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On Multivortices in the Electroweak Theory I: Existence of Periodic Solutions

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Abstract. In this paper we consider the bosonic sector of the electroweak theory. It has been shown in the work of Ambjorn and Olesen that when the Higgs mass equals to the mass of the Z boson, the model in two dimensions subject to the 't Hooft periodic boundary condition may be reduced to a Bogomol'nyi system and that the solutions of the system are vortices in a "dual superconductor". We shall prove using a constrained variational reformulation of the problem the existence of such vortices. Our conditions for the existence of solutions are necessary and sufficient when the vortex number $N = 1, 2$.

1. Introduction

Instantons, monopoles, and vortices form a rich spectrum of topologically elegant solutions of gauge field theories. Vortices arise in two-dimensional models in which the gauge symmetry is spontaneously broken via Higgs bosons. Such solutions represent string-like field configurations in higher dimensions and, in the context of the abelian Higgs theory, were first discovered in Abrikosov's poineering study [1] of the magnetic properties of superconducting materials. In recent years, due to their interesting roles in grand unified theories, especially in cosmology [10], nonabelian vortices have attracted a considerable amount of attention. It is well-known that one of the most important and successful nonabelian gauge field theories is the electroweak theory of Glashow, Salam and Weinberg, where the gauge group is $SU(2) \times U(1)$. In a series of papers, Ambjorn and Olesen [3–5] proposed that a class of periodic vortex-like solutions similar to those of Abrikosov occur in this electroweak theory (see also Skalozub [11, 12]). They showed that,

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when the coupling constants satisfy a critical condition, energetically stable solutions can be found from a Bogomol'nyi system. These solutions give rise to a distribution of vortex-lines and the total energy is proportional to the quantized flux or the vortex number. Moreover, the interesting structure of the equations allows one to derive a magnetic anti-screening phenomenon relevant to the quark confinement problem. Since the fermionic sector of the model is not responsible for the spontaneously broken symmetry, it suffices to consider the bosonic sector only. The periodicity may be realized by the 't Hooft boundary condition [13]. Ambjorn and Olesen used a perturbation analysis and numerical experiments to support the existence of such nonabelian vortices but they were unable to obtain a rigorous proof [4]. The major difficulty is that the Bogomol'nyi equations now take a more delicate form than in the classical abelian case [8, 14] due to the above mentioned anti-screening of the magnetic field. (For the abelian case, the structure of the Bogomol'nyi equations allows a complete resolution of the multivortex problem over a periodic cell realizing Abrikosov's solutions [14].) Indeed, such a significant difference has already been exhibited in an earlier study of Ambjorn and Olesen on the periodic vortices of a simplified $SO(3)$ theory [2] in which the W-bosons acquire masses through a Higgs mechanism but the Higgs fields are neglected from the Lagrangian. Here a system of the Bogomol'nyi type equations also occur but the reduced elliptic equation takes a similar form as those in the prescribed Gaussian curvature problem for compact surfaces with a positive Euler characteristic [9]. Hence in this situation one might only expect to find certain sufficient conditions for the existence of multivortex solutions [15].

In this paper we will study the existence of multivortex solutions in the full electroweak theory proposed by Ambjorn and Olesen. Our main strategy is to use a crucial change of field variables to transform the system into a "lower diagonal" form. Such an approach allows a multi-constrained variational solution of the problem if the given data in the problem satisfy certain restrictions. Under these restrictions, existence results will be established. When the vortex number $N = 1, 2$, our conditions for existence are both necessary and sufficient. Whether or not these conditions for the case $N \geqq 3$ may further be improved remains open.

The organization of the paper is as follows. In Sect. 2 we discuss the electroweak theory in the standard unitary gauge with a residual $U(1)$ symmetry and set up most of our preliminary notation. In Sect. 3 we show that a convenient 't Hooft periodic boundary condition (for an arbitrary lattice structure) in the electroweak theory is equivalent to that in the corresponding $U(1)$ model. Section 4 gives a characterization of the quantized flux by the vortex number, parallel to the situation in the abelian Higgs model [14]. In Sect. 5 we prove our main theorem (Theorem 5.6) for the existence of multivortices in the electroweak theory. Section 6 contains some concluding remarks.

2. The Electroweak Theory in the Unitary Gauge

We shall use $\{\tau_a\}_{a=1,2,3}$ to denote the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $t_a = \tau_a/2$, $a = 1, 2, 3$ is a set of generators of $SU(2)$ satisfying the commutation relation

$$[t_a, t_b] = i\varepsilon_{abc}t_c.$$

Let ϕ be a complex doublet. The gauge group $SU(2) \times U(1)$ transforms ϕ as follows:

$$\phi \mapsto \exp(-i\omega_a t_a)\phi, \quad \omega_a \in \mathbb{R}, \quad a = 1, 2, 3,$$
$$\phi \mapsto \exp(-i\xi t_0)\phi, \quad \xi \in \mathbb{R},$$

where

$$t_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a generator of $U(1)$ in the above matrix representation.

In the $(1+3)$-dimensional Minkowski space with the signature $(-+++)$, the $SU(2)$ and $U(1)$ gauge fields are denoted respectively by $A_\mu = A^a_\mu t_a$ (or $\mathbf{A}_\mu = (A^a_\mu)$ as an isovector) and B_μ. Both A^a_μ and B_μ are real 4-vectors. The field strength tensors and the $SU(2) \times U(1)$ gauge-covariant derivative are defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu],$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\nu,$$
$$D_\mu \phi = \partial_\mu \phi + igA^a_\mu t_a \phi + ig' B_\mu t_0 \phi,$$

where $g, g' > 0$ are coupling constants.

The Lagrangian density of the electroweak theory in the bosonic sector is

$$\mathscr{L} = -\tfrac{1}{4}(F^{\mu\nu} \cdot F_{\mu\nu} + G^{\mu\nu}G_{\mu\nu}) - (D^\mu \phi)^\dagger \cdot (D_\mu \phi) - \lambda(\varphi_0^2 - \phi^\dagger \phi)^2, \tag{2.1}$$

where and in what follows, "$\dagger$" always denotes the Hermitian conjugate, and λ, φ_0 are positive parameters.

