquad n\in\mathbb{Z},\tag{2.4a}$$

where $\hat{Y}_0(z)$ is holomorphic at $z=0$ [23]. The dominant behavior near $z=0$ is characterized by $Y_{0_z}\sim -\theta\sigma_1 Y_0/z$; thus $\hat{Y}_{0_z}(z)\sim -\frac{\theta}{z}(\sigma_1\hat{Y}_0(z)+\hat{Y}_0(z)\sigma_3)$ and

$$\sigma_1\hat{Y}_0(0)+\hat{Y}_0(0)\sigma_3=0.\tag{2.4b}$$

(ii) *Analysis near* $z=\infty$. The solution of Eq. (2.1a), for large z, possesses a formal expansion of the form $Y\sim\tilde{Y}$, $\tilde{Y}=\hat{Y}_\infty\exp\left[-i\left(\frac{4z^3}{3}+tz\right)\sigma_3\right]$, where $\hat{Y}_\infty$ is a formal power series. However, because $z=\infty$ is an irregular singular point, the actual asymptotic behavior of Y changes form in certain sectors of the complex z-plane. These sectors are determined by $\mathrm{Re}[i\frac{4}{3}z^3+itz]=0$; thus for large z the boundaries of the sectors, Σ_j, are asymptotic to the rays $\arg z=\frac{i\pi}{3}$, $0\leqq j\leqq 7$. Let S_j be the sector containing the boundaries Σ_j, i.e. if $z\in S_1$, $0\leqq\arg z<\frac{\pi}{3}$, etc. Then, according to the Stokes phenomenon, if $Y\sim\tilde{Y}$ as $z\to\infty$ in S_1, $Y\sim\tilde{Y}G_1G_2\ldots G_j$, as $z\to\infty$ in S_{j+1}, $1\leqq j\leqq 6$. The matrices

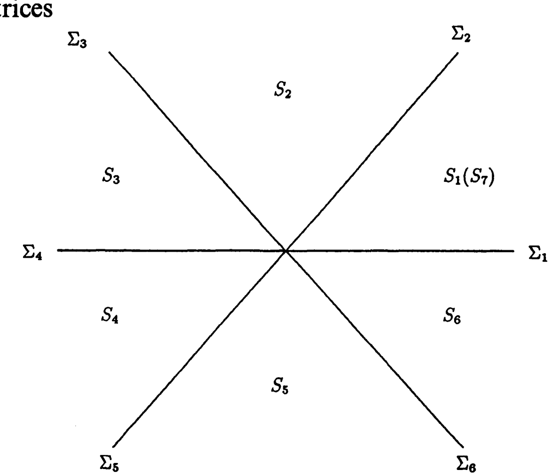


Fig. 2.1

G_j, $1 \leqq j \leqq 6$ are triangular and are called Stokes multipliers. Alternatively, for the formulation of the RH problem it is more convenient to introduce different solutions Y_j, $1 \leqq j \leqq 7$ such that Y_j is asymptotic to $\tilde{Y}$ in S_j. Then $Y_{j+1} = Y_j G_j$, $1 \leqq j \leqq 6$; also it can be shown [24] that $Y_1(z) = Y_7(ze^{2i\pi})$. Thus the nonsingular matrices Y_j satisfy:

$$Y_j(z) \sim \hat{Y}_\infty(z) e^{Q(z)}, \quad \text{as} \quad z \to \infty, \, z \in S_j, \, Q = -i(\tfrac{4}{3}z^3 + tz)\sigma_3, \tag{2.5}$$

where $\hat{Y}_\infty(z)$ is piecewise holomorphic (relative to the contours of Fig. 2.1) at $z = \infty$, with asymptotic expansion of the form $\hat{Y}_\infty(z) \sim I + O(1/z)$ as $z \to \infty$. They are related by

$$Y_{j+1}(z) = Y_j(z) G_j, \quad 1 \leqq j \leqq 5; \quad Y_1(z) = Y_6(ze^{2i\pi}) G_6, \tag{2.6}$$

where the Stokes multipliers are given by

$$\begin{aligned} G_1 &= \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \\ G_4 &= \begin{pmatrix} 1 & \hat{a} \\ 0 & 1 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 1 & 0 \\ \hat{b} & 1 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 1 & \hat{c} \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.7}$$

We note that although $Y_1 \sim \tilde{Y}$ as $z \to \infty$ on Σ_1, $Y_1 \sim \tilde{Y} G_1^{-1}$ as $z \to \infty$ on Σ_2, similarly $Y_2 \sim \tilde{Y} G_2^{-1}$ as $z \to \infty$ on Σ_3, etc.

(iii) *Connection between Y_0 and Y_1.* Since both Y_0 and Y_1 satisfy (2.1a) they are related by a constant matrix,

$$Y_1 = Y_0 E_0, \quad E_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det E_0 = 1. \tag{2.8}$$

The condition on the determinant follows from the fact that we have normalized Y_j, Y_0 to have unit determinant.

(iv) *Symmetry Relations.* For a complex matrix function f we denote

$$f^{\sigma_1}(ze^{i\pi}) = \sigma_1 f(z) \sigma_1. \tag{2.9}$$

Equations (2.2) imply that the A and B of Eqs. (2.1) satisfy

$$A^{\sigma_1} = -A, \quad B^{\sigma_1} = B. \tag{2.10}$$

Hence

$$Y_z^{\sigma_1} = A^{\sigma_1} Y^{\sigma_1}, \quad Y_t^{\sigma_1} = B^{\sigma_1} Y^{\sigma_1}. \tag{2.11}$$

Equations (2.11) and the fact that Y, Y^{σ_1} have the same asymptotics at $z = \infty$ imply $Y = Y^{\sigma_1}$,

$$Y_{j+3}(ze^{i\pi}) = \sigma_1 Y_j(z) \sigma_1, \quad j = 1, 2, 3. \tag{2.12}$$

Equations (2.12) imply

$$G_{j+3} = \sigma_1 G_j \sigma_1, \quad j = 1, 2, 3, \quad \text{i.e.} \quad \hat{a} = a, \, \hat{b} = b, \, \hat{c} = c. \tag{2.13}$$

Furthermore Eqs. (2.12) also imply [for an appropriate choice of $Y_0(z)$] the important relationship

$$-E_0^{-1} \sigma_3 e^{-i\pi\theta\sigma_3} E_0 = G_1 G_2 G_3 \sigma_1. \tag{2.14}$$

Indeed, the equations

$$Y_4(z)=Y_1(z)G_1G_2G_3 \quad \text{and} \quad Y_4(ze^{i\pi})=\sigma_1 Y_1(z)\sigma_1,$$

yield,

$$\sigma_1 Y_1(z)\sigma_1 = Y_1(ze^{i\pi})G_1G_2G_3.$$

Using $Y_1=Y_0E_0$, and $Y_0=\hat{Y}_0 z^{\theta\sigma_3}$, this equation becomes

$$z^{-\theta\sigma_3}\hat{Y}_0^{-1}(z)\sigma_1\hat{Y}_0(z)z^{\theta\sigma_3}=e^{i\pi\theta\sigma_3}E_0G_1G_2G_3\sigma_1E_0^{-1}. \tag{2.15}$$

Let $F \doteq e^{i\pi\theta\sigma_3}E_0G_1G_2G_3\sigma_1E_0^{-1}$; using Eq. (2.4b), it follows from the limit of (2.15) as $z\to 0$ that $(F)_{\mathrm{Diag}}=-\sigma_3$. By Eq. (2.4a), $\theta\notin\frac{1}{2}+\mathbb{Z}$. If $\theta=0$, or if $\theta\notin\mathbb{Z}$, then $F=F_{\mathrm{Diag}}=-\sigma_3$, and Eq. (2.14) follows. If $\theta\in\mathbb{Z}/\{0\}$, then F in general is upper (lower) triangular for $\mathrm{Re}\,\theta>0$ ($\mathrm{Re}\,\theta<0$); however, it follows from Eqs. (2.4a) and (2.8) that it is possible to choose a $Y_0(z)$ such that the corresponding F is also diagonal.

