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W-Algebras for Generalized Toda Theories

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Abstract. The generalized Toda theories obtained in a previous paper by the conformal reduction of WZNW theories possess a new class of W-algebras, namely the algebras of gauge-invariant polynomials of the reduced theories. An algorithm for the construction of base-elements for the W-algebras of all such generalized Toda theories is found, and the W-algebras for the maximal $SL(N,R)$ generalized Toda theories are constructed explicitly, the primary field basis being identified.

1. Introduction

In some previous papers [1] it was shown that Toda field theories [2] could be regarded as Wess–Zumino–Novikov–Witten (WZNW) theories [3], in which the Kac–Moody (KM) currents were subjected to some first-class linear constraints. Among the advantages obtained by regarding the Toda theories as reduced WZNW theories was a very natural interpretation of the W-algebras [4, 5] of Toda theories, namely, as the algebras of the gauge invariant polynomials of the constrained KM currents and their derivatives [1].

In a subsequent paper [6] it was shown that the WZNW-Toda reduction could be extended to yield a series of generalized Toda theories. These generalized Toda theories are a set of conformally-invariant integrable theories that interpolate between the WZNW theories and the Toda theories, and are partially-ordered in correspondence with the strata of group-orbits in the adjoint representation of the WZNW group G, the traditional Toda theories corresponding to the (unique) minimal stratum. To obtain these generalized Toda theories the KM currents of the WZNW theories are subjected to a more general set of first-class linear constraints, and thus, like the Toda theories, are gauge theories, the gauge group being just that generated by the constraints. As a result these Toda theories possess algebras of gauge-invariant polynomials of the constrained currents and their derivatives, where the multiplication is defined by the Poisson-brackets and commutators of the polynomials in the classical and quantum cases respectively.

As will be seen below, the algebras of gauge-invariant polynomials obtained in this way are W-algebras in the sense of Zamolodchikov [7], that is to say, they are non-linear extensions of the Virasoro algebra by primary fields. But they can also be regarded as non-linear extensions of KM algebras.

The purpose of the present paper is twofold, namely to give an algorithm for constructing a basis for all such W-algebras (Sect. 3 through 5), and to display the W-algebra itself for the maximal Toda theories of $SL(N,R)$ (Sect. 6 through 8).

The bases are not quite general in that they are constructed subject to a technical restriction on the ordering of subgroups in the WZNW reduction, but the procedure is such that it can readily be generalized to other orderings. To clarify the procedure we first consider the case of $SL(N,R)$ Toda theories before proceeding to the general case. All the results include, of course, the construction of gauge-invariant polynomials for the conventional (minimal) Toda theories.

It is evident from the structure of the W-algebra for the maximal $SL(N,R)$ Toda theories that they are polynomial extensions of KM-algebras, but because the fields involved are not all primary it is not immediately evident that they are Zamolodchikov algebras. However, we determine the non-tensorial properties of the fields, and, using this information, identify the Virasoro operator and the primary fields (Sects. 9 and 10).

2. Recall of Generalized WZNW Reduction

We begin by recalling the generalized WZNW-reduction. First the WZNW groups G used are the (maximally non-compact) simple groups generated by the *real* linear span of the canonical Cartan generators, i.e. by the generators (H_i, E^α) in conventional notation. For the A and D Lie algebras, for example, these are the simple groups $SL(N,R)$ and $SO(N,N)$.

The problem is that the KM currents $J^a(z)$ have conformal spin unity with respect to the conformal group generated by the Sommerfield–Sugawara energy–momentum tensors $L(z)=T_{zz}(z)$ and $\bar{L}(\bar{z})=\bar{T}_{\bar{z}\bar{z}}(\bar{z})$, i.e.

$$[L(z), J^a(w)] = -(\partial_w J^a(w))\delta(z-w) + J^a(w)\partial_z\delta(z-w), \tag{2.1}$$

and similarly for the barred quantities, and since the constraints that must be imposed in order to obtain the Toda theories involve setting some of the components of the KM currents equal to non-zero constants, this cannot be done without breaking the conformal symmetry generated by $L(z)$ and $\bar{L}(\bar{z})$. The solution is to replace the conformal group generated by the $L(z)$ and $\bar{L}(\bar{z})$ by another conformal group generated by modified generators, $\Lambda(z)$ and $\bar{\Lambda}(\bar{z})$ say, with respect to which the current components in question are scalars. The Λ's are defined as follows:

Let $\mathbf{m}_i$, $i=1,\dots,l$, where l is the rank of G, be the l fundamental coweights of G, select any subset $\mathbf{m}_a$, define a vector $\mathbf{w}$ as $\mathbf{w}=\sum \mathbf{m}_a$ and an element H of the Cartan subalgebra $\mathbf{H}$ as $H=\mathbf{w}\cdot\mathbf{H}$. Then the element H has the property that the simple root-vectors E^{α_i} are eigenvectors of H with eigenvalues zero or unity (depending on whether the α_i are dual or not to the weights $\mathbf{m}_a$ chosen). Thus

$$[H, E^{\alpha_i}] = hE^{\alpha_i}, \quad \text{where} \quad h=0,1, \quad i=1,2,\dots,l. \tag{2.2}$$

It is clear from (2.2) that H provides an integer grading of the whole Lie algebra,

$$[H, E^\alpha_h] = hE^\alpha_h, \quad \text{where} \quad h = h(\alpha) \in Z. \tag{2.3}$$

In particular the elements of the algebra of the little group of H, which we shall call B, have zero grade. It is not difficult to see that the set of little groups B for all possible choices of H are just the (non-compact versions of) the little groups in the adjoint representation of the compact form of G. Since these little groups are, by definition, in one-one correspondence with the strata of G-orbits in the adjoint representation of (the compact form of) G and the strata can be partially ordered [8] it follows that the WZNW-reductions can be partially-ordered in the same way. The minimal stratum is unique, and has as little group the Cartan subgroup of G. It occurs [1] for $\mathbf{w} = \mathbf{s}$, where $\mathbf{s}$ sum over *all* the simple coweights (= half the sum of the positive coroots), and the corresponding (minimal) Toda theory is just the conventional Toda theory. The maximal strata are not unique. For example for $SL(N, R)$ they occur when the reducing matrix H has only two distinct eigenvalues and thus corresponds to a two-block reduction, $SL(N, R) \to S(L(p, R) \times L(q, R))$, where $p + q = N$. A particularly interesting case is the reduction of $SL(2n, R) \to S(L(n, R) \times L(n, R))$ of $SL(2n, R)$ into two equidimensional blocks. This case is a natural generalization of the Liouville case, to which it reduces for $n = 1$ and, accordingly, we shall call the resultant $S(L(n, R) \times L(n, R))$ theory the generalized Liouville theory.

The extension of (2.3) to the (left- or right-handed) KM algebras of WZNW theories (or indeed of any KM theories)

$$[J^a(z), J^b(w)] = f^{ab}{}_c J^c(w)\delta(z - w) + kg^{ab}\partial_z\delta(z - w), \tag{2.4}$$

where $J^a(z) = \operatorname{tr}(J(z)\sigma^a)$ and the σ's are the generators of G, is evidently

$$[H(z), J^\alpha_h(w)] = hJ^\alpha_h(w)\delta(z - w), \tag{2.5}$$

where $H(z) = \operatorname{tr}(J(z)H)$. Note that the part $J^B_\perp$ of the current J^B corresponding to the little group B, which is orthogonal to H, commutes with $H(z)$, and that $H(z)$ has a nonvanishing commutator with itself,

$$[H(z), J^B_\perp(w)] = 0, \quad [H(z), H(w)] = k\partial_z\delta(z - w)\operatorname{tr} H^2. \tag{2.6}$$

This means that if we modify the Virasoro operators $L(z)$ of the WZNW theories to

$$\Lambda(z) = L(z) + \partial_z H(z), \tag{2.7}$$

then $\Lambda(z)$ again satisfies a Virasoro algebra (with centre $c \to c_{\mathrm{KM}} + 12k\operatorname{tr} H^2$), but since

$$\begin{aligned}
[\Lambda(z), H(w)] &= -(\partial_w H(w))\delta(z - w) + H(w)\partial_z\delta(z - w) + k\operatorname{tr} H^2\partial_z^2\delta(z - w), \\
[\Lambda(z), J^\alpha_h(w)] &= -(\partial_w J^\alpha_h(w))\delta(z - w) + (1 + h)J^\alpha_h(w)\partial_z\delta(z - w),
\end{aligned} \tag{2.8}$$