The new vector fields P_μ and Z_μ are a rotation of the pair A^3_μ and B_μ:

$$P_\mu = B_\mu \cos\theta + A^3_\mu \sin\theta,$$
$$Z_\mu = -B_\mu \sin\theta + A^3_\mu \cos\theta.$$

In terms of P_μ, Z_μ, D_μ is written

$$D_\mu = \partial_\mu + ig(A^1_\mu t_1 + A^2_\mu t_2) + iP_\mu(g\sin\theta t_3 + g'\cos\theta t_0) + iZ_\mu(g\cos\theta t_3 - g'\sin\theta t_0).$$

Requiring that the coefficient of P_μ be the charge operator $eQ = e(t_3 + t_0)$, where $-e$ is the charge of the electron, we obtain the relations

$$e = g\sin\theta = g'\cos\theta,$$
$$e = \frac{gg'}{(g^2 + g'^2)^{1/2}},$$
$$\cos\theta = \frac{g}{(g^2 + g'^2)^{1/2}}. \tag{2.2}$$

Such a θ is called the Weinberg (mixing) angle. In the sequel, we will always assume that θ is fixed this way. The D_μ takes the form

$$D_\mu = \partial_\mu + ig(A^1_\mu t_1 + A^2_\mu t_2) + iP_\mu eQ + iZ_\mu eQ',$$

where $Q' = \cot\theta t_3 - \tan\theta t_0$ is the neutral charge operator.

From (2.2), when we go to the unitary gauge in which

$$\phi = \begin{pmatrix} 0 \\ \varphi \end{pmatrix},$$

where φ is a real scalar field, there holds

$$D_\mu\phi = \begin{pmatrix} \frac{i}{2}g(A^1_\mu - iA^2_\mu)\varphi \\ \partial_\mu\varphi - \frac{ig}{2\cos\theta}Z_\mu\varphi \end{pmatrix}.$$

Define now the complex vector field

$$W_\mu = \frac{1}{\sqrt{2}}(A^1_\mu + iA^2_\mu)$$

and $\mathscr{D}_\mu = \partial_\mu - igA^3_\mu$. With the notation $P_{\mu\nu} = \partial_\mu P_\nu - \partial_\nu P_\mu$, $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, the Lagrangian (2.1) takes the form

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{2}(\mathscr{D}^\mu W^\nu - \mathscr{D}^\nu W^\mu)^\dagger(\mathscr{D}_\mu W_\nu - \mathscr{D}_\nu W_\mu) - \tfrac{1}{4}Z^{\mu\nu}Z_{\mu\nu} - \tfrac{1}{4}P^{\mu\nu}P_{\mu\nu} \\ &- \tfrac{1}{2}g^2([W^\mu W^\dagger_\mu]^2 - [W^\mu W_\mu][W^\nu W_\nu]^\dagger) - ig(Z^{\mu\nu}\cos\theta + P^{\mu\nu}\sin\theta)W^\dagger_\mu W_\nu \\ &- \tfrac{1}{2}g^2\varphi^2 W^\mu W^\dagger_\mu - \partial^\mu\varphi\partial_\mu\varphi - \frac{1}{4\cos^2\theta}g^2\varphi^2 Z^\mu Z_\mu - \lambda(\varphi_0^2 - \varphi^2)^2. \qquad (2.3)\end{aligned}$$

Thus the model is reformulated in the unitary gauge. The W and Z fields represent two massive vector bosons which eliminate the unphysical massless goldstone particle in the original setting. These fields mediate short-range (weak) interactions. The remaining massless gauge (photon) field P arising from the residual $U(1)$ symmetry mediates long-range (electromagnetic) interactions.

As in [4], we assume that the magnetic excitation is in the third direction. Thus, we arrive at the vortex ansatz

$$\begin{aligned}&A^a_0 = A^a_3 = B_0 = B_3 = 0, \\ &A^a_j = A^a_j(x_1, x_2), \quad B_j = B_j(x_1, x_2), \quad j = 1, 2, \\ &\phi = \phi(x_1, x_2). \qquad (2.4)\end{aligned}$$

As a consequence, if the corresponding W_1 and W_2 are represented by a complex scalar field W according to $W_1 = W$, $W_2 = iW$ (this implies the relation $A^1_2 = -A^2_1$, $A^2_2 = A^1_1$), the energy density associated with (2.3) takes the form

$$\begin{aligned}\mathscr{E} = &|\mathscr{D}_1 W + i\mathscr{D}_2 W|^2 + \tfrac{1}{2}P^2_{12} + \tfrac{1}{2}Z^2_{12} - 2g(Z_{12}\cos\theta + P_{12}\sin\theta)|W|^2 \\ &+ 2g^2|W|^4 + (\partial_j\varphi)^2 + \frac{1}{4\cos^2\theta}g^2\varphi^2 Z^2_j + g^2\varphi^2|W|^2 + \lambda(\varphi_0^2 - \varphi^2)^2. \qquad (2.5)\end{aligned}$$

The residual $U(1)$ symmetry of the model may clearly be seen from the invariance of (2.5) under the gauge transformation

$$W \mapsto \exp(i\zeta)W, \quad P_j \mapsto P_j + \frac{1}{e}\partial_j \zeta, \quad Z_j \mapsto Z_j, \quad \varphi \mapsto \varphi, \tag{2.6}$$

due to (2.2).

3. Equivalence of the 't Hooft Periodic Boundary Conditions

In this section, we discuss the 't Hooft periodic boundary conditions. Since we are interested in vortex-like solutions, only the two-dimensional case will be examined. Namely, we assume that the field configurations are in the form (2.4).

Consider a fundamental domain Ω of a lattice in $\mathbb{R}^2$ generated by independent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\Omega = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,|\, \mathbf{x} = s_1\mathbf{a}_1 + s_2\mathbf{a}_2, 0 < s_1, s_2 < 1\}.$$

Define

$$\Gamma_{\mathbf{a}_k} = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathbf{x} = s_k\mathbf{a}_k, 0 < s_k < 1\}, \quad k = 1, 2.$$

Then $\partial\Omega = \Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2} \cup \{\mathbf{a}_1 + \Gamma_{\mathbf{a}_2}\} \cup \{\mathbf{a}_2 + \Gamma_{\mathbf{a}_1}\} \cup \{\mathbf{0}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2\}$.