We note that, since $\sigma_1G_1G_2G_3\sigma_1=G_4G_5G_6$, the square of Eq. (2.14) yields

$$E_0^{-1}e^{2i\pi\theta\sigma_3}E_0\prod_{j=1}^{6}G_j=I. \tag{2.16}$$

This equation is a consistency condition (see [1]). The trace of Eq. (2.14) implies

$$a+b+c+abc=-2i\sin\theta\pi. \tag{2.17}$$

(v) *The Monodromy Data.* In previous investigations the components of the connection matrix E_0 were taken as part of the monodromy data. This is not necessary since E_0 can be determined from (a,b,c). Here we call monodromy data the set (a,b,c) defined on the algebraic variety (2.17). Since $\det G_1G_2G_3\sigma_1=-1$ and $\mathrm{tr}\, G_1G_2G_3\sigma_1=-2i\sin\pi\theta$, it follows that $G_1G_2G_3\sigma_1$ has eigenvalues $-e^{i\pi\theta}$ and $e^{-i\pi\theta}$. Also $\theta\notin\frac{1}{2}+\mathbb{Z}$, thus these two eigenvalues are unequal and $G_1G_2G_3\sigma_1$ is diagonalized to $-\sigma_3e^{i\pi\theta\sigma_3}$. Therefore there exists a matrix E_0, with $\det E_0=1$, such that Eq. (2.14) is valid. For the inverse problem we will define a RH problem for the matrices Y_j defined in Eqs. (2.5) and (2.6), where Y_1 satisfies $Y_1=\hat{Y}_0(z)z^{i\theta\sigma_3}E_0$, $\hat{Y}_0(z)$ is analytic at $z=0$, and E_0 is any matrix obtained from Eq. (2.14) ($\det E_0=1$). We note that the equation for Y_1 is well defined. Indeed, if $\hat{E}_0$ is another solution of (2.14), then $E_0\hat{E}_0^{-1}$ is diagonal and Y_1 satisfies a similar equation with $\hat{Y}_0(z)$ replaced by $\hat{Y}_0(z)E_0\hat{E}_0^{-1}$. Therefore, the different choices of E_0 do not affect $Y_1,\ldots,Y_6$.

If y evolves according to PII, then the monodromy data (a,b,c) are time independent. To show this we note that if Y satisfies (2.1a) and y satisfies PII, it follows that Y_t satisfies

$$Y_t=BY+PY, \tag{2.18}$$

where B is defined in Eq. (2.2) and $P=P(t)$ is some matrix independent of z. If $Y=Y_j$, $j=1,\ldots,6$, then $e^{Q}Pe^{-Q}$ must vanish as $z\to\infty$ from two adjacent sectors, which implies $P=0$. Therefore Y satisfies Eq. (2.1) and hence G_j, $j=1,\ldots,6$ are time independent.

2.2. PIV. The PIV equation

$$y_{tt}=\frac{1}{2y}y_t^2+\frac{3}{2}y^3+4ty^2+2(t^2-\alpha)y+\frac{\beta}{y}, \tag{2.19}$$

is associated with the Lax pair (2.1) with

$$A=(z+t)\sigma_3+\frac{\theta_0-v}{z}\sigma_3+F+\frac{G}{z},\qquad B=z\sigma_3+F,$$
$$\alpha=2\theta_\infty-1,\qquad \beta=-8\theta_0^2, \tag{2.20}$$

where

$$F=\begin{pmatrix} 0 & u \\ \frac{2}{u}(v-\theta_0-\theta_\infty) & 0 \end{pmatrix},\qquad G=\begin{pmatrix} 0 & \frac{-uy}{2} \\ \frac{2v}{uy}(v-2\theta_0) & 0 \end{pmatrix}. \tag{2.21}$$

In the above equation v, u, y are related via the following equations:

$$\frac{dy}{dt}=-4v+y^2+2ty+4\theta_0,\qquad \frac{du}{dt}=-u(y+2t),$$
$$\frac{dv}{dt}=-\frac{2}{y}v^2+\left(\frac{4\theta_0}{y}-y\right)v+(\theta_0+\theta_\infty)y. \tag{2.22}$$

(i) *Analysis near* $z=0$.

$$Y_0(z)=\hat{Y}_0(z)z^{\theta_0\sigma_3},\qquad \theta_0\neq\frac{n}{2},\qquad n\in\mathbb{Z}, \tag{2.22a}$$

where $\hat{Y}_0(z)$ is holomorphic at $z=0$ [25].

(ii) *Analysis near* $z=\infty$. The formal expansion $\tilde{Y}$ is now of the form

$$\tilde{Y}=\hat{Y}_\infty\exp\left[\left(\frac{z^2}{2}+tz\right)\sigma_3\right]\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3}.$$

The relevant sectors are given by $\mathrm{Re}\left[\frac{z^2}{2}+tz\right]=0$; thus for large z the boundaries of these sectors, Σ_j, are asymptotic to the rays $\arg z=-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4},\frac{-3\pi}{4}$.

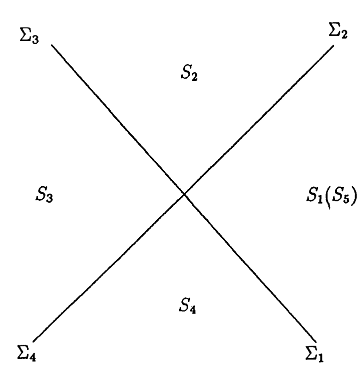


Fig. 2.2

As before $Y_{j+1}=Y_jG_j$, $1\leqq j\leqq 4$, and $Y_1(z)=Y_5(ze^{2i\pi})e^{2i\pi\theta_\infty\sigma_3}$. Thus

$$Y_j(z)\sim\hat{Y}_\infty(z)e^{Q(z)}\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3},\quad \text{as}\quad z\to\infty,\ z\in S_j,\ Q=\left(\frac{z^2}{2}+tz\right)\sigma_3, \tag{2.23}$$

where $\hat{Y}_\infty(z)$ is holomorphic at $z=\infty$, and $\hat{Y}_\infty(z)\sim I+0\left(\frac{1}{z}\right)$. Also

$$Y_{j+1}(z)=Y_j(z)G_j, \quad 1\leqq j\leqq 3, \quad Y_1(z)=Y_4(ze^{2i\pi})G_4e^{2i\pi\theta_\infty\sigma_3}, \tag{2.24}$$

where

$$G_1=\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}, \quad G_2=\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, \quad G_3=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}, \quad G_4=\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}. \tag{2.25}$$

(iii) *Connection between Y_1 and Y_1.* As before Y_0, Y_1 are related via (2.8).

(iv) *The Monodromy Data.* The trace of the consistency condition

$$E_0^{-1}e^{2i\pi\theta_0\sigma_3}E_0\prod_{j=1}^{4}G_je^{2i\pi\theta_\infty\sigma_3}=I, \tag{2.26}$$

implies

$$(1+bc)e^{2i\pi\theta_\infty}+[ad+(1+cd)(1+ab)]e^{-2i\pi\theta_\infty}=2\cos 2\pi\theta_0. \tag{2.27}$$

We further notice that (2.27) is invariant under the transformation $\hat{a}=r^2a$, $\hat{b}=b/r^2$, $\hat{c}=cr^2$, $\hat{d}=d/r^2$, $r\neq 0$. Also if we define $\hat{Y}$ by $\hat{Y}=R^{-1}YR$, where $R=\operatorname{diag}\left(r,\frac{1}{r}\right)$, $r\neq 0$, the Lax pair for $\hat{Y}$ is the same as the Lax pair for Y with the only change of replacing u by u/r^2; this does not affect y, the solution of P IV. We therefore choose to define the monodromy data for P IV as the equivalent classes induced by the transformation $\hat{}$, on the algebraic variety (2.27). These monodromy data imply E_0 as in case of P II: Since $\operatorname{tr}G_1G_2G_3G_4e^{2i\pi\theta_\infty}=e^{2i\pi\theta_0}+e^{-2i\pi\theta_0}$ and $\det G_1G_2G_3G_4e^{2i\pi\theta_\infty}=1$ it follows that the relevant eigenvalues are $e^{2i\pi\theta_0}$ and $e^{-2i\pi\theta_0}$. Using the fact that $\theta_0\notin\frac{1}{2}+\mathbb{Z}$ Eq. (2.26) follows. Then, for the solution of the inverse problem we demand that Y_1 satisfies $Y_1=\hat{Y}_1(z)z^{\theta\sigma_0}E_0$, where $\hat{Y}_1(z)$ is analytic at $z=0$ and E_0 is any solution of (2.26). Again different choices of E_0 do not affect $Y_1,\ldots,Y_4$. It is also straightforward to show that the monodromy data are time independent.

