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A Generalized Spectral Duality Theorem

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Dedicated to Professor Marek Burnat

Abstract. We establish a version of the spectral duality theorem relating the point spectrum of a family of *-representations of a certain covariance algebra to the continuous spectrum of an associated family of *-representations. Using that version, we prove that almost all the images of any element of a certain space of fixed points of some *-automorphism of an irrational rotation algebra via standard *-representations of the algebra in $l^2(\mathbb{Z})$ do not have pure point spectrum over any non-empty open subset of the common spectrum of those images. As another application of the spectral duality theorem, we prove that if almost all the Bloch operators associated with a real almost periodic function on $\mathbb{R}$ have pure point spectrum over a Borel subset of $\mathbb{R}$, then almost all the Schrödinger operators with potentials belonging to the compact hull of the translates of this function have, over the same set, purely continuous spectrum.

Introduction

Let $\Gamma = (\Omega, G, \theta, \mathbb{P})$ be a quadruple consisting of a metrizable compact space Ω; a separable locally compact Abelian group G; a continuous G-action θ on Ω, that is, a mapping $\theta : \Omega \times G \to \Omega$ such that $\theta(\omega, 0) = \omega$ and $\theta(\omega, g+h) = \theta(\theta(\omega, h), g)$ for $\omega \in \Omega$ and $g, h \in G$; and a Borel probability measure $\mathbb{P}$ on Ω that is θ_g-invariant for each $g \in G$, where θ_g is the homeomorphism of Ω given by

$$\theta_g(\omega) = \theta(\omega, g) \qquad (\omega \in \Omega).$$

Hereafter any such Γ will be called a dynamical system. If $\Gamma = (\Omega, G, \theta^{(\alpha)}, m_\Omega)$ is such that Ω is a metrizable compact Abelian group, G is a separable locally compact non-compact Abelian group, $\theta^{(\alpha)}$ has the form

$$\theta^{(\alpha)}(\omega, g) = \omega + \alpha(g) \qquad (\omega \in \Omega,\, g \in G),$$

where α is a continuous one-to-one homomorphism from G onto a dense subgroup of Ω, and m_Ω is the probabilistic Haar measure on Ω, then Γ will be called a special dynamical system.

With $\Gamma=(\Omega, G, \theta, \mathbb{P})$ a dynamical system, for $1 \leqq p<+\infty$, let $L^p(\Omega)$ (respectively $L^p(G)$) be the p^{th} Lebesgue space based on $\mathbb{P}$ (respectively m_G with m_G the Haar measure on G) with norm $\|\cdot\|_p$.

Given a topological space X, let $C(X)$ be the $\mathbb{C}$-algebra of all complex continuous functions on X, let $C_{\mathbb{R}}(X)$ be the $\mathbb{R}$-algebra of all real functions in $C(X)$, and let $\mathscr{K}(X)$ be the $\mathbb{C}$-algebra of all complex continuous functions on X with compact support.

For each $g \in G$, let $\tilde{\theta}_g$ be the *-automorphism of $C(\Omega)$ given by

$$\tilde{\theta}_g F = F \circ \theta_g \qquad (F \in C(\Omega)).$$

We denote by $\tilde{\theta}$ the mapping $g \to \tilde{\theta}_g$, which is a strongly continuous representation of G into the group of *-automorphisms of $C(\Omega)$.

For each $x \in \mathscr{K}(\Omega \times G)$ and each $g \in G$, let x_g be the element of $C(\Omega)$ given by

$$(x_g)(\omega)=x(\omega, g) \qquad (\omega \in \Omega).$$

Let $\|\cdot\|_\infty$ denote the supremum norm.

Equipped with a multiplication, involution, and norm defined by

$$\begin{gathered}
(x \circ y)(\omega, g)=\int_G x(\omega, h) y(\theta_h(\omega), g-h) dm_G(h), \\
x^*(\omega, g)=\overline{x(\theta_g(\omega),-g)}, \\
\|x\|_1=\int_G \|x_g\|_\infty dm_G(g) \\
(x, y \in \mathscr{K}(\Omega \times G), \omega \in \Omega, g \in G),
\end{gathered}$$

$\mathscr{K}(\Omega \times G)$ is a normed *-algebra. We denote by $L^1(\Gamma)$ the completion of $\mathscr{K}(\Omega \times G)$ in $\|\cdot\|_1$. $L^1(\Gamma)$ is a separable Banach *-algebra, but in general is not a C^*-algebra. Setting

$$\|x\|=\sup_\varrho \|\varrho(x)\| \qquad (x \in L^1(\Gamma)),$$

where ϱ ranges over all the Hilbert space representations of $L^1(\Gamma)$, defines a C^*-seminorm on $L^1(\Gamma)$. In fact, $\|\cdot\|$ is a norm (cf. [6, Theorems 7.7.4 and 7.7.7]) and $\|x\| \leqq \|x\|_1$ holds for all $x \in L^1(\Gamma)$ (cf. [29, Theorem 25.10]). The completion of $L^1(\Gamma)$ in $\|\cdot\|$ is a separable C^*-algebra called the *covariance algebra* of Γ or the *crossed product* of $C(\Omega)$ and G, and is denoted $C^*(\Gamma)$ or $C(\Omega) \times_{\tilde{\theta}} G$.

Given $x \in \mathscr{K}(\Omega \times G)$ and $\omega \in \Omega$, let $\kappa_\omega(x)$ be the operator in $L^2(G)$ defined by

$$(\kappa_\omega(x) \varphi)(g)=\int_G x(\theta_g(\omega), h) \varphi(g+h) dm_G(h) \qquad (\varphi \in L^2(G), g \in G).$$

It is easily verified that for each $\omega \in \Omega$ the mapping κ_ω $(x \to \kappa_\omega(x))$ is a *-representation of $\mathscr{K}(\Omega \times G)$ in $L^2(G)$. The unique continuous extension of κ_ω to a *-representation of $C^*(\Gamma)$ in $L^2(G)$ will also be denoted by κ_ω.

Let $\hat{G}$ be the dual group of G.

Given $x \in \mathscr{K}(\Omega \times G)$ and $\gamma \in \hat{G}$, let $\lambda_\gamma(x)$ be the operator in $L^2(\Omega)$ defined by

$$(\lambda_\gamma(x) F)(\omega)=\int_G x(\omega, g)(g, \gamma) F(\theta_g(\omega)) dm_G(g) \qquad (F \in L^2(\Omega), \omega \in \Omega).$$

It is easily verified that for each $\gamma \in \hat{G}$ the mapping λ_γ $(x \to \lambda_\gamma(x))$ is a *-representation of $\mathscr{K}(\Omega \times G)$ in $L^2(\Omega)$. The unique continuous extension of λ_γ to a *-representation of $C^*(\Gamma)$ in $L^2(\Omega)$ will also be denoted by λ_γ.

Given a *-algebra A, let A_{sa} be the self-adjoint part of A and A_+ be the positive part of A.

Given a C^*-algebra A, let $\mathscr{B}^s(A)$ be the C^*-algebra whose self-adjoint part is the strong sequential closure of A_{sa} on the universal Hilbert space for A (cf. [6, Subsect. 4.5.14]). As is well known, for each *-representation ϱ of A in a Hilbert space H, there is a unique sequentially normal *-representation ϱ'' of $\mathscr{B}^s(A)$ in H that extends ϱ (cf. [6, Theorem 3.7.7]). When $\mathscr{B}^s(A)$ contains a unit, which is the case, for example, when A is separable, then $f(x)\in\mathscr{B}^s(A)$ for every x in $\mathscr{B}^s(A)_{sa}$ and every bounded Borel function f on $\mathbb{R}$ (cf. [6, Theorem 4.5.7]). Moreover, still under the assumption that $\mathscr{B}^s(A)$ contains a unit, $\varrho''(f(x))=f(\varrho''(x))$ for every x in $\mathscr{B}^s(A)_{sa}$, every bounded Borel function f on $\mathbb{R}$, and every *-representation ϱ of A. In fact, given $x\in\mathscr{B}^s(A)_{sa}$ and a *-representation ϱ of A, the set of those bounded Borel functions f on $\mathbb{R}$ for which $\varrho''(f(x))=f(\varrho''(x))$ contains all bounded continuous functions on $\mathbb{R}$ and is strongly sequentially closed. Therefore it coincides with the set of all Borel functions on $\mathbb{R}$.

For each $x\in\mathscr{B}^s(C^*(\Gamma))$, the function $\gamma\to(\lambda''_\gamma(x)1,1)$ is Borel measurable. Indeed, the set of those x in $\mathscr{B}^s(C^*(\Gamma))$ for which the function $\gamma\to(\lambda''_\gamma(x)1,1)$ is Borel measurable is weakly sequentially closed and, since $\gamma\to(\lambda_\gamma(x)1,1)$ is continuous for each $x\in C^*(\Gamma)$, it contains $C^*(\Gamma)$. Thus, this set coincides with $\mathscr{B}^s(C^*(\Gamma))$.