Let $A_j = A_j^a t_a$, B_j, and ϕ be the gauge potentials and the Higgs boson fields respectively. The 't Hooft periodic boundary conditions are such that the triple (A_j, B_j, ϕ) are doubly periodic in $\mathbb{R}^2$ up to gauge transformations. For our purpose we impose this periodicity as follows:

$$(\exp(-i\xi_k(t_3 + t_0))\phi)(\mathbf{x} + \mathbf{a}_k) = (\exp(-i\xi_k(t_3 + t_0))\phi)(\mathbf{x}),$$

$$\left(\omega_k A_j \omega_k^{-1} - \frac{i}{g}\omega_k \partial_j \omega_k^{-1}\right)(\mathbf{x} + \mathbf{a}_k) = \left(\omega_k A_j \omega_k^{-1} - \frac{i}{g}\omega_k \partial_j \omega_k^{-1}\right)(\mathbf{x}),$$

$$\left(B_j + \frac{1}{g'}\partial_j \xi_k\right)(\mathbf{x} + \mathbf{a}_k) = \left(B_j + \frac{1}{g'}\partial_j \xi_k\right)(\mathbf{x}),$$

$$\mathbf{x} \in (\Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2}) - \Gamma_{\mathbf{a}_k}, \quad k = 1, 2, \tag{3.1}$$

where ξ_1, ξ_2 are real-valued smooth functions defined in a neighborhood of $\Gamma_{\mathbf{a}_2} \cup \{\mathbf{a}_1 + \Gamma_{\mathbf{a}_2}\}$, $\Gamma_{\mathbf{a}_1} \cup \{\mathbf{a}_2 + \Gamma_{\mathbf{a}_1}\}$, respectively, and

$$\omega_k(\mathbf{x}) = \exp(-i\xi_k(\mathbf{x})t_3) \in SU(2), \quad \exp(-i\xi_k(\mathbf{x})t_0) \in U(1).$$

Let us see what these conditions imply for the field configurations in the unitary gauge. It is easy to verify that the first relation in (3.1) says that φ is periodic:

$$\varphi(\mathbf{x} + \mathbf{a}_k) = \varphi(\mathbf{x}), \quad \mathbf{x} \in (\Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2}) - \Gamma_{\mathbf{a}_k}, \quad k = 1, 2. \tag{3.2}$$

To proceed further, we recall the following well-known Campbell–Hausdorff formula

$$\exp(-A)B\exp(A) = B + \frac{1}{1!}[B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots,$$

where A, B are $n \times n$ complex matrices. Therefore

$$\omega A_j \omega^{-1} = \exp(-i\xi t_3) A_j^a t_a \exp(i\xi t_3)$$
$$= \left(A_j^1 - \frac{1}{1!}\xi A_j^2 - \frac{1}{2!}\xi^2 A_j^1 + \frac{1}{3!}\xi^3 A_j^2 + \cdots \right) t_1$$
$$+ \left(A_j^2 + \frac{1}{1!}\xi A_j^1 - \frac{1}{2!}\xi^2 A_j^2 - \frac{1}{3!}\xi^3 A_j^1 + \cdots \right) t_2$$
$$+ A_j^3 t_3,$$
$$-\frac{i}{g}\omega \partial_j \omega^{-1} = \frac{1}{g}(\partial_j \xi) t_3.$$

Thus after some calculation we obtain, in the notation of Sect. 2,

$$\exp(i\xi_k(\mathbf{x} + \mathbf{a}_k)) W(\mathbf{x} + \mathbf{a}_k) = \exp(i\xi_k(\mathbf{x})) W(\mathbf{x}),$$
$$\left(A_j^3 + \frac{1}{g}\partial_j \xi_k \right)(\mathbf{x} + \mathbf{a}_k) = \left(A_j^3 + \frac{1}{g}\partial_j \xi_k \right)(\mathbf{x}),$$
$$\mathbf{x} \in (\Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2}) - \Gamma_{\mathbf{a}_k}, \quad k = 1, 2. \tag{3.3}$$

Combining the above equation with the last relation in the boundary condition (3.1) and using (2.2), we have

$$\left(P_j + \frac{1}{e}\partial_j \xi_k \right)(\mathbf{x} + \mathbf{a}_k) = \left(P_j + \frac{1}{e}\partial_j \xi_k \right)(\mathbf{x}),$$
$$Z_j(\mathbf{x} + \mathbf{a}_k) = Z_j(\mathbf{x}), \quad \mathbf{x} \in (\Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2}) - \Gamma_{\mathbf{a}_k}, \quad k = 1, 2. \tag{3.4}$$

We summarize the boundary conditions (3.2)–(3.4) we have obtained as follows:

$$\varphi(\mathbf{x} + \mathbf{a}_k) = \varphi(\mathbf{x}),$$
$$\exp(i\xi_k(\mathbf{x} + \mathbf{a}_k)) W(\mathbf{x} + \mathbf{a}_k) = \exp(i\xi_k(\mathbf{x})) W(\mathbf{x}),$$
$$\left(P_j + \frac{1}{e}\partial_j \xi_k \right)(\mathbf{x} + \mathbf{a}_k) = \left(P_j + \frac{1}{e}\partial_j \xi_k \right)(\mathbf{x}),$$
$$Z_j(\mathbf{x} + \mathbf{a}_k) = Z_j(\mathbf{x}), \quad \mathbf{x} \in (\Gamma_{\mathbf{a}_1} \cup \Gamma_{\mathbf{a}_2}) - \Gamma_{\mathbf{a}_k}, \quad k = 1, 2. \tag{3.5}$$

The relations (3.5) are exactly the 't Hooft periodic boundary conditions for the reduced $U(1)$ model (2.5) over the lattice with fundamental domain Ω (because in such a situation a gauge transformation is defined according to the formula (2.6)). Hence we have shown that the 't Hooft periodic conditions for the full $SU(2) \times U(1)$ theory and the theory in the residual $U(1)$ symmetry are in fact equivalent.