3. The Inverse Problem

The relations (2.6) and (2.24) indicate that the Inverse Monodromy Problems are RH problems on self-intersecting contours, with singularities at $z=0$ and $z=\infty$. In this section we first remove these singularities by using an analytic solution inside a circle around the origin and by performing a small clockwise rotation near $z=\infty$. We then use the method introduced by one of the authors (Zhou) [21] to study the resulting regular RH problems. Since the procedure is essentially the same for both P II and P IV we give details only for P IV.

We define Φ_j, $j=0,\ldots,4$ by

$$Y_j\doteqdot\Phi_je^Q\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3}, \quad j=1,\ldots,4, \quad Y_0\doteqdot\Phi_0e^Qz^{\theta_0\sigma_3}, \quad Q\doteqdot\left(\frac{z^2}{2}+tz\right)\sigma_3. \tag{3.1}$$

We then obtain a piecewise holomorphic function $\Phi=(\Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_4)$ and a contour $\Sigma_A\doteqdot\Sigma\cup\{\text{circle}\}$ indicated in Fig. 3.1. We note that the augmented

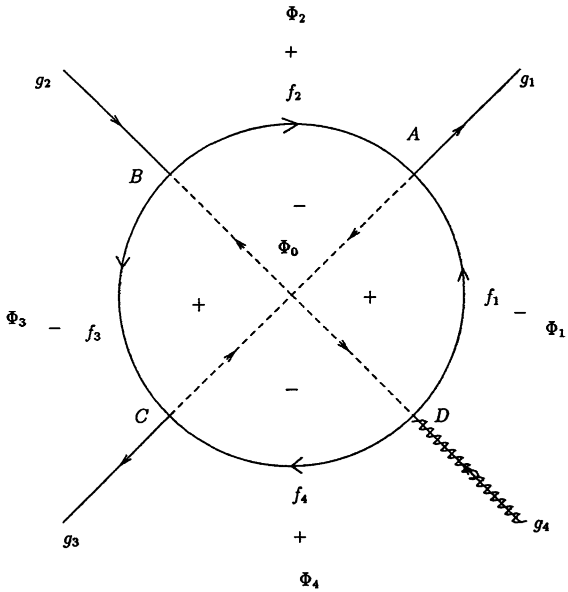


Fig. 3.1

contour, Σ_A, need not be precise except near $z=\infty$, where it is precisely defined by $\mathrm{Re}\left(\frac{z^2}{2}+tz\right)=0$.

Inside the circle, $\Phi=\Phi_0$ is holomorphic, hence the multiplicative jump of Φ across the dotted lines is I. We denote by $\Omega_\pm$ the union of all $\pm$ regions as marked in Fig. 3.1. The orientation used in Fig. 3.1 makes Σ the positively oriented boundary Σ^+ for Ω^+ and the negatively oriented boundary Σ^- for Ω^-. Let V denote the global jump on Σ_A, $V=I$ on the rays inside the circle, $V \doteqdot (f_1, f_2, f_3, f_4)$ on the circle and $V \doteqdot (g_1, g_2, g_2, g_4)$ on the rays outside the circle. V may be expressed in terms of the monodromy data:

$$\Phi_0=\Phi_1 e^Q f_1 e^{-Q}, \qquad f_1^{-1}=z^{\theta_0\sigma_3}E_0\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_2=\Phi_0 e^Q f_2 e^{-Q}, \qquad f_2=z^{\theta_0\sigma_3}E_0 G_1\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_0=\Phi_3 e^Q f_3 e^{-Q}; \qquad f_3^{-1}=z^{\theta_0\sigma_3}E_0 G_1 G_2\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_4=\Phi_0 e^Q f_4 e^{-Q}, \qquad f_4=z^{\theta_0\sigma_3}E_0 G_1 G_2 G_3\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3}.$$

$$\Phi_2=\Phi_1 e^Q g_1 e^{-Q}, \qquad g_1=\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3} G_1\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_2=\Phi_3 e^Q g_2 e^{-Q}, \qquad g_2=\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3} G_2^{-1}\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_4=\Phi_3 e^Q g_3 e^{-Q}, \qquad g_3=\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3} G_3\left(\frac{1}{z}\right)^{-\theta_\infty\sigma_3};$$

$$\Phi_4=\Phi_1 e^Q g_4 e^{-Q}, \qquad g_4=\left(\frac{1}{z}\right)_+^{\theta_\infty\sigma_3} G_4^{-1}\left(\frac{1}{z}\right)_+^{-\theta_\infty\sigma_3},$$

where $\left(\frac{1}{z}\right)_+^{\theta_\infty\sigma_3}$ means the boundary value of $\left(\frac{1}{z}\right)^{\theta_\infty\sigma_3}$ from the $+$ region. If we denote by $\Phi_\pm$ the boundary values of Φ from $\Omega^\pm$, we have

$$\Phi_+ = \Phi_- V_Q, \qquad V_Q \doteq e^Q V e^{-Q}. \tag{3.2}$$

The function V is smooth (in fact analytic) away from the intersections A, B, C, and D. At these intersections, it satisfies the following cyclic product conditions,

$$\begin{aligned} &g_1 f_2^{-1} f_1^{-1} = I \quad \text{near} \quad A, \qquad f_2 g_2^{-1} f_3 = I \quad \text{near} \quad B, \\ &f_3^{-1} g_3 f_4^{-1} = I \quad \text{near} \quad C, \qquad f_4 g_4^{-1} f_1 = I \quad \text{near} \quad D. \end{aligned} \tag{3.3}$$

The above products make sense because the relevant functions admit analytic continuations. The conditions in (3.3) follow directly from (3.2). We may also check their validity directly. For example consider the condition near D:

$$f_4(ze^{2i\pi}) g_4^{-1}(ze^{2i\pi}) f_1(z) = I, \quad \text{or} \quad e^{2i\pi\theta_0\sigma_3} E_0 \left(\prod_{j=1}^{4} G_j\right) e^{2i\pi\theta_\infty\sigma_3} E_0^{-1} = I,$$

which is the compatibility condition (2.18). Roughly speaking, the cyclic product conditions (3.3) represent the "smoothness" of V at the intersections A, B, C, D. Having established that the global V is smooth it is then possible to factorize V into a product of two functions in certain Sobolev spaces [see below (3.5)].

Now V satisfies all the conditions needed in the theory of [21] except that V_Q may oscillate or even grow as $z \to \infty$. Nevertheless we can still define a RH problem without ambiguity.

Definition 3.1. *Let Φ be holomorphic on $\mathbb{C} \backslash \Sigma_A$ with smooth extension $\Phi_\pm$ satisfying*

$$\Phi_+ = \Phi_- V_Q. \tag{3.4}$$

Then Φ is said to be a normalized solution of the RH *problem* (3.4) *if:*
1. *Φ has at worst polynomial growth at $z \to \infty$.*
2. *Φ tends to I as $z \to \infty$ in certain directions.*

Conditions 1 and 2 appear weak, but they actually imply the uniqueness of the solution:

Proposition 3.1. *The normalized solution in Definition 3.1 is unique and* $\det \Phi = 1$.

Proof. Since $\det V_Q = 1$, $\det \Phi_+ = \det \Phi_-$. Hence $\det \Phi$ is an entire function. By Conditions 1 and 2, $\det \Phi = 1$.

Let Φ' be another solution. Then it is easily checked that $\Phi' \Phi^{-1}$ is entire and hence by Conditions 1 and 2, equals I. QED

Remark. If Φ is a solution from the direct problem, then obviously Φ satisfies all the conditions enumerated in Definition 3.1. We may in fact derive better properties for Φ from the direct problem. For example, if the sector in Fig. 3.2(a) is slightly rotated clockwise to the sector in Fig. 3.2(b), then $\hat{\Phi}$ obtained from Φ_j by restriction and analytic extension approaches I as $z \to \infty$.
However we prefer to establish such properties directly from the inverse problem. This will show that all the necessary information to derive these properties is already contained in our characterization of the data V.