Given $x\in\mathscr{B}^s(C^*(\Gamma))_+$, let

$$\tau(x)=\int_{\hat{G}}(\lambda''_\gamma(x)1,1)\,dm_{\hat{G}}(\gamma)\,. \tag{1}$$

It is easily seen that τ is a σ-trace on $\mathscr{B}^s(C^*(\Gamma))$ which in general is not faithful (see [6, Sects. 5.1.1 and 5.2.1] for relevant definitions and [19, Lemma 3.3] for the proof).

As usual, we denote by 1_E the characteristic function of the set E.

Let $x\in\mathscr{B}^s(C^*(\Gamma))_{sa}$ be such that $\tau(1_{(a,b)}(x))<+\infty$ for $a,b\in\mathbb{R}\cup\{-\infty,+\infty\}$ with $a<b$. Then the spectral density function $N_x^{(a)}$ over (a,b) is defined by

$$N_x^{(a)}(\mu)=\tau(1_{(a,\mu]}(x))\qquad(\mu\in(a,b))\,.$$

$N_x^{(a)}$ is non-decreasing, and so the set $\mathscr{D}(N_x^{(a)})$ of points of discontinuity of $N_x^{(a)}$ is at most countable.

Let H be a Hilbert space, T be a self-adjoint (bounded or unbounded) operator in H, and E be a Borel subset of $\mathbb{R}$. We recall that T is said to have pure point spectrum over E if

$$1_E(T)=\sum_{\mu\in E}1_{\{\mu\}}(T)\,,$$

where the sum is to be interpreted in the sense of strong convergence; T is said to have pure point spectrum with finite multiplicity over E if the above identity is valid and, for each $\mu\in E$, the range space of the projection $1_{\{\mu\}}(T)$ is finite-dimensional; and T is said to have purely continuous spectrum over E if $1_{\{\mu\}}(T)=0$ for each $\mu\in E$. Denote by T_E the operator $1_E(T)T$. Using the identity

$$1_A(T_E)=1_{A\cap E}(T)+\delta_{\{0\}}(A)1_{\mathbb{R}\setminus E}(T)\qquad(A\text{ a Borel subset of }\mathbb{R})\,,$$

where $\delta_{\{0\}}$ denotes the Dirac measure concentrated at 0, one easily verifies that T has pure point spectrum over E if and only if T_E has pure point spectrum (over $\mathbb{R}$).

Bellissard and Testard [5] have presented the following spectral duality theorem.

Theorem A. *Let $\Gamma=(\Omega, G, \theta^{(\alpha)}, m_\Omega)$ be a special dynamical system, let $a, b \in \mathbb{R}$ be such that $a<b$, and let E be a Borel subset of (a, b). If $x \in \mathscr{B}^s(C^*(\Gamma))_{\mathrm{sa}}$ is such that $\tau(1_{(a,b)}(x))<+\infty$ and if, for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum with finite multiplicity over E, then, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\kappa''_\omega(x)$ has purely continuous spectrum over E.*

Kaminker and Xia [19] established another version of the spectral duality theorem, a slightly generalized variant of which, tailored to the setting of the present paper, goes as follows.

Theorem B. *Let $\Gamma=(\Omega, G, \theta, \mathbb{P})$ be a dynamical system, let $a, b \in \mathbb{R}$ be such that $a<b$, and let E be a Borel subset of $(a, b)\backslash\mathscr{D}(N_x^{(a)})$. If $x \in \mathscr{B}^s(C^*(\Gamma))_{\mathrm{sa}}$ is such that $\tau(1_{(a,b)}(x)) < +\infty$ and if, for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum over E, then, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\kappa''_\omega(x)$ has purely continuous spectrum over E.*

The main purpose of the present paper is to establish a version of the spectral duality theorem that simultaneously generalizes Theorems A and B. Using that version, we prove that almost all the images of any element of a certain space of fixed points of some *-automorphism of an irrational rotation algebra via standard *-representations of the algebra in $l^2(\mathbb{Z})$ do not have pure point spectrum over any non-empty open subset of the common spectrum of those images. As another application of the spectral duality theorem, we prove that if almost all the Bloch operators associated with a real almost periodic function on $\mathbb{R}$ have pure point spectrum over a Borel subset of $\mathbb{R}$, then almost all the Schrödinger operators with potentials belonging to the compact hull of the translates of this function have, over the same set, purely continuous spectrum.

1. The Main Result

We begin with a simple preliminary.

Proposition 1. *Let E be a Borel subset of $\mathbb{R}$, and let H be a Hilbert space. If T is a self-adjoint operator in H such that, for each $\xi \in H$,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(1_E(T)e^{itT}\xi, \xi)|^2\, dt = 0, \tag{2}$$

then T has purely continuous spectrum over E.

Proof. The proof proceeds along the same lines as that of a well known theorem of Wiener (cf. [24, Theorem 5.6.9]).

Given $\xi \in H$, let η_ξ be the spectral measure of T associated with ξ, that is,

$$\eta_\xi(A) = (1_A(T)\xi, \xi) \qquad (A \text{ a Borel subset of } \mathbb{R}).$$

Applying the operational calculus for normal operators (cf. [28, Theorem 11.12.3]) in conjunction with Fubini's theorem and adopting the convention that $\sin 0/0=1$, we find that, for each $T>0$,

$$\begin{aligned}\frac{1}{2T} \int_{-T}^{T} |(1_E(T)e^{itT}\xi, \xi)|^2\, dt &= \frac{1}{2T} \int_{T}^{T} \left[\int_{E\times E} e^{it(\mu-\mu')} d\eta_\xi \otimes \eta_\xi(\mu, \mu') \right] dt \\ &= \int_{E\times E} \frac{\sin T(\mu-\mu')}{T(\mu-\mu')}\, d\eta_\xi \otimes \eta_\xi(\mu, \mu'). \end{aligned} \tag{3}$$

If we let

$$\mathscr{D}_E = \{(s,t) \in E \times E : s = t\},$$

then, by (3) and Lebesgue's dominated convergence theorem,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(1_E(T)e^{itT}\xi, \xi)|^2 dt = (\eta_\xi \otimes \eta_\xi)(\mathscr{D}_E). \tag{4}$$

On the other hand, by Fubini's theorem,

$$(\eta_\xi \otimes \eta_\xi)(\mathscr{D}_E) = \int_E \eta_\xi(\{\mu\}) d\eta_\xi(\mu) = \sum_{\mu \in E} |\eta_\xi(\{\mu\})|^2 .$$

Hence, in view of (2) and (4),

$$\sum_{\mu \in E} |\eta_\xi(\{\mu\})|^2 = 0 ,$$

implying that $\eta_\xi(\{\mu\}) = 0$ for each $\mu \in E$. By the arbitrariness of ξ, $1_{\{\mu\}}(T) = 0$ for each $\mu \in E$.

The proof is complete.

The main result of this section is the following.

Theorem 2. *Let $\Gamma = (\Omega, G, \theta, \mathbb{P})$ be a dynamical system, and let E be a Borel subset of $\mathbb{R}$. If x is an element of $\mathscr{B}^s(C^*(\Gamma))_{sa}$ such that $\tau(1_{\{\mu\}}(x)) = 0$ for each $\mu \in E$ and if, for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum over E, then, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\kappa''_\omega(x)$ has purely continuous spectrum over E.*

Proof. Given $\varphi \in L^1(G)$, let $\hat{\varphi}$ be the Fourier transform of φ, that is,

$$\hat{\varphi}(\gamma) = \int_G \varphi(g)(g, -\gamma) dm_G(g) \qquad (\gamma \in \hat{G}) .$$

Adopting a standard convention, we assume that the Haar measure on $\hat{G}$ is normalized so that

$$\varphi(x) = \int_{\hat{G}} \hat{\varphi}(\gamma)(x, \gamma) dm_{\hat{G}}(\gamma) \qquad (x \in G)$$

whenever $\varphi \in L^1(G) \cap C(G)$ and $\hat{\varphi} \in L^1(\hat{G})$.

Let $A(G)$ be the space of the Fourier transforms of functions in $L^1(\hat{G})$.