For convenience, we momentarily denote the value of a function ξ at a point $\mathbf{x} = s_1 \mathbf{a}_1 + s_2 \mathbf{a}_2 \in \bar{\Omega}$ by $\xi(s_1, s_2)$. Since W is a single-valued complex scalar field, there must exist an integer $N \in \mathbb{Z}$ so that

$$\xi_1(1, 1^-) - \xi_1(1, 0^+) + \xi_1(0, 0^+) - \xi_1(0, 1^-)$$
$$+ \xi_2(0^+, 1) - \xi_2(1^-, 1) + \xi_2(1^-, 0) - \xi_2(0^+, 0) + 2\pi N = 0. \tag{3.6}$$

As a consequence of (3.5)–(3.6), there holds

$$\Phi = \int_{\Omega} P_{12} dx = \int_{\partial\Omega} P_j dx_j = \frac{2\pi N}{e}. \tag{3.7}$$

Namely, the total magnetic flux through Ω is quantized and independent of the size of Ω. On the other hand, it is easily seen that the flux through Ω induced by the massive vector boson Z is zero.

Using (3.7) and the boundary condition (3.5), we see that the energy density (2.5) leads to the energy lower bound as in Ambjorn and Olesen [4]:

$$\begin{aligned} E &= \int_{\Omega} \mathscr{E} dx \\ &= \int_{\Omega} dx \left\{ |\mathscr{D}_1 W + i\mathscr{D}_2 W|^2 + \frac{1}{2}\left(P_{12} - \frac{g}{2\sin\theta}\varphi_0^2 - 2g\sin\theta|W|^2 \right)^2 \right\} \\ &+ \int_{\Omega} dx \left\{ \frac{1}{2}\left(Z_{12} - \frac{g}{2\cos\theta}(\varphi^2 - \varphi_0^2) - 2g\cos\theta|W|^2 \right)^2 + \left(\frac{g\varphi}{2\cos\theta} Z_j + \varepsilon_{jk}\partial_k\varphi \right)^2 \right\} \\ &+ \int_{\Omega} dx \left\{ \left(\lambda - \frac{g^2}{8\cos^2\theta} \right)(\varphi_0^2 - \varphi^2)^2 - \frac{g^2}{8\sin^2\theta}\varphi_0^4 \right. \\ &\left. + \frac{g\varphi_0^2}{2\sin\theta} P_{12} - \frac{g\varphi_0^2}{2\cos\theta} Z_{12} - \frac{g}{2\cos\theta}\partial_k(\varepsilon_{jk} Z_j \varphi^2) \right\} \\ &\geqq \frac{g\varphi_0^2}{\sin\theta}\left(\frac{\pi N}{e} - \frac{g\varphi_0^2}{8\sin\theta}|\Omega| \right) \quad \text{for } \lambda \geqq \frac{g^2}{8\cos^2\theta}. \end{aligned}$$

In the critical case where

$$\lambda = \frac{g^2}{8\cos^2\theta}, \tag{3.8}$$

namely the Higgs mass equals to the mass of Z vector boson, the above energy lower bound may be saturated by the solutions of the following Bogomol'nyi system:

$$\begin{cases} \mathscr{D}_1 W + i\mathscr{D}_2 W = 0, \\ P_{12} = \dfrac{g}{2\sin\theta}\varphi_0^2 + 2g\sin\theta|W|^2, \\ Z_{12} = \dfrac{g}{2\cos\theta}(\varphi^2 - \varphi_0^2) + 2g\cos\theta|W|^2, \\ Z_j = -\dfrac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \ln\varphi, \end{cases} \tag{3.9}$$

subject to the 't Hooft periodic boundary condition (3.5).

It is straightforward to verify that solutions of (3.9) give rise to solutions of the original electroweak theory. From the second equation in (3.9) and (3.7) it is seen that the integer N in the relation (3.6) must be positive. The rest of the paper will be devoted to a construction of the solutions of (3.9)

4. Realization of Quantized Flux

This section discusses how the quantized flux is characterized by a smooth solution quartet (φ, W, P_j, Z_j) of the Bogomol'nyi equations (3.9). For simplicity, we assume that the field W does not vanish on $\partial\Omega$. The domain Ω may be viewed as a subset in the complex plane $\mathbb{C}$. A point in Ω will be denoted by $z = x_1 + ix_2$ and the set of zeros of W by $Z(W)$.

Under the notation

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial^\dagger = \frac{1}{2}(\partial_1 + i\partial_2), \quad \alpha = A_1^3 + iA_2^3,$$

the first equation in (3.9) takes the form

$$\partial^\dagger W = \tfrac{1}{2} ig\alpha W. \tag{4.1}$$

Such a relation implies that, locally in Ω, W is the product of a holomorphic function and a nonvanishing smooth function (see Jaffe and Taubes [8]). Let $z_0 \in Z(W)$. Then we have the representation

$$W(z) = (z - z_0)^{n_0} h_0(x_1, x_2) \tag{4.2}$$

in a neighborhood of $z = z_0$. Here h_0 is a nonvanishing complex-valued smooth function and the multiplicity n_0 of the zero $z_0 \in Z(W)$ a positive integer. This description implies in particular that $Z(W)$ is a finite set.

The unitary gauge assumption makes it necessary to impose that the real Higgs field φ has no zero.

To proceed further, we let $Z(W) = \{z_1, \ldots, z_m\}$ and assume that the multiplicity of the zero $z = z_l$ is $n_l > 0$, $l = 1, \ldots, m$. $z_1, \ldots, z_m$ are the vortex locations of the solution and $n_1, \ldots, n_m$ are commonly called the local vortex numbers. Hence $N = n_1 + \cdots + n_m$ is the total vortex number.