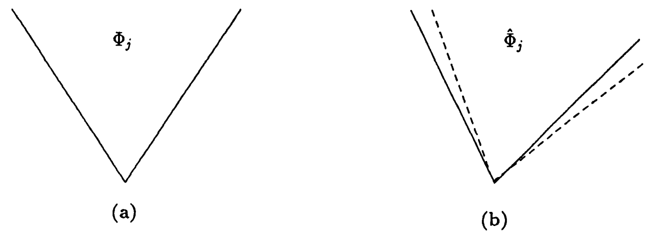


Fig. 3.2

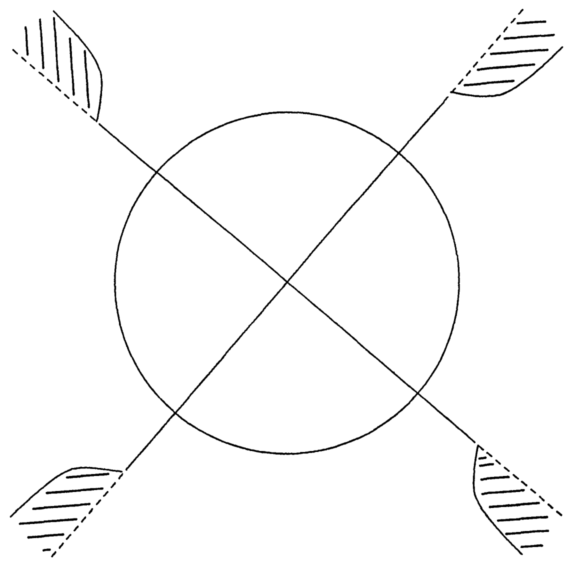


Fig. 3.3

We make a slight rotation of Σ near $z=\infty$ clockwise as in Fig. 3.3. Since the asymptotes of Σ are independent of t, the rotated contour, Σ_{AR}, can actually be made independent of t.

In general, a contour Γ for a RH problem may be deformed to another contour Γ' within the domain of holomorphicity of the jump functions. However, in our problem, V is not smooth at $z=\infty$. Therefore we used more careful argument for the equivalence of the RH problems before and after the rotations. We still denote our jump function by V after the rotation, since it is obtained by a direct analytic extension of the original V. Since V is triangular near $z=\infty$ it follows that $e^{-Q}Ve^{Q}-I$, and its derivatives, decay exponentially as $z\to\infty$, on the rotated contour Σ_{AR}. By condition (3.3), V can be factorized (see [21]),

$$V=(I-W^{-})^{-1}(I+W^{+}) \tag{3.5}$$

in such a way that

1. $\det(I\pm W^{\pm})$ does not vanish.
2. For every "$\pm$" region, the restriction of $I\pm W^{\pm}$ to its boundary is smooth.
3. $W^{-}=0$ away from a neighborhood of A, B, C, and D.

A matrix function f on Σ_{AR} belongs to $H^1(\Sigma_{AR}^{\pm})$ if $f/\partial w$ is H^1 for every $\pm$ region w (see [21]), H^1 is the space of functions f, such that f and its distributional derivative both belong to L_2. Clearly $W_Q^{\pm} \doteqdot e^{-Q}W^{\pm}e^{Q}\in H^1(\Sigma_{AR}^{\pm})$.

Define the Cauchy integral operators $C_\pm$ by

$$C_\pm f(z'')=\lim_{z'\to z''}\frac{1}{2\pi i}\int_{\Sigma_{AR}}\frac{f(z)dz}{z-z'},$$

where the nontangential limit $z'\to z''$ is taken from $\pm$ regions. The following results are given in [21]: $H^1(\Sigma_{AR}^\pm)$ are Banach algebras (under certain equivalent norms), and the Cauchy integral operators are bounded on them,

$$C_\pm : H^1(\Sigma_{AR}^\pm)\to H^1(\Sigma_{AR}^\pm), \tag{3.6}$$

and

$$C_\pm : H^1(\Sigma_{AR}^\mp)\to H^1(\Sigma_{AR})\doteqdot H^1(\Sigma_{AR}^+)\cap H^1(\Sigma_{AR}^-). \tag{3.7}$$

We therefore have a linear integral equation on the space $H^1(\Sigma_{AR})$,

$$Y=\nu I+C_+ W_Q^- + C_- W_Q^+ \doteqdot \nu I + C_W Y, \qquad \nu\in\mathbb{C}. \tag{3.8}$$

Equation (3.8) is in fact Fredholm of zero index, where the Fredholm index $(\mathrm{Id}-C_W)=\dim\ker(\mathrm{Id}-C_W)-\dim\operatorname{coker}(\mathrm{Id}-C_W)$. It is easily checked that Y is a solution of (3.8) iff

$$\hat{\Phi}_\pm \doteqdot Y(I\pm W^\pm)=\nu I + C_\pm Y(W_Q^- + W_Q^+), \tag{3.9}$$

is a solution of the RH problem

$$\hat{\Phi}_+ = \hat{\Phi}_- V_Q, \tag{3.10}$$

satisfying $\hat{\Phi}_\pm(\infty)=\nu I$. As usual we denote by $\hat{\Phi}$ the sectionally holomorphic function obtained from the analytic extension of $\hat{\Phi}_\pm$, and sometimes call $\hat{\Phi}$ the solution of the RH problem (3.10). The function $\hat{\Phi}$ is said to be a normalized solution, if $\nu=1$, and a vanishing solution if $\nu=0$. The normalized solution is always unique.

Proposition 3.2. *Let Φ be holomorphic on $\mathbb{C}\backslash\Sigma_A$ satisfying* (3.4) *and let $\hat{\Phi}$ which is holomorphic on $\mathbb{C}\backslash\Sigma_{AR}$ be obtained from Φ by restricting it from one side of each sector and analytically extending it into the shadowed area from the other side. Then Φ is a normalized solution of the* RH *problem described in* (3.4) *iff $\hat{\Phi}$ is the normalized solution of the* RH *problem* (3.10).

Proof. If $\hat{\Phi}$ happends to be a normalized solution for (3.10) in H^1 space, then obviously the corresponding Φ satisfies all the conditions enumerated in Definition 3.1.

Now assume that Φ is a normalized solution in the sense of Definition 3.1. We first show that $I-C_W$ is invertible by showing that there exist not nontrivial vanishing solutions for the RH problem (3.10). Let $\hat{\Phi}_0$ be a vanishing solution for that RH problem. Let $\hat{\Phi}$ be obtained from Φ by the restriction, extension procedure described above. Then it is clear that $\hat{\Phi}_0\hat{\Phi}^{-1}$ is entire. By the conditions in Definition 3.1 for Φ, $\hat{\Phi}_0\hat{\Phi}^{-1}=0$. This shows $\hat{\Phi}_0=0$. Then since the Fredholm index of $\mathrm{Id}-C_W$ is zero, it is invertible, and we obtain a solution $\hat{\Phi}'$ in H^1 space for (3.10). A similar argument as above shows $\hat{\Phi}=\hat{\Phi}'$. QED.

Remark. The above proposition shows that from the function Φ described in Definition 3.1 we obtain a solution $\hat{\Phi}$ in Sobolev space with $\lim_{z\to\infty}\hat{\Phi}(z)=I$. Therefore Φ has limit I on $z\to\infty$ in the unshadowed regions in Fig. 3.3, and behaves like $e^Q V e^{-Q}$ or its inverse in the shadowed regions.