only the KM current components $J^B_\perp$ are conformal vectors, the $H(z)$ being a spin-one *connection* and the $J^\alpha_h(z)$ being conformal tensors of weight $(1 + h)$. The physical meaning of $\Lambda(z)$ and the corresponding $\bar{\Lambda}(\bar{z})$ is that they are the components of the improved (i.e. traceless) energy-momentum tensor of the reduced theory and the physical meaning of the connection $H(z)$ is that it is a gravitational connection of the Polyakov type [9]. In fact if we define the field $h(z)$ as $H(z)(\operatorname{tr} H^2)^{-1}$, then

$$\mathscr{D}_z = \partial_z + \frac{s}{k} h(z), \tag{2.9}$$

is a covariant derivative for the current components of spin s, i.e.

$$[\Lambda(z), \mathscr{D}_w J^\alpha_s(w)] = -(\partial_w(\mathscr{D}_w J^\alpha_s(w)))\delta(z-w) + (1+s)(\mathscr{D}_w J^\alpha_s)\partial_z \delta(z-w). \tag{2.10}$$

Note that even in the classical case, for which $c_{\mathrm{KM}} = 0$, the centre c for Λ is not zero but $12k \operatorname{tr} H^2$.

From (2.8) it follows, in particular, that with respect to $\Lambda(z)$, *the current components of grade $h = -1$ transform as conformal scalars.* Because of this one can impose the constraints

$$J^\alpha_{-1}(z) = J^\alpha_{-1}(0) \neq 0 \quad \text{and} \quad J^\alpha_h(z) = 0, \quad h < -1, \tag{2.11}$$

without breaking conformal symmetry, or, more precisely, without breaking the conformal symmetry generated by $\Lambda(z)$. Note that, in general, the constraints (2.11) can be expressed as

$$J^{\text{constr.}}(z) = M_{-1} + J^{\text{pos}}(z), \tag{2.12}$$

where M_{-1} is a constant matrix of grade minus one and $J^{\text{pos}}(z)$ denotes the part of the current for which the components have zero or strictly positive grades. The constraints (2.11), or, equivalently, (2.12), are the constraints that define the reduced theory.

An intuitive feeling for the meaning of the constraints (2.11) or (2.12) may be obtained by considering the $G = SL(N, R)$ case, for which the constrained current $J(z)$ takes the form

$$J^{\text{constr.}}(z) = \begin{pmatrix} J_{11}(z) & J_{12}(z) & J_{13}(z) & \cdot & \cdot & J_{1n}(z) \\ M_{21} & J_{22}(z) & J_{23}(z) & \cdot & \cdot & J_{2n}(z) \\ 0 & M_{32} & J_{33}(z) & \cdot & \cdot & J_{3n}(z) \\ 0 & 0 & M_{43} & \cdot & \cdot & J_{4n}(z) \\ 0 & 0 & 0 & \cdot & \cdot & J_{5n}(z) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & M_{n,n-1} & J_{nn}(z) \end{pmatrix}, \tag{2.13}$$

where $M_{r+1,r} \equiv J_{r+1,r}(0)$ and the $M_{r+1,r}$ and $J_{ab}(z)$ denote submatrices of currents which in general are not single entries or even square matrices. (The single-entry case corresponds to the original Toda reduction.)

The constraints (2.11), (2.12) are obviously not invariant with respect to general KM transformations, $J(z) \to U(z)J(z)U^{-1}(z) + kU(z)\partial_z U^{-1}(z)$, but there exists a residual group of KM transformations with respect to which they are invariant. These are the KM transformations for which $U(z)$ lies in the subgroup A of G generated by the root vectors with strictly positive h, and correspond to the KM transformations that would be implemented by the constraints themselves. These residual KM transformations are then regarded as gauge transformations and only those functions, or functionals, of the constrained currents $J(z)$ which are invariant with respect to this gauge group are regarded as physical. Since there are $\dim A$ constraints and $\dim A$ gauge degrees of freedom, there are just $\dim G - 2\dim A = \dim B$ independent physical fields altogether. So there exist $\dim B$ independent gauge-invariant polynomials of the constrained currents and their derivatives. Furthermore, since the Poisson- or commutator-bracket of two gauge-invariant polynomials is again a gauge-invariant polynomial, it is clear that the Poisson- or commutator-bracket algebras of the $\dim B$ gauge-invariant polynomials will close. We define these Poisson- and commutator-bracket algebras to be the classical and quantum W-algebras of the generalized Toda theories.