The first equation in (3.9) or (4.1) may be rewritten

$$\alpha = -\frac{2i}{g}\partial^\dagger \ln W, \quad \text{away from } Z(W). \tag{4.3}$$

Therefore, outside $Z(W)$, Eqs. (3.15) may be reduced by virtue of (4.3) and $Z_{12} = (2\cos\theta/g)\Delta \ln \varphi$ to

$$\begin{cases} -\Delta \ln |W|^2 = g^2\varphi^2 + 4g^2|W|^2, \\ \Delta \ln \varphi = \dfrac{g^2}{4\cos^2\theta}(\varphi^2 - \varphi_0^2) + g^2|W|^2 \end{cases} \tag{4.4}$$

(cf. [4]). Since W has the representation (4.2) in a neighborhood of a point $\in Z(W)$, the substitution $|W|^2 = \exp(u)$, $\varphi^2 = \exp(w)$ allows us to rewrite (4.4) in the full domain Ω in the form

$$\begin{cases} -\Delta u = g^2 \exp(w) + 4g^2 \exp(u) - 4\pi \sum_{l=1}^{m} n_l \delta(z - z_l), \\ \Delta w = \dfrac{g^2}{2\cos^2\theta}(\exp(w) - \varphi_0^2) + 2g^2 \exp(u), \quad \text{in } \Omega, \\ u, w \quad \text{are periodic on } \partial\Omega. \end{cases} \tag{4.5}$$

Conversely, if (u, w) is a solution of (4.5), then we can define the quartet (φ, W, P_j, Z_j) according to

$$\varphi(z) = \exp\left(\tfrac{1}{2} w(z)\right),$$

$$W(z) = \exp\left(\tfrac{1}{2}[u(z) + i\Theta(z)]\right); \quad \Theta(z) = 2 \sum_{l=1}^{m} n_l \arg(z - z_l),$$

$$Z_j(z) = -\frac{2\cos\theta}{g} \varepsilon_{ij} \partial_k \ln \varphi(z),$$

$$P_j(z) = \csc\theta A_j^3(z) - \cot\theta Z_j(z), \tag{4.6}$$

where A_j^3 is determined through (4.3) (the definition of α may actually be extended smoothly to the full Ω; see [8]). It is not hard to check that (φ, W, P_j, Z_j) is a solution of the Bogomol'nyi system (3.9) satisfying the periodic boundary condition (3.5) so that the total vortex number in (3.6) is given by $N = n_1 + \cdots + n_m$.

In conclusion the quantized flux Φ is characterized as in the abelian Higgs model [14] by the vortex number and, to find a solution with flux $2\pi N/e$, it suffices to solve the coupled equations (4.5) with $n_1 + \cdots + n_m = N$. In the next section, we will present a resolution of this system of equations.

5. Existence of Multivortices

Since the boundary condition in (4.5) is periodic, it will be most convenient to view the problem as defined on the 2-torus $T(\Omega) = \mathbb{R}^2/\sim$ where $\mathbf{x} \sim \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ if $\mathbf{x} = \mathbf{y}$ mod$(\mathbf{a}_1)$ or mod$(\mathbf{a}_2)$. In the sequel no mention of the domain Ω will be made whenever there is no risk of confusion.

The following standard result will be useful for our background subtraction.

Lemma 5.1. *For $N = n_1 + \cdots + n_m$, there is a function u_0 which is smooth in the complement of the set $\{z_1, \ldots, z_m\}$ so that*

$$\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{l=1}^{m} n_l \delta(z - z_l). \tag{5.1}$$

Moreover, $u_0(z) - \ln|z - z_l|^{2n_l}$ is smooth in a small neighborhood of $z = z_l$.

A proof of this lemma may be found in Aubin [7].

Now define $v = u - u_0$. Obviously the function $U_0 = \exp(u_0)$ is smooth and nonnegative. Hence Eqs. (4.5) become

$$\begin{cases} \Delta v = \dfrac{4\pi N}{|\Omega|} - g^2 \exp(w) - 4g^2 U_0 \exp(v), \\[2ex] \Delta w = \dfrac{g^2}{2\cos^2\theta}(\exp(w) - \varphi_0^2) + 2g^2 U_0 \exp(v). \end{cases} \tag{5.2}$$

It may be hard to treat the above system directly. To proceed further we introduce the following transformation of dependent variables:

$$\begin{cases} \eta = v + 2w, \\ v = v. \end{cases} \tag{5.3}$$

Then (5.2) is equivalent to

$$\begin{cases} \Delta\eta = -H + g^2 \tan^2\theta \exp(\frac{1}{2}[\eta - v]), \\ \Delta v = \dfrac{4\pi N}{|\Omega|} - g^2 \exp(\frac{1}{2}[\eta - v]) - 4g^2 U_0 \exp(v), \end{cases} \tag{5.4}$$

where

$$H = \frac{g^2\varphi_0^2}{\cos^2\theta} - \frac{4\pi N}{|\Omega|}.$$

An integration by parts of the first equation in (5.4) yields a constraint for the solution:

$$\int \exp\left(\frac{1}{2}[\eta - v]\right) = C_1 \equiv \frac{|\Omega|}{g^2}(\cot^2\theta)H > 0. \tag{5.5}$$

On the other hand, using (5.5) and the second equation in (5.4), we obtain another constraint:

$$\int U_0 \exp(v) = C_2 \equiv \frac{|\Omega|}{4g^2\sin^2\theta}\left(\frac{4\pi N}{|\Omega|} - g^2\varphi_0^2\right) > 0. \tag{5.6}$$

These are constraints for both the solutions and the ranges of physical parameters. For convenience, we extract the above constraints for the parameters as follows:

$$g^2\varphi_0^2 < \frac{4\pi N}{|\Omega|} < \frac{g^2\varphi_0^2}{\cos^2\theta}. \tag{5.7}$$

Let $W^{1,2} = W^{1,2}(T(\Omega))$ be the usual Sobolev space (the set of $(\mathbf{a}_1, \mathbf{a}_2)$-periodic L^2 functions whose distributional derivatives are also in L^2, equipped with the standard inner product). Here $L^p = L^p(\Omega) = L^p(T(\Omega))$. The norm of L^p will be denoted by $\| \ \|_p$.

Lemma 5.2. *The mapping $W^{1,2} \to L^1$ given by $f \mapsto \exp(f)$ is well-defined and compact.*

Proof. See Theorem 2.46 in Aubin [7]. □

It will be seen that the modified system (5.4) leads to a variational reformulation of the problem. Let us first define the functionals I_σ, J_1, J_2 on $W^{1,2}$ by the expressions

$$I_\sigma(\eta, v) = \int \left\{ \frac{1}{2}|\nabla v|^2 + \frac{\sigma}{2}|\nabla\eta|^2 + \frac{4\pi N}{|\Omega|}v - \sigma H\eta \right\},$$

$$J_1(\eta, v) = \int \exp(\tfrac{1}{2}[\eta - v]),$$

$$J_2(\eta, v) = \int U_0 \exp(v).$$

Lemma 5.3. *Consider the following constrained minimization problem*

$$\min\{I_\sigma(\eta,v)|(\eta,v)\in W^{1,2},\ J_k(\eta,v)=C_k,\ k=1,2\}. \tag{5.8}$$

If $\sigma=\cot^2\theta$, then a solution of (5.8) is a smooth solution of Eqs. (5.4).