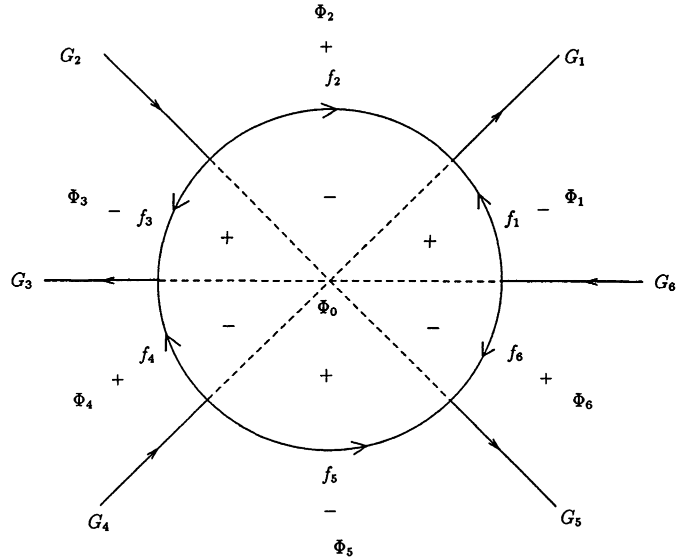


Fig. 3.4

For P II we define $\Phi_j, j=0,1,\dots,6$ by

$$Y_j \doteq \Phi_j e^Q, \quad j=1,\dots,6, \quad Y_0 \doteq \Phi_0 e^Q z^{\theta\sigma_3}, \quad Q \doteq -i\left(\frac{4z^3}{3}+tz\right)\sigma_3 .$$

Then again we obtain the RH (3.2), where the jumps are given by

$$\begin{array}{lll} \Phi_2=\Phi_1 e^Q G_1 e^{-Q}, & \Phi_2=\Phi_3 e^Q G_2^{-1} e^{-Q}, & \Phi_4=\Phi_3 e^Q G_3 e^{-Q}, \\ \Phi_4=\Phi_5 e^Q G_4^{-1} e^{-Q}, & \Phi_6=\Phi_5 e^Q G_5 e^{-Q}, & \Phi_6=\Phi_1 e^Q G_6^{-1} e^{-Q}, \\ \Phi_0=\Phi_1 e^Q f_1^{-1} e^{-Q}, & \Phi_2=\Phi_0 e^Q f_2 e^{-Q}, & \Phi_0=\Phi_3 e^Q f_3^{-1} e^{-Q}, \\ \Phi_4=\Phi_0 e^Q f_4 e^{-Q}, & \Phi_0=\Phi_5 e^Q f_5^{-1} e^{-Q}, & \Phi_6=\Phi_0 e^Q f_6 e^{-Q}, \end{array}$$

where

$$f_j = z^{\theta\sigma_3} E_0 G_1 \dots G_{j-1}, \quad j=1,\dots,6, \quad G_0 \doteq I .$$

Theorem 3.1. The Cauchy problems for P IV and P II always admit global meromorphic in t solutions. These solutions can be obtained by solving RH problems of the form $\Phi_+ = \Phi_- e^Q V e^{-Q}$ (see Figs. 3.1 and 3.4). For P IV the RH problem is specified in terms of the monodromy data (a,b,c,d) defined on the algebraic variety V_{IV} given by Eqs. (2.27) [E_0 is any solution of Eq. (2.26)]. For P II the RH problem is specified in terms of the monodromy data (a,b,c) defined on the algebraic variety V_{II} given by Eq. (2.17) [E_0 is my solution of Eq. (2.14)]. Having obtained Φ, the solutions of P IV and P II are constructed from

$$y = -2t - \partial_t \ln(\Phi_{-1})_{12} \quad \text{and} \quad y = 2(\Phi_{-1})_{12}$$

respectively, where

$$\Phi = I + \frac{\Phi_{-1}}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty .$$

For each $t_0 \in \mathbb{C}$, there exists an analytic variety V_{IV}^0, $\dim V_{IV}^0 < 2$, such that the monodromy transformation for P IV is a bijection, $\mathbb{C}^2 \to V_{IV}/V_{IV}^0$, $(y(t_0), y'(t_0)) \mapsto (a,b,c,d) \in V_{IV}/V_{IV}^0$ and $(a,b,c,d) \in V_{IV}^0$ iff the inverse monodromic transform does not exist at t_0. Similarly for P II.

Proof. Since $W_Q^\pm$ may be viewed as holomorphic functions in t from $\mathbb{C} \to H(\Sigma_{AR}^\pm)$, the operator $\mathrm{Id} - C_W$ is holomorphic in t. Hence if it is invertible at some t_0, then its inverse is meromorphic in t (see [21], Proposition 4.3). For the Cauchy problem, let $\Phi(t_0, z)$ be the solution obtained from the direct problem at $t = t_0$. As shown in the proof of Proposition 3.2, $\mathrm{Id} - C_W$ is invertible at t_0. Once $\hat{\Phi}$ is obtained one can use (2.1) to determine y, provided that one can prove that $\hat{\Phi}$ satisfies (2.1). In other words one needs to show that the solution of the inverse problem solves the direct problem:

(i) P IV. Recall that

$$Y_j = \Phi_j e^Q \left(\frac{1}{z}\right)^{\theta_\infty \sigma_3}, \quad j = 1,2,3,4 \quad \text{and} \quad Y_0 = \Phi_0 e^Q z^{\theta_0 \sigma_3}.$$

We denote by Y, Φ the solutions piecewise defined by Y_j, Y_0, Φ_j, Φ_0 and we define A by $A \doteqdot Y_z Y^{-1}$. Then

$$A = [\Phi_z + (z+t)\Phi\sigma_3 - \theta_\infty z^{-1}\Phi\sigma_3]\Phi^{-1}.$$

It follows from the fact that Y_z and Y both admit the same jump across Σ that A is holomorphic in $\mathbb{C} \backslash \{0\}$, thus $A = z^{-1}A_{-1} + A_0 + zA_1$ (note that the oscillation and the possible polynomial growth of Φ near Σ is insignificant to the determination of the growth of A as $z \to \infty$), hence

$$\Phi_z + (z+t)\Phi\sigma_3 - \theta_\infty z^{-1}\Phi\sigma_3 = (z^{-1}A_{-1} + A_0 + zA_1)\Phi, \tag{3.11}$$

where z is outside the circle. Let

$$\Phi = I + \frac{\Phi_{-1}}{z} + \frac{\Phi_{-2}}{z^2} + O\left(\frac{1}{z^3}\right), \quad \text{is} \quad z \to \infty,$$

in any sector but away from its boundary. Then the large z asymptotics of Eq. (3.11) implies

$$\begin{gathered} A_1 = \sigma_3, \quad A_0 = t\sigma_3 + [\Phi_{-1}, \sigma_3], \\ A_{-1} = -\theta_\infty \sigma_3 + [\Phi_{-2} + t\Phi_{-1}, \sigma_3] + [\sigma_3, \Phi_{-1}]\Phi_{-1}. \end{gathered} \tag{3.12}$$

Similarly, considering z inside the circle we find

$$\Phi_z + (z+t)\Phi\sigma_3 + \theta_0 z^{-1}\Phi\sigma_3 = (z^{-1}A_{-1} + A_0 + zA_1)\Phi.$$

The asymptotics of this equation as $z \to 0$ yields

$$A_{-1} = \theta_0 \Phi(0,t)\sigma_3 \Phi^{-1}(0,t) \Rightarrow \det A_{-1} = -\theta_0^2. \tag{3.13}$$

Let us define u, v by

$$u = -2(\Phi_{-1})_{12}, \quad v = \theta_0 + \theta_\infty - 2(\Phi_{-1})_{12}(\Phi_{-1})_{21}. \tag{3.14}$$

Then Eq. (3.12b) implies

$$A_0 = t\sigma_3 + \begin{pmatrix} 0 & u \\ \frac{2}{u}(v - \theta_0 - \theta_\infty) & 0 \end{pmatrix}. \tag{3.15}$$

To determine A_{-1} we note that Eq. (3.12c) yields

$$(A_{-1})_{\mathrm{diag}}=(2(\Phi_{-1})_{12}(\Phi_{-1})_{21}-\theta_{\infty})\sigma_3=(\theta_0-v)\sigma_3\,. \tag{3.16}$$

If we let $(A_{-1})_{21}=-\frac{uy}{2}$, then Eqs. (3.16), (3.13) imply that $(A_{-1})_{21}=\frac{2v}{uy}(v-2\theta_0)$. Thus

$$A_{-1}=\begin{pmatrix} \theta_0-v & \frac{-uy}{2} \\ \frac{2v}{uy}(v-2\theta_0) & -\theta_0+v \end{pmatrix}. \tag{3.17}$$

This concludes the proof that the inverse problem solves (2.1a). In a similar way it can be shown that it also solves Eq. (2.1b) and hence P IV.