For each $x \in \mathscr{K}(\Omega \times G)$, each $\varphi \in A(G) \cap \mathscr{K}(G)$, and each $g \in G$, we have

$$\begin{aligned}
&\int_\Omega |(\kappa_\omega(x)\varphi)(g)|^2 d\mathbb{P}(\omega) \\
&= \int_{\Omega \times G \times G} x(\theta_g(\omega), h)\overline{x(\theta_g(\omega), h')}\varphi(g+h)\overline{\varphi(g+h')} d\mathbb{P} \otimes m_{G\times G}(\omega, h, h') \\
&= \int_{G \times G} \left[\int_\Omega x(\omega, h)\overline{x(\omega, h')} d\mathbb{P}(\omega)\right] \varphi(g+h)\overline{\varphi(g+h')} dm_{G\times G}(h, h') \\
&= \int_{G \times G \times \hat{G} \times \hat{G}} \left[\int_\Omega x(\omega, h)\overline{x(\omega, h')} d\mathbb{P}(\omega)\right] (g+h, \gamma)(g+h', -\gamma') \\
&\quad \times \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')} dm_{G\times G\times \hat{G}\times \hat{G}}(h, h', \gamma, \gamma') \\
&= \int_{\hat{G} \times \hat{G}} (\lambda_\gamma(x)1, \lambda_{\gamma'}(x)1)(g, \gamma - \gamma')\hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')} dm_{\hat{G}\times\hat{G}}(\gamma, \gamma') .
\end{aligned} \tag{5}$$

Note that the use of Fubini's theorem is legitimate since $\hat{\varphi}$ is in $L^1(\hat{G}) \cap C(\hat{G})$ and hence the function $(\gamma, \gamma') \to \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')}$ is in $L^1(\hat{G} \times \hat{G}) \cap C(\hat{G} \times \hat{G})$. Let K be a compact subset of G containing the support of φ. Then, for each $\omega \in \Omega$,

$$|(\kappa_\omega(x)\varphi, \varphi)|^2 \leqq m_G(K)\, \|\varphi\|_\infty^2 \int_K |(\kappa_\omega(x)\varphi)(g)|^2 dm_G(g).$$

This together with (5) yields

$$\int_\Omega |(\kappa_\omega(x)\varphi, \varphi)|^2 d\mathbb{P}(\omega) \leqq m_G(K)\, \|\varphi\|_\infty^2 \int_{K\times\hat{G}\times\hat{G}} (\lambda_\gamma(x)1, \lambda_{\gamma'}(x)1)(g, \gamma-\gamma') \times \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')}\, dm_{G\times\hat{G}\times\hat{G}}(g,\gamma,\gamma'). \tag{6}$$

The latter inequality implies in turn that, for each $x \in \mathscr{B}^s(C^*(\Gamma))$,

$$\int_\Omega |(\kappa''_\omega(x)\varphi, \varphi)|^2 d\mathbb{P}(\omega) \leqq m_G(K)\, \|\varphi\|_\infty^2 \int_{K\times\hat{G}\times\hat{G}} (\lambda''_\gamma(x)1, \lambda''_{\gamma'}(x)1)(g, \gamma-\gamma') \times \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')}\, dm_{G\times\hat{G}\times\hat{G}}(g,\gamma,\gamma'). \tag{7}$$

In fact, by the previous argument, for each $x \in \mathscr{B}^s(C^*(\Gamma))$ the functions $\omega \to (\kappa''_\omega(x)\varphi, \varphi)$ and $(\gamma, \gamma') \to (\lambda''_\gamma(x)1, \lambda''_{\gamma'}(x)1)$ are Borel measurable. Moreover, by Lebesgue's dominated convergence theorem, the set of those x in $\mathscr{B}^s(C^*(\Gamma))$ for which (7) holds is strongly sequentially closed, and, by (6), contains $C^*(\Gamma)$. It therefore coincides with $\mathscr{B}^s(C^*(\Gamma))$.

Let E be Borel subset of $\mathbb{R}$ and $x \in \mathscr{B}^s(C^*(\Gamma))_{\mathrm{sa}}$ be such that $\tau(1_{\{\mu\}}(x)) = 0$ for each $\mu \in E$ and, for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum over E. We claim that, for $m_{\hat{G}\times\hat{G}}$-almost all $(\gamma, \gamma') \in \hat{G} \times \hat{G}$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\lambda''_\gamma(1_E(x)e^{itx})1, \lambda''_{\gamma'}(1_E(x)e^{itx})1)\, dt = 0. \tag{8}$$

Let Δ be the set of those $\gamma \in \hat{G}$ for which $\lambda''_\gamma(x)$ has pure point spectrum over E. For each $\gamma \in \Delta$, let $(X_{\gamma,i})_{i\in\mathscr{I}_\gamma}$ be a complete system of eigenvectors of the restriction of $\lambda''_\gamma(x)$ to the range space $\mathscr{R}(1_E(\lambda''_\gamma(x)))$ of the projection $1_E(\lambda''_\gamma(x))$ with a corresponding system $(\mu_{\gamma,i})_{i\in\mathscr{I}_\gamma}$ of eigenvalues, where the index set $\mathscr{I}_\gamma$ has the cardinality equal to the orthogonal dimension of $\mathscr{R}(1_E(\lambda''_\gamma(x)))$. Given $\gamma \in \Delta$ and $\varepsilon > 0$, let $I_{\gamma,\varepsilon}$ be a finite subset of $\mathscr{I}_\gamma$ such that

$$\Big\| 1_E(\lambda''_\gamma(x))1 - \sum_{i\in I_{\gamma,\varepsilon}} (1_E(\lambda''_\gamma(x))1, X_{\gamma,i}) X_{\gamma,i} \Big\|_2 < \varepsilon.$$

Notice that $(1_E(\lambda''_\gamma(x))1, X_{\gamma,i}) = (1, X_{\gamma,i})$ whatever $\gamma \in \Delta$ and $i \in \mathscr{I}_\gamma$. Thus, for any $\gamma, \gamma' \in \Delta$, any $\varepsilon > 0$, and any $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\Big| (\lambda''_\gamma(1_E(x)e^{itx})1, \lambda''_{\gamma'}(1_E(x)e^{itx})1) \\
&\quad - \sum_{(i,j)\in I_{\gamma,\varepsilon}\times I_{\gamma',\varepsilon}} e^{it(\mu_{\gamma,i}-\mu_{\gamma',j})}(1, X_{\gamma,i})\overline{(1, X_{\gamma',j})}(X_{\gamma,i}, X_{\gamma',j}) \Big| \\
&\leqq \Big| \Big(\lambda''_\gamma(1_E(x)e^{itx}) \Big(1_E(\lambda''_\gamma(x))1 - \sum_{i\in I_{\gamma,\varepsilon}} (1, X_{\gamma,i})X_{\gamma,i} \Big), \lambda''_{\gamma'}(1_E(x)e^{itx})1 \Big) \Big| \\
&\quad + \Big| \Big(\sum_{i\in I_{\gamma,\varepsilon}} (1, X_{\gamma,i})\lambda''_\gamma(1_E(x)e^{itx})X_{\gamma,i}, \\
&\quad \lambda''_{\gamma'}(1_E(x)e^{itx}) \Big(1_E(\lambda''_{\gamma'}(x))1 - \sum_{j\in I_{\gamma',\varepsilon}} (1, X_{\gamma',j})X_{\gamma',j} \Big) \Big) \Big| \\
&\leqq \varepsilon + \varepsilon \Big\| \sum_{i\in I_{\gamma,\varepsilon}} (1, X_{\gamma,i})X_{\gamma,i} \Big\|_2 \leqq \varepsilon(2+\varepsilon).
\end{aligned} \tag{9}$$

Given $\mu \in E$ and $\gamma \in \Delta$, let

$$J_{\mu,\gamma} = \{i \in \mathscr{J}_\gamma : \mu = \mu_{\gamma,i}\}.$$

Plainly, for each $\mu \in E$ and each $\gamma \in \Delta$,

$$(\lambda''_\gamma(1_{\{\mu\}}(x))1, 1) = \|\lambda''_\gamma(1_{\{\mu\}}(x))1\|_2^2 = \|1_{\{\mu\}}(\lambda''_\gamma(x))1\|_2^2 = \sum_{i \in J_{\mu,\gamma}} |(1, X_{\gamma,i})|^2.$$

Hence, by (1) and the assumption, for each $\mu \in E$ the set

$$\Gamma_\mu = \{\gamma \in \Delta : (1, X_{\gamma,i}) = 0 \text{ for } i \in J_{\mu,\gamma}\}$$

is of full measure in $\hat{G}$. Given $\gamma \in \Delta$ and $\varepsilon > 0$, let

$$\Delta_{\gamma,\varepsilon} = \bigcap_{i \in I_{\gamma,\varepsilon}} \Gamma_{\mu_{\gamma,i}}.$$

Clearly, $\Delta_{\gamma,\varepsilon}$ is also of full measure in $\hat{G}$.

