Commun. Math. Phys. 144, 107–148 (1992)

Communications in
Mathematical
Physics

The K-Property of Four Billiard Balls

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Received April 19, 1991; in revised form July 23, 1991

Abstract. A further step is achieved toward establishing the celebrated Boltzmann-Sinai ergodic hypothesis: for systems of four hard balls on the ν-torus ($\nu > 2$) it is shown that, on the submanifold of the phase specified by the trivial conservation laws, the system is a K-flow. All parts of our previous demonstration providing the analogous result for three hard balls are simplified and strengthened. The main novelties are: (i) A refinement of the geometric-algebraic methods used earlier helps us to bound the codimension of the arising implicitly given set of degeneracies even if we can not calculate their exact dimension that was possible for three-billiards. As a matter of fact, it is this part of our arguments, where further understanding and new ideas are necessary before attacking the general ergodic problem; (ii) In the "pasting" part of the proof, which is a sophisticated version of Hopf's classical device, the arguments are so general that it is hoped they work in the general case, too. This is achieved for four balls, in particular, by a version of the Transversal Fundamental Theorem which, on one hand, is simpler and more suitable for applications than the previous one and, on the other hand, as we have discovered earlier, is the main tool to prove global ergodicity of semi-dispersing billiards; (iii) The verification of the Chernov-Sinai ansatz is essentially simplified and the new idea of the proof also promises to work in the general case.

Table of Contents

* Research partially supported by the Hungarian National Foundation for Scientific Research, grant No. 104052
** Research partially supported by the Hungarian National Foundation for Scientific Research, grant No. 1815

1. Introduction

Our knowledge about the phase picture of classical Hamiltonian systems with several particles (more than two, say) is surprisingly meager despite the central and old interest in the subject. So far as we know, the only completely understood examples are:

1. some one-dimensional completely integrable systems where the pair interaction has a special, so-called reflection-symmetric form [cf. M(1981)];
2. the ergodic system of three elastic balls [cf. K-S-Sz(1991)];
3. special models of Hamiltonian systems with an arbitrary number of particles elaborated by Bunimovich-Liverani-Pellegrinotti-Sukhov [B-L-P-S (1991)] whose ergodicity could have been shown by enhancing the methods of K-S-Sz (1991); though these are not very realistic, they are the first Hamiltonian systems with an arbitrary number of particles whose ergodicity has been established.

As to the case with two degrees of freedom, different families of potentials have been found that result in an ergodic system. Here we suggest the reader to consult the recent paper of Donnay-Liverani (1991) containing the most complete results (for instance, most remarkably, ergodic systems given by smooth potentials) and a good survey of earlier achievements.

The celebrated Boltzmann-Sinai ergodic hypothesis says that the Hamiltonian systems of an arbitrary ($N \geqq 2$) number of elastic hard balls on the $\nu(\geqq 2)$ dimensional torus are ergodic on connected components of submanifolds of the phase space where the trivial integrals of motion are constant.

An instructive way to think about the problem is the following. The hyperbolicity property needed to at least demonstrate local ergodicity of these systems is only provided for a trajectory if it gathers all possible degrees of freedom of the system. For two balls, one collision of these particles obviously introduces all necessary degrees of freedom. This is, however, evidently not so for a higher number of balls and, in K-S-Sz(1991), we suggested a strategy to tackle this difficulty and could also successfully apply it to the case of three balls.

Now, in the present paper, we are able to establish the K-property of systems of four balls. In fact – by simplifying and strengthening the tools of the previous work – we could reach such a general understanding of the ergodic problem that, as it seems to us at present, the main obstacle in proving the K-property for an arbitrary number of balls is a question of combinatorial flavor. This question was relatively easy for $N=3$, is tractable for $N=4$ and – apart from some conjectures to be discussed in a forthcoming paper – is quite new and completely open for $N \geqq 5$.

Let us go now to a technical formulation of our result. Assume that, in general, a system of $N(\geqq 2)$ balls of radii $r>0$ are given on $\mathbf{T}^\nu$, the ν-dimensional unit torus ($\nu \geqq 2$). Denote the phase point of the i^{th} ball by $(q_i, v_i) \in \mathbf{T}^\nu \times \mathbf{R}^\nu$. The configuration space $\tilde{\mathbf{Q}}$ of the N balls is a subset of $\mathbf{T}^{N \cdot \nu}$: from $\mathbf{T}^{N \cdot \nu}$ we cut out $\binom{N}{2}$ cylindric scatters:

$$\tilde{C}_{i,j} = \{Q=(q_1, \ldots, q_N) \in \mathbf{T}^{N \cdot \nu} : |q_i - q_j| < 2r\},$$

$1 \leqq i < j \leqq N$. The energy $H = \frac{1}{2}\sum_1^N v_i^2$ and the total momentum $P = \sum_1^N v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $H=\frac{1}{2}$ and $P=0$ and, moreover, that the sum of spatial components $B=\sum_1^N q_i = 0$ (if $P \neq 0$, then the center of mass has an additional conditionally periodic or periodic motion). For these values of H, P and B, the phase space of the system reduces to $M := \mathbf{Q} \times \mathscr{S}_{N \cdot \nu - \nu - 1}$, where

$$\mathbf{Q} := \left\{ Q \in \tilde{\mathbf{Q}} : \sum_1^N q_i = 0 \right\}$$

with $d := \dim \mathbf{Q} = N \cdot \nu - \nu$, and where $\mathscr{S}_k$ denotes, in general, the k-dimensional unit sphere. It is easy to see that the dynamics of the N balls, determined by their uniform motion with elastic collisions on one hand, and the billiard flow $[S^t : t \in \mathbf{R}\}$ in $\mathbf{Q}$ with specular reflections at $\partial \mathbf{Q}$ on the other hand, are isomorphic and they conserve the Liouville measure $d\mu = \text{const} \cdot dq \cdot dv$.

In this paper we prove the following

Main Theorem. *For $N=4$, $\nu \geqq 3$ and $r < 1/8$ the dynamical system $(M, \{S^t\}, \mu)$ is a K-flow.*

Earlier, for $N=2$, Sinai (1970) [see also Bunimovich-Sinai (1973) and Gallavotti (1975)] and Chernov and Sinai (1987) settled the cases $\nu=2$, $r \neq 1/4$ and $\nu \geqq 3$, $r \neq 1/4$ respectively [for a detailed formulation see also K-S-Sz (1990)] while, for $N=3$, $\nu \geqq 2$ and $r < 1/6$ the present authors could establish the K-property [cf. K-S-Sz (1991)]. The role of the restriction $r < 1/8$ is the same as that of the condition $r < 1/6$ in the case $N=3$. Its explanation and the discussion of the case $r > 1/8$, $N=4$ is analogous to what has been said in the case of three billiards and for details we refer to K-S-Sz (1991). On the other hand, the description of the difficulties arising in the case $\nu=2$, $N=4$ is essential and can be found in Sect. 4.

Next we describe the main parts of the proof by parallelly detailing the organization of the paper. As explained above, our hard ball systems can be represented as billiards in a higher-dimensional flat manifold with a special CW-complex as its boundary. (In the strict sense, the configuration space is not a *manifold with boundary*.)

As we have already mentioned, for two balls – apart from a well tractable small set – the trajectories of the isomorphic billiard system are manifestly hyperbolic since, in technical terms, the boundary components of the configuration space are strictly convex from outside (i.e. the billiard is a dispersing one).

In the case of three or more balls, however, the boundaries, in fact cylinders, are just convex (i.e. the billiard system is a semi-dispersing one). To obtain hyperbolicity of a trajectory or, as we said, to get all degrees of freedom of the system involved, one should assume that the collision history of the orbit is sufficiently rich.

The approach of our previous papers, K-S-Sz (1989) and K-S-Sz (1991) is based on the so-called fundamental theorem for semi-dispersing billiards [cf. S-Ch (1987) and K-S-Sz (1990)] which, in a 0^{th} approximation, sounds as follows:

Fundamental "Theorem" for Semi-Dispersing Billiards. *If x is a sufficient phase point, then it has an open neighborhood $U(x)$ that belongs to one ergodic component.*

Here *sufficiency* of a phase point x or, in fact, of its orbit is a dynamical notion implying that almost every point of its sufficiently small neighborhood is hyperbolic (though x itself may have zero exponents). The basic necessary notions and notations about semidispersing billiards including those connected with sufficiency are summarized in Sect. 2. The exact and, in fact, new and most powerful formulation of the Fundamental Theorem is given in Sect. 3.

If, for a semi-dispersing billiard, one can verify the statement that *non-sufficient points form a codimension two subset of the phase space*, then the Fundamental Theorem almost implies the ergodicity of the system. The proof of the above statement for our system of four balls forms the main body of the proof but further difficulties also arise, for the Fundamental Theorem (Theorem 3.4) has additional conditions (this is why its given formulation is only a 0^{th} approximation), and the presence of the first of them, in fact, leads to further difficulties. This assumption is a *global condition*, the so-called Chernov-Sinai Ansatz. (For hard ball systems, the verification of the additional two regularity conditions 3.2 and 3.3 is not difficult.)

Anyway, having the Fundamental Theorem at hand, the aforementioned strategy consists of the following main steps:

1. Find an appropriate concept of *richness* of a trajectory or a finite trajectory segment.

The definition need not be given in dynamical terms but rather in terms of the symbolic collision sequence of the orbit. Also, it is worth noting that the appropriate definition of richness may depend on the dimension of the torus.

2. *Geometric-algebraic considerations.*

According to our understanding, in a sufficiently small neighborhood of a phase point x with a rich trajectory segment, non-sufficient points form a CW-complex since they are solutions of certain implicitly given equations. Though we can not show exactly this statement, we can still prove that non-sufficient points form subsets of certain CW-complexes of codimension at least two. This is, of course, amply suitable for our purposes.

This part of the proof uses geometric and algebraic considerations. For the case of three balls, once the language of the problem had been found, this part was quite elementary.

The suitable concept of *richness* is introduced in Sect. 4 and, moreover, Main Theorem 4.3 expressing the statement of step 2 is also proved there.

3. *Prove that non-rich phase points form a codimension two subset.*

This part, the content of Sect. 5, is basically a sophisticated version of Hopf's classical method, a fundamental tool for establishing local or global ergodicity of hyperbolic dynamical systems. In fact, in K-S-Sz (1991) we proposed the method of "pasting," i.e. a variant of the Hopf method where the unstable and stable foliations to be used to connect typical phase points arise from different dynamical systems and these transversal foliations are complemented by a suitable, also transversal "neutral" foliation. It is this part of the proof where the present stronger though, in fact, equivalent formulation of the Fundamental Theorem to be given in Sect. 3 is needed in our proof.

4. *Singular orbits.*
In Parts 2 and 3, it is convenient to first restrict the arguments to neighborhoods of non-singular points. The treatment of points with a single singularity and the verification of the Chernov-Sinai Ansatz (Condition 3.1, an important assumption of the fundamental theorem) are closely related since both regard the notion of sufficiency of points with exactly one singularity on their trajectory. The analysis of these questions is executed in Sect. 6. Compared to our proof for the case of three billiards, the demonstration is new and essentially simpler.

Having these steps elaborated, the proof of our theorem will be easily composed from them in Sect. 7. Sections 3–7 containing the arguments are preceded by Sect. 2 collecting prerequisites about semi-dispersing billiards and their singularities, the underlying notions of neutral subspaces and sufficiency, the necessary facts from topological dimension theory and some further concepts: advance of an orbit, decompositions into subsystems, sub-billiards, etc. The paper is supplemented by an Appendix where we show that sets of phase points whose trajectories have three-class partitions on infinite time intervals are of codimension not less than two.

2. Prerequisites

Semi-Dispersing Billiards

In our previous paper K-S-Sz (1990), we formulated a self-contained summary of some basic notions about semi-dispersing billiards. For convenience and brevity, we will throughout use the concepts and notations of Sect. 2 of the aforementioned paper. Here we only summarize some further notions from K-S-Sz (1989) and K-S-Sz (1991) necessary to our forthcoming arguments.

As to the definitions of *semi-dispersing and dispersing billiards* $(M, \{S^t\}, \mu)$ on a d-dimensional billiard table $\mathbf{Q}$, their *Poincaré maps* $T_+ : M \to \partial M$, $T : \partial M \to \partial M$, *local stable and unstable invariant manifolds* $\gamma^{s,u}(x)$ we refer to the aforementioned work. Fix a semi-dispersing billiard system $(M, \{S^t\}, \mu)$ and let $x \in M \setminus \partial M$. Choose a C^2-smooth, codimension 1 submanifold $\tilde{\Sigma} \subset \mathbf{Q} \setminus \partial \mathbf{Q}$ such that $Q \in \tilde{\Sigma}$ and V is a unit normal vector to $\tilde{\Sigma}$ at Q. We shall throughout use the notation $x = (Q, V)$ for a point $x \in M$ where $V = p(x)$ is the velocity and $Q = \pi(x)$ is the spatial component of x. $\tilde{\Sigma}$ can be lifted to M to obtain a unique $\Sigma \subset M$ by requiring that

(i) the projection $\pi : \Sigma \to \tilde{\Sigma}$ be a C^1-diffeomorphism;
(ii) $x \in \Sigma$;
(iii) for every $y \in \Sigma$ the vector $p(y)$ is a normal vector to $\tilde{\Sigma}$.

We call Σ a *local orthogonal manifold* with support $\tilde{\Sigma}$. The unit normal vector V attached to $Q \in \tilde{\Sigma}$ by Σ is denoted by $V(Q)$.

The *second fundamental form* $B_\Sigma(x)$ of Σ (or $\tilde{\Sigma}$) at x is defined through

$$V(Q + dQ) - V(Q) = B_\Sigma(x) dQ + o(\|dQ\|)$$

and is a selfadjoint operator acting in the $(d-1)$-dimensional tangent hyperplane $J(x)$ of $\tilde{\Sigma}$ at x, where $d=\dim \mathbf{Q}$. For $q \in \partial Q$ denote by $K(q)$ the second fundamental form of ∂Q at q (it is always non-negative semi-definite since our billiard is semi-dispersing). In this definition, of course, ∂Q is supplied with the field of normal vectors pointing inwards Q.

Sometimes it will be convenient to denote $x^t = S^t x$ and $\Sigma^t = S^t \Sigma$. An orthogonal manifold Σ is called *convex* if $B_\Sigma(y) \geqq 0$ for every $y \in \Sigma$. For a finite or infinite interval $I \subset \mathbf{R}$ and $x \in M$, $S^I x$ will throughout denote the trajectory segment $\{S^t x : t \in I\}$.

Singularities

We denote by M^* the set of points $x \in M$ whose trajectory contains infinitely many collisions such that at most one of these collisions is *singular* (i.e. a multiple or a tangential collision). A collision at a point $x \in \partial M$ such that, in $\pi(x)$, at least two smooth pieces of $\partial \mathbf{Q}$ meet is called a *multiple* collision. A collision is called *tangential* if $x \in \partial M$ and $p(x) \in \mathbf{T}_{\pi(x)} \partial \mathbf{Q}$. We shall frequently use the collection $\mathscr{SR}^+$ of all singular reflections:

Definition 2.1. The set $\mathscr{SR}^+$ is the collection of all phase points $x \in \partial M$ for which the reflection, occurring at x, is singular (tangential or multiple) and, in the case of a multiple collision, x is supplied with the outgoing velocity V^+.

The reason for the last requirement is that, in the case of a multiple collision, there is no collision law for the velocities. It is not hard to see that $\mathscr{SR}^+$ is a compact cell-complex in M and $\dim(\mathscr{SR}^+) = \dim M - 2 = 6v - 3$.

In the case of a singular collision, in general, there are two branches of the trajectory of x (the dynamics $\{S^t\}$ has a discontinuity according to the order of the collisions) and, if $x \in M^*$, then the notions, to be introduced below, make sense for both of them. The important property of the set M^* is that its complement $M \setminus M^*$ is a residual set, see Definition 2.6. The subset of points $x \in M^*$ whose trajectory has no singular collision is denoted by M^0; then let $M^1 = M^* \setminus M^0$.

Finally, for arbitrary $n \in N$, let Δ_n be the set of *double singularities* of order not higher than n. Δ_n consists of points $x \in \partial M$ for which there exist two different integers k_1 and k_2 ($|k_1|, |k^2| \leqq n$) such that either $T^{k_i} x$ or $-T^{k_i} x$ belongs to $\mathscr{SR}^+$ $(i=1,2)$.

Neutral Subspaces and Sufficiency of Trajectory Segments

Let us introduce a convention. Since the configuration space $\mathbf{Q}$ has a natural parallelization, i.e. all of its tangent spaces $\mathscr{T}_Q \mathbf{Q}$ $(Q \in \mathbf{Q})$ can naturally be identified with a d-dimensional vector space $\mathscr{Z}$, the sums of the form $Q + W$ $(Q \in \mathbf{Q},\ W \in \mathscr{Z})$ are meaningful if the length $\|W\|$ is small enough.

Now we can define the neutrality of a direction $W \in \mathscr{Z}$.

Definition 2.2. Let $x \in M \setminus \partial M$. We say that the vector $W \in \mathscr{Z}$ is a neutral vector for the trajectory segment $S^{[a,b]} x$ at the point $x = (Q, V)$ $(a<0,\ b>0)$ and we assume that a, b are not moments of collision) if, for some $\varepsilon > 0$, there is no strict dilation effect of the mappings S^a and S^b restricted to the short line segment

$\{(Q+sW, V):|s|<\varepsilon\}$. The linear space $W_0(S^{[a,b]}x)$ of the neutral vectors is called the neutral space of $S^{[a,b]}x$ at the point x.

Of course, the neutral subspace can be defined at every point $S^t x$ of the trajectory segment and – in the spirit of the gluing of orthogonal hyperplanes as described in Sect. 2 of K-S-Sz (1989) – it is an invariant of the segment hence its dimension is also an invariant. If necessary, we will denote the neutral subspace of a path segment as $W(S^{(a,b)}x)$.

For different values of $t\in[a,b]$, however, the representations of $W(S^{[a,b]}x)$ at $S^t x$ as subspaces of the common linear space $\mathscr{X}$ are, in general, different. Thus, by writing $W_t(S^{[a,b]}x)$, we will always mean the local representation of $W(S^{[a,b]}x)$ at $S^t x\notin \partial M$. Finally, one more notation will be used throughout: for $w\in\mathscr{X}$ and $t\in\mathbb{R}$, $Q_t^w(x)$ will denote the point $S^{-t}y$ where $y=(Q^t+w, V^t)$ with $x^t=(Q^t, V^t)$.

The infinitesimal perturbation of the trajectories will play a crucial role in our considerations. However, for proving some elementary properties of the neutral directions it will be more convenient to work with small finite perturbations.

Lemma 2.3 (Characterization of a neutral direction.). *A vector $W\in\mathscr{X}$ is a neutral direction for the trajectory segment $\{S^t x: t_1<t<t_2\}$ at the point x ($t_1\leqq 0$ and $t_2\geqq 0$) if and only if there exist two positive numbers ε_1 and ε_2 such that, for every $\varepsilon<\varepsilon_1$, $p(S^t(Q,V))=p(S^t(Q+\varepsilon W, V))$ for every $t\in(t_1,t_2)$ except for the ε_2-neighborhoods of the reflection moments $t_1'<\ldots<t_k'$ of the interval (t_1,t_2). (The endpoints t_1, t_2 are supposed not to be moments of collision.)*

Now we are able to introduce the fundamental concept of sufficiency of a trajectory segment. It is worth stressing that the definition to be given is though equivalent but formally different from the formulations of this notion of our previous papers.