(ii) P II. Recall that $Y=\Phi e^{Q}z^{\theta\sigma_3}$ inside the circle and $Y=\Phi e^{Q}$ outside the circle. Introducing the notation

$$f^{\sigma_1}(e^{i\pi}z)=\sigma_1 f(z)\sigma_1\,, \tag{3.18}$$

the equations $G_{j+3}=\sigma_1 G_j\sigma_1, j=1,2,3$ can be written in the compact form $G=G^{\sigma_1}$. This implies

$$\Phi=\Phi^{\sigma_1}\,,\qquad z \text{ outside the circle}. \tag{3.19}$$

The symmetry condition (3.19) plays a crucial role in establishing that the solution of the inverse problem also solves the direct problem. Let us first consider Eq. (2.1b): Following the same arguments as in (i) above it follows that

$$\Phi_t-iz\Phi\sigma_3=(B_0+zB_1)\Phi\,. \tag{3.20}$$

The large z asymptotics of (3.20) yields

$$B_1=-i\sigma_3\,,\qquad B_0=i[\sigma_3,\Phi_{-1}]\,. \tag{3.21}$$

However, we need to show that $B_0=y\sigma_1$ for some y. This can be achieved by using (3.19):

$$\Phi_{-1}=-\Phi_{-1}^{\sigma_1}\Rightarrow B_0^{\sigma_1}=i[\sigma_3^{\sigma_1},\Phi_{-1}^{\sigma_1}]=i[\sigma_3,\Phi_{-1}]=B_0\,,$$

which implies

$$B_0=i[\sigma_3,\Phi_{-1}]=y\sigma_1\,,\quad \text{for some } y\,. \tag{3.22}$$

We now consider Eq. (2.1 a),

$$\Phi_z-i(4z^2+t)\Phi\sigma_3+\theta z^{-1}\Phi\sigma_3=(z^{-1}A_{-1}+A_0+zA_1+z^2A_2)\Phi\,, \tag{3.23}$$

where z is inside the circle. Hence

$$A_{-1}=\theta\Phi(0,t)\sigma_3\Phi^{-1}(0,t)\,. \tag{3.24}$$

However, we need to show that $A_{-1}=-\theta\sigma_1$. This can be achieved by using the symmetry relation. This relation implies (for z inside the circle),

$$\Phi^{\sigma_1}=-i\Phi e^{Q}\sigma_2 e^{-Q}\Rightarrow\Phi^{\sigma_1}(0,t)=-i\Phi(0,t)\sigma_2\,. \tag{3.25}$$

Equations (3.24), (3.25) yield

$$A_{-1}=-\theta\sigma_1\,.$$

To determine A_0, A_1, A_2 we use

$$\Phi_z - i(4z^2+t)\Phi\sigma_3 = (z^{-1}A_{-1}+A_0+zA_1+z^2A_2)\Phi, \tag{3.26}$$

for z outside the circle. The large z asymptotics of (3.26) yields

$$\begin{aligned} A_2 &= -4i\sigma_3, \qquad A_1 = 4i[\sigma_3, \Phi_{-1}] = 4y\sigma_1, \\ A_0 &= -it\sigma_3 - 4y\sigma_1\Phi_{-1} + 4i[\sigma_3, \Phi_{-2}]. \end{aligned} \tag{3.27}$$

Equation (3.20) implies

$$\Phi_{-1_t} = -i[\sigma_3, \Phi_{-2}] + y\sigma_1\Phi_{-1}. \tag{3.28}$$

Using Eqs. (3.28) and (3.22) we find

$$(\Phi_{-1})_{\text{off-diag}} = \frac{y}{2}\sigma_2, \qquad (\Phi_{-1_t})_{\text{diag}} = \frac{iy^2}{2}\sigma_3. \tag{3.29}$$

Using the relations (3.29) in (3.27c) we obtain

$$A_0 = -it\sigma_3 - 4\Phi_{-1_t} = -it\sigma_3 - 2y_t\sigma_2 - 2iy^2\sigma_3. \tag{3.30}$$

Finally, from the direct problem at $t=t_0$ there exist (a,b,c,d) such that $\mathrm{Id}-C_W$ is invertible at t_0. Then the analytic Fredholm theorem on irreducible varieties implies that the set $V_{\mathrm{IV}}^0 \doteqdot \{(a,b,c,d) \in V_{\mathrm{IV}}/\text{corresponding } \mathrm{Id}-C_W \text{ at } t_0 \text{ is not invertible}\}$ is an analytic variety with $\dim V_{\mathrm{IV}}^0 < \dim V_{\mathrm{IV}} = 2$. QED.

In the last section, we will obtain certain conditions on the monodromy data. Under those conditions, the solutions of P II and P IV have no poles when t is restricted to Stokes lines. In this respect the following lemma will be useful.

Lemma 3.1. *Every vanishing solution $\hat{\Phi}$ of the* RH *problem has strong decay,*

$$\hat{\Phi} = O\left(\frac{1}{z}\right).$$

Proof. By (3.8),

$$\hat{\Phi}(z) = \frac{1}{2\pi_i} \int_{\Sigma_{AR}} \frac{d\xi}{\xi - z} Y(\xi)(W_Q^-(\xi) + W_Q^+(\xi)), \qquad z \in \mathbb{C}\backslash\Sigma_{AR},$$

where Φ is the vanishing solution for the equation corresponding to $\hat{\Phi}$. Therefore

$$\begin{aligned} z\hat{\Phi}(z) = {} & \frac{1}{2\pi i} \int \frac{d\xi}{\xi - z} Y(\xi)\xi(W_Q^-(\xi) + W_Q^+(\xi)) \\ & - \frac{1}{2\pi i} \int d\xi Y(\xi)(W_Q^-(\xi) + W_Q^+(\xi)). \end{aligned}$$

The first integral is bounded because $Y\xi(W_Q^- + W_Q^+)$ is a H^1 function and the second is bounded because $Y(W_Q^- + W_Q^+)$ is an L^1 function.

Because of the analytic structure of the jump matrices, it is possible to define an equivalent RH on fewer contours. Consider for example P II. Since $Y_2 = Y_1G_1$, $Y_3 = Y_2G_1$ and Y_1, Y_2, Y_3 tend to e^Q on $\Sigma_1, \Sigma_2, \Sigma_3$ respectively (see Fig. 2.1), it follows that Y_2 can be defined in $S_1 \cup S_2$ and $Y_2 \sim e^Q$ away from $\Sigma_1 \cup \Sigma_3$, $Y_2 \sim e^Q G_1$ on Σ_1, $Y_2 \sim e^Q G_2^{-1}$ on Σ_3. Similar considerations apply to the other Y_j's and to P IV.

We shall use this fact to obtain vanishing lemmas. Suppose $\hat{\Phi}$ is a vanishing solution. We can use it, to construct via analytic continuation a vanishing solution

Y. From Lemma 3.1, the asymptotic behavior of Y_j, as $z \to \infty$ in sector $S_j \cup S_{j-1}$ is given as follows:

For P II:

$$Y_j \sim \begin{cases} 0\left(\frac{1}{z}\right) e^Q & \text{away from} \quad \Sigma_{j+1} \cup \Sigma_{j-1} \\ 0\left(\frac{1}{z}\right) e^Q G_j^{-1} & \text{near} \quad \Sigma_{j+1} \\ 0\left(\frac{1}{z}\right) e^Q G_{j-1} & \text{near} \quad \Sigma_{j-1}, \end{cases} \tag{3.31a}$$

where $Q = -i(\frac{4}{3}z^3 + tz)\sigma_3$, and $j = 1, \ldots, 6 \bmod 6$.

For P IV:

$$Y_j \sim \begin{cases} 0\left(\frac{1}{z}\right) e^Q \left(\frac{1}{z}\right)^{\theta_\infty \sigma_3} & \text{away from} \quad \Sigma_{j+1} \cup \Sigma_{j-1} \\ 0\left(\frac{1}{z}\right) e^Q \left(\frac{1}{z}\right)^{\theta_\infty \sigma_3} \hat{G}_j^{-1} & \text{near} \quad \Sigma_{j+1} \\ 0\left(\frac{1}{z}\right) e^Q \left(\frac{1}{z}\right)^{\theta_\infty \sigma_3} \hat{G}_{j-1} & \text{near} \quad \Sigma_{j-1}, \end{cases} \tag{3.31b}$$

where $Q = \left(\frac{z^2}{2} + tz\right)\sigma_3$, $\hat{G}_j = G_j$ for $j = 1, 2, 3$, $\hat{G}_4 = G_4 e^{2i\pi\theta_\infty}$, and $j = 1, \ldots, 4$, mod 4.