3. W-Bases: Generalized Toda Theories with $G = SL(N, R)$ and B Maximal

We consider now the problem of constructing the $\dim B$ gauge-invariant polynomials in the constrained currents and their derivatives explicitly. To illustrate the idea in its simplest form we begin with the case of maximal $SL(N, R)$ Toda theories, which, as discussed in Sect. 2, are just the 2-block reductions of WZNW theory. In this case the reduction matrix $H = \mathbf{w}\cdot\mathbf{H}$ takes the form $H = \frac{1}{N}\mathrm{diag}(qI_p, -pI_q)$, where $p + q = N$ and I_p and I_q denote the unit matrices in p and q dimensions respectively. The constrained current (2.12) reduces to

$$J^{\mathrm{constr.}} = \begin{pmatrix} K(z) & R(z) \\ M & C(z) \end{pmatrix}, \tag{3.1}$$

where the entries K, R and C are p^2, pq and q^2 block-sub-matrices, respectively. The gauge group A of residual KM transformations discussed in the preceding section evidently consists of all matrices of the form

$$g(a) = \begin{pmatrix} I & a(z) \\ 0 & I \end{pmatrix}, \tag{3.2}$$

where $a(z)$ is a block-matrix and thus contains pq parameters. We shall assume that $p \geqq q$ and that the constant matrix M is minimally degenerate, i.e. that $\mathrm{rank}(MM^t) = q$. This means that there exists a matrix $\tilde{M}$ such that $M\tilde{M} = I_q$ (and the $\tilde{M}M$ is a rank-q projection on a space of p dimensions). It also means that we can choose a basis so that

$$J^{\mathrm{constr.}} = \begin{pmatrix} A & X & S \\ B & Y & T \\ 0 & I & C \end{pmatrix}, \tag{3.3}$$

where Y and C are square matrices of dimension q and A is a square matrix of dimension $p - q$. This basis will be called the canonical basis.

The gauge transformations of the constrained current with respect to $g(a)$ are

$$J^{\mathrm{constr.}} \to gJ^{\mathrm{constr.}}g^{-1} + g\partial g^{-1}, \tag{3.4}$$

where $g(a) \in A$, and we have set $k = 1$ to simplify the notation. It will be convenient to write (3.4) as

$$J^{\text{constr.}} \to \mathrm{Adj}(g) J^{\text{constr.}}, \tag{3.5}$$

where $\mathrm{Adj}(g)$ is the usual $\mathrm{adj}(g)$ of Lie group theory supplemented with the derivative term. The important point is that, as is easily verified from (3.4), $\mathrm{Adj}(g)$ satisfies the group property $\mathrm{Adj}(g_1)\mathrm{Adj}(g_2) = \mathrm{Adj}(g_1 g_2)$. It is easy to see that for the sub-block K of (3.1) the gauge transformations induce the transformations

$$K \to K + aM. \tag{3.6}$$

It follows that if we define a pq-block j as

$$j = K\tilde{M}, \tag{3.7}$$

then the gauge-transformation of j is simply

$$j \to j + a. \tag{3.8}$$

(Thus j absorbs all of the gauge-transformation.) Then, if we define $g(j)$ as the matrix $g(a)$ with a replaced by j we have

$$g(j) \to g(j + a). \tag{3.9}$$

Let us now define the current

$$J^{(2)} = \mathrm{Adj}(g^{-1}(j)) J^{\text{constr.}}. \tag{3.10}$$

We see at once that under a gauge transformation

$$\begin{aligned} J^{(2)} &\to \mathrm{Adj}(g^{-1}(j+a))\,\mathrm{Adj}(g(a)) J^{\text{constr.}} \\ &= \mathrm{Adj}(g^{-1}(j+a)g(a)) J^{\text{constr.}} = \mathrm{Adj}(g^{-1}(j)) J^{\text{constr.}} = J^{(2)}. \end{aligned} \tag{3.11}$$