Proof. Lemma 5.2 implies that J_1, J_2 are well-defined in $W^{1,2}$. Note also that the Fréchet derivatives J_1', J_2' of the constraint functionals are linearly independent.

Given $\sigma>0$, let (η,v) be a solution of (5.8). Then by standard elliptic regularity theory (η,v) must be smooth and there exist Lagrange multipliers $\lambda_\sigma, \mu_\sigma$ depending of course on σ so that

$$\begin{cases} \Delta\eta=-H+\dfrac{\lambda_\sigma}{2\sigma}\exp\left(\dfrac{1}{2}[\eta-v]\right), \\ \Delta v=\dfrac{4\pi N}{|\Omega|}-\dfrac{\lambda_\sigma}{2}\exp\left(\dfrac{1}{2}[\eta-v]\right)+\mu_\sigma U_0\exp(v). \end{cases} \tag{5.9}$$

Integrating the first equation in (5.9) and using $J_1(\eta,v)=C_1$, we obtain

$$\lambda_\sigma=2\sigma g^2\tan^2\theta,$$

which means that (η,v) verifies the first equation in (5.4) for any $\sigma>0$.

To recover the second equation in (5.4), we choose $\sigma=\cot^2\theta$. Therefore, by virtue of $\lambda_\sigma=2g^2$ and integrating the second equation in (5.9), we have $\mu_\sigma=-4g^2$. In particular, (η,v) solves the second equation in (5.4) as well. The lemma is proved. □

In the rest of this section, we fix $\sigma=\cot^2\theta$ and suppress the subscript of I_σ for simplicity. The admissible set of the variational problem (5.8) will be denoted by

$$\mathscr{S}=\{(\eta,v)\in W^{1,2}|J_k(\eta,v)=C_k,\ k=1,2\}.$$

When (5.7) is satisfied, $C_1, C_2>0$, and thus $\mathscr{S}\neq\emptyset$.

Lemma 5.4. *For $f\in W^{1,2}$ with $\int f=0$ and given $\varepsilon>0$, there holds the following optimal estimate:*

$$\int\exp(f)\leqq C(\varepsilon)\exp\left(\left[\frac{1}{16\pi}+\varepsilon\right]\|\nabla f\|_2^2\right), \tag{5.10}$$

where $C(\varepsilon)>0$ is a constant depending only on ε.

The above lemma is a special case of a result in Aubin [6].

We now state our existence result for Eqs. (4.5) as follows.

Lemma 5.5. *If, in addition to (5.7), there holds the inequality*

$$1>\frac{|\Omega|}{8\pi\sin^2\theta}\left(\frac{4\pi N}{|\Omega|}-g^2\varphi_0^2\right), \tag{5.11}$$

then for any distribution $z_1,\dots,z_m\in\Omega$ and $n_1,\dots,n_m\in\mathbf{Z}_+$ with $n_1+\dots+n_m=N$, the system (4.5) has a solution.

Proof. It suffices to prove that (5.2) or (5.4) has a solution. However, by virtue of

Lemma 5.3, it is sufficient to show the existence of a minimizer of the constrained optimization problem (5.8)

We first prove that, under the condition (5.11), the objective functional I is bounded from below on $\mathscr{S}$. For this purpose we rewrite each $f \in W^{1,2}$ as follows:

$$f = \mathscr{M}(f) + f',$$

where $\mathscr{M}(f)$ denotes the integral mean of f: $\mathscr{M}(f) = (\int f)/|\Omega|$ and $\mathscr{M}(f') = 0$. Hence I may be put for $(\eta, v) \in \mathscr{S}$ in the form

$$I(\eta, v) = \int \left\{ \frac{1}{2} |\nabla v'|^2 + \frac{\sigma}{2} |\nabla \eta'|^2 \right\} + 4\pi N \mathscr{M}(v) - \sigma H |\Omega| \mathscr{M}(\eta). \tag{5.12}$$

Let us now evaluate

$$\Lambda(\eta, v) \equiv 4\pi N \mathscr{M}(v) - \sigma H |\Omega| \mathscr{M}(\eta)$$

in (5.12) in terms of η', v', and the constraints.

From (5.6), we have

$$\exp(\mathscr{M}(v)) \int U_0 \exp(v') = C_2.$$

Thus

$$\mathscr{M}(v) = \ln C_2 - \ln(\textstyle\int U_0 \exp(v')). \tag{5.13}$$

On the other hand, (5.5) implies in a similar manner

$$\mathscr{M}(\eta) = \mathscr{M}(v) + 2 \ln C_1 - 2 \ln(\textstyle\int \exp(\frac{1}{2}[\eta' - v'])). \tag{5.14}$$

As a consequence,

$$\Lambda(\eta, v) = (4\pi N - \sigma H |\Omega|) \mathscr{M}(v) + 2\sigma H |\Omega| \ln(\textstyle\int \exp(\frac{1}{2}[\eta' - v'])) + C_3,$$

where $C_3 = -2\sigma H |\Omega| \ln C_1$.

The second term in the expression of $\Lambda(\eta, v)$ above has a lower bound as may be seen from the convexity of the exponential function and Jensen's inequality:

$$\ln\left(\int \exp\left(\frac{1}{2} [\eta' - v'] \right) \right) \geqq \ln\left(|\Omega| \exp\left(\frac{1}{|\Omega|} \int \frac{1}{2} [\eta' - v'] \right) \right) = \ln |\Omega|.$$

Therefore, using (5.13),

$$\begin{aligned} \Lambda(\eta, v) &\geqq C_4 \mathscr{M}(v) + C_3 + 2\sigma H |\Omega| \ln |\Omega| \\ &= C_4 \ln C_2 + C_3 + 2\sigma H |\Omega| \ln |\Omega| - C_4 \ln(\textstyle\int U_0 \exp(v')), \end{aligned} \tag{5.15}$$

where

$$C_4 \equiv 4\pi N - \sigma H |\Omega| = \frac{|\Omega|}{\sin^2 \theta} \left(\frac{4\pi N}{|\Omega|} - g^2 \varphi_0^2 \right) > 0$$

due to the condition (5.7).