4. Vanishing Lemmas

In this section we show that under certain constraints on the monodromy data, every vanishing solution of the RH problems associated with P II and P IV is zero. As described earlier, a vanishing solution is an element in $K_{er}(I_d - C_w)$. Therefore the vanishing lemma shows that $I_d - C_w$ is invertible and the RH problem is then uniquely solvable.

We denote by $f^\dagger(z) \doteqdot f^*(\bar{z})$ the Schwarz reflection of a matrix function f. Consider the RH problem $\varphi^+ = \varphi^- V$ on the contour Σ containing the real axis. Let $V \wedge \mathbb{R}$ and $V \wedge \Sigma \backslash \mathbb{R}$ denote V on the real axis and on the rest of Σ respectively. It is shown in [21] that if V satisfies

$$V \wedge \Sigma \backslash \mathbb{R} \quad \text{is Schwarz reflection invariant} \tag{4.1a}$$

and

$$\operatorname{Re} V \wedge \mathbb{R} > 0, \tag{4.1b}$$

then every vanishing solution to the above RH problem must be zero. This result is obtained as follows: Let $H \doteqdot \varphi\varphi^\dagger$, where φ is a vanishing solution. Then Eq. (4.1a) implies that H is holomorphic on $\mathbb{C} \backslash \mathbb{R}$, and on $\mathbb{R}$

$$H^+ = \varphi^+(\varphi^-)^\dagger = \varphi^- V(\varphi^-)^* .$$

Since $H=0\left(\frac{1}{z^2}\right)$ as $z\to\infty$,

$$\int_{\mathbb{R}} H^+ = \int_{\mathbb{R}} \varphi^- V(\varphi^-)^* = 0 .$$

Then Eq. (4.1b) implies that $\varphi^- \equiv 0$.

Using the above result and Eqs. (3.31) it is straightforward to obtain vanishing lemmas for P II and P IV. The only difference here is that the relevant solutions for P II and P IV may possess singularities at $z=0$ and $z=\infty$. We therefore also impose appropriate constants on θ, θ_0, θ_∞ to ensure that such singularities for H^+ are integrable.

It is interesting that a direct application of the above result to the original RH problems for P II and P IV, i.e. before the reduction of the number of contours, fails to yield vanishing lemmas. However, it is possible to obtain vanishing lemmas for the original RH problems provided one uses $H=\varphi h \varphi^\dagger$, instead of $H=\varphi\varphi^\dagger$, where h is an appropriately chosen piecewise constant matrix. This is shown in the appendix.

Lemma 4.1. *(Vanishing Lemma for* P II*). Assume that the matrix function Y, holomorphic on $\mathbb{C}/\Sigma$, has the following properties:*

(i) *possesses the jumps G_j, $1\leqq j\leqq 6$ on Σ given by* (2.6),
(ii) *behaves near $z=0$ according to Eq.* (2.4) *with $|\mathrm{Re}\,\theta|<\frac{1}{2}$,*
(iii) *the analytic continuation of Y behaves near $z=\infty$ according to* (3.31a).

Then $Y(t)\equiv 0$,

$$\text{for} \quad t\in\mathbb{R} \quad \text{if} \quad b=\bar{b}, \quad |a-\bar{c}|<2, \tag{4.2a}$$

$$\text{for} \quad t\in e^{\frac{2\pi i}{3}}\mathbb{R} \quad \text{if} \quad c=\bar{c}, \quad |b-\bar{a}|<2, \tag{4.2b}$$

$$\text{for} \quad t\in e^{\frac{-2\pi i}{3}}\mathbb{R} \quad \text{if} \quad a=\bar{a}, \quad |c-\bar{b}|<2. \tag{4.2c}$$

Proof.

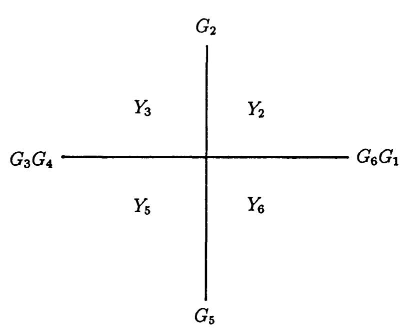


Fig. 4.1

As it is illustrated in Fig. 4.1, the conditions (4.1) yield

$$G_2=G_5^*, \qquad \mathrm{Re}\,G_3G_4>0, \qquad \mathrm{Re}\,G_6G_1>0. \tag{4.3}$$

Equation $G_2=G_5^*$ implies $b=\bar{b}$, while Eqs. (4.3b) and (4.3c) imply $|a-\bar{c}|<2$. The conditions (ii) and (iii) in the lemma guarantee that the singularities of H at $z=0,\infty$ are integrable. Hence the Vanishing Lemma follows.

The cases (4.2b) and (4.2c) are obtained in a similar way after performing suitable rotations of the $\mathbb{C}$-plane.

Lemma 4.2. *(Vanishing Lemma for* P IV*). Assume that the matrix function* Y*, holomorphic on* $\mathbb{C}/\Sigma$*, has the following properties:*
(i) *possesses jumps* $\hat{G}_j$, $1\leqq j\leqq 4$ *on* Σ, $\hat{G}_j=G_j$, $j=1,2,3$, $\hat{G}_4=G_4 e^{2i\pi\theta_\infty\sigma_3}$,
(ii) *behaves near* $z=0$ *according to Eq.* (2.22a) *with* $|\mathrm{Re}\,\theta_0|<\frac{1}{2}$,
(iii) *the analytic continuation of* Y *satisfies* (3.31b) *with* $|\mathrm{Re}\,\theta_\infty|<\frac{1}{4}$.
Then $Y(t)\equiv 0$,

$$\textit{for}\quad t\in e^{-\frac{\pi i}{4}}\mathbb{R},\quad \textit{if}\quad |b-\bar{c})<2,$$

and

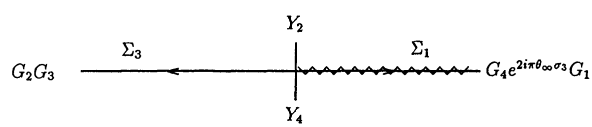
$$e^{2\pi I_m\theta_\infty}|a-\bar{d}|<2|\cos 2\pi\,\mathrm{Re}\,\theta_\infty|, \tag{4.4a}$$

$$\textit{for}\quad t\in e^{\frac{\pi i}{4}}\mathbb{R},\quad \textit{if}\quad |c-\bar{d}|<2,$$

and

$$e^{-2\pi I_m\theta_\infty}|a-\bar{b}|<2|\cos 2\pi\,\mathrm{Re}\,\theta_\infty|. \tag{4.4b}$$

Proof. Performing the rotation $z'=e^{\frac{i\pi}{4}}z$, Σ_1 is mapped on to the real axis and q becomes $q=-i\left(\frac{z'^2}{3}+te^{\frac{i\pi}{4}}z'\right)$, thus for $te^{\frac{i\pi}{4}}\in\mathbb{R}$, $Q^\dagger=-Q$,

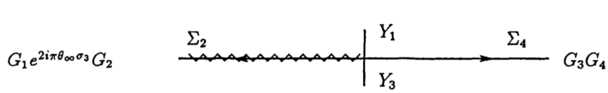
Fig. 4.2


As it is illustrated in Fig. 4.2, Eq. (4.1b) implies

$$\mathrm{Re}\,G_2G_3>0,\qquad \mathrm{Re}\,G_4e^{2i\pi\theta_\infty\sigma_3}G_1>0. \tag{4.5}$$

Let $M\doteqdot G_4e^{2i\pi\theta_\infty\sigma_3}G_1$. One of the diagonal entries of $M+M^*$ is $e^{2i\pi\bar{\theta}_\infty}+e^{-2i\pi\theta_\infty}$; positivity of this entry is equivalent to $|\mathrm{Re}\,\theta_\infty|<\frac{1}{4}$. Also demanding that $\det(M+M^*)>0$ we find $e^{2\pi I_m\theta_\infty}|a-\bar{d}|<2|\cos 2\pi\,\mathrm{Re}\,\theta_\infty|$. Again conditions (ii) and (iii) in the lemma guarantee that the singularities of H at $z=0$, ∞ are integrable.