Fix $\gamma \in \Delta$ and $\gamma' \in \Delta_{\gamma,\varepsilon}$ arbitrarily. Note that, if $\mu_{\gamma,i} = \mu_{\gamma',j}$ for some $(i,j) \in I_{\gamma,\varepsilon} \times I_{\gamma',\varepsilon}$, then, since $\gamma' \in \Gamma_{\mu_{\gamma,i}}$, we have that $(1, X_{\gamma',j}) = 0$. Therefore, if we let

$$\mathscr{A}_{\gamma,\gamma',\varepsilon} = \{(i,j) \in I_{\gamma,\varepsilon} \times I_{\gamma',\varepsilon} : \mu_{\gamma,i} \neq \mu_{\gamma',j}\},$$

then, for each $t \in \mathbb{R}$,

$$\sum_{(i,j) \in I_{\gamma,\varepsilon} \times I_{\gamma',\varepsilon}} e^{it(\mu_{\gamma,i} - \mu_{\gamma',j})}(1, X_{\gamma,i})\overline{(1, X_{\gamma',j})}(X_{\gamma,i}, X_{\gamma',j})$$
$$= \sum_{(i,j) \in \mathscr{A}_{\gamma,\gamma',\varepsilon}} e^{it(\mu_{\gamma,i} - \mu_{\gamma',j})}(1, X_{\gamma,i})\overline{(1, X_{\gamma',j})}(X_{\gamma,i}, X_{\gamma',j}).$$

This together with (6) implies that, for each $\gamma \in \Delta$ and each $\gamma' \in \Delta_{\gamma,\varepsilon}$,

$$\limsup_{T \to \infty} \left| \frac{1}{2T} \int_{-T}^{T} (\lambda''_\gamma(1_E(x)e^{itx})1, \lambda''_{\gamma'}(1_E(x)e^{itx})1)\,dt \right| \leqq \varepsilon(2+\varepsilon).$$

Let

$$\tilde{\Delta}_\gamma = \bigcap_{n \in \mathbb{N}} \Delta_{\gamma,1/n}.$$

Clearly, $\tilde{\Delta}_\gamma$ is of full measure in $\hat{G}$. Moreover, (8) holds for all $\gamma \in \Delta$ and all $\gamma' \in \tilde{\Delta}_\gamma$. Let N be the Borel set of those $(\gamma, \gamma') \in \hat{G} \times \hat{G}$ for which (8) holds. For each $\gamma \in \hat{G}$, let

$$N_\gamma = \{\gamma' \in \hat{G} : (\gamma, \gamma') \in N\}.$$

Since $\tilde{\Delta}_\gamma \subset N_\gamma$ for every $\gamma \in \Delta$, it follows from Fubini's theorem that N has full measure in $\hat{G} \times \hat{G}$. The claim is thus established.

The function $(\gamma, \gamma') \to \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')}$ is in $L^1(\hat{G} \times \hat{G}) \cap C(\hat{G} \times \hat{G})$ and K is compact, so, by (8) and Lebesgue's dominated convergence theorem,

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[\int_{K \times \hat{G} \times \hat{G}} (\lambda''_\gamma(1_E(x)e^{itx})1, \lambda''_{\gamma'}(1_E(x)e^{itx})1)(g, \gamma - \gamma') \right.$$
$$\left. \times \hat{\varphi}(\gamma)\overline{\hat{\varphi}(\gamma')}\,dm_{G \times \hat{G} \times \hat{G}}(g, \gamma, \gamma') \right] dt = 0.$$

This jointly with (7) implies that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left[\int_{\Omega} |(\kappa''_\omega(1_E(x)e^{itx})\varphi, \varphi)|^2 \,d\mathbb{P}(\omega) \right] dt = 0.$$

Since, for each $T>0$,

$$\frac{1}{2T}\int_{-T}^{T}|(\kappa''_\omega(1_E(x)e^{itx})\varphi,\varphi)|^2\,dt \leqq \|\varphi\|_2^2,$$

it follows from Lebesgue's dominated convergence theorem that

$$\int_\Omega\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|(\kappa''_\omega(1_E(x)e^{itx})\varphi,\varphi)|^2\,dt\right]d\mathbb{P}(\omega)=0.$$

Thus, for $\mathbb{P}$-almost all $\omega\in\Omega$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|(\kappa''_\omega(1_E(x)e^{itx})\varphi,\varphi)|^2\,dt=0.$$

The space $A(G)\cap\mathcal{K}(G)$ is dense in $L^2(G)$ and the latter space is separable. Therefore there exists a $\mathbb{P}$-null subset N of Ω such that, for all $\varphi\in L^2(\Omega)$ and all $\omega\in\Omega\backslash N$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|(1_E(\kappa''_\omega(x))e^{it\kappa''_\omega(x)}\varphi,\varphi)|^2\,dt=0.$$

In view of Proposition 1, for each $\omega\in\Omega\backslash N$, $\kappa''_\omega(x)$ has purely continuous spectrum over E.

The proof is complete.

2. Some Consequences

It is clear that Theorem 2 implies Theorem B. The proof of the fact that Theorem 2 implies Theorem A is based on the following.

Proposition 3. *Let $\Gamma=(\Omega,G,\theta^{(\alpha)},m_\Omega)$ be a special dynamical system, and let E be a Borel subset of $\mathbb{R}$. If $x\in\mathcal{B}^s(C^*(\Gamma))_{sa}$ is such that, for $m_{\hat{G}}$-almost all $\gamma\in\hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum with finite multiplicity over E, then $\tau(1_{\{\mu\}}(x))=0$ for each $\mu\in E$.*

Proof. Let $\hat{\alpha}$ be the homomorphism from $\hat{\Omega}$ to $\hat{G}$ given by

$$(g,\hat{\alpha}(\zeta))=(\alpha(g),\zeta)\qquad(\zeta\in\hat{\Omega},\,g\in G).$$

Since α is one-to-one, $\hat{\alpha}(\hat{\Omega})$ is dense in $\hat{G}$. Since G is non-compact and σ-compact, $\hat{G}$ is non-discrete and metrizable (cf. [23, Theorems 12 and 29]). Thus there exists a sequence $(\zeta_k)_{k\in\mathbb{N}}$ of pairwise different elements of $\hat{\Omega}$ such that $\lim_{k\to\infty}\hat{\alpha}(\zeta_k)=0$.

For a measure space $(X,\mathfrak{M},\mu)$ and $f\in L^\infty(X,\mu)$, we denote by M_f the operator in $L^2(X,\mu)$ given by

$$M_f\varphi=f\varphi\qquad(\varphi\in L^2(X,\mu)).$$

For a non-negative operator S in a Hilbert space, we denote by $\mathrm{Tr}(S)$ the trace of S.

For a set E, $\#E$ denotes the cardinality of E.

Given a subset E of an Abelian group A and an element a of A, we let

$$E+a=\{b\in A: b=e+a,\,e\in E\}.$$

A direct computation shows that, for each $\zeta\in\hat{\Omega}$, each $\gamma\in\hat{G}$, and each $y\in\mathcal{B}^s(C^*(\Gamma))$,

$$M_{-\zeta}\lambda''_\gamma(y)M_\zeta=\lambda''_{\gamma+\hat{\alpha}(\zeta)}(y).\tag{10}$$

Hence, if $x \in \mathscr{B}^s(C^*(\Gamma))_{sa}$ is such that $\lambda''_\gamma(x)$ has pure point spectrum with finite multiplicity over E for every γ in a set Δ of full measure in $\hat{G}$, then, for each $\mu \in E$, each $\gamma \in \Delta$, and each $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} (\lambda''_{\gamma+\hat{\alpha}(\zeta_k)}(1_{\{\mu\}}(x))1, 1) = \sum_{k=1}^{n} (\lambda''_\gamma(1_{\{\mu\}}(x))\zeta_k, \zeta_k) \leqq \mathrm{Tr}(\lambda''_\gamma(1_{\{\mu\}}(x))) = \# J_{\mu,\gamma}.$$

In particular, for each $\mu \in E$ and each $\gamma \in \Delta$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\lambda''_{\gamma+\hat{\alpha}(\zeta_k)}(1_{\{\mu\}}(x))1, 1) = 0. \tag{11}$$

Let K be a compact subset of $\hat{G}$. Since $\lim_{k\to\infty} \hat{\alpha}(\zeta_k) = 0$, it follows that, for each $\mu \in E$,

$$\begin{aligned}\int_K (\lambda''_\gamma(1_\mu(x))1, 1)\,dm_{\hat{G}}(\gamma) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \int_{K-\hat{\alpha}(\zeta_k)} (\lambda''_\gamma(1_\mu(x))1, 1)\,dm_{\hat{G}}(\gamma)\\ &= \lim_{n\to\infty} \int_K \frac{1}{n} \sum_{k=1}^{n} (\lambda''_{\gamma+\hat{\alpha}(\zeta_k)}(1_\mu(x))1, 1)\,dm_{\hat{G}}(\gamma).\end{aligned}$$

On the other hand, by (11) and Lebesgue's dominated convergence theorem, for each $\mu \in E$, we have

$$\lim_{n\to\infty} \int_K \frac{1}{n} \sum_{k=1}^{n} (\lambda''_{\gamma+\hat{\alpha}(\zeta_k)}(1_\mu(x))1, 1)\,dm_{\hat{G}}(\gamma) = 0.$$

Hence, for each $\mu \in E$, $(\lambda''_\gamma(1_\mu(x))1, 1) = 0$ for $m_{\hat{G}}$-almost all $\gamma \in K$ and, in view of the arbitrariness of K, $(\lambda''_\gamma(1_\mu(x))1, 1) = 0$ for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$.