Definition 2.4. Assume that the finite or infinite trajectory $S^{(a,b)}x$ contains not more than one singular collision. We say that it is sufficient if $\dim W(S^{[a,b]}x)=1$, i.e. no neutral directions exist apart from the trivial one: the direction of the flow. If $S^{(a,b)}x$ does contain a singular collision, then the property just formulated is required to hold for both branches of the segment.

Of course, if $I\subset I'$ and $S^{I'}x$ is sufficient, then $S^{I'}x$ is also sufficient.

Decompositions and Sub-Billiards

It is intuitively clear that trajectory segments along which the system of balls decomposes into at least two non-interacting subclasses can not gather all degrees of freedom or, in a more technical wording, the segment can not be sufficient. Throughout the paper we will be using the following notations: assume that a system $\mathscr{N}=\{1,2,\ldots,N\}$ of N ($\geqq 2$) balls is decomposed into two non-empty classes: $\mathscr{N}=\mathscr{N}_1\cup\mathscr{N}_2$, $\mathscr{N}_1\cap\mathscr{N}_2=\emptyset$, $|\mathscr{N}_i|\neq 0$ ($i=1,2$). We say that the trajectory segment $S^{(a,b)}x$ *is partitioned* by $P=\{\mathscr{N}_1,\mathscr{N}_2\}$ if all its collisions occur among particles of the same class of P, only. In such a case, the action of S^t on $x=\{\tilde{x}_{\mathscr{N}_1},\tilde{x}_{\mathscr{N}_2}\}$ ($t\in[a,b]$) is certainly the product of two independent *subdynamics*: $\tilde{S}^t_{\mathscr{N}_1}\tilde{x}_{\mathscr{N}_1}$ and $S^t_{\mathscr{N}_2}\tilde{x}_{\mathscr{N}_2}$.

To be more exact, we express $S^t x$ in detail as a direct product. To this end decompose

$$x=\{\{(q_j,v_j):j\in\mathscr{N}_1\},\{(q_j,v_j):j\in\mathscr{N}_2\}\}:=\{\tilde{x}_{\mathscr{N}_1},\tilde{x}_{\mathscr{N}_2}\}.$$

Denote

$$\sum_{j\in\mathcal{N}_i} q_j=\bar{q}_i, \qquad \sum_{j\in\mathcal{N}_i} v_j=I_i, \qquad \frac{1}{2}\sum_{j\in\mathcal{N}_i} v_j^2=E_i$$

for $i=1,2$. By a standard change of coordinates $K_{(\bar{q}_i,I_i,E_i)}$, to every $\tilde{x}_{\mathcal{N}_i}$ there corresponds an element $x_{\mathcal{N}_i}=K_{(\bar{q}_i,I_i,E_i)}\tilde{x}_{\mathcal{N}_i}\in M_{\mathcal{N}_i}$, (the phase space of a billiard with $|\mathcal{N}_i|$ balls), and $S^t x$ can be understood as

$$S^t x=\{K^{-1}_{(\bar{q}_i+tI_i,I_i,E_i)}S^t_{\mathcal{N}_i}K_{(\bar{q}_i,I_i,E_i)}\tilde{x}_{\mathcal{N}_i}:\ i=1,2\}.$$

The dynamics $S^t_{\mathcal{N}_i}$ or $K^{-1}_{(\bar{q}_i+tI_i,I_i,E_i)}S^t_{\mathcal{N}_i}K_{(\bar{q}_i,I_i,E_i)}$ acting on $\tilde{x}_{\mathcal{N}_i}$ will be called *subdynamics* or a *sub-billiard*. The properties of a particular form of a three particle sub-billiard are analyzed e.g. in and after (5.52).

Advance of a Subsystem

Next we introduce the notion of *advance* or time-shift for a system of balls. It will be used for sub-billiards but we prefer to introduce it for our full dynamical system $(M,\{S^t\},\mu)$ as it was defined in Sect. 1. As mentioned earlier in the present section, if $x=(Q,V)\notin\partial M$, then the flow direction is always a neutral direction at x for any trajectory segment $S^{(a,b)}x$: $a<0<b$. Thus, if $S^a x, S^b x\notin\partial M$ and α is a sufficiently small real number such that

$$S^{(-|\alpha|,|\alpha|)}x\cap\partial M=\emptyset, \qquad S^{(a-|\alpha|,a+|\alpha|)}x\cap\partial M=\emptyset, \qquad S^{(b-|\alpha|,b+|\alpha|)}x\cap\partial M=\emptyset,$$

then the neutral subspaces of $S^{(a,b)}x$ at x and of $S^{(a,b)}(S^\alpha x)$ at $S^\alpha x$ will be the same, and, moreover, $S^\alpha x$ will be a perturbation of x in the configuration space, namely

$$S^\alpha x=(Q+\alpha V, V). \tag{2.5}$$

In fact, if, in addition, $S^{(a,b)}x$ is a sufficient trajectory segment, then the only neutral directions at x are of the form $\{\alpha V:\alpha\in\mathbf{R}\}$. This follows from Definitions 2.2 and 2.4 and this observation will be used throughout. Having pure spatial perturbations of the form (2.5) (i.e. perturbations to the trivial neutral direction of the flow), the number α is called the *advance* or *time shift* of the billiard system (M,S^t,μ) with respect to the neutral perturbation (2.5). It is also useful to express (2.5) in local coordinates. For instance, if the trajectory $S^{[a,b]}x$ is partitioned by the partition $\{P_1,\{i,j\}\}$, then the advance of the subsystem $\{i,j\}$ with respect to the spatial perturbations $dQ\in\mathcal{Z}$ satisfying

$$dq_i-dq_j=\alpha(v_i-v_j)$$

is just α.

Some Facts from Topological Dimension Theory

Here we outline the necessary statements from general topology. As to a broader exposition of the issues, see E (1978) or Sect. 2 of K-S-Sz (1991). We start with the notion of residuality. Note that the dimension $\dim A$ of a separable metric space A

is one of the three classical notions of dimension: the covering, the small inductive, or the large inductive dimension. As it is known from general topology, all of them are the same for separable metric spaces.

Definition 2.6. A subset A of M is called residual iff A can be covered by a countable family of codimension 2 closed sets of μ-measure zero [cf. Definition 2.12 of K-S-Sz (1991)].

Remark 2.7. Any countable union of codimension 2 smooth submanifolds is residual.

Lemma 2.8. *A subset $A \subset M$ is residual if and only if, for every $x \in A$, there exists a neighborhood U of x such that $U \cap A$ is residual. (Locality, cf. Lemma 2.14 of K-S-Sz (1991).)*

The following lemmas characterize codimension-two and codimension-one sets.

Lemma 2.9. *For any closed subset $S \subset M$ the following three conditions are equivalent:*
(i) $\dim S \leqq \dim M - 2$*;*
(ii) $S \neq M$ *and, for every open connected set $G \subset M$, the difference set $G \backslash S$ is also connected;*
(iii) *For every point $x \in M$ and for any neighborhood V of x in M there exists a smaller neighborhood $W \subset V$ of the point x such that, for every pair of points $y, z \in W \backslash S$, there is a continuous curve γ in the set $V \backslash S$ connecting the points y and z and, moreover,* $\operatorname{int} S = \emptyset$.
[See Theorem 1.8.13 and Problem 1.8.E of E (1978).]

Property 2.10. *For any subset $S \subset M$ the condition* $\dim S \leqq \dim M - 1$ *is equivalent to* $\operatorname{int} S = \emptyset$. *[See Theorem 1.8.10 of E (1978).]*

We recall an elementary, but important lemma [Lemma 4.15 from K-S-Sz (1991)]. Let R_2 the set of phase points $x \in M \backslash \partial M$ such that the trajectory $\{S^t x\}$ has a singular collision both for $t < 0$ and $t > 0$.

Lemma 2.11. *The set R_2 is a countable union of codimension 2 of submanifolds of M.*

Finally, we cite the most important property of residual sets which gives us the fundamental geometric tool to connect the open ergodic components of billiard flows.

Lemma 2.12. *The complement of a residual set $A \subset M$ always contains an arcwise connected, dense set $B \subset M$ with full measure [See Property 3 of Sect. 4.1 of K-S-Sz (1989).]*

3. New Formulation of the Transversal Fundamental Theorem

The importance of fundamental theorems for demonstrating stochastic properties of hyperbolic systems with discontinuities is twofold:
(i) they, in their *original form* initiated by Chernov and Sinai (1987), first of all

guarantee *local ergodicity* in neighborhoods of hyperbolic – in our case, of sufficient – points;
(ii) moreover, the *transversal versions* of the fundamental theorems as suggested by K-S-Sz (1990 and 1991) make possible to apply the method of *pasting* which is basic for establishing global ergodicity of semi-dispersing billiards.

The present form of the theorem is its "stable version" which guarantees the existence of a bulk of "not too short" local *stable* invariant manifolds in suitably small neighborhoods of sufficient phase points. As a matter of fact, the present formulation of the fundamental theorem contains three modifications – formal improvements – which are listed in the second remark following the theorem. Although these changes are important in the applications, it is not hard to see that this version of the theorem is equivalent to its earlier formulations in S-Ch (1987) or K-S-Sz (1990). For more details see Remark 2 below.

Assume we are given a semi-dispersing billiard flow $(M, \{S^t\}, \mu)$ satisfying the following conditions:

Condition 3.1 (Chernov-Sinai Ansatz). *Vor* $\nu_{\mathscr{SR}^+}$ *– almost every point* $x \in \mathscr{SR}^+$, *the positive semi-trajectory* $S^{(0,\infty)}x$ *is sufficient, where* $\nu_{\mathscr{SR}^+}$ *denotes the Riemannian volume on the codimension one* CW*-complex* $\mathscr{SR}^+$.

Condition 3.2 (Regularity of the set of degenerate tangencies). *The set*

$$\{x=(q,v)\in \partial Q\times S_{d-1}:(v,n(q))=0 \;\; and \;\; K(q)v=0\}$$

is a finite union of compact smooth submanifolds of ∂M *(usually with boundary), i.e. this set is a* CW*-subcomplex of* ∂M. *Recall that* $K(q)$ *is the second fundamental form of* ∂Q *at* $q\in\partial Q$.

We remark that, in general, Condition 3.2 trivially holds for semi-dispersing billiards with solely cylindric scatterers (e.g. systems of billiard balls).

Our last regularity condition concerns the sets Δ_n of double singularities:

Condition 3.3 (Regularity of double singularities). *For every* $n\in\mathbf{N}$, *the set* Δ_n *is a finite union of compact smooth submanifolds of* ∂M.

Assume, moreover, that the following objects are given:
(i) a *sufficient* base point $x_0\in M\backslash\partial M$ such that x_0 has at most one singularity on its trajectory and this singularity, if it exists at all, occurs in the past,
(ii) a codimension-one smooth submanifold $\mathscr{J}\ni x_0$ of M which is *transversal* to all the possible local stable leaves and singularity manifolds corresponding to singular reflections in the future [the role of $\mathscr{J}$ is played by $\pi_{2,3,4}(\Phi_{I,E})$ in Sect. 5],
(iii) constants $C>0$ and $0<\alpha<1$.

Theorem 3.4 (Transversal Fundamental Theorem). *Under Conditions 3.1–3.3 and in the setup just given there is an open neighborhood* $W_{C,\alpha}(x_0)$ *of* x_0 *in* $\mathscr{J}$, *such that*

$$\mu_{\mathscr{J}}(\{z\in W_{C,\alpha}(x_0):\mu_{\mathscr{J}}(B_g(z,\delta))<\alpha\mu_{\mathscr{J}}(B(z,\delta))\})=o(\delta)$$
$$(\textit{small order of } \delta,\ \delta\to 0),$$

where

$$B(z,\delta):=\{z'\in\mathscr{J}:\varrho(z,z')<\delta\},$$

$$B_g(z,\delta):=\{z'\in B(z,\delta): \textit{the inner radius of the leaf } \gamma^s(z') \textit{ is greater than } C\delta\},$$

$\mu_{\mathcal{J}}$ is the Riemannian volume in $\mathcal{J}$, ϱ is the natural metric on $\mathcal{J}$ and $\gamma^s(z')$ denotes the intersection with $\mathcal{J}$ of the local stable invariant manifold of the flow $\{S^t\}$ through the point $z' \in M$.

Remarks. 1. We note that (ii) is always fulfilled by any codimension-one submanifold $\mathcal{J} \subset M$ which is defined purely in terms of the velocities only. It is a routine task to show the needed transversality in that case.

2. The reader can find *three changes* in this formulation compared to that one appeared in K-S-Sz (1990). The first one is just the presence of the high percentage α (arbitrarily close to 1) of the good points $z' \in B_g(z, \delta)$ in $B(z, \delta)$. A thorough analysis of the original proof shows that, indeed, arbitrarily high percentage of good points (with stable leaf greater than $C\delta$) can be guaranteed by choosing the free parameters of the proof carefully.

The second change is that here we got rid of the obnoxiously technical notion of regular coverings and parallelepipeds but, instead, we use the well-shaped ball neighborhoods $B(z, \delta)$. A simple geometric argument – operating with different values of α, C and δ – shows that, actually, the present formulation is equivalent to the original one using regular coverings.

Finally, the third change is that, in the new version, the statement of the theorem refers to the codimension-one submanifold $\mathcal{J}$ of M. This modification of the fundamental theorem plays an important role in the application 5.57. The main reason for its validity is the transversality property (ii) of the submanifold $\mathcal{J}$. Indeed, we can repeat the whole proof of the fundamental theorem dealing, on one hand, with the original geometric objects (such as singularity manifolds, balls, tubular neighborhoods, local orthogonal manifolds etc.) defined in a "solid" neighborhood $U(x_0)$ of x_0 in M and, on the other hand, with the intersections of these geometric objects with the codimension-one submanifold $\mathcal{J} \cap U(x_0)$. Due to the transversality assumption (ii), all the occurring numerical estimates of measures [their proofs heavily use the measure-preserving property of the flow $(M, \{S^t\}, \mu)$] can be translated word by word to the corresponding estimate on $\mathcal{J}$ regarding the Riemannian volume of the submanifold $\mathcal{J}$. It is important to note that no relative version (with respect to $\mathcal{J}$) of the Chernov-Sinai Ansatz is required for this improvement but just the original form 3.1 concerning the flow $(M, \{S^t\}, \mu)$.
3. Theorem 3.4 remains, of course, true without (ii), that is, in every sufficiently small "solid" neighborhood $W_{C,\alpha}(x_0)$ of x_0 in M. This "more original" form of the theorem was used for the subsystems $\{1, 2\}$ and $\{3, 4\}$ in the simultaneous application in Proposition 5.57.

Corollary 3.5. *Let $(M, \{S^t\}, \mu)$ be a semi-dispersing billiard system satisfying the conditions of Theorem 3.4. Every sufficient point $x \in M^*$ has a neighborhood belonging to one ergodic component of the system, and the system is a K-flow on such ergodic components.*

The corollary can be derived from Theorem 3.4 and its dual (the unstable version) in the same way as Corollary 3.12 of K-S-Sz (1990) was obtained from the Transversal Fundamental Theorem via the Zig-zag Theorem (Corollary 3.10 of the same paper) and the absolute continuity properties elaborated in Kat-Str (1986).

4. Geometric-Algebraic Lemmas on the Codimension of Manifolds Corresponding to Non-Sufficient Trajectory Segments with a Rich Collision Structure

As said before, non-sufficient points of the phase space will be treated in two steps. In this section we will show that, in neighborhoods of points with a rich collision structure to be defined soon, non-sufficient points form CW-complexes whose codimension can be bounded from below. If this lower bound is at least 2, we are done, and the main aim of the forthcoming analysis is to find such a lower bound. The basic idea is that, as a matter of fact, non-sufficiency means degeneracy of certain equations. They should be first found or, at least, characterized and then, since they are implicit, their degeneracies should be properly analyzed.

In this section we shall study finite, *non-singular* trajectory segments $S^{(a,b)}x = \{S^t x : a < t < b\}$ with collision structure $(\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\})$. We always assume that the endpoints a, b of the time interval are not moments of collision. Recall that the symbol $\{i_l, j_l\}$ denotes a non-empty sequence of consecutive collisions between the i_l-th and j_l-th particles where $i_l, j_l \in \{1,2,3,4\}$, $i_l \neq j_l$. We do not assume that the non-ordered pair $\{i_l, j_l\}$ is different from the neighboring pair $\{i_{l+1}, j_{l+1}\}$ $(l = 1, 2, \ldots, k-1)$. The sequence of subsequent collisions corresponding to the symbol $\{i_l, j_l\}$ is often called an *island*. The following operations among symbolic collision structures are called *permitted operations*:

(a) Unification of two neighboring islands $\{i_l, j_l\}$ and $\{i_{l+1}, j_{l+1}\}$ which correspond to the same pair of particles;
(b) Splitting an island $\{i_l, j_l\}$ into two neighboring islands with the same pair of particles $\{i_l, j_l\}$
[this is just the inverse of the operation (a)];
(c) Interchanging two *disjoint* islands $\{i_l, j_l\}$ and $\{i_{l+1}, j_{l+1}\}$ where disjointness means that the sets of particles $\{i_l, j_l\}$ and $\{i_{l+1}, j_{l+1}\}$ are disjoint;
(d) Re-labeling the particles $i_1, j_1, \ldots, i_k, j_k$ by a permutation of the numbers $1, 2, 3, 4$;
(e) Time reversal, that is replacing the sequence $(\{i_1, j_1\}, \ldots, \{i_k, j_k\})$ by the sequence $(\{i_k, j_k\}, \ldots, \{i_1, j_1\})$.

We say that the collision structures of two trajectory segments are *equivalent* iff the symbolic collision structure of the first segment can be transformed into that of the second one by finitely many applications of permitted operations (a)–(e). Throughout this section we shall often use the following, basic principle on the dynamical behavior of trajectory segments with equivalent collision structures:

Principle 4.1. *Two trajectory segments $S^{(a,b)}x$ and $S^{(c,d)}y$ with equivalent collision structures are completely equivalent from the point of view of all dynamical methods occurring in Sect. 4 and 5 of this paper, therefore, in the investigations we can confine ourselves to the study of trajectory segments with mutually non-equivalent collision structures.*

Before drawing up the main result of this section we need one more definition:

Definition 4.2. We say that the trajectory segment $S^{(a,b)}x$ decomposes iff there is a non-collision moment $t \in (a,b)$ and there are two partitions P^- and P^+ of the

particle set $\{1,2,3,4\}$ with the following properties:
(i) Both partitions P^- and P^+ consist of two non-empty classes;
(ii) all the collisions in the time interval (a,t) take place between particles from the same class of P^-, i.e. $S^{(a,t)}x$ is partitioned by P^-;
(iii) $S^{(t,b)}x$ is partitioned by P^+.

It is important to observe that a trajectory segment with collision structure $(\{i_1,j_1\},\ldots,\{i_k,j_k\})$ does not decompose if and only if there is an index $l\in\{1,2,\ldots,k\}$ such that the "collision graph" on the vertices 1, 2, 3, 4 with edges $\{i_1,j_1\},\ldots,\{i_l,j_l\}$ is connected and, similarly, the collision graph with edges $\{i_l,j_l\},\ldots,\{i_k,j_k\}$ is also connected on the vertex set $\{1,2,3,4\}$.