In the case of $t=e^{\frac{i\pi}{4}}\mathbb{R}$, we take the branch cut along Σ_2. Then the rotation $z'=e^{\frac{3i\pi}{4}}z$ yields the jumps indicated in Fig. 4.3 and Eq. (4.4b) follows similarly.

Fig. 4.3


Theorem 4.1. *For each Stokes line, there exists a set of conditions on the monodromy data given by Eqs.* (4.2)*, such that for* t *on a Stokes line, the solution for* P II *with* $|\mathrm{Re}\,\theta|<\frac{1}{2}$ *has no poles.*

Remark 4.1. If the Cauchy data is purely imaginary and t is real, then Eq. (4.2a) is always satisfied. Indeed, if q and θ are purely imaginary, then A defined by (2.2a) satisfies $\sigma_2\bar{A}(\bar{z})\sigma_2=A(z)$. Thus if $Y(z)$ is a solution of (2.1a), $\sigma_2\bar{Y}(\bar{z})\sigma_2$ is also a solution. Hence

$$\sigma_2\bar{Y}_2(\bar{z})\sigma_2=Y_6(z), \quad \sigma_2\bar{Y}_3(\bar{z})\sigma_2=Y_5(z), \quad \sigma_2\bar{Y}_1(\bar{z})\sigma_2=Y_1(z). \tag{4.6}$$

But

$$G_5=Y_5^{-1}(z)Y_6(z)=\sigma_2(\bar{Y}_3(\bar{z}))^{-1}\bar{Y}_2(\bar{z})\sigma_2=\sigma_2(\bar{G}_2)^{-1},$$

thus $b=\bar{b}$. Also

$$G_6=Y_6^{-1}(z)Y_1(z)=\sigma_2(\bar{Y}_2(\bar{z}))^{-1}\bar{Y}_1(\bar{z})\sigma_2=\sigma_2(\bar{G}_1)^{-1}\sigma_2,$$

thus $\bar{a}=c$.

Remark 4.2. Unfortunately, we do not have such a theorem for P IV. The reason for this is that y has a pole when Φ_{-1} has a zero. But the vanishing lemma for P IV is still useful, for instance, regarding the long time asymptotic behavior of P IV.

Appendix

We now show how the vanishing lemmas can be derived on the original contours. Let h be a piecewise constant matrix. We consider $H \doteqdot YhY^{\dagger}$ and we choose h, G_j's in such a way that H is holomorphic in $\mathbb{C}^+$, and H can be reduced on $\mathbb{R}$ to a quadratic form with positive definite real part.

P II.

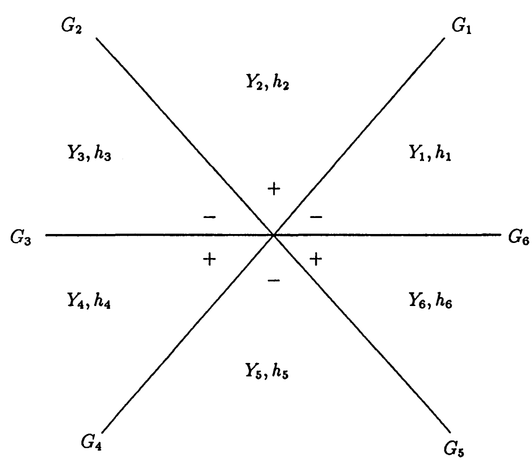


Fig. A.1

We note that for $t\in\mathbb{R}$, $Q^{\dagger}=-Q$, thus in order for H to be bounded, the large z behavior of Y suggests that h must be lower triangular in Sects. I, III, V and upper triangular in the complement. Also continuity across Σ_3 implies

$$H^+=Y_2h_2Y_5^{\dagger}=Y_2h_2G_4^*Y_4^{\dagger}=H^-=Y_3h_3Y_4^{\dagger}=Y_2G_2h_3Y_4^{\dagger}.$$

Thus $h_2G_4^*=G_2h_3$ and the uniqueness of upper-lower triangular factorization yields

$$h_2=G_2, \quad h_3=G_4^*. \tag{A.1}$$

Continuity across Σ_2 implies

$$H^+ = Y_2 h_2 Y_5^\dagger = Y_1 G_1 h_2 Y_5^\dagger = H^- = Y_1 h_1 Y_6^\dagger = Y_1 h_1 G_5^* Y_5^\dagger .$$

Thus

$$h_1 = G_1, \quad h_2 = G_5^* . \tag{A.2}$$

Equations (A.1), (A.2) yield the constraint $G_2 = G_5^*$, i.e. $b = \bar{b}$. On Σ_4, Σ_1, H^- becomes:

On $\Sigma_4 : H^- = Y_3 h_3 Y_4^\dagger = Y_3 G_4^* G_3^* Y_3^\dagger$.

On $\Sigma_1 : H^- = Y_1 h_1 Y_6^\dagger = Y_6 G_6 G_1 Y_6^\dagger$.

H is holomorphic in $\mathbb{C}^+$ with continuous extension to $\mathbb{R}$, except possibly at $z=0$ and ∞. But for $|\mathrm{Re}\,\theta| < \frac{1}{2}$ the possible singularities at 0 and ∞ are integrable. Thus

$$\int_{\mathbb{R}} \mathrm{tr} H^- dz = 0, \ \Rightarrow \int_{\mathbb{R}} \mathrm{tr}(H^- + H^{-*}) dz = 0 . \tag{A.3}$$

Let

$$L \doteq G_3 G_4, \quad M = G_6 G_1 . \tag{A.4}$$

If the eigenvalues of the Hermitian matrices $L + L^*$, $M + M^*$ are positive then Eq. (A.3b) implies $Y = 0$. Since both these matrices have 2 as one of their diagonal entries, it follows that the positivity of the eigenvalues is equivalent to the positivity of their determinants. Demanding that $\det(L + L^*) > 0$ and $\det(M + M^*) > 0$ we find $|a - \bar{c}| < 2$. □

P IV.

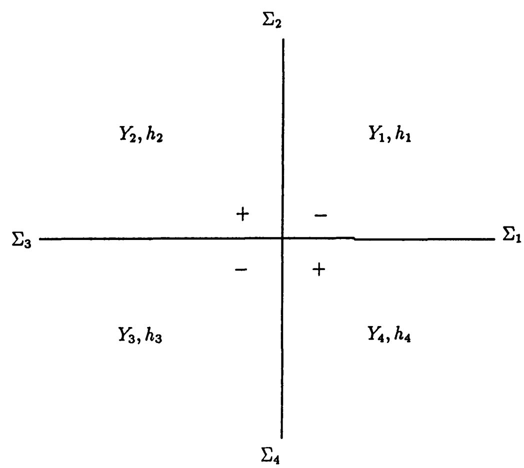


Fig. A.2

Continuity of $H = Y h Y^\dagger$ implies

$$H^- = Y_1 h_1 Y_4^\dagger = Y_1 h_1 G_3^* Y_3^\dagger = H^+ = Y_2 h_2 Y_3^\dagger = Y_1 G_1 h_2 Y_3^\dagger ,$$

i.e.

$$h_1 = G_1, \quad h_2 = G_3^* . \tag{A.5}$$

On Σ_3, Σ_1, H^+, H^- become

$$Y_2 G_3^* Y_3^\dagger = Y_2 G_3^* G_2^* Y_2^\dagger \quad \text{and} \quad Y_1 G_1 Y_4^\dagger = Y_4 G_4 e^{2i\pi\theta_\infty\sigma_3} G_1 Y_4^\dagger$$

respectively. The positivity of the eigenvalues of $G_2 G_3 + (G_2 G_3)^*$ implies $|b - \bar{c}| < 2$. Let $M \doteq G_4 e^{2i\pi\theta_\infty\sigma_3} G_1$. One of the diagonal entries of $M + M^*$ is $e^{2i\pi\bar{\theta}_\infty} + e^{-2i\pi\theta_\infty}$; positivity of this entry is equivalent to $|\mathrm{Re}\,\theta_\infty| < \frac{1}{4}$. Also demanding that $\det(M + M^*) > 0$ we find $e^{2\pi I_m \theta_\infty}|a - \bar{d}| < 2|\cos 2\pi\, \mathrm{Re}\,\theta_\infty|$.

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