The proof is complete.

As a consequence of Theorem 2 and Proposition 3, we have the following generalization of Theorem A.

Theorem 4. *Let $\Gamma = (\Omega, G, \theta^{(\alpha)}, m_\Omega)$ be a special dynamical system, and let E be a Borel subset of $\mathbb{R}$. If $x \in \mathscr{B}^s(C^*(\Gamma))_{sa}$ is such that, for $m_{\hat{G}}$-almost all $\gamma \in \hat{G}$, $\lambda''_\gamma(x)$ has pure point spectrum with finite multiplicity over E, then, for $\mathbb{P}$-almost all $\omega \in \Omega$, $\kappa''_\omega(x)$ has purely continuous spectrum over E.*

3. Some Covariant Representations

Let $\Gamma = (\Omega, G, \theta, \mathbb{P})$ be a dynamical system. A *covariant representation* of Γ is a triple $(\mathfrak{H}, \pi, U)$ in which $\mathfrak{H}$ is a Hilbert space, π is a *-representation of $C(\Omega)$ in $\mathfrak{H}$, and U is a strongly continuous unitary representation of G in $\mathfrak{H}$ such that, for each $F \in C(\Omega)$ and each $g \in G$,

$$\pi(\tilde{\theta}_g F) = U(g)\pi(F)U(-g). \tag{12}$$

With any covariant representation $(\mathfrak{H}, \pi, U)$ of Γ there is associated a non-degenerate *-representation $\varrho_{\pi,U}$ of $C^*(\Gamma)$ in $\mathfrak{H}$ uniquely determined by

$$\varrho_{\pi,U}(x) = \int_G \pi(x_g)U(g)\,dm_G(g) \qquad (x \in \mathscr{K}(\Omega \times G)),$$

the integral being taken in the strong-operator topology. It turns out that every non-degenerate *-representation of $C^*(\Gamma)$ arises as the *-representation associated with a certain covariant representation of Γ (cf. [6, Proposition 7.6.4]). We illustrate this fact by giving any of the *-representations κ_ω ($\omega \in \Omega$) and λ_γ ($\gamma \in \hat{\Gamma}$) the form of a *-representation associated with a covariant representation of Γ.

Given a function f on a group G and an element a of G, let $T_a f$ be the translate of f by a, that is,

$$T_a f(b) = f(a+b) \qquad (b \in G).$$

For each $\omega \in \Omega$, let π_ω be the *-representation of $C(\Omega)$ in $L^2(G)$ defined by

$$(\pi_\omega(F)\varphi)(g) = F(\theta_g(\omega))\varphi(g) \qquad (F \in C(\Omega),\ \varphi \in L^2(G),\ g \in G).$$

Let T be the strongly continuous unitary representation of G in $L^2(G)$ given by

$$T(g)\varphi = T_g\varphi \qquad (\varphi \in L^2(G),\ g \in G).$$

Then, for each $\omega \in \Omega$, $(L^2(G), \pi_\omega, T)$ is a covariant representation of Γ and $\kappa_\omega = \varrho_{\pi_\omega, T}$.

For each $\gamma \in \hat{G}$, let U_γ be the strongly continuous unitary representation of G in $L^2(\Omega)$ defined by

$$U_\gamma(g)H = (g, \gamma)\tilde{\theta}_g H \qquad (H \in L^2(\Omega),\ g \in G).$$

Let $\mathscr{P}$ be the *-representation of $C(\Omega)$ in $L^2(\Omega)$ given by

$$\mathscr{P}(F)H = M_F H \qquad (F \in C(\Omega),\ H \in L^2(\Omega)).$$

Then, for each $\gamma \in \hat{G}$, $(L^2(\Omega), \mathscr{P}, U_\gamma)$ is a covariant representation of Γ and $\lambda_\gamma = \varrho_{\mathscr{P}, U_\gamma}$.

For the remainder of the present section, let $\Gamma = (\Omega, G, \theta^{(\alpha)}, m_\Omega)$ be a special dynamical system. Let bG be the Bohr compactification of G, β be the canonical monomorphism from G into bG, and η be the homomorphism from bG onto Ω such that $\alpha = \eta \circ \beta$ (cf. [25, Definition 14.7.3]).

Let $J_\eta : L^2(\Omega) \to L^2(bG)$ be the operator given by

$$J_\eta F = F \circ \eta \qquad (F \in L^2(\Omega)).$$

Since $m_\Omega(A) = m_{bG}(\eta^{-1}(A))$ for any Borel subset A of Ω, it follows that J_η is an isometry.

Let $\wp$ be the *-representation of $C(\Omega)$ in $L^2(bG)$ given by

$$\wp(F)H = M_{F \circ \eta} H \qquad (F \in C(\Omega),\ H \in L^2(bG)).$$

Let $\mathscr{U}$ be a strongly continuous unitary representation of G in $L^2(bG)$ defined by

$$\mathscr{U}(g)F = T_{\alpha(g)} J_\eta F \qquad (F \in L^2(bG),\ g \in G).$$

Then $(L^2(bG), \wp, \mathscr{U})$ is a covariant representation of Γ. As we shall see shortly, the associated *-representation $\varrho_{\wp, \mathscr{U}}$ contains information simultaneously about all the *-representations λ_γ ($\gamma \in \hat{\Gamma}$).

For each $\gamma \in \hat{G}$, let χ_γ be the element of $\widehat{bG}$ such that

$$(\beta(g), \chi_\gamma) = (g, \gamma) \qquad (g \in G).$$

Of course, the set $\{\chi_\gamma : \gamma \in \hat{G}\}$ coincides with $\widehat{bG}$, and as such forms an orthonormal basis of $L^2(bG)$. Since $\alpha(G)$ is dense in Ω, $\hat{\alpha}$ is injective. For each $\xi \in \hat{\alpha}(\hat{\Omega})$, let $\zeta_\xi = \hat{\alpha}^{-1}(\xi)$; then, clearly, $\zeta_\xi \circ \eta = \chi_\xi$. For each $\gamma \in \hat{G}$, let $\mathfrak{H}_\gamma$ be the closed linear subspace of $L^2(bG)$ spanned by $\{\chi_{\gamma+\xi} : \xi \in \hat{\alpha}(\hat{\Omega})\}$. Note that if $\gamma - \gamma' \in \hat{\alpha}(\hat{\Omega})$, then $\mathfrak{H}_\gamma = \mathfrak{H}_{\gamma'}$, and if $\gamma - \gamma' \in \hat{G} \setminus \hat{\alpha}(\hat{\Omega})$, then $\mathfrak{H}_\gamma$ and $\mathfrak{H}_{\gamma'}$ are mutually orthogonal. Let $\mathscr{S}$ be a selector of the quotient group $\hat{G}/\hat{\alpha}(\hat{\Omega})$, that is, a subset of $\hat{G}$ whose intersection with any coset of $\hat{\alpha}(\hat{\Omega})$ is a singleton. Plainly

$$L^2(bG) = \bigoplus_{\gamma \in \mathscr{S}} \mathfrak{H}_\gamma .$$

Given $x \in \mathscr{K}(\Omega \times \mathbb{R})$, $\gamma \in \hat{G}$, and $\xi \in \hat{\Omega}$, we have

$$\varrho_{\wp,\mathscr{U}}(x)\chi_{\gamma+\xi} = M_{\chi_\gamma} J_\eta \lambda_\gamma(x)\zeta_\xi .$$

Hence, for each $\gamma \in \hat{G}$, $\mathfrak{H}_\gamma$ is an invariant subspace for $\varrho_{\wp,\mathscr{U}}$ and the restriction of $\varrho_{\wp,\mathscr{U}}$ to $\mathfrak{H}_\gamma$ is unitarily equivalent to λ_γ. Accordingly, up to a unitary equivalence,

$$\varrho_{\wp,\mathscr{U}} = \bigoplus_{\gamma \in \mathscr{S}} \lambda_\gamma . \tag{13}$$

Note that this representation does not depend on the choice of the selector $\mathscr{S}$ as, in view of (7), λ_γ and $\lambda_{\gamma'}$ are unitarily equivalent whenever $\gamma - \gamma' \in \hat{\Omega}$.