We now estimate the last term on the right-hand-side of (5.15). Let p, q be a pair of conjugate exponents: $1 < p, q < \infty$, $1/p + 1/q = 1$. From the Schwarz inequality and Lemma 5.4 it follows that

$$\begin{aligned}\ln(\int U_0 \exp(v')) &\leqq \frac{1}{p}\ln(\int U_0^p) + \frac{1}{q}\ln(\int \exp(qv')) \\ &\leqq \frac{1}{p}\ln(\int U_0^p) + \frac{1}{q}\ln C(\varepsilon) + \left(\frac{1}{16\pi} + \varepsilon\right) q \|\nabla v'\|_2^2. \qquad (5.16)\end{aligned}$$

By virtue of (5.15)–(5.16) we obtain the lower bound

$$\begin{aligned} I(\eta, v) &= \tfrac{1}{2}\|\nabla v'\|_2^2 + \frac{\sigma}{2}\|\nabla \eta'\|_2^2 + \Lambda(\eta, v) \\ &\geqq \kappa(q,\varepsilon)\|\nabla v'\|_2^2 + \frac{\sigma}{2}\|\nabla \eta'\| + C_3 + 2\sigma H|\Omega|\ln|\Omega| \\ &\quad + C_4\left(\ln C_2 - \frac{1}{p}\ln[\int U_0^p] - \frac{1}{q}\ln C(\varepsilon)\right), \qquad (5.17)\end{aligned}$$

where

$$\begin{aligned}\kappa(q,\varepsilon) &= \frac{1}{2} - C_4\left(\frac{1}{16\pi} + \varepsilon\right) q \\ &= \frac{1}{2}\left(1 - \frac{2|\Omega|}{\sin^2\theta}\left[\frac{4\pi N}{|\Omega|} - g^2\varphi_0^2\right]\left[\frac{1}{16\pi} + \varepsilon\right] q\right).\end{aligned}$$

Using (5.11), it is seen that the constants $q > 1$ and $\varepsilon > 0$ can be suitably chosen to make $\kappa(q,\varepsilon) > 0$. Hence I has a lower bound on $\mathscr{S}$.

Finally, let $\{(\eta_j, v_j)\} \subset \mathscr{S}$ be a minimizing sequence of the variational problem (5.8). The inequality (5.17) implies that $\{(\eta_j', v_j')\}$ is bounded in $W^{1,2}$. On the other hand, the relations (5.13)–(5.14) and Lemma 5.2 say that $\{\mathscr{M}(v_j)\}$ and $\{\mathscr{M}(\eta_j)\}$ are bounded sequences as well. Hence $\{(\eta_j, v_j)\}$ itself is bounded in $W^{1,2}$. For simplicity, we assume that $(\eta_j, v_j) \to$ some $(\eta, v) \in W^{1,2}$ weakly as $j \to \infty$. As a consequence of Lemma 5.2, there holds $(\eta, v) \in \mathscr{S}$. However, the weak lower semicontinuity of the functional I over $W^{1,2}$ enables us to make the comparison $I(\eta, v) \leqq \liminf I(\eta_j, v_j)$. Thus (η, v) solves (5.8) and the proof of the lemma is complete. □

From Lemma 5.5 and the discussion of Sect. 4, we are immediately led to the following existence result for multivortex solutions of the electroweak theory.

Theorem 5.6. *For any $z_1, \dots, z_m \in \Omega$ and $n_1, \dots, n_m \in \mathbf{Z}_+$ with $n_1 + \dots + n_m = N$ satisfying (5.7) and (5.11), the Bogomol'nyi system (3.9) subject to the 't Hooft periodic boundary condition has a smooth vortex-line solution (φ, W, P_j, Z_j) so that $\varphi > 0$ and $Z(W) = \{z_1, \dots, z_m\}$, the multiplicity of the zero $z = z_l$ of W is $n_l, l = 1, \dots, m$, and the total flux $\Phi = 2\pi N/e$.*

Corollary 5.7. *In the case that $N = 1, 2$, (5.7) is a necessary and sufficient condition for the existence of a vortex solution described in Theorem 5.6.*

Proof. We may rewrite (5.11) in the form

$$0 > 4\pi(N - 2\sin^2\theta) - g^2|\Omega|\varphi_0^2. \qquad (5.18)$$

It is easy to see that (5.18) is contained in (5.7) for $N = 1, 2$. □

6. Concluding Remarks

Remark 6.1. Our existence theorem is obtained through the transformation (5.3) which reformulates the problem into a "lower diagonal" system so that a constrained variational solver may be used. We do not know at this moment whether or not the sufficient condition (5.11) may further be improved. At first glance, the condition (5.11) seems to depend on our special choice of the change of variables (5.3). For example, the transformation

$$\begin{cases} \gamma = \dfrac{v}{2\cos^2\theta} + w, \\ w = w \end{cases} \tag{6.1}$$

also reduces the system (5.2) into a variational problem which makes one think that a different set of sufficient conditions for the existence of multivortices of the model might be worked out and a possible improvement upon (5.11) would result. The following brief discussion provides a negative answer to this speculation.