We note that the decomposability is invariant under the permitted operations except for (c).

Now we are in the position of formulating the main result of this section.

Main Theorem 4.3. *If the dimension ν of the underlying torus is at least three and the collision structure of the non-singular trajectory segment $S^{(a,b)}x$ is not equivalent to any decomposing collision structure, then there is an open neighborhood $U(x)$ of x in M and a closed subset N of $U(x)$ with the following properties:*
(1) *For every point $y\in U(x)\backslash N$ the trajectory segment $S^{(a,b)}y$ is sufficient;*
(2) *N is a finite union of smooth submanifolds of $U(x)$ (that is, a* CW*-complex in $U(x)$);*
(3) *the codimension of N is at least $\nu-1$ $(\geqq 2)$, i.e.* $\dim N\leqq 5\nu$.

For the purposes of the present paper *richness* of the collision structure of a non-singular trajectory segment is just the condition formulated in the main theorem: the indecomposability of any collision structure equivalent to the given one. [E.g. the collision structure $(\{13\}\ \{34\}\ \{12\}\ \{34\}\ \{13\})$ is though indecomposable but not rich since the collision structure $(\{13\}\ \{12\}\ \{34\}\ \{13\})$, equivalent to it, is decomposable.] If we intended to cover the 2-dimensional case, too, we would be forced to accept a more stringent notion of richness (cf. Remark 4.28).

All the remaining part of this section is devoted to the proof of the main lemma.

First of all, according to Principle 4.1, we can assume that the symbolic collision structure $(\{i_1,j_1\},\ldots,\{i_k,j_k\})$ of the trajectory segment $S^{(a,b)}x$ is *minimal*, that is, the number k of islands is the minimum number of islands in the equivalence class of this collision structure. A simple but a bit tedious combinatorial enumeration shows that any minimal collision structure, which satisfies the condition of the Main Theorem 4.3, is *equivalent* to a collision structure that *contains a block* (sequence of consecutive islands) from the following list of minimal, mutually non-equivalent collision sequences:

1. $(\{13\},\{14\},\{12\},\{13\},\{14\})$
2. $(\{23\},\{24\},\{12\},\{13\},\{14\})$
3. $(\{34\},\{14\},\{12\},\{13\},\{34\})$
4. $(\{34\},\{24\},\{12\},\{13\},\{34\})$
5. $(\{24\},\{13\},\{12\},\{34\},\{13\})$
6. $(\{23\},\{14\},\{12\},\{34\},\{13\})$
7. $(\{34\},\{14\},\{12\},\{13\},\{14\})$
8. $(\{34\},\{24\},\{12\},\{13\},\{14\})$
9. $(\{1i\},\{234\}_{\geqq 3},\{1j\})\quad i,j\in\{2,3,4\}$;
10. $(\{234\}_{\geqq 3},\{12\},\{13\})$
11. $(\{234\}_{\geqq 3},\{12\},\{23\})$.

Here the symbol $\{234\}_{\geqq 3}$ denotes any collision sequence among the particles 2, 3, 4 that contains at least three *different* islands. For the sake of brevity, we omit the complete proof of this combinatorial classification and leave it to the reader. However, we hint to the reader that the starting point of this classification is the collision $\{i_l, j_l\}$, for which both collision graphs $\{i_1, j_1\}, \ldots, \{i_l, j_l\}$ and $\{i_l, j_l\}, \ldots, \{i_k, j_k\}$ are connected on the vertex set $\{1, 2, 3, 4\}$. Next, we can classify the collision graphs $\{i_l, j_l\}, \ldots, \{i_k, j_k\}$ according to the growth of these connected graphs from the starting edge $\{i_l, j_l\}$ up to the first edge $\{i_m, j_m\}$ which makes this graph connected. There will be five non-equivalent cases:

$$(\{12\}, \{34\}, \{13\}),$$

$$(\{12\}, \{13\}, \{14\}),$$

$$(\{12\}, \{13\}, \{24\}),$$

$$(\{12\}, \{13\}, \{34\}), \quad \text{and}$$

$$(\{123\}_{\geqq 3}, \{i4\}) \qquad i \in \{1, 2, 3\}.$$

Finally, we proceed in a similar manner when classifying the connected graphs $\{i_1, j_1\}, \ldots, \{i_l, j_l\}$ and then we consider all the possible, non-equivalent matchings of these cases.

If the symbolic collision sequence of the trajectory segment $S^{(a,b)}x$ contains a block from the above list of eleven collision sequences, then the same is true for every trajectory segment $S^{(a,b)}y$, provided the point y is close enough to the base point x. It turns out that this block is *rich enough* in order to show the sufficiency required in Main Theorem 4.3, that is, sufficiency outside of a closed CW-complex N with codimension at least $v-1$ ($v \geqq 3$). Thus the verification of the main lemma splits into eleven cases according to the blocks 1–11. As a matter of fact, these proofs are very similar to each other in the groups of cases $\{1, 2, 3, 4, 7, 8\}$, $\{5, 6\}$, and $\{10, 11\}$, therefore, we shall only present the proof for the cases 5, 7, 9, and 11.

Before going into details we outline the basic ideas of the analysis. As noted earlier, the non-sufficiency of trajectory segments $S^{(a,b)}y$ in some neighborhood $y \in U(x)$ of a rich point x should, in fact, be thought of as the degeneracy of certain equations. Since our notion of richness is, necessarily, rather implicit, even minimal rich sequences can be rather long; so, in general, it seems to be a too complicated task to calculate these equations for all sequences. [For the case of $N=3$ balls or the case of $N=4$ balls on $\mathbf{T}^v$ ($v \geqq 4$) this is still possible.] As a matter of fact, we can and do obtain these equations in cases 9 and 11 while in cases 5 and 7 we give these types of equation for shorter subsets of the collision structure. This part is quite elementary and is based on our calculus with neutral subspaces and the collision laws. Then, in the second part of the proof, we use appropriate perturbational arguments combined with the dispersing property of smaller billiard subsystems to obtain the desired bound for the codimensions.

Case 7. $(\{34\}, \{14\}, \{12\}, \{13\}, \{14\})$. In the study of some neutral spaces we will use the following notations:

T^-: certain (not specified) moment between the first and second islands;
t^-: certain moment between the second and third islands;
t^+: certain moment between the third and fourth islands;
T^+: certain moment between the fourth and fifth islands;
α: the advance of the collisions at the second island;
β: The advance of the collisions at the fourth island.

(Islands can be thought of as two-billiard subsystems on appropriate time intervals.)

It is a natural wish to get rid of the trivial neutral direction: the direction of the flow $\{S^\tau : \tau \in \mathbf{R}\}$. We can achieve this goal by simply requiring that the advance of the central island $\{12\}$ be zero. This assumption automatically implies that the reduced neutral spaces $W^0_{t^-}(S^{(t^-,t^+)}y)$ and $W^0_{t^+}(S^{(t^-,t^+)}y)$ are identical:

$$\begin{cases} W^0_{t^-}(S^{(t^-,t^+)}y) = W^0_{t^+}(S^{(t^-,t^+)}y) \\ \qquad = \left\{ (w_1, w_2, w_3, w_4) \in \mathbb{R}^{4\nu} : \sum_{i=1}^{4} w_i = 0 \ \& \ w_1 = w_2 \right\} \\ \qquad = \{ (-\tfrac{1}{2}(\xi+\eta), -\tfrac{1}{2}(\xi+\eta), \xi, \eta) : \xi, \eta \in \mathbb{R}^\nu \} . \end{cases} \tag{4.4}$$

Here the phase point y is any point close enough to x. The superscript 0 refers to the assumption of having zero advance at the central island $\{12\}$. *Throughout this section we assume the convention of having zero advance at the central (third) island.* We know from the preliminaries that neutrality with respect to the first island $\{14\}$ with advance α exactly means that

$$\tfrac{1}{2}\xi + \tfrac{3}{2}\eta = \alpha(v_4^{t^-} - v_1^{t^-}) . \tag{4.5}$$

Similarly, neutrality with respect to the island $\{13\}$ with advance β is just equivalent to the equation

$$\tfrac{3}{2}\xi + \tfrac{1}{2}\eta = \beta(v_3^{t^+} - v_1^{t^+}) . \tag{4.6}$$

Here ξ and η $(\in \mathbf{R}^\nu)$ are the linear parameters of the space $W^0_{t^-}(S^{(t^-,t^+)}y)$ [or $W^0_{t^+}(S^{(t^-,t^+)}y)$] from (4.4). It is, of course, true that

$$v_3^{t^-} = v_3^{t^+} \quad \& \quad v_4^{t^-} = v_4^{t^+} . \tag{4.7}$$

The equations (4.5)–(4.6) obviously imply that

$$\begin{cases} \xi = -\tfrac{1}{4}\alpha(v_4^{t^-} - v_1^{t^-}) + \tfrac{3}{4}\beta(v_3^{t^+} - v_1^{t^+}) , \\ \eta = \tfrac{3}{4}\alpha(v_4^{t^-} - v_1^{t^-}) - \tfrac{1}{4}\beta(v_3^{t^+} - v_1^{t^+}) . \end{cases} \tag{4.8}$$

Thus the reduced neutral space $W^0_{t^-}(S^{(T^-,T^+)}y)$ is two-dimensional; two independent linear parameters are α and β. From the collision equations for the island $\{13\}$ we have

$$\begin{cases} w_2^{T^+} = w_2^{t^+} = -\tfrac{1}{2}(\xi+\eta) , \\ w_4^{T^+} = w_4^{t^+} = \eta , \\ w_1^{T^+} + w_3^{T^+} = w_1^{t^+} + w_3^{t^+} = \tfrac{1}{2}(\xi - \eta) , \\ w_3^{T^+} - w_1^{T^+} = \beta(v_3^{T^+} - v_1^{T^+}) . \end{cases} \tag{4.9}$$

Using (4.8) and (4.9) we obtain the following formula for the relative displacement $w_4^{T^+} - w_1^{T^+}$:

$$w_4^{T^+} - w_1^{T^+} = \alpha(v_4^{t^-} - v_1^{t^-}) + \tfrac{1}{2}\beta[(v_3^{T^+} - v_1^{T^+}) - (v_3^{t^+} - v_1^{t^+})] . \tag{4.10}$$

We want to characterize the set N^+ of points $y \in U(x)$ [$U(x)$ is some open neighborhood of x in M, small enough – the whole Sect. 4 deals with *local* objects], for which the trajectory segment $S^{(T^-,b)}y$ is *not sufficient*, that is, we want to describe the set

$$\begin{cases} N^+ = \{y \in U(x): W_{t^+}^0(S^{(T^-,b)}y) \neq \{0\}\} \\ \quad = \{y \in U(x): \gamma(v_4^{T^+} - v_1^{T^+}) \\ \quad = \alpha(v_4^{t^-} - v_1^{t^-}) + \frac{1}{2}\beta[(v_3^{T^+} - v_1^{T^+}) - (v_3^{t^+} - v_1^{t^+})] \\ \quad \text{can be solved for } \alpha, \beta, \gamma \text{ with } \alpha^2 + \beta^2 \neq 0\} \\ \quad = \{y \in U(x): \text{ the vectors } v_4^{t^-} - v_1^{t^-},\ v_4^{T^+} - v_1^{T^+}, \text{ and} \\ \quad v_3^{T^+} - v_1^{T^+} - v_3^{t^+} + v_1^{t^+} \text{ are linearly dependent}\}. \end{cases} \tag{4.11}$$

Here we can observe that

$$\begin{cases} v_4^{T^+} - v_1^{T^+} = v_4^{t^-} - v_1^{T^+} \\ \qquad = (v_4^{t^-} - v_1^{t^-}) + (v_1^{t^-} - v_1^{t^+}) + (v_1^{t^+} - v_1^{T^+}) \end{cases} \tag{4.12}$$

and, by the collision equations for the island $\{13\}$,

$$v_1^{t^+} - v_1^{T^+} = \tfrac{1}{2}[(v_3^{T^+} - v_1^{T^+}) - (v_3^{t^+} - v_1^{t^+})], \tag{4.13}$$

therefore, the linear dependence of the vectors $v_4^{t^-} - v_1^{t^-}$, $v_4^{T^+} - v_1^{T^+}$, and $v_3^{T^+} - v_1^{T^+} - v_3^{t^+} + v_1^{t^+}$ is equivalent to that of the vectors $v_4^{t^-} - v_1^{t^-}$, $v_1^{t^-} - v_1^{t^+}$, and $v_1^{t^+} - v_1^{T^+}$. The characterization of the set of degeneracy $N^+ \subset U(x)$ is contained in the following lemma:

Lemma 4.14. *The set $N^+ \subset U(x)$ defined above is a closed* CW-*complex (the union of finitely many smooth submanifolds of $U(x)$) in $U(x)$ with codimension at least $\nu - 2$, that is, $\dim N^+ \leqq 5\nu + 1$. Moreover, the perturbations of the form $Q_{t^+}^w(\cdot)$ with $w = (\xi, \xi, -3\xi, \xi)$ $(\xi \in \mathbf{R}^\nu)$ are transversal to every cell of N^+ with codimension $\nu - 2$ and the perturbations of the form $Q_{t^-}^w(\cdot)$ with $w = (\xi, \xi, \xi, -3\xi)$ $(\xi \in \mathbf{R}^\nu)$ are tangential to every cell of N^+.*

Proof. We know from (4.13) that the closed set of degeneracy $N^+ \subset U(x)$ can be defined by the formula

$$\begin{cases} N^+ = \{y \in U(x): \text{ the vectors } v_4^{t^-}(y) - v_1^{t^-}(y),\ v_1^{t^-}(y) - v_1^{t^+}(y), \text{ and} \\ \qquad v_1^{t^+}(y) - v_1^{T^+}(y) \text{ are linearly dependent}\}. \end{cases} \tag{4.15}$$

First we observe that

A) the perturbations of the form $Q_{t^-}^w(\cdot)$ with $w = (\xi, \xi, \xi, -3\xi)$ $(\xi \in \mathbf{R}^\nu)$ leave all the three vectors in (4.15) fixed, so the statement of Lemma 4.14 regarding the tangency of these perturbations is obvious. Furthermore, it is clear that the relative velocity $v_4^{t^-} - v_1^{t^-}$ is not the zero vector:

$$v_4^{t^-}(y) - v_1^{t^-}(y) \neq 0 \quad \text{for all} \quad y \in U(x). \tag{4.16}$$

Next,

B) the perturbations $Q_{t^+}^w(\cdot)$ with $w = (\xi, \xi, -3\xi, \xi)$ $(\xi \in \mathbf{R}^\nu)$ leave the vectors $v_4^{t^-} - v_1^{t^-}$ and $v_1^{t^-} - v_1^{t^+}$ fixed, while the third vector $v_1^{t^+} - v_1^{T^+}$ varies on an open piece of a $(\nu - 1)$-dimensional sphere in $\mathbf{R}^\nu$ and this sphere contains the origin of $\mathbf{R}^\nu$.

As the third observation, we claim that

C) the perturbations of the form $Q^{w}_{t^-}(\cdot)$ with $w=(\xi,-\xi,0,0)$ $(\xi\in\mathbf{R}^{\nu})$ leave the nonzero vector $v_4^{t^-}-v_1^{t^-}$ fixed, while the vector $v_1^{t^-}-v_1^{t^+}$ varies on an open piece of a $(\nu-1)$-dimensional sphere in $\mathbf{R}^{\nu}$ and this sphere contains the origin.

Now it is time to define the four cells of the complex N^+,

$$\begin{cases} N_1^+ := \{y\in U(x): v_1^{t^-}(y)=v_1^{t^+}(y)\}\,, \\ N_2^+ := \{y\in U(x): v_1^{t^-}(y)\neq v_1^{t^+}(y) \ \& \ \text{the vectors} \\ \qquad v_4^{t^-}(y)-v_1^{t^-}(y) \ \text{and} \ v_1^{t^-}(y)-v_1^{t^+}(y) \ \text{are parallel}\}\,, \\ N_3^+ := \{y\in U(x): v_1^{t^+}(y)=v_1^{T^+}(y)\}\,, \\ N_4^+ := \ N^+\backslash(N_1^+\cup N_2^+\cup N_3^+)\,. \end{cases} \tag{4.17}$$

It is obvious that the set N^+ is the union of the sets N_1^+, N_2^+, N_3^+, and N_4^+. The observation C) implies now that the set N_1^+ is either empty or it is a codimension-$(\nu-1)$ smooth submanifold of $U(x)$. Similarly, C) and (4.16) again imply that the set N_2^+ is either the empty set or it is a codimension $\nu-1$ smooth submanifold in $U(x)$. An argument, completely analogous to that regarding the set N_1^+, shows that N_3^+ is either empty or it is a codimension $\nu-1$ smooth submanifold of $U(x)$. Finally, it follows from the observation B) that N_4^+ is either empty or it is a codimension $\nu-2$ submanifold of $U(x)$. Summing up these results, we see that the set N^+ is a CW-complex in $U(x)$ with $\dim N^+\leqq 5\nu+1$.

The remaining part of Lemma 4.14 (the transversality) follows easily from B). Hence the proof of Lemma 4.14.

Now we are going to describe the set N^- of points $y\in U(x)$ for which the trajectory segment $S^{(a,T^+)}y$ is not sufficient. From the collision equations for the first island $\{14\}$,

$$\begin{aligned} w_2^{T^-}&=w_2^{t^-}=-\tfrac{1}{2}(\xi+\eta)\,, \\ w_3^{T^-}&=w_3^{t^-}=\xi\,, \\ w_1^{T^-}+w_4^{T^-}&=w_1^{t^-}+w_4^{t^-}=\tfrac{1}{2}(\eta-\xi)\,, \\ w_4^{T^-}-w_1^{T^-}&=\alpha(v_4^{T^-}-v_1^{T^-}) \end{aligned} \tag{4.18}$$

and from (4.8) we get the following formula for the relative displacement $w_4^{T^-}-w_3^{T^-}$:

$$w_4^{T^-}-w_3^{T^-}=\tfrac{1}{2}\alpha[(v_4^{t^-}-v_1^{t^-})+(v_4^{T^-}-v_1^{T^-})]-\beta(v_3^{t^+}-v_1^{t^+})\,. \tag{4.19}$$

Thus the closed subset $N^-\subset U(x)$ can be defined in the following way:

$$\begin{cases} N^- = \{y\in U(x): W^0_{t^-}(S^{(a,T^+)}y)\neq\{0\}\} \\ \quad =\{y\in U(x):\ \text{the equation}\ \gamma(v_4^{T^-}-v_3^{T^-}) \\ \quad =\tfrac{1}{2}\alpha[(v_4^{t^-}-v_1^{t^-})+(v_4^{T^-}-v_1^{T^-})]-\beta(v_3^{t^+}-v_1^{t^+}) \\ \qquad \text{can be solved for}\ \alpha,\beta,\gamma\ \text{with}\ \alpha^2+\beta^2\neq 0\} \\ \quad =\{y\in U(x):\ \text{the vectors}\ v_4^{T^-}-v_3^{T^-},\, v_3^{t^+}-v_1^{t^+},\ \text{and} \\ \qquad (v_4^{t^-}-v_1^{t^-})+(v_4^{T^-}-v_1^{T^-})\ \text{are linearly dependent}\}\,. \end{cases} \tag{4.20}$$

Here we observe that

$$\begin{aligned} v_4^{T^-} - v_3^{T^-} = \tfrac{1}{2}[(v_4^{t^-} - v_1^{t^-}) + (v_4^{T^-} - v_1^{T^-})] \\ + (v_1^{t^-} - v_1^{t^+}) + (v_1^{t^+} - v_3^{t^+}). \end{aligned} \tag{4.21}$$

(In the proof we used the conservation of the momentum: $v_1^{T^-} + v_4^{T^-} = v_1^{t^-} + v_4^{t^-}$.) Thus the set of degeneracy $N^- \subset U(x)$ can also be defined by the formula

$$\begin{cases} N^- = \{y \in U(x)\colon \text{ the vectors } v_3^{t^+}(y) - v_1^{t^+}(y),\, v_1^{t^-}(y) - v_1^{t^+}(y), \text{ and} \\ \quad v_4^{t^-}(y) - v_1^{t^-}(y) + v_4^{T^-}(y) - v_1^{T^-}(y) \text{ are linearly dependent}\}. \end{cases} \tag{4.22}$$

The next lemma (characterization of the set N^-) and its proof are very similar to Lemma 4.14 and its proof but, as a matter of fact, the situation here is not completely isomorphic to the situation of Lemma 4.14, since there, for instance, the first particle participates in all of the collisions.