4. A Remark

Let Γ be a special dynamical system of the form $(\Omega, \mathbb{R}, \theta^{(\alpha)}, m_\Omega)$, and $(\mathfrak{H}, \pi, U)$ be a covariant representation of Γ. Denote by $\mathbb{T}$ the set of all complex numbers with unit modulus. Let $Y: \Omega \times \mathbb{R} \to \mathbb{T}$ be a continuous cocycle on Ω, that is, a continuous function satisfying the cocycle relation

$$Y(\omega, s+t) = Y(\omega, s)\, Y(\omega + \alpha(s), t) \qquad (\omega \in \Omega,\ s,\ t \in \mathbb{R}). \tag{14}$$

Given $t \in \mathbb{R}$, let Y_t be the element of $C(\Omega)$ defined by

$$Y_t(\omega) = Y(\omega, t) \qquad (\omega \in \Omega).$$

For each $t \in \mathbb{R}$, set

$$G_{\pi,U,Y}(t) = \pi(Y_t)\, U(t).$$

In view of (12), (14), the unitarity and the norm continuity of the function $\mathbb{R} \ni t \to Y_t \in C(\Omega)$, the mapping $G_{\pi,U,Y}\,(t \to G_{\pi,U,Y}(t))$ is a strongly continuous unitary one-parameter group in $\mathfrak{H}$. By Stone's theorem (cf. [28, Corollary 9.9.2]), the infinitesimal generator of $G_{\pi,U,Y}$ has the form $iA_{\pi,U,Y}$, where $A_{\pi,U,Y}$ is self-adjoint. Clearly, $A^2_{\pi,U,Y}$ is self-adjoint, positive, and, as an easy application of the operational calculus for normal operators reveals, for each $\mu < 0$, the resolvent $R(\mu, A^2_{\pi,U,Y})$ of $A^2_{\pi,U,Y}$ at μ satisfies

$$R(\mu, A^2_{\pi,U,Y}) = -\frac{1}{2\sqrt{-\mu}} \int_{\mathbb{R}} e^{-\sqrt{-\mu}|s|} G_{\pi,U,Y}(s)\, ds = \varrho_{\pi,U}(m_{\mu,Y}), \tag{15}$$

where $m_{\mu,Y}$ is the element of $L^1(\Gamma)_{\mathrm{sa}}$ given by

$$m_{\mu,Y}(\omega, s) = -\frac{1}{2\sqrt{-\mu}} e^{-\sqrt{-\mu}|s|}\, Y(\omega, s) \qquad (\omega \in \Omega,\ s \in \mathbb{R}).$$

With each $Q \in C_{\mathbb{R}}(\Omega)$ there is associated the continuous cocycle $Y^{(Q)}$ on Ω given by

$$Y^{(Q)}(\omega, t) = \exp\left(i \int_0^t Q(\omega + \alpha(s))\, ds \right) \qquad (\omega \in \Omega,\ t \in \mathbb{R}).$$

If we denote by D_U the infinitesimal generator of the unitary one-parameter group U, then, as one directly verifies,

$$A_{\pi, U, Y^{(Q)}} = i^{-1} D_U + \pi(Q).$$

A fundamental fact is that there exist functions Q in $C_{\mathbb{R}}(\Omega)$ such that, for each $\gamma \in \hat{\mathbb{R}}$, $A_{\mathscr{P}, U_\gamma, Y^{(Q)}}$ has purely continuous spectrum. More precisely, there exist functions Q in $C_{\mathbb{R}}(\Omega)$ such that, for each $\gamma \in \hat{\mathbb{R}}$, $A_{\mathscr{P}, U_\gamma, Y^{(Q)}}$ has purely Lebesgue spectrum; and there exist functions Q in $C_{\mathbb{R}}(\Omega)$ such that, for each $\gamma \in \hat{\mathbb{R}}$, $A_{\mathscr{P}, U_\gamma, Q}$ has purely singularly continuous spectrum. The truth of the fact is seen as follows. Let $(\Theta, \mathfrak{M}, \mu)$ be a probability space carrying a sequence $(X_n)_{n \in \mathbb{N}}$ of Ω-valued independent random variables, each uniformly distributed on Ω. Let f be a unitary continuous function on $\mathbb{T}$ with at least two non-zero Fourier coefficients. Then, by a result of [11], there exists a sequence $(\zeta_n)_{n \in \mathbb{N}}$ in $\hat{\Omega}$ with $(\hat{\alpha}(\zeta_n))_{n \in \mathbb{N}}$ tending to 0 as fast as we please such that, for each $(\theta, \omega, t) \in \Theta \times \Omega \times \mathbb{R}$, the product

$$\prod_{n=1}^{\infty} f((\omega + X_n(\theta), \zeta_n)) \overline{f((\omega + X_n(\theta) + \alpha(t), \zeta_n))}$$

converges (with uniform convergence in θ and ω, and with local uniform convergence in t) and, for any fixed $\theta \in \Theta$, defines a continuous cocycle $Y_{\theta, f}$ on Ω such that, for μ-almost all $\theta \in \Theta$, all the operators $A_{\mathscr{P}, U_\gamma, Y_{\theta, f}}$ $(\gamma \in \hat{\mathbb{R}})$ have purely Lebesgue spectrum (respectively purely singularly continuous spectrum). Let g be a real non-constant continuous function on $\mathbb{T}$ such that $(2\pi)^{-1} \int_0^{2\pi} g(e^{iu}) du$ is an integer, and, for each $s \in [0, 2\pi)$, set

$$f(e^{is}) = \exp\left(-i \int_0^s g(e^{iu}) du \right).$$

Then f is a unitary continuous function on $\mathbb{T}$ with at least two non-zero Fourier coefficients. Now, as indicated above, one can choose a sequence $(\zeta_n)_{n \in \mathbb{N}}$ in $\hat{\Omega}$ so that, if, for each $n \in \mathbb{N}$, α_n is such that

$$e^{i\alpha_n t} = (\alpha(t), \zeta_n) \qquad (t \in \mathbb{R}),$$

then $\sum_{n=1}^{\infty} |\alpha_n| < +\infty$ and, if, for each $\theta \in \Theta$, the function Q_θ in $C(\Omega)$ is given by

$$Q_\theta(\omega) = \sum_{n=1}^{\infty} \alpha_n g((\omega + X_n(\theta), \zeta_n)) \qquad (\omega \in \Omega),$$

then $Y_{\theta, f} = Y^{(Q_\theta)}$ and, for μ-almost all $\theta \in \Theta$, all the operators $A_{\mathscr{P}, U_\gamma, Y^{(Q_\theta)}}$ $(\gamma \in \hat{\mathbb{R}})$ have purely Lebesgue spectrum (respectively purely singularly continuous spectrum).

Note that, for each $Q \in C_{\mathbb{R}}(\Omega)$ and each $\omega \in \Omega$, $A_{\pi_\omega, T, Y^{(Q)}}$ coincides with the operator $i^{-1}(d/dx) + q_\omega(x)$, defined on the Sobolev space $H^1(\mathbb{R})$, where $q_\omega = (T_\omega Q) \circ \alpha$. For each $x \in \mathbb{R}$, set

$$u_{\omega, Q}(x) = \exp\left(-i \int_0^x q_\omega(s) ds \right).$$

It is readily verified that

$$M_{u_{\omega, Q}} A_{\pi_\omega, T, Q} M_{u_{\omega, Q}}^{-1} = \frac{1}{i} \frac{d}{dx},$$

so that $A_{\pi_\omega, T, Y(Q)}$ and $i^{-1}(d/dx)$ are unitarily equivalent. Accordingly, $A_{\pi_\omega, T, Y(Q)}$ has purely Lebesgue spectrum.

Let $Q \in C_{\mathbb{R}}(\Omega)$ be such that, for each $\gamma \in \hat{\mathbb{R}}$, $A_{\mathscr{P}, U_\gamma, Y(Q)}$ has purely continuous spectrum. Fix arbitrarily $\mu < 0$. Then, for each $\omega \in \Omega$, $A^2_{\pi_\omega, T, Y(Q)}$ has purely Lebesgue spectrum and hence, by (15), so does $\kappa_\omega(m_{\mu, Y(Q)})$. Moreover, for each $\gamma \in \hat{\mathbb{R}}$, $A_{\mathscr{P}, U_\gamma, Q}$ has purely continuous spectrum, and so, by (15), $\lambda_\gamma(m_{\mu, Y(Q)})$ has purely continuous spectrum. We thus see there exist elements of $C^*(\Gamma)_{sa}$ whose images by the κ_ω $(\omega \in \Omega)$ have purely continuous spectrum without the images by the λ_γ $(\gamma \in \hat{\mathbb{R}})$ having pure point spectrum.