In fact, substituting (6.1) into (5.2), we have

$$\begin{cases} \Delta\gamma = H' - 2g^2\tan^2\theta U_0 \exp(2\cos^2\theta[\gamma - w]), \\ \Delta w = \alpha(\exp(w) - \varphi_0^2) + 2g^2 U_0 \exp(2\cos^2\theta[\gamma - w]), \end{cases} \tag{6.2}$$

where

$$H' = \frac{1}{2\cos^2\theta}\left(\frac{4\pi N}{|\Omega|} - g^2\varphi_0^2\right), \quad \alpha = \frac{g^2}{2\cos^2\theta}.$$

Integrating (6.2), we find the constraints for a solution as follows:

$$\int U_0 \exp(2\cos^2\theta[\gamma - w]) = C_1' \equiv \cot^2\theta \frac{H'|\Omega|}{2g^2}, \tag{6.3}$$

$$\int \exp(w) = C_2' \equiv \cot^2\theta \frac{|\Omega|}{g^2}\left(\frac{g^2\varphi_0^2}{\cos^2\theta} - \frac{4\pi N}{|\Omega|}\right). \tag{6.4}$$

It can be shown as before that, if $\sigma = \cot^2\theta$, a minimizer of the constrained optimization problem

$$\min\{I(\gamma, w) | (\gamma, w) \in \mathscr{S}\}, \tag{6.5}$$

$$I(\gamma, w) = \int \left\{\frac{\sigma}{2}|\nabla\gamma|^2 + \frac{1}{2}|\nabla w|^2 + \sigma H'\gamma - \beta w\right\}, \quad \beta = \alpha\varphi_0^2,$$

$$\mathscr{S} \equiv \{(\gamma, w) \in W^{1,2} | (\gamma, w) \text{ satisfies (6.3)–(6.4)}\}$$

is a smooth solution of the system (6.2).

With the notation of Sect. 5, we have, for $(\gamma, w) \in \mathscr{S}$ the decomposition

$$\gamma = \mathscr{M}(\gamma) + \gamma', \quad w = \mathscr{M}(w) + w'.$$

Therefore we may rewrite $I(\gamma, w)$ in the form

$$I(\gamma, w) = \frac{\sigma}{2}\|\nabla\gamma'\|_2^2 + \tfrac{1}{2}\|\nabla w'\|_2^2 + |\Omega|(\sigma H'\mathscr{M}(\gamma) - \beta\mathscr{M}(w))$$

$$= \frac{\sigma}{2} \| \nabla \gamma' \|_2^2 + \tfrac{1}{2} \| \nabla w' \|_2^2 + \beta' \ln(\textstyle\int \exp(w'))$$

$$- |\Omega| \sigma H' \frac{1}{2\cos^2\theta} \ln(\textstyle\int U_0 \exp(2\cos^2\theta[\gamma' - w'])) + C_3',$$

where

$$\beta' = |\Omega|(\beta - \sigma H') = \frac{|\Omega|}{2\sin^2\theta}\left(\frac{g^2\varphi_0^2}{\cos^2\theta} - \frac{4\pi N}{|\Omega|}\right) > 0,$$

$$C_3' = -\beta' \ln C_2' + \frac{1}{2\cos^2\theta} |\Omega| \sigma H' \ln C_1'.$$

Jensen's inequality again implies that $\ln(\int \exp(w')) \geqq \ln |\Omega|$.

Let p, q be a pair of conjugate exponents as in Sect. 5. From the Schwarz inequality and (5.10) we obtain the following lower bound for $I(\gamma, w)$:

$$I(\gamma, w) \geqq \kappa' \| \nabla \gamma' \|_2^2 + \kappa'' \| \nabla w' \|_2^2 + C_4', \tag{6.6}$$

where

$$\kappa' = \sigma\left(\frac{1}{2} - 2|\Omega| H' q \cos^2\theta \left[\frac{1}{16\pi} + \varepsilon\right]\left[1 + \frac{1}{r}\right]\right),$$

$$\kappa'' = \left(\frac{1}{2} - 2|\Omega| H' q \cos^2\theta \left[\frac{1}{16\pi} + \varepsilon\right] \sigma [1 + r]\right),$$

$r > 0$ is a constant,

$$C_4' = C_3' - \frac{|\Omega| \sigma H'}{2\cos^2\theta}\left(\frac{1}{p} \ln[\textstyle\int U_0^p] + \frac{1}{q} \ln C(\varepsilon)\right) + \beta' \ln |\Omega|.$$

Suppose now there is a suitable $r > 0$ to make

$$\begin{cases} 1 > \dfrac{|\Omega|}{8\pi}\left(\dfrac{4\pi N}{|\Omega|} - g^2\varphi_0^2\right)\left(1 + \dfrac{1}{r}\right), \\[2ex] 1 > \dfrac{|\Omega|}{8\pi}\left(\dfrac{4\pi N}{|\Omega|} - g^2\varphi_0^2\right)\cot^2\theta (1 + r). \end{cases} \tag{6.7}$$

As a consequence of this condition, it is immediate to see that we can choose suitable $q > 1$ and $\varepsilon > 0$ so that $\kappa', \kappa'' > 0$. Thus (6.6) implies that (6.5) has a minimizer and the existence of multivortex solutions again follows.

However, the two conditions (5.11) and (6.7) are actually equivalent.

To see this, we first assume that (5.11) is true. Let $r = \tan^2\theta$. It is seen that both requirements in (6.7) are verified. Hence (5.11) implies (6.7). Suppose now (6.7) holds for some $r > 0$. If $r \geqq \tan^2\theta$, then the second inequality in (6.7) implies (5.11); while if $r < \tan^2\theta$, or $1/r > \cot^2\theta$, then (5.11) follows from the first inequality in (6.7). Thus (6.7) implies (5.11) as well.

Remark 6.2. Let T denote the temperature and $T_c > 0$ a critical temperature. The dependence of the electroweak theory on T may be switched on by adding the term $2\lambda\varphi_0^2(T/T_c)^2\varphi^2$ to the static energy (2.5) (see Ambjorn and Olesen [5]). Therefore the vortex equations (3.9) become

$$\begin{cases} \mathscr{D}_1 W + i\mathscr{D}_2 W = 0, \\ P_{12} = \dfrac{g}{2\sin\theta}\varphi_0^2\left(1 - \left[\dfrac{T}{T_c}\right]^2\right) + 2g\sin\theta|W|^2, \\ Z_{12} = \dfrac{g}{2\cos\theta}\left(\varphi^2 - \varphi_0^2\left[1 - \left(\dfrac{T}{T_c}\right)^2\right]\right) + 2g\cos\theta|W|^2, \\ Z_j = -\dfrac{2\cos\theta}{g}\varepsilon_{jk}\partial_k \ln\varphi. \end{cases}$$

It is clear that for $T \geqq T_c$, this system has no solution, while for $T < T_c$, an N-vortex solution exists provided that N satisfies (5.7) and (5.11) in which φ_0^2 is replaced by $\varphi_0^2(1 - [T/T_c]^2)$.

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