Lemma 4.23. *The set N^- is a closed* CW-*complex in $U(x)$ with codimension at least $\nu-2$, that is, $\dim N^- \leqq 5\nu + 1$. Moreover, the perturbations of the form $Q_{t^-}^w(\cdot)$ with $w = (\xi, \xi, \xi, -3\xi)$ $(\xi \in \mathbf{R}^\nu)$ are transversal to every cell of N^- which has the maximum allowed dimension $5\nu+1$ and the perturbations of the form $Q_{t^+}^w(\cdot)$ with $w = (\xi, \xi, -3\xi, \xi)$ $(\xi \in \mathbf{R}^\nu)$ are tangential to every cell of N^-.*

Proof of Lemma 4.23. We note first that one of the three vectors occuring in (4.22) is never zero:

$$v_3^{t^+}(y) \neq v_1^{t^+}(y) \quad \text{for every point} \quad y \in U(x). \tag{4.24}$$

Furthermore, we make three observations, analogous to A)–C) in Lemma 4.14:
A′) The perturbations of the form $Q_{t^+}^w(\cdot)$ with $w = (\xi, \xi, -3\xi, \xi)$ $(\xi \in \mathbf{R}^\nu)$ leave all the three vectors occurring in (4.22) fixed, so the statement of Lemma 4.23 regarding these perturbations is obvious.
B′) The perturbations of the form $Q_{t^-}^w(\cdot)$ with $w = (\xi, \xi, \xi, -3\xi)$ $(\xi \in \mathbf{R}^\nu)$ leave the vectors $v_3^{t^+} - v_1^{t^+}$ and $v_1^{t^-} - v_1^{t^+}$ fixed, while the third vector $v_4^{t^-} - v_1^{t^-} + v_4^{T^-} - v_1^{T^-}$ varies on an open piece of a $(\nu-1)$-dimensional sphere in $\mathbf{R}^\nu$ and this sphere contains the origin of $\mathbf{R}^\nu$.
C′) The perturbations $Q_{t^+}^w(\cdot)$ with $w = (\xi, -\xi, 0, 0)$ $(\xi \in \mathbf{R}^\nu)$ leave the non-zero vector $v_3^{t^+} - v_1^{t^+}$ fixed, while the vector $v_1^{t^-} - v_1^{t^+}$ varies on an open piece of a $(\nu-1)$-dimensional sphere in $\mathbf{R}^\nu$ and this sphere contains the origin.

The decomposition of the closed set N^- into cells looks as follows:

$$\begin{cases} N_1^- := \{y \in U(x)\colon v_1^{t^-}(y) = v_1^{t^+}(y)\}, \\ N_2^- := \{y \in U(x)\colon v_1^{t^-}(y) \neq v_1^{t^+}(y) \;\&\; \text{the vectors} \\ \qquad v_1^{t^-}(y) - v_1^{t^+}(y) \text{ and } v_3^{t^+}(y) - v_1^{t^+}(y) \text{ are parallel}\}, \\ N_3^- := \{y \in U(x)\colon v_4^{t^-}(y) - v_1^{t^-}(y) + v_4^{T^-}(y) - v_1^{T^-}(y) = 0\}, \\ N_4^- := N^- \setminus (N_1^- \cup N_2^- \cup N_3^-). \end{cases} \tag{4.25}$$

It is clear that $N^- = \bigcup_{j=1}^{4} N_j^-$. Furthermore, in virtue of C′) the closed set N_1^- is either empty or it is a smooth submanifold of $U(x)$ with codimension $\nu-1$. Secondly, by C′) again, the set N_2^- is also either empty or it is a codimension $\nu-1$

smooth submanifold of $U(x)$. Similarly, observation B′) implies that the closed set N_3^- is again either empty or it is a codimension $\nu-1$ smooth submanifold of $U(x)$. Finally, again B′) yields that the set N_4^- is either empty or it is a codimension $\nu-2$ smooth submanifold of $U(x)$. The statement of the lemma regarding the perturbations $Q_{t^-}^w(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ is also a consequence of B′). Hence the lemma.

Now it is time to combine Lemmas 4.14 and 4.23 to complete the proof of the main lemma in the case ({34}, {14}, {12}, {13}, {14}). The closed subset $N\subset U(x)$ required in the main lemma will be $N^-\cap N^+$:

$$N:=N^-\cap N^+=\bigcup_{j,k=1}^{4}(N_j^-\cap N_k^+). \tag{4.26}$$

It follows from the definition of the sets $N^\pm$ that for every point $y\in U(x)\backslash N$ the trajectory segment $S^{(a,b)}y$ is sufficient. We note, however, that even the set N may contain points y with a sufficient trajectory segment $S^{(a,b)}y$. Statement (2) of the main theorem follows easily from the nature of the definition of N. We saw above that $\dim(N_j^-\cap N_k^+)\leqq 5\nu$ unless $j=k=4$. As far as the intersection $N_4^-\cap N_4^+$ is concerned, the statements of Lemmas 4.14 and 4.23 concerning the perturbations $Q_{t^-}^w(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ show that at each point $y\in N_4^-\cap N_4^+$ the manifolds N_4^- and N_4^+ are *transversal*, therefore the set $N_4^-\cap N_4^+$ is either empty or it is a $(4\nu+3)$-dimensional submanifold of $U(x)$ and $4\nu+3\leqq 5\nu$, provided $\nu\geqq 3$. Hence the proof of the Main Theorem 4.3 in the case of the collision structure ({34}, {14}, {12}, {13}, {14}).

Remark 4.27. The upper estimate 5ν for $\dim(N^-\cap N^+)$, in general, cannot be improved because the set $N_1^-=N_1^+\subset N^-\cap N^+$ can be a non-empty manifold of dimension 5ν.

Remark 4.28. It is clear from the proof of Main Theorem 4.3 that the present method breaks down for $\nu=2$. Indeed, in this case both sets N^- and N^+ fill out the entire neighborhood $U(x)$ because every triplet of vectors in $\mathbf{R}^2$ is linearly dependent. Therefore, in the case $\nu=2$ we need a more subtle investigation of the set $N'\subset U(x)$ of points $y\in U(x)$ for which the segment $S^{(a,b)}y$ is not sufficient. Despite the fact that the set N' is much slimmer than $N=N^+\cap N^-=U(x)$, unfortunately, in general it is a codimension-one cell complex, and we would have to use one more island in the study in order to guarantee two codimensions. However, the number of mutually non-equivalent symbolic collision sequences of six islands would be so high, and the study of the set of points with non-sufficient trajectories containing 6 islands would be technically so involved, that we decided not to work it out.

Case 5. ({24}, {13}, {12}, {34}, {13}). First we note that the definitions of T^-, t^-, t^+, T^+, α, β, N^-, and N^+ are the same as earlier. We again assume that the advance of collisions at the third island is zero. A computation, analogous to that one giving us (4.11), shows that

$$\begin{aligned}N^+=\{y\in U(x)\colon\ &\text{the vectors } v_3^{t^-}(y)-v_1^{t^-}(y),\ v_3^{T^+}(y)-v_1^{T^+}(y), \text{ and}\\ &v_3^{T^+}(y)-v_4^{T^+}(y)-v_3^{t^+}(y)+v_4^{t^+}(y) \text{ are linearly dependent}\}.\end{aligned} \tag{4.29}$$

From the collision laws for the fourth island and from the equations $v_1^{t^+}=v_1^{T^+}$, $v_3^{t^-}=v_3^{t^+}$ we get

$$\begin{aligned} v_3^{T^+}-v_1^{T^+} &= (v_3^{t^-}-v_1^{t^-})-(v_1^{t^+}-v_1^{t^-})+(v_3^{T^+}-v_3^{t^+}) \\ &= (v_3^{t^-}-v_1^{t^-})-(v_1^{t^+}-v_1^{t^-})+\tfrac{1}{2}[(v_3^{T^+}-v_4^{T^+})-(v_3^{t^+}-v_4^{t^+})]. \end{aligned} \tag{4.30}$$

Therefore, we have

$$\begin{aligned} N^+=\{y\in U(x)\colon & \text{ the vectors } v_3^{t^-}(y)-v_1^{t^-}(y),\ v_1^{t^+}(y)-v_1^{t^-}(y), \text{ and} \\ & v_3^{T^+}(y)-v_3^{t^+}(y) \text{ are linearly dependent}\}. \end{aligned} \tag{4.31}$$

The lemma describing the set of degeneracy N^+ is analogous to Lemma 4.14 and has the following actual form:

Lemma 4.32. *The set $N^+\subset U(x)$ is a closed cell complex in $U(x)$ with codimension at least $v-2$, that is, $\dim N^+\leqq 5v+1$. The perturbations $Q_{t^+}^w(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ $(\xi\in\mathbf{R}^v)$ are transversal to every cell of N^+ with codimension $v-2$ and the perturbations $Q_{t^-}^w(\cdot)$ with $w=(\xi,\xi,-\xi,-\xi)$ $(\xi\in\mathbf{R}^v)$ are tangential to every cell of N^+.*

Proof. Since the proof is very similar to that of Lemma 4.14, we shall only list the observations A″)–C″) analogous to A)–C) and the definitions of the manifolds $N_j^{+\prime}$ analogous to N_j^+.
A″) The perturbations $Q_{t^-}^w(\cdot)$ with $w=(\xi,\xi,-\xi,-\xi)$ $(\xi\in\mathbf{R}^v)$ leave all the three vectors occurring in (4.31) fixed.
B″) The perturbations $Q_{t^+}^w(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ $(\xi\in\mathbf{R}^v)$ leave the vectors $v_3^{t^-}-v_1^{t^-}$ and $v_1^{t^+}-v_1^{t^-}$ fixed while the vector $v_3^{T^+}-v_3^{t^+}$ varies on an open piece of a $(v-1)$-dimensional sphere in $\mathbf{R}^v$ and this sphere contains the origin.
C″) The perturbations $Q_{t^-}^w(\cdot)$ with $w=(\xi,-\xi,0,0)$ leave the vector $v_3^{t^-}-v_1^{t^-}$ fixed while the vector $v_1^{t^+}-v_1^{t^-}$ varies on an open piece of a $(v-1)$-dimensional sphere and this sphere contains the origin

$$\left\{\begin{aligned} & N_1^{+\prime}=\{y\in U(x)\colon v_1^{t^-}(y)=v_1^{t^+}(y)\}, \\ & N_2^{+\prime}=\{y\in U(x)\colon v_1^{t^-}(y)\neq v_1^{t^+}(y) \ \&\ \text{the vectors} \\ & \qquad v_3^{t^-}(y)-v_1^{t^-}(y) \text{ and } v_1^{t^+}(y)-v_1^{t^-}(y) \text{ are parallel}\}, \\ & N_3^{+\prime}=\{y\in U(x)\colon v_3^{t^+}(y)=v_3^{T^+}(y)\}, \\ & N_4^{+\prime}=N^+\setminus(N_1^{+\prime}\cup N_2^{+\prime}\cup N_3^{+\prime}). \end{aligned}\right. \tag{4.33}$$

The structure of the proof of Lemma 4.32 is the same as that of Lemma 4.14, therefore we omit the details here.

Let us switch to the brief study of the set of degeneracy N^-. A routine calculus, similar to that one giving us (4.11), shows that

$$\begin{aligned} N^-=\{y\in U(x)\colon & \text{ the vectors } v_3^{t^-}(y)-v_1^{t^-}(y),\ v_4^{t^-}(y)-v_3^{t^-}(y), \text{ and} \\ & v_2^{t^-}(y)-v_4^{t^-}(y) \text{ are linearly dependent}\}. \end{aligned} \tag{4.34}$$

A bit surprising to us, this description of the set N^- has a nature, significantly different from that of (4.22), because in (4.34) all the velocities are considered at the same time t^-. Corresponding to this difference, the geometric description of the set $N^-\subset U(x)$ is also different from Lemma 4.23:

Lemma 4.35. *The set N^- is a closed cell complex in $U(x)$ with codimension at least $\nu-2$, that is* $\dim N^- \leqq 5\nu+1$. *The perturbations $Q^w_{t^+}(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ are tangential to every cell of N^-.*

Proof. It is easy to see that the set $\{(\xi_1,\xi_2,\xi_3)\in\mathbf{R}^{3\nu}$: the vectors $\xi_j\in\mathbf{R}^\nu$ are linearly dependent$\}$ is a closed cell-complex in $\mathbf{R}^{3\nu}$ with codimension $\nu-2$ and this set is invariant under multiplications by non-zero scalars. This yields the first part of the lemma. As far as the second part is concerned, all the three vectors from (4.34) are invariant under perturbations $Q^w_{t^+}(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ ($\xi\in\mathbf{R}^\nu$). Hence the lemma.

Using Lemmas 4.32 and 4.35, we can complete the proof of the main theorem in the case

$$(\{24\},\{13\},\{12\},\{34\},\{13\})$$

just the same way as in the case

$$(\{34\},\{14\},\{12\},\{13\},\{14\}).$$

Here the perturbations $Q^w_{t^+}(\cdot)$ with $w=(\xi,\xi,\xi,-3\xi)$ will show the transversality of the manifolds $N_4^{+\prime}$ and $N_4^{-\prime}$.

Case 9. $(\{1i\},\{234\}_{\geqq 3},\{1j\})$. $(i,j\in\{2,3,4\})$. It turns out that the treatment of the Cases 9–11 is much easier than the study of the Cases 1–8. Even our usual notations become simpler:

t^-: certain (not specified) moment between the blocks $\{1i\}$ and $\{234\}_{\geqq 3}$;
t^+: certain moment between the blocks $\{234\}_{\geqq 3}$ and $\{1j\}$;
α: the advance of the collisions at the island $\{1i\}$;
β: the advance of the collisions at the island $\{1j\}$.

We note first that, as it follows from our former result concerning the three-particle-billiards [cf. K-S-Sz (1991)], generalization of Lemma 4.12 for the ν-dimensional torus), the set N^0 of points $y\in U(x)$ for which the trajectory segment $S^{(t^-,t^+)}_{2,3,4}y$ of the sub-billiard $\{2,3,4\}$ is not sufficient, is a closed cell complex of $U(x)$ having codimension at least $\nu-1$. Therefore, it is enough to study the smallness (codimension is at least $\nu-1$) of the set $N\backslash N^0$, where

$$N=\{y\in U(x):\ \text{the segment } S^{(a,b)}y \text{ is not sufficient}\}. \tag{4.36}$$

This means technically that we can assume the base point x not belong to N^0 or, what is basically the same, the set $N^0\subset U(x)$ be empty. The set equality $N^0=\emptyset$ actually means that for every point $y\in U(x)$ and for every vector $w\in W_{t^\pm}(S^{(t^-,t^+)}y)$ the advance is the *same* for all the collisions in the block $\{234\}_{\geqq 3}$. We can again get rid of the trivial neutral direction of the flow by simply taking the advance of these collisions zero. We denote the corresponding neutral spaces by $W^0_{t^\pm}(\cdot)$. It is now straightforward that

$$\begin{cases} W^0_{t^-}(S^{(t^-,t^+)}y)=W^0_{t^+}(S^{(t^-,t^+)}y)\\ \qquad\qquad =\{(-3\xi,\xi,\xi,\xi):\xi\in\mathbf{R}^\nu\}\quad\text{for all}\quad y\in U(x).\end{cases} \tag{4.37}$$

This yields the following simple description of the neutral space $W^0_{t^-}(S^{(a,b)}y)$:

$$\begin{cases} W^0_{t^-}(S^{(a,b)}y)=W^0_{t^+}(S^{(a,b)}y)\\ \qquad =\{(-3\xi,\xi,\xi,\xi):\ \text{the vector } \xi\in\mathbf{R}^\nu \text{ is parallel both with}\\ \qquad\quad v_i^{t^-}(y)-v_1^{t^-}(y) \text{ and } v_j^{t^+}(y)-v_1^{t^+}(y)\}\quad\text{for all}\quad y\in U(x).\end{cases} \tag{4.38}$$

Thus $y \in N$ if and only if $v_i^{t^-}(y) - v_1^{t^-}(y)$ is parallel with $v_j^{t^+}(y) - v_1^{t^+}(y)$:

$$N = \{y \in U(x)\colon \text{ the vectors } v_i^{t^-}(y) - v_1(y) \text{ and } v_j^{t^+}(y) - v_1(y) \text{ are parallel}\}. \tag{4.39}$$

Here $v_1(y)$ denotes the common velocity $v_1^{t^-}(y) = v_1^{t^+}(y)$. We want to show that the closed cell-complex N has codimension at least $v-1$, that is $\dim N \leqq 5v$. The procedure is as follows: For every point $y \in U(x)$ we consider the configuration perturbations $Q_{t^-}^w(y) \in U(x)$ with $w = (0, w_2, w_3, w_4)$ $(w_j \in \mathbf{R}^v, \sum w_j = 0)$. The effect of these perturbations on the compound velocity $(v_2^{t^+}, v_3^{t^+}, v_4^{t^+})$ is that the compound vector $(v_2^{t^+} + v_1/3, v_3^{t^+} + v_1/3, v_4^{t^+} + v_1/3)$ varies on (fills out) an open piece $P(y)$ of a $(2v-1)$-dimensional sphere $S^{2v-1}(y)$ in the $2v$-dimensional euclidean space

$$H = \{(v_2, v_3, v_4)\colon v_j \in \mathbf{R}^v \ \& \ \textstyle\sum v_j = 0\}.$$

This is a well-known fact from the theory of billiards. Here we used the assumption that the trajectory segment $S_{2,3,4}^{(t^-,t^+)}y$ of the sub-billiard system $\{2, 3, 4\}$ is sufficient. In such a sphere the linear dependence of the vectors $v_i^{t^-} - v_1$ and $v_j^{t^+} - v_1$ just means that the vector $(v_2^{t^+} + v_1/3, \ v_3^{t^+} + v_1/3, \ v_4^{t^+} + v_1/3)$ belongs to certain codimension $v-1$ affine subspace $A(y)$ of H. The intersection $P(y) \cap A(y)$ is either empty, or a singleton, or a v-dimensional submanifold of $P(y)$. This easily implies that the codimension of N in $U(x)$ is at least $v-1$.