5. Some Applications

5.1. Consider $\mathbb{T}$ as a compact group with multiplication as group operation, and let $\Gamma = (\mathbb{T}, \mathbb{Z}, \theta^{(\alpha)}, m_{\mathbb{T}})$ be a special dynamical system in which the homomorphism $\alpha: \mathbb{Z} \to \mathbb{T}$ is given by

$$\alpha(n) = e^{2\pi i \xi n} \qquad (n \in \mathbb{Z})$$

with $\xi \in \mathbb{R} \backslash \mathbb{Q}$. Let u and v be the elements of $\mathscr{K}(\mathbb{T} \times \mathbb{Z})$ defined by

$$u(\omega, n) = 1_{\{1\}}(n) \quad \text{and} \quad v(\omega, n) = \omega 1_{\{0\}}(n) \qquad (\omega \in \mathbb{T}, n \in \mathbb{Z}).$$

Considered as elements of $C^*(\Gamma)$, u and v are unitaries satisfying the twisted commutation relation

$$u \circ v = e^{2\pi i \xi} v \circ u .$$

A direct computation shows that $C^*(\Gamma)$ coincides with the C^*-algebra generated by u and v. It is well known that there exists exactly one, up to *-isomorphism, C^*-algebra generated by two unitaries satisfying the above twisted commutation relation (cf. [7; 26, p. 117]). That C^*-algebra is called the *irrational rotation algebra* and is usually denoted by $\mathscr{A}_\xi$. Accordingly, $C^*(\Gamma)$ is a realisation of $\mathscr{A}_\xi$.

Given an operator T in a Banach space or an element T of a Banach algebra, denote by $\sigma(T)$ the spectrum of T.

Since $C^*(\Gamma)$ is simple (cf. [26, Theorem 4.3.3]), all the *-representations of $C^*(\Gamma)$ are faithful. Hence, in particular, $\sigma(\kappa_\omega(x)) = \sigma(x)$ for each $x \in C^*(\Gamma)$ and each $\omega \in \mathbb{T}$.

Given a *-algebra A, let $\mathrm{Aut}(A)$ be the group of all *-automorphisms of A. For a subset E of A and $\mathfrak{a} \in \mathrm{Aut}(A)$, let $E^{\mathfrak{a}}$ be the set of all fixed points of $\mathfrak{a}$ in E. For each $\mathfrak{a} \in \mathrm{Aut}(A)$ and each $x \in A$, let $\mathfrak{a}^0(x) = x$ and, by induction, let $\mathfrak{a}^n(x) = \mathfrak{a}(\mathfrak{a}^{n-1}(x))$ for each $n \in \mathbb{N}$. If $\mathfrak{a} \in \mathrm{Aut}(A)$ is such that $\mathfrak{a}^n = \mathrm{id}_A$ for some $n \in \mathbb{N}$, then setting

$$\pi_{\mathfrak{a}}(x) = \frac{1}{n} \sum_{k=0}^{n-1} \mathfrak{a}^k(x) \qquad (x \in A)$$

defines a projection $\pi_{\mathfrak{a}}$ from A onto $A^{\mathfrak{a}}$.

Let K be the *-subalgebra of $C^*(\Gamma)$ generated by u and v. Clearly, if $\mathfrak{a} \in \mathrm{Aut}(C^*(\Gamma))$ is such that $\mathfrak{a}^n = \mathrm{id}_{C^*(\Gamma)}$ for some $n \in \mathbb{N}$, then $\pi_{\mathfrak{a}}$ maps K_{sa} onto $K^{\mathfrak{a}}_{sa}$.

For any $s \in \mathrm{SL}(2, \mathbb{Z})$ and any $m, n \in \mathbb{Z}$, denote by (m_s, n_s) the image of (m, n) under the standard action of s on $\mathbb{Z} \times \mathbb{Z}$. As shown by Brenken [7], the exists a representation $s \to \mathfrak{a}_s$ of $\mathrm{SL}(2, \mathbb{Z})$ in $\mathrm{Aut}(C^*(\Gamma))$ such that

$$\mathfrak{a}_s(v^m u^n) = e^{\pi i \xi (m_s n_s - mn)} v^{m_s} u^{n_s} \qquad (s \in \mathrm{SL}(2, \mathbb{Z}), m, n \in \mathbb{Z}).$$

Let

$$s_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the sequel, the automorphism $\mathfrak{a}_{s_0}$ will play a special rôle and will be denoted briefly as $\mathfrak{b}$. It is easy to see that $\mathfrak{b}$ is uniquely determined by the identities

$$\mathfrak{b}(u) = v^* \quad \text{and} \quad \mathfrak{b}(v) = u.$$

Obviously, as $s_0^4 = e$, where e is the neutral element of $\mathrm{SL}(2, \mathbb{Z})$, we have $\mathfrak{b}^4 = \mathrm{id}_{C^*(\Gamma)}$.
One of the elements of $K_{\mathrm{sa}}^{\mathfrak{b}}$ is

$$h = 2\pi_{\mathfrak{b}}(u + u^*) = u + u^* + v + v^*.$$

The corresponding operators $\kappa_\omega(h)\,(\omega \in \mathbb{T})$ arise as hamiltonians in the Harper model of a two-dimensional crystal with square symmetry in a magnetic field. If $\omega = e^{2\pi i\theta}$ with $\theta \in [0, 1)$, then, as one easily verifies,

$$(\kappa_\omega(h)\varphi)(n) = \varphi(n+1) + \varphi(n-1) + 2\cos 2\pi(\theta + \xi n)\varphi(n), \quad (\varphi \in l^2(\mathbb{Z}) = L^2(\mathbb{Z}), n \in \mathbb{Z}).$$

The spectral properties of the $\kappa_\omega(h)\,(\omega \in \mathbb{T})$ and of related operators have long been investigated by physicists and mathematicians (cf. [1, 13–17, 27] and the bibliographies therein). A still unproved conjecture asserts that, for each $\omega \in \mathbb{T}$, $\kappa_\omega(h)$ has purely singular continuous spectrum and that $\sigma(h)$, which, as indicated above, coincides with $\sigma(\kappa_\omega(h))$ for each $\omega \in \mathbb{T}$, is of zero Lebesgue measure. Using an argument due to Aubry and André ([1]; cf. also [2, 4]), we shall establish a result (Theorem 6) concerning the entire space $K_{\mathrm{sa}}^{\mathfrak{b}}$, which, when applied to h, partially substantiates the conjecture.

Theorem 5. *For every $x \in K_{\mathrm{sa}}^{\mathfrak{b}}$ and every Borel subset E of $\mathbb{R}$, either $(\kappa_\omega(x))_E = 0$ for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, or, for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ has no pure point spectrum over E.*

Proof. Let $\mathscr{F}$ be the Fourier transformation from $L^2(\mathbb{T})$ onto $l^2(\mathbb{Z})$ given by

$$(\mathscr{F}F)(n) = \int_{\mathbb{T}} F(\omega)\bar{\omega}^n dm_{\mathbb{T}}(\omega) \qquad (F \in L^2(\mathbb{T})).$$

As is well known, $\mathscr{F}$ sets up a unitary equivalence between $L^2(\mathbb{T})$ and $l^2(\mathbb{Z})$. Identifying $\hat{\mathbb{Z}}$ with $\mathbb{T}$ in a standard way, one directly verifies that for each $\omega \in \mathbb{T}$, each $\varphi \in l^2(\mathbb{Z})$, and each $n \in \mathbb{Z}$,

$$\begin{aligned} (\kappa_\omega(u)\varphi)(n) &= (\mathscr{F}\lambda_\omega(v^*)\mathscr{F}^{-1}\varphi)(n) = \varphi(n+1), \\ (\kappa_\omega(v)\varphi)(n) &= (\mathscr{F}\lambda_\omega(u)\mathscr{F}^{-1}\varphi)(n) = \omega e^{2\pi i\xi n}\varphi(n). \end{aligned} \tag{16}$$

Consequently, for each $y \in C^*(\Gamma)$ and each $\omega \in \mathbb{T}$,

$$\kappa_\omega(y) = \mathscr{F}\lambda_\omega(\mathfrak{b}(y))\mathscr{F}^{-1}.$$

Now, if $x \in K_{\mathrm{sa}}^{\mathfrak{b}}$ and $\omega \in \mathbb{T}$, then

$$\kappa_\omega(x) = \mathscr{F}\lambda_\omega(x)\mathscr{F}^{-1}. \tag{17}$$

Hence, for each $\omega \in \mathbb{T}$ and each bounded continuous function f on $\mathbb{R}$,

$$\kappa_\omega(f(x)) = \mathscr{F}\lambda_\omega(f(x))\mathscr{F}^{-1},$$

and further, by the sequential normality of κ''_ω and λ''_ω, for each bounded Borel function f on $\mathbb{R}$,

$$\kappa''_\omega(f(x)) = \mathscr{F}\lambda''_\omega(f(x))\mathscr{F}^{-1}.$$

In particular, for each $\omega \in \mathbb{T}$ and each Borel subset E of $\mathbb{R}$,

$$\kappa''_\omega(x_E) = \mathscr{F}\lambda''_\omega(x_E)\mathscr{F}^{-1}. \tag{18}$$

It is also easy to see that, for each $\omega \in \mathbb{T}$,

$$\kappa_{\theta^{(\alpha)}(\omega, n)}(x) = T(n)\kappa_\omega(x)T(-n),$$

whence, by a similar argument, for each $\omega \in \mathbb{T}$ and each Borel subset E of $\mathbb{R}$,

$$\kappa''_{\theta^{(\alpha)}(\omega, n)}(x_E) = T(n)\kappa''_\omega(x_E)T(-n),$$

where, of course, x_E denotes the element $1_E(x)x$ of $\mathscr{B}^s(C^*(\Gamma))_{\mathrm{sa}}$. Since the function $\omega \to \kappa_\omega(x)$ is strongly continuous, it follows, by a standard argument, that for every bounded Borel function f on $\mathbb{R}$ the function $\omega \to \kappa''_\omega(f(x))$ is weakly Borel measurable. In particular, for each Borel subset E of $\mathbb{R}$, the function $\omega \to \kappa''_\omega(x_E)$ is weakly Borel measurable. Now, since the dynamical system Γ is ergodic, it follows from a theorem of Kunz-Soullaird ([22]; cf. also [20]) that, for each Borel subset E of $\mathbb{R}$, the set of those $\omega \in \mathbb{T}$ for which $\kappa''_\omega(x_E)$ has pure point spectrum is either $m_{\mathbb{T}}$-null or of full measure in $\mathbb{T}$.