Thus $J^{(2)}$ is gauge-invariant and its entries are the required gauge-invariant polynomials. That they form a complete set follows from the fact that $J^{\text{constr.}}$ has $\dim G - \dim A$ independent components and since $J^{(2)}$ is obtained from it by a gauge-transformation with $\dim A$ parameters (which are completely absorbed according to (3.8)) it must have $(\dim G - \dim A) - \dim A = \dim B$ independent components, which, as discussed in Sect. 2, is the total number of independent gauge-invariant polynomials. On computing $J^{(2)}$ explicitly we obtain

$$J^{(2)} = \begin{pmatrix} K^{(2)} & R^{(2)} \\ M & C^{(2)} \end{pmatrix} = \begin{pmatrix} K - jM & R + Kj - jC - jMj + j' \\ M & C + Mj \end{pmatrix}. \tag{3.12}$$

Thus the gauge-invariant polynomials in the diagonal blocks are actually linear in the original current components and the gauge-invariant polynomials in the off-diagonal block are bilinear. Note that since $J^{(2)}$ has only $\dim B$ independent entries it must satisfy $\dim A$ constraints, and it is easy to check that these are

$$K^{(2)}\tilde{M} = 0. \tag{3.13}$$

In the Toda reduction a gauge in which the gauge-invariant polynomials are current components themselves is called a Drinfeld–Sokolov (DS) [10] gauge. It is clear that the gauge defined by $J^{(2)}$ has this property. Thus DS gauges exist for the generalized Toda theories and we may write

$$J^{\text{DS}} = J^{(2)}. \tag{3.14}$$

The content of the gauge-invariant current $J^{(2)}$ becomes more explicit in the canonical basis, in which (3.12) reduces to

$$J^{(2)} = \begin{pmatrix} A^{(2)} & 0 & S^{(2)} \\ B^{(2)} & 0 & T^{(2)} \\ 0 & I & C^{(2)} \end{pmatrix}, \tag{3.15}$$

Note that in the generalized Liouville case, i.e. the case in which the two diagonal blocks are equidimensional ($p = q$), we have, in the canonical basis,

$$\begin{pmatrix} K & R \\ M & C \end{pmatrix} \to \begin{pmatrix} Y & T \\ I & C \end{pmatrix} \quad \text{so} \quad J^{(2)} = \begin{pmatrix} 0 & T - YC + Y' \\ I & C + Y \end{pmatrix}. \tag{3.16}$$

In particular, in the conventional Liouville case ($p = q = 1$) one finds that, by the traceless condition $C + Y = 0$, (3.16) reduces further to

$$J^{(2)} = \begin{pmatrix} 0 & \Lambda \\ 1 & 0 \end{pmatrix}, \tag{3.17}$$

where $\Lambda = T + Y^2 + Y' = \mathrm{tr}(\frac{1}{2}J^2 + HJ')$ is just the Virasoro operator [1] of that theory.

4. W-Bases: Generalized Toda Theories with $G = SL(N, R)$ and Arbitrary B

Let us next consider the reduction of $SL(N, R)$ WZNW theory corresponding to any subgroup B, i.e. corresponding to any number of sub-blocks. In this case the current takes the block-form shown in Eq. (2.13). It will, however, be convenient to label the entries by their weights with respect to the reducing matrix H of Sect. 2 and their rows. For $SL(N, R)$ this means that we use the rows and the lines parallel to the diagonal, rather than the conventional rows and columns. Thus we write

$$J^{\mathrm{constr.}} = \begin{pmatrix} J_{01} & J_{11} & J_{21} & \cdot & \cdot & J_{n-2,1} & J_{n-1,1} \\ M_{-1,2} & J_{02} & J_{12} & \cdot & \cdot & J_{n-3,2} & J_{n-2,2} \\ 0 & M_{-1,3} & J_{03} & \cdot & \cdot & J_{n-4,3} & J_{n-3,3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & J_{0,n-2} & J_{1,n-2} & J_{2,n-2} \\ 0 & 0 & 0 & \cdot & M_{-1,n-1} & J_{0,n-1} & J_{1,n-1} \\ 0 & 0 & 0 & \cdot & 0 & M_{-1,n} & J_{0,n} \end{pmatrix}. \tag{4.1}$$

It will also be convenient to parametrize the elements $g(\alpha)$ of the gauge group G, which is the group generated by all real strictly upper-triangular matrices, as

$$g(\alpha) = c_1(a_1)c_2(a_2)\cdots c_{n-1}(a_{n-1}), \tag{4.2}$$

where the $c_h(a_h)$ are the matrices

$$c_h(a_h) = \begin{pmatrix} I & 0 & 0 & \cdots & a_{h,1} & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 & a_{h,2} & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & a_{h,3} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot & \cdot & \cdots & a_{h,n-h} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & \cdot & \cdot & \cdot & \cdots & I \end{pmatrix}, \tag{4.3}$$

and α denotes the collection of parameters $a_{h,r}$. It will also be convenient to consider the family of nested subgroups G_h of G defined as those with elements

$$g_h(\alpha_h) = c_h(a_h)c_{h+1}(a_{h+1})\cdots c_{n-1}(a_{n-1}), \tag{4.4}$$

where α_h denotes the collection of parameters $a_h, a_{h+1}\cdots a_{n-1}$. Note that the $c_h(a_h)$ may be regarded as representatives of the cosets G_h/G_{h+1}.