Case 11. $(\{234\}_{\geqq 3}, \{12\}, \{23\})$. Once again, let t^- separate the blocks $\{234\}_{\geqq 3}$ and $\{12\}$ while $t^{\mp}$ separate the islands $\{12\}$ and $\{23\}$ and, moreover, let α, β be the advances of the collisions at the islands $\{12\}$ and $\{23\}$, respectively. For the same reason as before (in the treatment of Case 9), we can assume that for every point $y \in U(x)$ the trajectory segment $S_{2,3,4}^{(a,t^-)}y$ of the sub-billiard system $\{2, 3, 4\}$ is sufficient. We again get rid of the trivial neutral direction of the flow by taking the advance of the collisions at the block $\{234\}_{\geqq 3}$ zero. We denote the corresponding neutral spaces by $W_{t^\pm}^0(\cdot)$. It is obvious that

$$W_{t^-}^0(S^{(a,t^-)}y) = \{(-3\xi, \xi, \xi, \xi)\colon \xi \in \mathbf{R}^v\} \quad \text{for all} \quad y \in U(x). \tag{4.40}$$

The neutrality at the island $\{12\}$ with advance α means that $4\xi = \alpha(v_2^{t^-} - v_1^{t^-})$, that is

$$\xi = \tfrac{1}{4}\alpha(v_2^{t^-} - v_1^{t^-}). \tag{4.41}$$

From the collision laws for the island $\{12\}$ we can easily obtain the following formula for the relative displacements $w_2^{t^+}$ and $w_3^{t^+}$:

$$\begin{cases} w_2^{t^+} = \tfrac{1}{2}\alpha(v_2^{t^+} - v_1^{t^+}) - \tfrac{1}{4}\alpha(v_2^{t^-} - v_1^{t^-}), \\ w_3^{t^+} = w_3^{t^-} = \xi = \tfrac{1}{4}\alpha(v_2^{t^-} - v_1^{t^-}). \end{cases} \tag{4.42}$$

Thus the neutrality with respect to the island $\{23\}$ means that the vectors $v_2^{t^+} - v_3^{t^+}$ and

$$w_2^{t^+} - w_3^{t^+} = \tfrac{1}{2}\alpha(v_2^{t^+} - v_1^{t^+} - v_2^{t^-} + v_1^{t^-}) = \alpha(v_2^{t^+} - v_2^{t^-})$$

are parallel, therefore the set $N \subset U(x)$ of points $y \in U(x)$ for which the segment $S^{(a,b)}y$ is not sufficient can be described in the following way:

$$N = \{y \in U(x)\colon \text{ the vectors } v_2^{t^+}(y) - v_3^{t^+}(y) \text{ and } v_2^{t^+}(y) - v_2^{t^-}(y) \text{ are parallel}\}. \tag{4.43}$$

We can now observe that applying perturbations of the form $Q^w_{t^+}(\cdot)$ with $w=(\eta, -\eta, 0, 0)$ $(\eta \in \mathbf{R}^\nu)$, the vector $v_2^{t^+}-v_3^{t^+}$ remains fixed while the vector $v_2^{t^+}-v_2^{t^-}$ fills out an open piece of a $(\nu-1)$-dimensional sphere in $\mathbf{R}^\nu$ and this sphere contains the origin. Using the presented arguments from the study of the preceding cases we can conclude that $\dim N \leqq 5\nu$.

The proof of the Main Theorem 4.3 is now complete.

5. Trajectories with a Poor Collision Structure

The result of the preceding section guarantees the sufficiency of non-singular trajectories $S^{(-\infty,\infty)}x_0$ for "almost every" point $x_0 \in M$, provided this trajectory contains a bounded segment whose symbolic collision sequence (structure) is not equivalent to any decomposing one. Here the phrase "almost every" means that the set of exceptional points x_0 (for which the implication is false) is a countable union of smooth, codimension two (at least two) submanifolds of M. The goal of the present section is to prove the following theorem:

Main Theorem 5.1. *The set M_d is residual, where M_d is the collection of all non-singular points $x_0 \in M$ (a non-singular point is a point with non-singular trajectory) whose trajectory*
(i) *is not sufficient*
and
(ii) *decomposes, that is, the collision sequence of every finite segment $S^{(a,b)}x_0$ is equivalent to some decomposing one.*

For our purposes it is important to show that for every point $x_0 \in M_d$ the trajectory of x_0 decomposes *uniformly*, that is, we can find a non-collision moment $t_0 \in \mathbb{R}$ and two partitions P_1, P_2 of the particles $1, 2, 3, 4$ such that both partitions consist of two classes and, after finitely many permitted operations (a)–(e) (see the beginning of Sect. 4) on the symbolic collision sequence of $S^{(-\infty,\infty)}x_0$, the trajectory segment $S^{(-\infty,t_0)}x_0$ is partitioned by P_1 and, similarly, $S^{(t_0,\infty)}x_0$ is partitioned by P_2. A classical and fundamental lemma (König's Lemma) from combinatories yields the required uniform decomposability. Before drawing up this lemma, we need a well-known definition from combinatories.

A tree is a partially ordered set $(T, <)$ such that for every element $a \in T$ the subset $\{t \in T : t < a\}$ is well ordered *by $<$. We say that $a \in T$ is at the α^{th} level (α is an ordinal number)* iff *the order type of the set $\{t \in T : t < a\}$ is α. The* height *of T is the least ordinal number β which is strictly greater then all the ordinals corresponding to non-empty levels. In this case we say that T is β-high.*

The combinatorial result just mentioned is the following statement [see Lemma 10.3 in E-H-M-R (1984)].

Lemma 5.2 (König's Lemma). *If we are given an ω-high tree T such that all the levels of T are finite, then T contains an infinite ordered subset (a branch).*

We note that the order type of such a branch is necessarily ω.

Having this result at hand, it is a routine task to show that for every point $x_0 \in M_d$ the trajectory $S^{(-\infty,\infty)}x_0$ decomposes uniformly, i.e. there is a non-collision moment $t_0 \in \mathbb{R}$ and there is a pair (P_1, P_2) of two-class partitions on

$\{1, 2, 3, 4\}$ such that, after *finitely many* permitted operations on the symbolic collision sequence of the trajectory of x_0, the trajectory segment $S^{(-\infty, t_0)}x_0$ is partitioned by P_1 and, similarly, $S^{(t_0, \infty)}x_0$ is partitioned by P_2. We note that the only required permitted operations in this process are the unification of two neighboring identical islands (a) and the interchanging of two neighboring, disjoint islands (c). We do not intend to go into the details of this argument – showing the uniform decomposability of the trajectory of points $x_0 \in M_d-$, we only give a hint to the reader. Let us denote the double infinite sequence of islands of $S^{(-\infty, \infty)}x_0$ by $\{\ldots, I_{-1}, I_0, I_1, \ldots\}$. Then the elements of the three T are 5-tuples of the form $(N, PO(N), k, P_1, P_2)$, where N is an arbitrary natural number, $PO(N)$ is the composition of finitely many permitted operations (a) and (c) on the sequence $\{I_{-N}, I_{-N+1}, \ldots, I_N\}$, k is an integer between 0 and n $(0 \leqq k \leqq n)$ where n is the length of the transformed sequence

$$PO(N)[I_{-N}, I_{-N+1}, \ldots, I_N] = \{J_1, J_2, \ldots, J_n\},$$

P_1 and P_2 are two-class partitions of the set $\{1, 2, 3, 4\}$, and we also require that all the collisions of $J_1, J_2, \ldots, J_k$ take only place inside of some classes of P_1 and, similarly, all the collisions of $J_{k+1}, \ldots, J_n$ only occur inside of some classes of P_2. The partial ordering $<$ on T is defined as the appropriate notion of natural restriction. It is now clear that any infinite branch

$$F_1 < F_2 < \ldots, \qquad F_N = (N, PO(N), k_N, P_1, P_2)$$

of T defines a composition of permitted operations (a) and (c) which shows the uniform decomposability of $\{\ldots, I_{-1}, I_0, I_1, \ldots\}$. It may happen that this composition contains infinitely many operations, but even in this case, *finitely many* of them will do. The reason is that for all N large enough the incoming islands I_N (I_{-N}) represent collisions between particles of the same class of P_2 (P_1), provided that the trajectory $S^{(t, \infty)}x_0$ $(S^{(-\infty, t)}x_0)$ is not partitioned by any three-class partition for every $t \in \mathbb{R}$ and, in virtue of the Appendix, this can be assumed for x_0.

Keeping in mind Principle 4.1 and the previous argument, in order to prove the residuality of the set M_d it is enough to show that the set M_{P_1, P_2} is residual (P_1 and P_2 are two-class partitions of the set $\{1, 2, 3, 4\}$), where

$$M_{P_1, P_2} = \{x \in M \backslash \partial M \colon S^{(-\infty, 0)}x \text{ is partitioned by } P_1 \text{ and } S^{(0, \infty)}x \text{ is partitioned by } P_2\}. \tag{5.3}$$

The two classes of the partition P_i are denoted by $C_1(P_i)$ and $C_2(P_i)$ $(i = 1, 2)$. We fix an arbitrary point $x_0 \in M_{P_1, P_2}$ and show that the set $M_{P_1, P_2} \cap U(x_0)$ is residual for some open neighborhood $U(x_0)$ of x_0. [Residuality is a local property, see Lemma 2.14 in K-S-Sz (1991).] Throughout the whole section we can assume that, for the fixed base point x_0, the *sub-billiard semi-trajectories* $S^{(-\infty, 0)}_{C_i(P_1)}x_0$ and $S^{(0, \infty)}_{C_i(P_2)}x_0$ $(i = 1, 2)$ are sufficient trajectories, where the symbol $S^{(0, \infty)}_{C_i(P_2)}x_0$ denotes the positive semi-trajectory of the billiard of particles from $C_i(P_2)$, observing this trajectory from the moving co-ordinate system associated with one of the centers of mass of the particles from $C_i(P_2)$. The trajectory of a "one-particle billiard system" is always thought to be sufficient! The notion of sufficiency for $S^{(-\infty, 0)}_{C_i(P_1)}x_0$ is analogous. The argument, showing the possibility of the assumption just mentioned, is the following one: By the result of the Appendix of the present paper we can assume that the sub-billiard semi-trajectories $S^{(0, \infty)}_{C_i(P_2)}$ and $S^{(-\infty, 0)}_{C_i(P_1)}$ $(i = 1, 2)$ contain *infinitely many* islands, provided the cardinality of $C_i(P_2)$ $(C_i(P_1))$ is greater than one. According to the Main Lemma 4.13 of K-S-SZ (1991) (applied to the general case $v \geqq 2$), the existence of four islands guarantees the required sufficiency modulo a codimension-two submanifold. Actually, in our case $v \geqq 3$, even the presence of three islands is enough.

Due to the continuity of the flow S_t it is clear that there exists an open, ball neighborhood $U(x_0)$ of x_0 not intersecting the boundary ∂M that has the following property: For every point $y \in M_{P_1,P_2} \cap U(x_0)$ the sub-billiard semi-trajectories $S_{C_i(P_1)}^{(-\infty,0)} y$ and $S_{C_i(P_2)}^{(0,\infty)} y$ $(i=1,2)$ are sufficient.

Now we encounter five, non-isomorphic possibilities for the pair (P_1, P_2) to be discussed:

(a) $P_1 = (\{1\}, \{2,3,4\})$, $P_2 = (\{1,2\}, \{3,4\})$;
(b) $P_1 = (\{1\}, \{2,3,4\})$, $P_2 = (\{2\}, \{1,3,4\})$;
(c) $P_1 = (\{1,2\}, \{3,4\})$, $P_2 = (\{1,3\}, \{2,4\})$;
(d) $P_1 = P_2 = (\{1\}, \{2,3,4\})$;
(e) $P_1 = P_2 = (\{1,2\}, \{3,4\})$.

It turns out that these cases are significantly different, namely, the most complicated one is the first. Cases (b) and (c) are simple, since, as we shall show it, there is actually *no point* $x_0 \in M_{P_1,P_2}$ for which all the four mentioned sub-billiard semi-trajectories are sufficient. Note that non-sufficiency of $S^{(-\infty,\infty)} x_0$ is required for points $x_0 \in M_d$. Finally, the last two cases can be treated by "integrating up codimension – two sets," see the finishing part of this section or K-S-Sz (1989) (proof of sublemma 2).

We are going to prove in each case that the set $M_{P_1,P_2} \cap U(x_0)$ is *residual*. We shall follow the basic strategy of the article K-S-Sz (1991), using the so-called pseudo-stable and pseudo-unstable manifolds, their transversality and, finally, the "zig-zag lemma" on then. In the proof of this zig-zag lemma the strongest, and most up to date version of the fundamental theorem for the sub-billiard systems $S_{C_i(P_j)}^t$ will heavily be used.

Discussion of Case (a). $P_1 = (\{1\}, \{2,3,4\})$, $P_2 = (\{1,2\}, \{3,4\})$.

We define four closed subsets $F_\pm$ and $F'_\pm$ of $U(x_0)$ as follows:

$$F_- = \{y \in U(x_0)\text{: the semi-trajectory } \{S^t y : t<0\} \text{ contains no proper collision between the classes of } P_1\}, \tag{5.4}$$

$$F_+ = \{y \in U(x_0)\text{: the semi-trajectory } \{S^t y : t>0\} \text{ contains no proper collision between the classes of } P_2\}, \tag{5.5}$$

$$F'_- = \{y \in U(x_0)\text{: cancelling the interaction between the classes of } P_1\text{, in the modified semi-trajectory } S_*^{(-\infty,0)} y \text{ the distance between centers of particles from different classes of } P_1 \text{ is never less than } 2r-\varepsilon_0\}, \tag{5.6}$$

$$F'_+ = \{y \in U(x_0)\text{: cancelling the interaction between the classes of } P_2\text{, in the modified semi-trajectory } S_*^{(0,\infty)} y \text{ the distance between centers of particles from different classes of } P_2 \text{ is never less than } 2r-\varepsilon_0\}. \tag{5.7}$$

Important Remark. If some of the semi-trajectories in the definitions (5.4)–(5.7) is not uniquely defined (caused by a multiple collision), then the text in these definitions reads "some of the branches of the semi-trajectory contains no proper collision...." This convention is important in order to have *closed* sets F_+, F_-, F'_+, and F'_-.

We note that r is the common radius of the particles and $\varepsilon_0 > 0$ is a fixed, small number. In these formulas the cancellation of the indicated interactions means that the non-interacting particles are allowed to overlap each other without any force between them. In the course of the proof we shall see that, after fixing the value of ε_0, the open neighborhood $U(x_0)$ must be chosen small enough in order that the proof can go on.

From the definition of these closed subsets of $U(x_0)$ and from the mixing property of the two- and three-billiards on the torus $\mathbb{T}^\nu$ [established in S-Ch (1987) and K-S-Sz (1991)], one can easily deduce the following properties of the closed sets $F_\pm$ and $F'_\pm$:

$$F_- \subset F'_- \quad \text{and} \quad F_+ \subset F'_+ ; \tag{5.8}$$

$$\mu(F' \cup F'_+) = 0 ; \tag{5.9}$$

$$M_{P_1, P_2} \cap U(x_0) \subset F_- \cap F_+ . \tag{5.10}$$

In order to establish the residuality of the set $F_- \cap F_+$, we need to define the so-called pseudo-stable manifolds $\gamma_0^s(y)$, $\gamma_{1,2}^s(y)$, $\gamma_{3,4}^s(y)$, $\gamma_e^s(y)$ and the pseudo-unstable manifolds $\gamma_0^u(y)$, $\gamma_e^u(y)$ for generic points y of $U(x_0)$:

$$\gamma_0^s(y) := \Bigg\{ z \in U(x_0) : V(z) = V(y) \quad \&$$

$$Q(z) - Q(y) = \begin{pmatrix} w \\ w \\ -w \\ -w \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} v_1(y) - v_2(y) \\ v_2(y) - v_1(y) \\ 0 \\ 0 \end{pmatrix}$$

$$+ \frac{\mu}{2} \begin{pmatrix} 0 \\ 0 \\ v_3(y) - v_4(y) \\ v_4(y) - v_3(y) \end{pmatrix} ; \quad w \in \mathbb{R}^\nu, \lambda, \mu \in \mathbb{R} \Bigg\}, \tag{5.11}$$

$$\begin{aligned} \gamma_{1,2}^s(y) := CC_y\{ z \in U(x_0) :\ & q_3(z) = q_3(y),\ q_4(z) = q_4(y), \\ & v_3(z) = v_3(y),\ v_4(z) = v_4(y) \ \&\ \operatorname{dist}(S_{1,2}^t z, S_{1,2}^t y) \to 0 \\ & \text{exponentially quickly as } t \to +\infty \}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \gamma_{3,4}^s(y) := CC_y\{ z \in U(x_0) :\ & q_1(z) = q_1(y),\ q_2(z) = q_2(y), \\ & v_1(z) = v_1(y),\ v_2(z) = v_2(y) \ \&\ \operatorname{dist}(S_{3,4}^t z, S_{3,4}^t y) \to 0 \\ & \text{exponentially quickly as } t \to +\infty \}, \end{aligned} \tag{5.13}$$

$$\gamma_e^s(y) := \bigcup_{z \in \gamma_{1,2}^s(y)} \gamma_{3,4}^s(z), \tag{5.14}$$

$$\gamma_0^u(y) := \left\{ z \in U(x_0):\ V(z) = V(y)\ \&\ Q(z) - Q(y) = \begin{pmatrix} -3w \\ w \\ w \\ w \end{pmatrix} + \lambda \begin{pmatrix} v_1(y) \\ v_2(y) \\ v_3(y) \\ v_4(y) \end{pmatrix};\quad w \in \mathbb{R}^\nu, \lambda \in \mathbb{R} \right\}, \tag{5.15}$$

$$\begin{aligned} \gamma_e^u(y) := CC_y\{z \in U(x_0):\ & q_1(z) = q_1(y), \\ & v_1(z) = v_1(y)\ \&\ \operatorname{dist}(S^t_{2,3,4}z, S^t_{2,3,4}y) \to 0 \\ & \text{exponentially quickly as } t \to -\infty\}. \end{aligned} \tag{5.16}$$

Here the subscript "e" stands for "exponential" and CC_y denotes the operation of taking the arcwise connected component of a set containing the point y.