Suppose that, for some Borel subset E of $\mathbb{R}$, the set of those $\omega \in \mathbb{T}$ for which $\kappa_\omega(x)$ has no pure point spectrum over E is not of full measure in $\mathbb{T}$. Since, for each $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ has pure point spectrum over E if and only if $(\kappa_\omega(x))_E = \kappa''_\omega(x_E)$ has pure point spectrum, it follows from the preceding paragraph that, for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa''_\omega(x_E)$ has pure point spectrum. Now, by (18), $\lambda''_\omega(x_E)$ has also pure point spectrum for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, and hence $\lambda_\omega(x)$ has pure point spectrum over E for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$. In view of (16), for each $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ is a difference operator of finite order, and so every eigenvalue of $\kappa_\omega(x)$ has finite multiplicity. Accordingly, by (17), for each $\omega \in \mathbb{T}$, every eigenvalue of $\lambda_\omega(x)$ has finite multiplicity. Applying now Theorem 4, we find that, for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ has purely continuous spectrum over E. Finally, the fact that, for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ has simultaneously pure point and purely continuous spectrum over E implies that $\kappa''_\omega(x_E) = 0$ for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$.

The proof is complete.

Theorem 6. *For every $x \in K^{\mathrm{b}}_{\mathrm{sa}}$ and $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ does not have pure point spectrum over any non-empty open subset of $\sigma(x)$.*

Proof. Let $x \in K^{\mathrm{b}}_{\mathrm{sa}}$. Since the topology of $\sigma(x)$ has a countable basis, it suffices to prove that, for each non-empty open subset of $\sigma(x)$ and for $m_{\mathbb{T}}$-almost all $\omega \in \mathbb{T}$, $\kappa_\omega(x)$ has no pure point spectrum over that subset.

Let U be a non-empty open subset of $\sigma(x)$ and $f:\mathbb{R} \to [0,1]$ be a non-zero continuous function with support in U. Then $0 < f(x)x^2 \leqq x_U^2$. Hence, by the faithfulness of the κ_ω ($\omega \in \mathbb{T}$), for each $\omega \in \mathbb{T}$, $0 < \kappa_\omega(f(x)x^2) \leqq (\kappa''_\omega(x_U))^2$ and so $\kappa''_\omega(x_U) \neq 0$. Now the theorem follows upon applying Theorem 5.

5.2. Let $\Gamma = (\Omega, \mathbb{R}, \theta^{(\alpha)}, m_{\mathbb{R}})$ be a special dynamical system, $(\mathfrak{H}, \pi, U)$ be a covariant representation of Γ, and D_U be the infinitesimal generator of U. Then $-D_U^2$ is self-adjoint, positive, and, for each $\mu < 0$,

$$R(\mu, -D_U^2) = -\frac{1}{2\sqrt{-\mu}} \int_{\mathbb{R}} e^{-\sqrt{-\mu}|s|}U(s)ds. \tag{19}$$

Given $Q \in C_{\mathbb{R}}(\Omega)$, let $H_{\pi,U,Q}$ be the self-adjoint operator defined by

$$H_{\pi,U,Q} = -D_U^2 + \pi(Q)$$

with domain coinciding with that of D_U^2. Clearly, $H_{\pi,U,Q}$ is self-adjoint and bounded below by $-\|Q\|_\infty$.

For each $\mu < 0$ and each $F \in C(\Omega)$, let $x_{\mu,F}$ be the element of $L^1(\Gamma)$ given by

$$x_{\mu,F}(\omega,s) = -\frac{1}{2\sqrt{-\mu}} e^{-\sqrt{-\mu}|s|} T_{\alpha(s)} F \qquad (\omega \in \Omega, s \in \mathbb{R}).$$

Clearly, $\|x_{\mu,F}\|_1 = \|F\|_\infty / |\mu|$. Moreover, in view of (12) and (19),

$$R(\mu, -D_U^2)\pi(F) = \varrho_{\pi,U}(x_{\mu,F}). \tag{20}$$

Since, for $\mu < -\|Q\|_\infty$,

$$\|x_{\mu,1}\|_1 + \|x_{\mu,1}\|_1 \sum_{n=1}^{\infty} \|x_{\mu,Q}\|_1^n = \frac{1}{|\mu|} + \frac{1}{|\mu|} \sum_{n=1}^{\infty} \left(\frac{\|Q\|_\infty}{|\mu|}\right)^n = \frac{1}{|\mu| - \|Q\|_\infty},$$

it follows that the series

$$x_{\mu,1} + x_{\mu,1} \circ \sum_{n=1}^{\infty} (x_{\mu,Q})^{\circ n}$$

converges in $L^1(\Gamma)$. Let $r_{\mu,F}$ be its sum. Since, for $\mu < -\|Q\|_\infty$,

$$R(\mu, H_{\pi,U,Q}) = R(\mu, -D_U^2)\left(I + \sum_{n=1}^{\infty} (R(\mu, -D_U^2)\pi(Q))^n\right),$$

it follows from (20) that

$$R(\mu, H_{\pi,U,Q}) = \varrho_{\pi,U}(r_{\mu,Q}). \tag{21}$$

A moment's reflection shows that $r_{\mu,Q}$ is self-adjoint.

The argument used in the proof of (21) goes back to Bellissard and Testard [5] (see also [3, Theorem 3.1]).

Note that, for each $\omega \in \Omega$, $H_{\pi_\omega,T,Q}$ is the Schrödinger operator $(-d^2/dx^2) + q_\omega(x)$ with the almost periodic potential $q_\omega = (T_\omega Q) \circ \alpha$, defined on the Sobolev space $H^2(\mathbb{R})$. Each $H_{\mathscr{P},U_\gamma,Q}$ $(\gamma \in \hat{\mathbb{R}})$ is a so-called Bloch operator. The operator $H_{\wp,\mathscr{U},Q}$ was first introduced and studied by Burnat ([8]; cf. also [9, 10, 18, 21]) and we shall accordingly call $H_{\wp,\mathscr{U},Q}$ the Burnat operator.

The main result of this subsection is the following.

Theorem 7. *Let $\Gamma = (\Omega, \mathbb{R}, \theta^{(\alpha)}, m_\Omega)$ be a special dynamical system, let Q be an element of $C_{\mathbb{R}}(\Omega)$, and let E be a Borel subset of $\mathbb{R}$. If, for $m_{\hat{\mathbb{R}}}$-almost all $\gamma \in \hat{\mathbb{R}}$, the Bloch operator $H_{\mathscr{P},U_\gamma,Q}$ has pure point spectrum over E, then, for m_Ω-almost all $\omega \in \Omega$, the Schrödinger operator $H_{\pi_\omega,T,Q}$ has purely continuous spectrum over E.*

Proof. Fix arbitrarily $\mu < -\|Q\|_\infty$. Let

$$F = \{f \in \mathbb{R} : f = (\mu - e)^{-1}, e \in E\}.$$

In view of (21), for $m_{\hat{\mathbb{R}}}$-almost all $\gamma \in \mathbb{R}$, $\lambda_\gamma''(r_{\mu,Q})$ has pure point spectrum over F. By the result of [12], every eigenvalue of the Burnat operator $H_{\wp,\mathscr{U},Q}$ is at most double. Hence, in view of (20), every eigenvalue of $\varrho''_{\wp,\mathscr{U}}(r_{\mu,Q})$ is at most double. Now, by (13), every eigenvalue of $\lambda_\gamma''(r_{\mu,Q})$ is at most double whatever $\gamma \in \hat{\mathbb{R}}$. By virtue of Theorem 4, for m_Ω-almost all $\omega \in \Omega$, $\kappa''_\omega(r_{\mu,Q})$ has purely continuous spectrum over F, and hence, for m_Ω-almost all $\omega \in \Omega$, $H_{\pi_\omega, T, Q}$ has purely continous spectrum over E.

The proof is complete.

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