At this point we have to make an assumption concerning the non-degeneracy of the matrix M_{-1}, consisting of all the submatrices $M_{-1,r}$. This is the assumption that the diagonal blocks $J_{0,r}$ in $J^{\text{constr.}}$ are arranged in order of non-increasing dimension and that the rank of the submatrices $M_{-1,r}M^t_{-1,r}$ is $\dim J_{0,r}$. As in the previous section, this means that there exists a set of matrices $\tilde{M}_{1,r}$ such that $M_{-1,r}\tilde{M}_{1,r} = I$, where I is the unit matrix for the block $J_{0,r}$ (and the same matrices multiplied in the reverse order form a projection of rank-$\dim J_{0,r}$ for the block $J_{0,r-1}$). In a canonical basis $M_{-1,r}$ takes the form $(0 \quad I)$.

Suppose now that $J^{(h)}$ is any current of the constrained form (4.1) for which the gauge transformation induced by the general gauge transformation $J^{\text{constr.}} \to \text{Adj}(g(\alpha))J^{\text{constr.}}$ is only with respect to the subgroup G_h, i.e.

$$J^{(h)} \to \text{Adj}(g_h(\alpha_h))J^{(h)}. \tag{4.5}$$

It is easy to see that the block-components of $J^{(h)}$ with weights less than $h-1$ are left invariant and that the block-components of weight $h-1$ undergo the simple translations

$$J^{(h)}_{h-1,r} \to J^{(h)}_{h-1,r} + [c_h, M_{-1}]_{h-1,r}. \tag{4.6}$$

More explicitly, for $1 \leqq r \leqq n-h+1$, they are

$$J^{(h)}_{h-1,r} \to J^{(h)}_{h-1,r} + a_{h,r}M_{-1,h+r} - M_{-1,r}a_{h,r-1}, \tag{4.7}$$

where we have defined $a_{h,0} = M_{-1,n+1} = 0$. It is easy to verify from (4.6) and (4.7) that if we construct linear combinations $j_{h,r}$ of the $J^{(h)}_{h-1,r}$ by the iterative process

$$j_{h,r} = (M_{-1,r}j_{h,r-1} + J^{(h)}_{h-1,r})\tilde{M}_{1,h+r}, \tag{4.8}$$

starting from $j_{h,0} = 0$, they transform according to

$$j_{h,r} \to j_{h,r} + a_{h,r}, \tag{4.9}$$

for $1 \leqq r \leqq n-h$ (and fully absorb the coset G_h/G_{h+1} part of the gauge transformation).

Let us now define the currents

$$J^{(h+1)} = \mathrm{Adj}(c_h^{-1}(j_h))J^{(h)}, \tag{4.10}$$

where $c_h(j_h)$ denotes the coset matrix $c_h(a_h)$ with $a_{h,r}$ replaced by $j_{h,r}$. Then the gauge transformation of $J^{(h+1)}$ induced by that of $J^{(h)}$ is evidently

$$\begin{aligned} J^{(h+1)} &\to \mathrm{Adj}(c_h^{-1}(j_h + a_h))\,\mathrm{Adj}(g_h(\alpha_h))J^{(h)} \\ &= \mathrm{Adj}(c_h^{-1}(j_h + a_h)g_h(\alpha_h)c_h(j_h))J^{(h+1)}. \end{aligned} \tag{4.11}$$