For every point $y \in U(x_0)$ the "neutral" manifolds $\gamma_0^s(y)$ and $\gamma_0^u(y)$ are well defined pieces of affine subspaces in $U(x_0)$ not terminating in $U(x_0)$, while, for almost every point $y \in U(x_0)$, the exponentially contracting (dilating) manifolds $\gamma_{1,2}^s(y)$, $\gamma_{3,4}^s(y)$, $\gamma_e^s(y)$, and $\gamma_e^u(y)$ are only smooth manifolds of class C^1 (actually of class C^∞) containing y as an interior point and having dimensions $\nu-1$, $\nu-1$, $2\nu-2$, and $2\nu-1$ respectively. The set of such points y is denoted by G (good points). It is known [see Ch (1982)] that $\mu(G) = \mu(U(x_0))$ but, unfortunately, the size of these manifolds can be arbitrarily small. We introduce one more neutral manifold "generated" by γ_0^s and γ_0^u:

$$\gamma_0(y) := \bigcup_{z \in \gamma_0^s(y)} \gamma_0^u(z). \tag{5.17}$$

The neutral manifolds $\gamma_0(y)$ are also the intersections of $U(x_0)$ with affine subspaces, just like in the case of $\gamma_0^s(y)$ and $\gamma_0^u(y)$. The manifold $\gamma_0^s(y)$ has dimension $\nu+2$ if $v_1(y) \neq v_2(y)$, $v_3(y) \neq v_4(y)$ and the manifold $\gamma_0^u(y)$ has dimension $\nu+1$ unless $v_2(y) = v_3(y) = v_4(y)$. By our assumption on the base point x_0, the required inequalities hold for $y = x_0$ and, by choosing the neighborhood $U(x_0)$ small enough, for every $y \in U(x_0)$, too. From the formulas (5.11) and (5.15) we see that the intersection of $\gamma_0^s(y)$ and $\gamma_0^u(y)$ is 2-dimensional for every point $y \in U(x_0)$ or, more precisely,

$$\gamma_0^s(y) \cap \gamma_0^u(y) = \left\{ z \in U(x_0:\ V(z) = V(y)\ \&\ Q(z) - Q(y) = \alpha \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} (v_1(y) - v_2(y)) + \lambda \begin{pmatrix} v_1(y) \\ v_2(y) \\ v_3(y) \\ v_4(y) \end{pmatrix};\quad \alpha, \lambda \in \mathbb{R} \right\}. \tag{5.18}$$

Thus we get, as a consequence, that for every $y \in U(x_0)$ $\dim(\gamma_0(y)) = 2v+1$, therefore,

$$\begin{aligned}\dim(\gamma_0(y)) + \dim(\gamma_e^s(y)) + \dim(\gamma_e^u(a)) &= 2v+1+2v-2+2v-1\\ &= 6v-2 = \dim M - 1\end{aligned}$$

for all points $y \in G$. One can expect that these three manifolds intersect transversally at the point y, that is, the tangent space $\mathscr{T}_y M$ of M at y contains a codimension one subspace L_y such that

$$L_y = \mathscr{T}_y \gamma_0(y) + \mathscr{T}_y \gamma_e^s(y) + \mathscr{T}_y \gamma_e^u(y). \tag{5.19}$$

Here $+$ denotes the direct sum (not necessarily orthogonal) of linear spaces. If turns out that this expectation is justified:

Lemma 5.20. *For every point $y \in G$ there exists a codimension one subspace L_y of the tangent space $\mathscr{T}_y M$ such that* (5.19) *holds.*

Proof. Since we deal with transversality properties of linear subspaces of $\mathscr{T}_y M$ which are the tangent spaces of some submanifolds through y, we shall use the notion of infinitesimal perturbations dq_j and dv_j $(j = 1, 2, 3, 4)$ of the data q_j and v_j, where these quantities dq_j, dv_j are v-dimensional, first order infinitesimally small vectors. First of all, we put forward a sublemma:

Sublemma 5.21. *For every point $y \in G$ the manifolds $\gamma_e^s(y)$ and $\gamma_e^u(y)$ are transversal.*

Proof. Let the compound perturbation $\{(dq_j, dv_j): h = 1, 2, 3, 4\}$ be a common *non-zero* element of the tangent spaces $\mathscr{T}_y \gamma_e^s(y)$ and $\mathscr{T}_y \gamma_e^u(y)$. By the negative definiteness of the second fundamental form of local stable invariant manifolds of the systems $\{S_{1,2}^t\}$ and $\{S_{3,4}^t\}$ and by the positive definiteness of the second fundamental form of local unstable invariant manifolds of the flow $\{S_{2,3,4}^t\}$, we have the following inequalities:

$$\sum_{j=1}^{4} dq_j \cdot dv_j < 0, \tag{5.22}$$

$$\sum_{j=1}^{4} dq_j \cdot dv_j > 0, \tag{5.23}$$

which is impossible. Sublemma 5.21 is proved. [The multiplication in (5.22)–(5.23) and in the further formulas denotes the standard scalar product in $\mathbb{R}^v$.] Now we return to the proof of Lemma 5.20.

By the sublemma, we have to prove the equality $\mathscr{T}_y \gamma_0(y) \cap [\mathscr{T}_y \gamma_e^s(y) + \mathscr{T}_y \gamma_e^u(y)] = 0$. Let the perturbation $\Pi = \{(dq_j, dv_j): j = 1, 2, 3, 4\}$ be a common element of $\mathscr{T}_y \gamma_0(y)$ and $\mathscr{T}_y \gamma_e^s(y) + \mathscr{T}_y \gamma_e^u(y)$. According to the sublemma, the perturbation Π can be split into the components $\Pi^- = \{(dq_j^-, dv_j^-): j = 1, 2, 3, 4\} \in \mathscr{T}_y \gamma_e^u(y)$ and $\Pi^+ = \{(dq_j^+, dv_j^+): j = 1, 2, 3, 4\} \in \mathscr{T}_y \gamma_e^s(y)$. By the conservation laws for the subsystems $\{S_{1,2}^t\}$, $\{S_{3,4}^t\}$, and $\{S_{2,3,4}^t\}$ (conservation of the kinetic energy, center of mass, and total momentum) we get the following system of equations:

$$\sum_{j=2}^{4} dq_j^- = 0, \tag{5.24}$$

$$dq_1^- = 0, \tag{5.25}$$

$$dq_1^+ + dq_2^+ = 0, \tag{5.26}$$

$$dq_3^+ + dq_4^+ = 0, \tag{5.27}$$

$$\sum_{j=2}^{4} dv_j^- = 0, \tag{5.28}$$

$$dv_1^- = 0, \tag{5.29}$$

$$dv_1^+ + dv_2^+ = 0, \tag{5.30}$$

$$dv_3^+ + dv_4^+ = 0, \tag{5.31}$$

$$\sum_{j=2}^{4} dv_j^- \cdot v_j = 0, \tag{5.32}$$

$$dv_1^+ \cdot (v_1 - v_2) = 0, \tag{5.33}$$

$$dv_3^+ \cdot (v_3 - v_4) = 0. \tag{5.34}$$

By the negative definiteness of the second fundamental form of the local stable invariant manifolds (of the flows $\{S_{1,2}^t\}$ and $\{S_{3,4}^t\}$) and by the positive definiteness of the second fundamental form of the local unstable invariant manifolds (of the flow $\{S_{2,3,4}^t\}$) we again get

$$\sum_{j=2}^{4} dq_j^- \cdot dv_j^- \geqq 0, \tag{5.35}$$

$$dq_1^+ \cdot dv_1^+ \leqq 0, \tag{5.36}$$

$$dq_3^+ \cdot dv_3^+ \leqq 0. \tag{5.37}$$

We note that in (5.35) strict inequality holds whenever $\Pi^- \neq 0$ and either in (5.36) or in (5.37) strict inequality holds if $\Pi^+ \neq 0$. Finally, using the fact $\Pi \in \mathcal{T}_y \gamma_0(y)$ is a pure spatial perturbation (that is, $dv_j = 0$), we also have

$$dv_j^- + dv_j^+ = 0 \qquad (j = 2, 3, 4), \tag{5.38}$$

$$dv_1^+ = 0. \tag{5.39}$$

From (5.30), (5.39), and (5.38) we obtain

$$dv_2^+ = 0, \tag{5.40}$$

$$dv_2^- = 0. \tag{5.41}$$

Next, (5.39), (5.40), and the non-degeneracy of the second fundamental form of local invariant manifolds yield

$$dq_1^+ = 0, \tag{5.42}$$

$$dq_2^+ = 0. \tag{5.43}$$

At this stage of the proof we again take advantage of the relation $\Pi^- + \Pi^+ \in \mathcal{T}_y \gamma_0(y)$, that is, $dq_3 - dq_4$ must be a scalar multiple of the relative velocity $v_3 - v_4$: $dq_3 - dq_4 = \alpha(v_3 - v_4)$ or, equivalently,

$$dq_2^- + 2dq_3^- + 2dq_3^+ = \alpha(v_3 - v_4) \tag{5.44}$$

for some $\alpha \in \mathbb{R}$. Here we used the formulas (5.24) and (5.27). Multiplying both sides of (5.44) by dv_3^+ and using (5.34) we get

$$2dv_3^+ \cdot dq_3^+ + dv_3^+ \cdot (dq_2^- + 2dq_3^-) = 0. \tag{5.45}$$

In the last equality the factor $dq_2^- + 2dq_3^-$ is clearly $dq_3^- - dq_4^-$. By (5.37), (5.35), (5.41), (5.28), and (5.38) we get that both terms on the left-hand side of (5.45) are non-positive numbers, therefore, by (5.45), equality holds in (5.35) and (5.37). Using the remark after (5.37) and the relation $dq_1^+ \cdot dv_1^+ = 0$, we obtain that $\Pi^- = \Pi^+ = 0$, i.e. $\Pi = 0$. Lemma 5.20 is proved.

We continue the proof of the residuality for the set $M_{P_1,P_2} \cap U(x_0)$ [cf. (5.3)] in case (a). By the set inequality (5.10) it is enough to prove that $\dim(F_- \cap F_+) \leq \dim M - 2 = 6\nu - 3$. To that end, we need an auxiliary foliation of the open ball $U(x_0)$ by the codimension $\nu + 1$ smooth submanifolds $\Phi_{I,E}$ as follows:

$$\Phi_{I,E} := \{y \in U(x_0):\ v_1(y) + v_2(y) = I \ \ \&\ \ v_1^2(y) + v_2^2(y) = 2E\}$$
$$(I \in \mathbb{R}^\nu, E \in \mathbb{R}_+). \tag{5.46}$$

According to property 4 in Sect. 4.1 of K-S-Sz (1989), the inequality $\dim(F_- \cap F_+) \leq 6\nu - 3$ would be implied by the following lemma:

Main Lemma 5.47. *For each submanifold $\Phi_{I,E} \subset U(x_0)$ the dimension of the closed set $F_- \cap F_+ \cap \Phi_{I,E}$ is at most $5\nu - 4 = \dim \Phi_{I,E} - 2$.*

All the remaining part of the discussion of case (a) is devoted to the proof of this Main Lemma. The base of this proof is a simultaneous application of the transversal fundamental theorem for the sub-billiard systems $S_{1,2}^t$ $(t > 0)$, $S_{3,4}^t$ $(t > 0)$, and $S_{2,3,4}^t$ $(t < 0)$. A series of lemmas leads to the proof of the Main Lemma.

Lemma 5.48. *For every point $y \in G$ the manifolds $\gamma_0(y)$, $\gamma_{1,2}^s(y)$, and $\gamma_{3,4}^s(y)$ are contained in the manifold $\Phi(y) := \Phi_{I(y),E(y)}$ and the intersection $\gamma_e^u(y) \cap \Phi(y)$ has dimension $\nu - 1$.*

Proof. The statements on $\gamma_0(y)$, $\gamma_{1,2}^s(y)$, and $\gamma_{3,4}^s(y)$ are obviously true. The last claim of this lemma follows from the

(i) non-degeneracy of the second fundamental form of local unstable manifolds of the sub-billiard flow $S_{2,3,4}^t$

and from the fact that

(ii) intersecting the manifold $\gamma_e^u(y)$ with $\Phi(y)$ means that, when making the exponential unstable perturbation in the subsystem $\{2, 3, 4\}$, the velocity v_2 must be kept fixed so, in terms of the velocities in this subsystem, this allows to us $\nu - 1$ degrees of freedom. [The velocities v_3 and v_4 can be varied in such a way that $v_3 + v_4$ and $v_3^2 + v_4^2$ are fixed, which defines a $(\nu - 1)$-dimensional sphere.] As a consequence of 5.20 and 5.48, we get

Lemma 5.49. *For every point $y \in G$ the tangent space $\mathcal{T}_y\Phi(y)$ is the direct sum of the tangent spaces $\mathcal{T}_y\gamma_0(y)$, $\mathcal{T}_y\gamma_{1,2}^s(y)$, $\mathcal{T}_y\gamma_{3,4}^s(y)$, and $\mathcal{T}_y(\gamma_e^u(y) \cap \Phi(y))$.*

Proof. Simple computation with the dimension.

In order to prove the Main Lemma, based on the previous sublemma, we want to use the famous and successful zig-zag argument [essentially due to Hopf and Hedlund, see the proof of Lemma 5.3 in K-S-Sz (1991)] in each manifold $\Phi_{I,E}$. However, this argument requires that the intersection $(F'_- \cup F'_+) \cap \Phi_{I,E}$ be a zero set with respect to the measure μ_Φ induced by the inherited Riemannian metric on $\Phi_{I,E}$.

Lemma 5.50. *For every pair (I, E) the intersection $(F'_- \cup F'_+) \cap \Phi_{I,E}$ is a zero set with respect to the measure μ_Φ.*

Proof. The relation $\mu_\Phi(F'_+ \cap \Phi_{I,E}) = 0$ follows easily from the mixing property of the subsystem $S^t_{1,2}$ $(t > 0)$ [or, equivalently, from the mixing property of the flow $S^t_{3,4}$ $(t > 0)$] as a "weak lemma on avoiding of balls," see Lemma 2.16 in K-S-Sz (1991). Namely, this lemma, applied to the system $S^t_{3,4}$ $(t > 0)$, yields that for every manifold $\Phi(q_1, q_2, v_1, v_2)$ (which is a submanifold of $\Phi_{v_1+v_2, \frac{1}{2}(v_1^2+v_2^2)}$) the intersection $F'_+ \cap \Phi(q_1, q_2, v_1, v_2)$ has measure zero with respect to the Riemannian volume in

$$\Phi(q_1, q_2, v_1, v_2) := \{y \in U(x_0):\ q_1(y) = q_1,\ q_2(y) = q_2,\ v_1(y) = v_1, \text{ and } v_2(y) = v_2\}.$$

The proof of the relation $\mu_\Phi(F'_- \cap \Phi_{I,E}) = 0$ is more subtle. For each pair (I, E) (for which $\Phi_{I,E}$ is non-empty) we define an auxiliary foliation $\Phi_{I,E}$ with the smooth submanifolds

$$\Phi(q_1, v_1, v_2) := \{y \in U(x_0):\ q_1(y) = q_1,\ v_1(y) = v_1 \text{ and } v_2(y) = v_2\}$$
$$(q_1 \in \mathbb{T}^\nu,\ v_1,\ v_2 \in \mathbb{R}^\nu,\ v_1 + v_2 = I,\ v_1^2 + v_2^2 = 2E).$$

We shall actually prove that for every q_1 for almost all v_2 [such that $v_1^2 + (I - v_1)^2 = 2E$] the intersection $F'_- \cap \Phi(q_1, v_1, I - v_1)$ has measure zero with respect to the Riemannian volume in $\Phi(q_1, v_1, I - v_1)$. For this purpose, we define (locally) the canonical projection $\pi_{2,3,4}: M \to M_3$ of the phase space M into the phase space M_3 of the sub-billiard system $S^t_{2,3,4}$ as follows:

$$\pi_{2,3,4}(\{(q_j, v_j): j = 1, 2, 3, 4\}) := \{(Q_j, V_j): j = 2, 3, 4\}, \tag{5.51}$$

where

$$\begin{cases} Q_j = q_j + \frac{1}{3} q_1 & (j = 2, 3, 4), \\ V_j = (1 - \frac{4}{3} v_1^2)^{-1/2} (v_j + \frac{1}{3} v_1) & (j = 2, 3, 4). \end{cases} \tag{5.52}$$

Here the normalizing factor $(1 - \frac{4}{3} v_1^2)^{-1/2}$ guarantees that the usual energy normalization

$$\sum_{j=2}^{4} V_j^2 = 1$$

holds. This definition is only correct locally because we multiply the position $q_1 \in \mathbb{T}^\nu$ by 1/3. It is clear that the projection $\pi_{2,3,4}$ restricted to $\Phi(q_1, v_1, I - v_1)$ is a diffeomorphism of $\Phi(q_1, v_1, I - v_1)$ onto a codimension ν smooth submanifold $\pi_{2,3,4}(\Phi(q_1, v_1, I - v_1))$ of M_3 and this submanifold can be defined by the equation $V_2 = (1 - \frac{4}{3} v_1^2)^{-1/2} (I - \frac{2}{3} v_1) = \text{const}$. The next result is an important characterization of the action of $\pi_{2,3,4}$ on the velocities. We define the projection $\pi^V_{2,3,4}$ of the $(3\nu - 1)$-dimensional sphere of velocities

$$S^{3\nu-1} = \left\{(v_1, v_2, v_3, v_4) \in \mathbb{R}^{4\nu}:\ \sum_{j=1}^{4} v_j = 0\ \&\ \sum_{j=1}^{4} v_j^2 = 1\right\}$$

onto the $(2\nu - 1)$-dimensional sphere of velocities

$$S^{2\nu-1} = \left\{(V_2, V_3, V_4) \in \mathbb{R}^{3\nu}:\ \sum_{j=2}^{4} V_j = 0\ \&\ \sum_{j=2}^{4} V_j^2 = 1\right\}$$

by assigning to any compound velocity (v_1, v_2, v_3, v_4) the three-particle compound velocity $(1 - \frac{4}{3} v_1^2)^{-1/2} (v_2 + \frac{1}{3} v_1,\ v_3 + \frac{1}{3} v_1,\ v_4 + \frac{1}{3} v_1)$.

Sublemma 5.53. *At any point* $(v_1, v_2, v_3, v_4) \in S^{3\nu-1}$ *the mapping* $\pi^V_{2,3,4}$; $S^{3\nu-1} \to S^{2\nu-1}$ *restricted to the subset*

$$S_I^{3\nu-1} := \{(v_1', v_2', v_3', v_4') \in S^{3\nu-1} : v_1' + v_2' = v_1 + v_2 \ (:= I)\}$$

(which is a $(2\nu-1)$*-dimensional smooth submanifold in a neighborhood of* (v_1, v_2, v_3, v_4)*) maps a suitable neighborhood of* (v_1, v_2, v_3, v_4) *in* $S_I^{3\nu-1}$ *diffeomorphically onto an open subset of* $S^{2\nu-1}$*, provided that*

(i) $v_1 \neq v_2$,

(ii) $v_3 \neq v_4$,

and

(iii) $v_1 \cdot (v_1 + v_2) \neq \frac{1}{2}$.

We note that the first two conditions are natural and obviously valid in a neighborhood of the base point $x_0 \in M$.

Proof. It is clear that the set $\{(v_1', v_2', v_3', v_4') \in S^{3\nu-1} : v_1' + v_2' = v_1 + v_2\}$ is a $(2\nu-1)$-dimensional smooth manifold in a neighborhood of (v_1, v_2, v_3, v_4). We want to show that the derivative of $\pi^V_{2,3,4}$ at (v_1, v_2, v_3, v_4) is non-degenerate on the tangent space of $S_I^{3\nu-1}$ at (v_1, v_2, v_3, v_4). Assume that the tangent vector $(dv_1, -dv_1, dv_3, -dv_3)$ of $S_I^{3\nu-1}$ at (v_1, v_2, v_3, v_4) is mapped into 0 by $D\pi^V_{2,3,4}$. This means that

$$\begin{aligned} dV_j &= (1-\tfrac{4}{3}v_1^2)^{-1/2}(dv_j + \tfrac{1}{3}dv_1) \\ &\quad + \tfrac{4}{3}(1-\tfrac{4}{3}v_1^2)^{-3/2}(v_1 \cdot dv_1)(v_j + \tfrac{1}{3}v_1) = 0 \end{aligned} \tag{5.54}$$

for $j = 2, 3, 4$. For $j = 2$ (5.54) is equivalent to the equation

$$dv_1 = \frac{2}{1 - \frac{4}{3}v_1^2}(v_1 \cdot dv_1)\left(\frac{1}{3}v_1 + v_2\right). \tag{5.55}$$

(Keep in mind that $dv_2 = -dv_1$ and $dv_4 = -dv_3$.)

If we multiply (5.55) by v_1 and make the straightforward simplifications, then we get

$$(v_1 \cdot dv_1)(v_1 \cdot (v_1 + v_2) - \tfrac{1}{2}) = 0. \tag{5.56}$$

By the assumption (iii) this implies $v_1 \cdot dv_1 = 0$. Using (5.55) we obtain the equation $dv_1 = 0$ and, therefore, $dv_3 = 0$ by (5.54) with $j = 3$. Hence the sublemma.

Let us return to the proof of Lemma 5.50. Assume that, on the contrary, for some $q_1^0 \in \mathbb{T}^\nu$ there is a *positive set of velocities* v_1 on the sphere $v_1^2 + (I - v_1)^2 = 2E$ [$I = v_1 + v_2$ and $E = \frac{1}{2}(v_1^2 + v_2^2)$ are fixed!] such that the intersection $F'_- \cap \Phi(q_1^0, v_1, I - v_1)$ has positive measure with respect to the Riemannian volume in $\Phi(q_1^0, v_1, I - v_1)$. We can obviously assume that for these velocities v_1 property (iii) of the sublemma holds. These assumptions and the sublemma yield that $\pi_{2,3,4}(F'_- \cap \Phi_{I,E}(q_1^0))$ has positive measure in the codimension one smooth submanifold (locally!) $\pi_{2,3,4}(\Phi_{I,E}) \subset M_3$. Here

$$\Phi_{I,E}(q_1^0) := \{y \in U(x_0) \colon \ q_1(y) = q_1^0, \ v_1(y) + v_2(y) = I \ \text{and} \ v_1^2(y) + v_2^2(y) = 2E\}.$$

The characterization of the projected manifolds $\pi_{2,3,4}(\Phi(q_1^0, v_1, I - v_1))$ shows that these manifolds are (uniformly) transversal to the local unstable leaves in M_3 and, therefore, similar transversality holds for the codimension one submanifold

$$\pi_{2,3,4}(\Phi_{I,E}(q_1^0))=\pi_{2,3,4}(\Phi_{I,E}).$$

The transversal fundamental theorem for M_3 implies that for almost all points z of $\pi_{2,3,4}(\Phi_{I,E}(q_1^0))$ the local unstable leaf $\gamma^u_{M_3}(z)$ exists. This statement, together with the latest form of our indirect assumption imply that for some v_1^0 the set

$$A:=\{z\in\pi_{2,3,4}(\Phi(q_1^0,v_1^0,I=v_1^0)\cap F'_-):\gamma^u_{M_3}(z) \text{ exists}\}$$

has positive measure in the codimension v smooth submanifold $\pi_{2,3,4}(\Phi(q_1^0,\, v_1^0,\, I-v_1^0))$ of M_3. Using the transversality between $\pi_{2,3,4}(\Phi(q_1^0,\, v_1^0,\, I-v_1^0))$ and the leaves $\gamma^u_{M_3}(z)$ we have that the set

$$\{z\in M_3:\exists z'\in A \text{ such that } z\in\gamma^u_{M_3}(z') \text{ and } \varrho(S^t_{2,3,4}z,S^t_{2,3,4}z')<\varepsilon_0 \text{ for all } t<0\}$$

has positive measure in M_3. However, all the points of this set avoid a moving open subset of the configuration space of M_3 and this open set moves according to a conditionally periodic motion on the torus. But the positivity and the latest statement contradict to the "weak lemma on avoiding of balls," see Lemma 2.16 in K-S-Sz (1991). The contradiction completes the proof of Lemma 5.50.

Now we are not too far from the end of the proof of Main Lemma. Consider a submanifold $\Phi_{I,E}\subset U(x_0)$ and an arbitrary point $y\in F_-\cap F_+\cap\Phi_{I,E}$ for which property (iii) of Sublemma 5.53 holds. We shall fix the point y until the end of the proof of 5.47. By the locality of being a codimension two set, our goal is to show that there is an open neighborhood $V(y)$ of y in $\Phi_{I,E}$ such that the dimension of the set $V(y)\cap F_-\cap F_+$ is at most $5v-4=\dim\Phi_{I,E}-2$. After the proof of the existence of such a neighborhood $V(y)$ we shall point out what to do if $v_1(y)\cdot I=\frac{1}{2}$.

The argument, just coming up, is the point where we apply the strong form of the transversal fundamental theorem for the subsystems $S^t_{1,2}$, $S^t_{3,4}$ $(t>0)$, and $S^t_{2,3,4}$ $(t<0)$. Using the fact that the trajectories $S^{(0,\infty)}_{1,2}y$, $S^{(0,\infty)}_{3,4}y$, and $S^{(-\infty,0)}_{2,3,4}y$ are sufficient and the codimension one smooth submanifold $\pi_{2,3,4}(\Phi_{I,E})\subset M_3$ can be defined purely in terms of the velocities (locally, cf. Sublemma 5.53 and the first remark after Theorem 3.4), the strong version of the fundamental theorem (see Sect. 3) applied simultaneously to the flows $S^t_{1,2}, S^t_{3,4}$ $(t>0)$ and $S^t_{2,3,4}$ $(t<0)$ says the following statement:

Proposition 5.57. *For every $C>0$ and $0<\alpha<1$ there is an open neighborhood $W_{C,\alpha}(y)$ of y in $\Phi_{I,E}$ with the following property:*

$$\mu_\Phi(\{z\in W_{C,\alpha}(y):\mu_\Phi(B_g(z,\delta))<\alpha\mu_\Phi(B(z,\delta))\})=o(\delta)$$
$$(\textit{small order of } \delta,\ \delta\to 0)$$

where

$$B(z,\delta):=\{z'\in\Phi_{I,E}:\varrho(z,z')<\delta\},$$

$$B_g(z,\delta):=\{z'\in B(z,\delta)\text{: the inner radii of the leaves } \gamma^s_{1,2}(z'),\ \gamma^s_{3,4}(z'), \text{ and } \gamma^u_e(z')\cap\Phi_{I,E} \text{ are greater than } C\delta\},$$

and μ_Φ is the Riemannian volume in $\Phi_{I,E}$.

Proof. It follows easily from the strong version of the transversal fundamental theorem for the subsystems $S^t_{1,2}$, $S^t_{3,4}$, and $S^t_{2,3,4}$, see Sect. 3.

Taking into account the well known fact that the conditional measures of μ_Φ with respect to the partitions $\gamma_0^s(\cdot)$, $\gamma_0^u(\cdot)$, $\gamma_{1,2}^s(\cdot)$, $\gamma_{3,4}^s(\cdot)$, and $\gamma_e^u(\cdot)\cap\Phi_{I,E}$ are almost everywhere *equivalent* to the Riemannian volumes on these submanifolds [absolute continuity, cf. Theorem 4.1 of Kat-Str (1986) and the transversality proved in Lemma 5.20], and the fact that the tangent spaces of these submanifolds depend *continuously* on the base point, by a successive use of a "Fubini type" argument we can conclude as follows:

Corollary 5.58. *For every* $0<\alpha<1$ *there exists a small open ball* $V_\alpha(y)\subset\Phi_{I,E}$ *around* y *in* $\Phi_{I,E}$ *with the following property:*

$$\mu_\Phi(\{z\in V_\alpha(y):\ B(z,\delta) \text{ does not have the } \alpha\text{-zig-zag property}\})=o(\delta)\qquad(\delta\to 0)\tag{5.59}$$

where the ball $B(z,\delta)\subset\Phi_{I,E}$ *has, by definition, the* α*-zig-zag property* iff *for every zero set* $N\subset\Phi_{I,E}$ *there is a set* $A_N\subset B(z,\delta)\setminus N$ *such that*
i) $\mu_\Phi(A_N)\geqq\alpha\mu_\Phi(B(z,\delta))$,
(ii) *for every pair of points* $w_1, w_2\in A_N$ *there exists a sequence* $\gamma_1,\gamma_2,\dots,\gamma_k$ *of submanifolds of* $\Phi_{I,E}$ *with the following properties:*
(a) γ_j *is either of type* γ_0^s, γ_0^u, $\gamma_{1,2}^s$, $\gamma_{3,4}^s$, *or* $\gamma_e^u\cap\Phi_{I,E}$ $(j=1,2,\dots,k)$,
(b) $w_1\in\gamma_1$, $w_2\in\gamma_k$,
(c) $\gamma_j\cap\gamma_{j+1}\cap(\Phi_{I,E}\setminus N)\neq\emptyset$ $(j=1,2,\dots,k-1)$.

Proof. A standard geometric argument.

Our final observation before finishing the proof of 5.47 is the analog of Lemma 5.8 from K-S-Sz (1991):

Proposition 5.60. *If* $U(x_0)$ *is small enough (compared to* ε_0*, see* (5.6)*), then for every* $z\in U(x_0)$

$$\begin{aligned}
\gamma_0^s(z)\cap F_+\neq\emptyset &\quad \textit{implies} \quad \gamma_0^s(z)\subset F'_+\,,\\
\gamma_{1,2}^s(z)\cap F_+\neq\emptyset &\quad \textit{implies} \quad \gamma_{1,2}^s(z)\subset F'_+\,,\\
\gamma_{3,4}^s(z)\cap F_+\neq\emptyset &\quad \textit{implies} \quad \gamma_{3,4}^s(z)\subset F'_+\,,\\
\gamma_e^u(z)\cap F_-\neq\emptyset &\quad \textit{implies} \quad \gamma_e^u(z)\subset F'_-\,,\quad \textit{and}\\
\gamma_0^u(z)\cap F_-\neq\emptyset &\quad \textit{implies} \quad \gamma_0^u(z)\subset F'_-\,.
\end{aligned}$$

The proof is straightforward.

We want to show that for the considered (and fixed) point $y\in\Phi_{I,E}\cap F_-\cap F_+$ there is a suitable open ball $V(y)\subset\Phi_{I,E}$ around y in $\Phi_{I,E}$, for which

$$\dim(V(y)\cap F_-\cap F_+)\leqq 5v-4\,.$$

Taking into account 5.58 and 5.60 with the zero set $N=(F'_-\cup F'_+)\cap\Phi_{I,E}$ (see Lemma 5.50), we see that for some α (close enough to 1) and for every open connected set $U\subset V_\alpha(y)$ the set $U\setminus(F_-\cap F_+)$ is also connected. As it is known from topological dimension theory, the last statement just means that

$$\dim(V_\alpha(y)\cap F_-\cap F_+)\leqq\dim\Phi_{I,E}-2=5v-4\,.$$

The only part, remaining from the proof of 5.47, is dealing with points $y\in\Phi_{I,E}$ for which (iii) of Sublemma 5.53 is false. Unfortunately, these points can fill in a codimension one (if not empty) submanifold H of $\Phi_{I,E}$. For such points we cannot say that the projected manifold $\pi_{2,3,4}(\Phi_{I,E}) \subset M_3$ is, at least locally, a codimension one, smooth submanifold of M_3. Therefore, in this case we are not able to apply the strong version of the fundamental theorem to the sub-billiard flow $S^t_{2,3,4}$ $(t<0)$. Anyhow, the fundamental theorem applies to the subsystems $S^t_{1,2}$ and $S^t_{3,4}$ $(t>0)$. Let us consider a small, open ball $V(y)$ around y in $\Phi_{I,E}$ which is split into two open connected parts by the manifold $H \cap V(y)$. By the preceding argument – proving 5.47 in some neighborhood of any point $z \in \Phi_{I,E} \setminus H$ – the set $F_- \cap F_+ \cap (V(y) \setminus H)$ can have dimension at most $5\nu - 4$. Thus our only task is to show that $\dim(F_- \cap F_+ \cap V(y) \cap H) \leqq 5\nu - 4$. Since $\dim(V(y) \cap H) = 5\nu - 3$, by property 2.10, it is enough to show that the set $F_- \cap F_+ \cap V(y) \cap H$ has an *empty interior* in $V(y) \cap H$. We can actually prove that even the set $F_+ \cap V(y) \cap H$ has an empty interior in $V(y) \cap H$. Assume the contrary:

$$U := \mathrm{int}_{V(y) \cap H}(F_+ \cap V(y) \cap H) \neq \emptyset .$$

Here we observe that all the existing leaves $\gamma^s_{1,2}(z) \subset \Phi_{I,E}$ $(z \in U)$ are *transversal* to U whenever $v_1(z)$ is not parallel with $I = v_1(z) + v_2(z)$, because during the perturbations of type $\gamma^s_{1,2}$ the velocity $v_1(z') - \frac{1}{2}I$ varies on a sphere centered at the origin of $\mathbb{R}^\nu$. The event $v_1(z) \| v_2(z)$, however, occurs on a set of codimension $\nu - 1$ $(\geqq 2)$, so we can assume that $v_1(z)$ is not parallel with I in $V(y)$. By the strong version of the fundamental theorem for the flow $S^t_{1,2}$ $(t>0)$ (see Sect. 3), for μ_H-almost every point $z \in U$ the local stable leaf $\gamma^s_{1,2}(z)$ exists with positive inner radius. By the transversality mentioned above, the set

$$\bigcup_{z \in U} \gamma^s_{1,2}(z) \subset \Phi_{I,E}$$

has positive μ_Φ-measure and this is impossible, because by Proposition 5.60, this set is a subset of $F'_+ \cap \Phi_{I,E}$ that has measure zero in $\Phi_{I,E}$ by Lemma 5.50. Hence the Main Lemma 5.47. The discussion of case (a) $(P_1 = (\{1\}, \{2,3,4\}). P_2 = (\{1,2\}, \{3,4\}))$ is now complete.

Let us continue the study of cases (a)–(e). It turns out that the discussion of cases (b)–(e) is much easier than that of (a).

Case (b). $P_1 = (\{1\}, \{2,3,4\})$, $P_2 = (\{2\}, \{1,3,4\})$. It follows easily from the assumed sufficiency of the sub-billiard trajectories $S^{(-\infty,0)}_{2,3,4}x_0$ and $S^{(0,\infty)}_{1,3,4}x_0$ that in case (b) the neutral linear spaces $W_0(S^{(-\infty,0)}x_0)$ and $W_0(S^{(0,\infty)}x_0)$ (their definition see in Sect. 4) are as follows:

$$W_0(S^{(-\infty,0)}x_0) = \left\{ \lambda \begin{pmatrix} v_1(x_0) \\ v_2(x_0) \\ v_3(x_0) \\ v_4(x_0) \end{pmatrix} + \begin{pmatrix} -3x \\ x \\ x \\ x \end{pmatrix} : \lambda \in \mathbb{R}, x \in \mathbb{R}^\nu \right\}, \tag{5.61}$$

$$W_0(S^{(0,\infty)}x_0) = \left\{ \mu \begin{pmatrix} v_1(x_0) \\ v_2(x_0) \\ v_3(x_0) \\ v_4(x_0) \end{pmatrix} + \begin{pmatrix} y \\ -3y \\ y \\ y \end{pmatrix} : \lambda \in \mathbb{R}, y \in \mathbb{R}^\nu \right\} \tag{5.62}$$

[compare with (5.15)]. Taking a closer look at the relative displacement $\Delta q_4 - \Delta q_3$ of any common element $(\Delta q_1, \Delta q_2, \Delta q_3, \Delta q_4)^*$ of the spaces $W_0(S^{(-\infty,0)}x_0)$ and $W_0(S^{(0,\infty)}x_0)$, we see that $\lambda = \mu$ and, therefore, $x = y$, $x = -3y$. Thus the common element must be of the form

$$\lambda \begin{pmatrix} v_1(x_0) \\ v_2(x_0) \\ v_3(x_0) \\ v_4(x_0) \end{pmatrix},$$

that is, the trajectory $S^{(-\infty,\infty)}x_0$ is sufficient. The conclusion is that if

$$P_1 = (\{1\}, \{2,3,4\}) \quad \text{and} \quad P_2 = (\{2\}, \{1,3,4\}),$$

then the set M_{P_1,P_2} is *residual*, because the only possibility to have points $x_0 \in M_{P_1,P_2}$ (non-sufficiency is required in M_{P_1,P_2}) is that one of the sub-billiard semi-trajectories $S_{2,3,4}^{(-\infty,0)}x_0$ and $S_{1,3,4}^{(0,\infty)}x_0$ is not sufficient.

Case (c). $P_1 = (\{1,2\}, \{3,4\})$, $P_2 = (\{1,3\}, \{2,4\})$. The argument is, in many respects, similar to that in case (b). From the assumed sufficiency of the four sub-billiard semitrajectories, an easy computation shows that the neutral spaces $W_0(S^{(-\infty,0)}x_0)$ and $W_0(S^{(0,\infty)}x_0)$ are as follows:

$$W_0(S^{(-\infty,0)}x_0) = \Big\{(\Delta q_1, \Delta q_2, \Delta q_3, \Delta q_4)^* \in \mathbb{R}^{4\nu}: \sum_{j=1}^{4} \Delta q_j = 0,$$
$$\Delta q_2 - \Delta q_1 = \alpha(v_2(x_0) - v_1(x_0)),\ \Delta q_4 - \Delta q_3 = \beta(v_4(x_0) - v_3(x_0))$$
$$\text{for some } \alpha, \beta, \in \mathbb{R}\Big\},$$

$$W_0(S^{(0,\infty)}x_0) = \Big\{(\Delta q_1, \Delta q_2, \Delta q_3, \Delta q_4)^* \in \mathbb{R}^{4\nu}: \sum_{j=1}^{4} \Delta q_j = 0,$$
$$\Delta q_3 - \Delta q_1 = \gamma(v_3(x_0) - v_1(x_0)),\ \Delta q_4 - \Delta q_2 = \delta(v_4(x_0) - v_2(x_0))$$
$$\text{for some } \gamma, \delta \in \mathbb{R}\Big\}.$$

A simple calculation shows that for any common element

$$(\Delta q_j)_{j=1}^4 \in W_0(S^{(-\infty,0)}x_0) \cap W_0(S^{(0,\infty)}x_0)$$

the equation

$$(\alpha-\gamma)v_1(x_0) + (\delta-\alpha)v_2(x_0) + (\gamma-\beta)v_3(x_0) + (\beta-\delta)v_4(x_0) = 0 \tag{5.63}$$

holds. If the only linear dependence among the velocities $v_j(x_0)$ $(j=1,2,3,4)$ is

$$\sum_{j=1}^{4} v_j(x_0) = 0,$$

then we have $\alpha-\gamma = \delta-\alpha = \gamma-\beta = \beta-\delta$, therefore, $\alpha=\beta=\gamma=\delta$ and the trajectory $S^{(-\infty,\infty)}x_0$ is sufficient, which is not allowed for points $x_0 \in M_{P_1,P_2}$. Thus $M_{P_1,P_2} \subset D \cup N$, where N is the set of points $x_0 \in M_{P_1,P_2}$ for which one of the four sub-billiard semitrajectories is *not sufficient*, while D is the set of points $y \in M$ for which the span of the velocities $v_j(y)$ is not 3-dimensional. (D stands for "dependence," while N for "not sufficient.") It is easy to see that D is a compact cell-complex in M and $\dim D \leqq 5\nu + 1 = \dim M - (\nu - 2)$. The only trouble can be caused when $\nu = 3$

and having some cells C in D with codimension one in M. However, we can show that the closed set $F_- \cap C$ has an empty interior in the cell C. [The same is true, of course, for $F_+ \cap C$. For the definition $F_\pm$ see (5.4)–(5.5).] This will be enough, because then the set $F_- \cap C$ is residual in M [cf. Theorem 1.8.10 of E (1978), showing that $\dim(F_- \cap C) \leqq \dim C - 1 = \dim - 2$] and it contains the set $M_{P_1,P_2} \cap C$. Assume, on the contrary, that $\mathrm{int}_C(F_- \cap C) \neq \emptyset$. It is obvious that for every point $y \in C$ the span of any pair of the four velocities $v_j(y)$ $(j=1,\dots,4)$ is the same 2-dimensional subspace of $\mathbb{R}^3$ depending, of course, on y. This yields that (i) for such points y the invariant manifold $\gamma^u_{3,4}(y)$ is transversal to C and (ii) for almost every point $y \in C$ the manifold $\gamma^u_{3,4}(y)$ exists. These two properties are contradictory to $\mathrm{int}_C(F_- \cap C) \neq \emptyset$, because the set

$$\bigcup_{y \in F_- \cap C} [\gamma^u_{3,4}(y) \cap U(x_0)]$$

must have positive measure and, on the other hand, it should be a subset of the zero set F'_-, provided $U(x_0)$ is small enough. Thus M_{P_1,P_2} is residual in case (c), too.

Cases (d) *and* (e). We do not intend to investigate both cases in detail because, on one hand, their discussions are quite similar and, on the other hand, the method has been well developed in K-S-Sz (1989) (the proof of Sublemma 2). Therefore, we shall only sketch the proof of the codimension two property for the set $F_- \cap F_+$ when $P_1 = P_2 = (\{1\}, \{2,3,4\})$. We want to apply property 4 in 4.1 of K-S-Sz (1989) (the possibility of "integrating up" codimension two closed sets) in order to obtain the required lower bound for the codimension of $F_- \cap F_+$. The auxilary foliation of the neighborhood $U(x_0)$ corresponds to all the data of the first particle: q_1 and v_1. For each submanifold $\Phi_{q_1,v_1} = \{y \in U(x_0): q_1(y) = q_1 \text{ and } v_1(y) = v_1\}$ the intersection $F_- \cap F_+ \cap \Phi_{q_1,v_1}$ is a codimension two closed set in Φ_{q_1,v_1} by the lemma on avoiding of balls [Lemma 3 in K-S-Sz (1989)]. By virtue of property 4 of the same paper we have $\dim(F_- \cap F_+) \leqq \dim U(x_0) - 2 = 6v - 3$. Hence the proof of residuality of M_{P_1,P_2} in all cases (a)–(e).

6. Singular Trajectories

The purpose of this supplementary section is the proof of the residuality of the set of all singular, non-sufficient phase points. We remind the reader that the residuality (basically, the codimension-two property) of the set of all *singular*, non-sufficient phase points was not covered by Sects. 4 and 5. The main result of this section – implying the residuality of the set of all singular, non-sufficient phase points and even the Chernov-Sinai Ansatz – is the following theorem:

Theorem 6.1. *For every cell C of maximum dimension $(6v-3)$ in $\mathscr{SR}^+$ the set $C_{\mathrm{ed}} \subset C$ of all "eventually decomposing" phase points can be covered by a countable family of closed, zero subsets (with respect to the surface measure in C) of C.*

A point $x \in \mathscr{SR}^+$ is said to be *eventually decomposing* if
(a) the semi-trajectory $S^{(0,\infty)}x$ is not singular and
(b) there is a number $t_0 > 0$ and a two-class partition P of the particles such that the trajectory segment $S^{(t_0,\infty)}x$ is partitioned by P (see Sect. 2).
This theorem, together with
(i) Main Theorem 4.3,
(ii) the result stating that the set of points with at least two singularities has two codimensions [Lemma 4.1 in K-S-Sz (1990)],

(iii) property 2.10,
(iv) the result on the uniform decomposability of any decomposing trajectory segment (the consequence of König's Lemma 5.2)
yields that
(A) for almost every point $x \in \mathcal{S}\mathcal{R}^+$ the positive trajectory $S^{(0,\infty)}x$ is sufficient [Chernov-Sinai Ansatz, see Condition 3.1 in K-S-Sz (1990)],
(B) the set of all points $S^t x$ $(x \in \mathcal{S}\mathcal{R}^+, t>0)$ for which the segment $S^{(0,\infty)}x$ is *not sufficient* is residual.

Proof of 6.1. We can, of course, assume that the number $t_0 > 0$ and the partition P are fixed; the corresponding subset of C_{ed} is denoted by $C_{\text{ed}}(t_0, P)$:

$$C_{\text{ed}}(t_0, P) := \{x \in C_{\text{ed}} \colon S^{(t_0,\infty)}x \text{ is partitioned by } P\}. \tag{6.2}$$

We claim that the closure $Cl_C(C_{\text{ed}}(t_0, P))$ of the above set has measure zero in C. Assume the contrary, that is

$$\mu_C(Cl_C(C_{\text{ed}}(t_0, P))) > 0. \tag{6.3}$$

Denote by C_{ds} the collection of all points $x \in C$ for which there is at least one singularity on the positive trajectory $S^{(0,\infty)}x$. (Points with double singularity.) It is now clear that the set $Cl_C(C_{\text{ed}}(t_0, P)) \setminus C_{\text{ds}}$ is exactly $C_{\text{ed}}(t_0, P)$ and the sets $Cl_C(C_{\text{ed}}(t_0, P))$ and $C_{\text{ed}}(t_0, P)$ have the same μ_C-measure, which is, according to the indirect assumption (6.3), positive. Consider now a Lebesgue point (point with density one) $x_0 \in C_{\text{ed}}(t_0, P)$ of the set $C_{\text{ed}}(t_0, P)$. Choose a number $t_* > t_0$ for which $S^{t_*}x_0 \notin \partial M$, that is t_* is not a moment of collision. It is clear that the point $y_0 = S^{t_*}x_0$ is a Lebesgue point of the set $\bigcup_{t>0} S^t(C_{\text{ed}}(t_0, P))$ with respect to the surface measure of the codimension – one smooth submanifold $N = \bigcup_{t>0} S^t(C)$. We fix a small, open ball $U(y_0) = B(y_0, \delta_0) \subset M \setminus \partial M$ with radius $\delta_0 < t_* - t_0$ centered at y_0. In the course of the present proof we shall see how small the radius δ_0 should be. Actually, we first require that δ_0 be small enough depending on the number ε_0 [cf. the note after (5.7)], so that the implications of Proposition 5.60 (regarding the pseudo-stable manifolds) hold for $U(y_0) = B(y_0, \delta_0)$ instead of $U(x_0)$. For the sake of brevity, from the time being we assume that $P = (\{1,2\}, \{3,4\})$; the discussion of the case $P = (\{1\}, \{2,3,4\})$ is similar and we leave it to the reader.

The definitions of the closed sets $F_+ \subset U(y_0)$ and $F'_+ \subset U(y_0)$ are formally the same as (5.5) and (5.7); the only difference is that the base neighborhood $U(x_0)$ is now replaced by $U(y_0) = B(y_0, \delta_0)$. The remark after (5.7) applies again to the definitions of F_+ and F'_+, ensuring the closedness of these sets. It is, of course, again true that

$$F_+ \subset F'_+ \quad \text{and} \quad \mu(F'_+) = 0. \tag{6.4}$$

The definitions of the pseudo-stable invariant manifolds $\gamma^s_0(\cdot)$, $\gamma^s_{1,2}(\cdot)$, and $\gamma^s_{3,4}(\cdot)$ are again formally the same as (5.11)–(5.13) with $U(x_0)$ replaced by $U(y_0)$. By the smallness of δ_0, the first three implications of 5.60 remain true. For an arbitrary point $y \in B(y_0, \delta_0)$ we can try to construct the generate $\gamma^s_g(y)$ of all the pseudo-stable leaves in the following way:

$$\gamma^s_g(y) := \bigcup_{z \in \gamma^{sp}_0(y)} \gamma^s_e(z) = \bigcup_{z \in \gamma^s_e(y)} \gamma^{sp}_0(z), \tag{6.5}$$

where the exponentially contracting leaves $\gamma^s_e(\cdot)$ are defined by (5.14) and

$$\gamma^{sp}_0(y) := \{z \in \gamma^s_0(y) \colon Q(z) - Q(y) \perp V(y)\}. \tag{6.6}$$

In terms of the parameters w, λ, μ of (5.11), the points $z \in \gamma_0^{sp}(y)$ can be described by the equation

$$2w \cdot (v_1 + v_2) + \frac{\lambda}{2}(v_1 - v_2)^2 + \frac{\mu}{2}(v_3 - v_4)^2 = 0. \qquad (6.7)$$

It is important to note that the coefficients $v_1 + v_2$, $(v_1 - v_2)^2$, and $(v_3 - v_4)^2$ in (6.7) do not vary while making perturbations of the form $\gamma_0^s(\cdot)$ or $\gamma_e^s(\cdot)$. The meaning of the parameters is as follows:

w is the displacement of the center of mass of the subsystem $\{1,2\}$,

λ is the time shift (advance) of the subsystem $\{1,2\}$ while μ is the time shift of the subsystem $\{3,4\}$.

The observations just mentioned imply the commutativity

$$\bigcup_{z \in \gamma_0^{sp}(y)} \gamma_e^s(z) = \bigcup_{z \in \gamma_e^s(y)} \gamma_0^{sp}(z)$$

appearing in (6.5). This commutativity together with the transversality of $\gamma_0^s(\cdot)$ and $\gamma_e^s(\cdot)$ yield that for almost all points $y \in B(y_0, \delta_0)$ the set $\gamma_g^s(y)$ is an *orthogonal manifold (of dimension $3\nu - 1$) containing y as an interior point.* Moreover, the orthogonal manifold $\gamma_g^s(y)$ is *concave*, which follows simply from the concavity of the manifolds $\gamma_0^s(\cdot)$ and $\gamma_e^s(\cdot)$. Since the concave orthogonal manifolds remain concave under the action of the flow S^t $(t<0)$, by Sublemma 4.2 of K-S-Sz (1990) we get that the manifolds $\gamma_g^s(y)$ are transversal to the codimension one manifold $CC_{y_0}(B(y_0, \delta_0) \cap N)$ for all points $y \in CC_{y_0}(B(y_0, \delta_0) \cap N)$. Here, again, $CC_{y_0}(A)$ denotes the arcwise connected component of the set A that contains the point y_0. We note that the transversality just mentioned is meaningful even in the case when $\gamma_g^s(y)$ does not exist, since the tangent space of this "non-existing" leaf $\gamma_g^s(y)$ can be constructed by the continued function method [see, for instance, Ch (1982), formulas (1) and (3)] and the transversality is the consequence of the continuity of the value of this continued fraction. We arrive at the conclusion that, by choosing the radius δ_0 small enough, at least one of the following statements is true:

1. for all points $y \in CC_{y_0}(B(y_0, \delta_0) \cap N)$ the manifold $\gamma_0^{sp}(y)$ is transversal to N;
2. for all points $y \in CC_{y_0}(B(y_0, \delta_0) \cap N)$ the manifold $\gamma_{1,2}^s(y)$ (if exists) is transversal to N;
3. for all points $y \in CC_{y_0}(B(y_0, \delta_0) \cap N)$ the manifold $\gamma_{3,4}^s(y)$ (if exists) is transversal to N.

We shall finish the proof assuming (2); the treatment of (3) is absolutely similar and the first case is even much simpler, since, in that case, there is no problem with the existence of the leaves $\gamma_0^{sp}(\cdot)$.

In virtue of the transversal fundamental theorem for the subsystem $\{1,2\}$, for almost every point $y \in CC_{y_0}(B(y_0, \delta_0) \cap N)$ (with respect to the Riemannian volume ν_N in N) the invariant manifold $\gamma_{1,2}^s(y)$ exists, that is, it contains y as an interior point. We chose y_0 as a Lebesgue point of the set $\bigcup_{t>0} S^t(C_{\mathrm{ed}}(t_0, P))$, therefore, the set

$$\left(\bigcup_{t>0} S^t(C_{\mathrm{ed}}(t_0, P))\right) \cap CC_{y_0}(B(y_0, \delta_0) \cap N) \qquad (6.8)$$

has positive ν_N-measure. The set in (6.8) is clearly a subset of

$$F_+ \cap CC_{y_0}(B(y_0, \delta_0) \cap N) := A.$$

We consider the following union of invariant manifolds:

$$B := \bigcup_{y \in A} \gamma^s_{1,2}(y).$$

The ν_N-positivity of A and the existence and transversality of the leaves $\gamma^s_{1,2}(y)$ (stated above) imply the μ-positivity of $B : \mu(B) > 0$. On the other hand, by the second implication of 5.60, the set B must be a subset of F'_+, but F'_+ is a zero set by (6.4). This contradiction finishes the proof of 6.1.

7. Proof of Theorem

The proof is, of course, based on the Fundamental Theorem (Theorem 3.4 and its Corollary 3.5) and therefore we start it by checking the Conditions 3.1–3.3. Condition 3.1 was verified in Sect. 6. Condition 3.2, as a matter of fact, trivially holds for semi-dispersing billiards with solely cylindric scatterers. Condition 3.3 certainly follows from Lemma 2.11.

Proof of Main Theorem. Now Corollary 3.5 implies that every sufficient point $x_0 \in M^*$ has an open neighborhood belonging to one ergodic component of the system. Further, by Lemma 2.11 and Remark 2.7, we see that $M \setminus M^*$ is a countable union of codimension-two submanifolds of M, a residual set. Thus we can focus on showing that N, the subset (in M^*) of non-sufficient points, is a residual set.

By Lemma 2.8, residuality can be verified locally, in neighborhoods of points of N. If $x \in N \cap M^1$, then the necessary statement is just corollary (B) of Theorem 6.1. If $x \in N \cap M^0$, then there are two possibilities: x is rich or x is non-rich. By Theorem 5.1, the set of all non-rich, non-sufficient points of M^0 is a residual set. Finally, by Theorem 4.3, in a sufficiently small neighborhood U of any rich point of $N \cap M^0$, $N \cap U$ is contained in a CW-complex with codimension at least two. Hence the Main Theorem of this paper.

Appendix. Three-Class Partitions of Particles on Infinite Time Intervals

The set-up of this section is more general than that throughout the whole paper: All the billiard flows $\{M, S^t, \mu)$ of N ($\geqq 3$) particles on the torus can be considered, provided that for all $k \leqq N-2$ the k-billiard on the torus is ergodic. It is known that this is the case if $N \leqq 5$. The result of the appendix is as follows:

Theorem. *Consider the flow $\{M, S^t, \mu\}$ of N ($\geqq 3$) billiard balls on the torus. Assume that for all $k \leqq N-2$ the k-billiard on the torus is ergodic (and, therefore, it is a K-system). Let $P = \{C_1, C_2, C_3\}$ be a partition of the N particles into three-non-empty classes C_1, C_2, and C_3. Then we claim that the codimension of the closed set*

$$F = \{x \in M: \textit{ in the trajectory } S^{(0,\infty)}x \textit{ there is no proper collision between different classes of } P\}$$

is at least two.

Proof. We shall distinguish between two cases, namely, when the number of classes C_i with $|C_i| \geqq 2$ ($|\cdot|$ denotes the cardinality of a set) is greater than one, or, when it is at most one. (The first case can not occur, of course, if $N \leqq 4$.)

Case I. $|\{i:|C_i|\geqq 2\}|\geqq 2$. Assume that $|C_1|\geqq 2$ and $|C_2|\geqq 2$. We foliate the phase space M according to all the initial data q_j, v_j $(j\in C_3)$, and all the outer parameters

$$\sum_{j\in C_1} q_j, \quad \sum_{j\in C_1} v_j, \quad \frac{1}{2}\sum_{j\in C_1} v_j^2,$$

$$\sum_{j\in C_2} q_j, \quad \sum_{j\in C_2} v_j, \quad \frac{1}{2}\sum_{j\in C_2} v_j^2$$

of the subsystems $S^t_{C_1}$ and $S^t_{C_2}$. For fixed such values, the corresponding submanifold Φ of M can be identified with the direct product $M_{C_1}\times M_{C_2}$ in a natural way where M_{C_i} denotes the phase space of the subsystem $S^t_{C_i}$. By the weak lemma on avoiding of balls ([Lemma 2.16 of K-S-Sz (1991)] and by the assumed K-property of the k-billiards for $k\leqq N-2$, the projection $\pi_{C_i}(F\cap\Phi)$ is contained in a suitable *closed zero subset* F_i of M_{C_i}. (The set F_i is defined via avoiding the balls from C_3.) Thus $F\cap\Phi\subset F_1\times F_2$ and

$$\begin{aligned}\dim(F\cap\Phi)&\leqq\dim(F_1\times F_2)\leqq\dim F_1+\dim F_2\\ &\leqq\dim M_{C_1}-1+\dim M_{C_2}-1=\dim\Phi-2\end{aligned}$$

by Theorems 1.5.16 and 1.8.10 of E (1978). Therefore, by property 4 of Sect. 4.1 in K-S-Sz (1989), the codimension of the closed set F is at least two.

Case II. $|\{i:|C_i|\geqq 2\}|\leqq 1$. Assume that $|C_2|=1$ and $|C_3|=1$. If, moreover, $|C_1|=1$ too, then a simple geometric argument shows that $\operatorname{codim}(F)=3$. (The condition that a fixed pair $\{i,j\}$ does not collide means that the relative velocity v_i-v_j belongs to the union of finitely many hyperplanes and these conditions, corresponding to different pairs of particles, are of course, independent.) Thus we suppose that $|C_1|\geqq 2$. We fix the values of q_j and v_j of the two particles from $C_2\cup C_3$. The corresponding submanifold of M, which is naturally isomorphic to the phase space M_{C_1} of the subsystem $S^t_{C_1}$, is again denoted by Φ. In order to have a non-empty intersection $F\cap\Phi$ it is necessary that the relative velocity v_i-v_j $(i\in C_2, j\in C_3)$ be contained in a finite collection of hyperplanes of $\mathbb{R}^\nu$. This is "one equation" that makes the codimension of F at least one. By the weak lemma on avoiding of ballls [Lemma 2.16 of K-S-Sz (1991)], applied to the subsystem $S^t_{C_1}$, we see that the measure of the intersection $F\cap\Phi$ in Φ is zero, thus $F\cap\Phi$ has an empty interior in Φ. By virtue of Theorem 1.8.10 of E (1978), this implies that $\operatorname{codim}(F)\geqq 2$. Hence the theorem of the Appendix.

Acknowledgements. D. Szász expresses his sincere gratitude to Elliott Lieb, John Mather, Tom Spencer, and Arthur Wightman for their interest toward this work and their kind hospitality at the Department of Mathematics of Princeton University and at the School of Mathematics of the Institute for Advanced Study in the Spring of 1989 and during the academic year 1990/91 when part of this research was done.

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Communicated by T. Spencer