But, since $g_h(\alpha_h) = g_h(a_h, \alpha_{h+1})$, it is evident from the nilpotent structure of the gauge group that the argument of Adj in (4.11) is an element of the subgroup G_{h+1}. Thus

$$J^{(h+1)} \to \mathrm{Adj}(g_{h+1}(\alpha_{h+1}))J^{(h+1)}, \tag{4.12}$$

where α_{h+1} is some function of α_h and j_h. Since $G_n \equiv 1$, it then follows by induction, starting from $J^{(1)} = J^{\mathrm{constr.}}$, that the current $J^{(n)}$ is gauge-invariant. We have thus shown that the components of the $(n)^{\mathrm{th}}$ current in the sequence

$$J^{(h+1)} = \mathrm{Adj}(\pi_h^{-1}(j))J^{(1)}, \tag{4.13}$$

where

$$\pi_h(j) = c_1(j_1)c_2(j_2)\cdots c_h(j_h), \tag{4.14}$$

and the j's are defined by (4.8), are gauge-invariant polynomials. Furthermore, they form a complete set because, as before, $J^{(1)} = J^{\mathrm{constr.}}$ contains $\dim G - \dim A$ independent components, and since $J^{(n)}$ is obtained from it by gauge transformations with $\dim A$ parameters (which are completely absorbed according to (4.9)) it must contain $\dim G - 2\dim A = \dim B$ independent components, which is the total number of gauge-invariant polynomials. This implies, of course, that the components of $J^{(n)}$ are subject to $\dim A$ constraints. To see this explicitly, we first note that from (4.7), (4.8) and (4.10) the block-components of weight $h-2$ of $J^{(h)}$ can be written as

$$J^{(h)}_{h-2,r} = (J^{(h-1)}_{h-2,r} + M_{-1,r}j_{h-1,r-1})(I - \tilde{M}_{1,h+r-1}M_{-1,h+r-1}), \tag{4.15}$$

from which we obtain the constraints $J^{(h)}_{h-2,r}\tilde{M}_{1,h+r-1} = 0$ for $1 \leqq r \leqq n-h+1$. Since the block-components $J^{(h)}_{k,r}$ of weight $k < h-2$ are equal to $J^{(k+2)}_{k,r}$ which fulfill the above constraints, the constraints on $J^{(h)}$ can be collected as

$$J^{(h)}_{k,r}\tilde{M}_{1,k+r+1} = 0 \quad \text{for} \quad 0 \leqq k \leqq h-2, \quad 1 \leqq r \leqq n-k-1. \tag{4.16}$$

Then we find that for $J^{(n)}$ the total number of the constraints in the entries of (4.16) is exactly $\dim A$. Finally, since $J^{(n)}$ is a current whose components are the gauge-invariant polynomials it is, by definition, a Drinfeld–Sokolov current,

$$J^{(n)} = J^{\mathrm{DS}}. \tag{4.17}$$

We conclude this section by considering the case when the dimensions of all the diagonal blocks are equal (as happens, for example, in the original Toda case where they are all of dimension one). In this case the matrices $M_{-1,r}$ can be chosen to be unit matrices and then we see from the definition that the subcurrents j_h are

$$\begin{aligned} j_{h,1} &= J^{(h)}_{h-1,1}, \\ j_{h,2} &= J^{(h)}_{h-1,1} + J^{(h)}_{h-1,2}, \\ j_{h,3} &= J^{(h)}_{h-1,1} + J^{(h)}_{h-1,2} + J^{(h)}_{h-1,3}, \end{aligned} \tag{4.18}$$

and so on. This means that (apart from the constant M_{-1}-blocks) the blocks to the left of the h^{th} vertical column in each $J^{(h)}$ vanish, which is also clear from (4.16). In particular, all the blocks in $J^{(n)}$ vanish except those in the last column. Thus the entries in the last column of $J^{(n)}$ are the gauge invariant polynomials for the equidimensional $SL(N, R)$ reduction.

For example, for the 3-block (Toda) reduction of $SL(3, R)$ WZNW theory one easily computes that

$$J^{(2)} = \begin{pmatrix} 0 & J_{11} + J_{01}^2 + J'_{01} & J_{21} - J_{01}J_{12} - (J_{11} - J_{01}J_{02})J_{03} + J_{01}J'_{03} \\ 1 & 0 & J_{12} - J_{02}J_{03} - J'_{03} \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.19}$$

using $J_{01} + J_{02} + J_{03} = 0$, and hence that

$$J^{(3)} = \begin{pmatrix} 0 & 0 & W_3 \\ 1 & 0 & W_2 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.20}$$

where

$$W_2 = J_{01}^2 + J_{01}J_{03} + J_{03}^2 + J_{11} + J_{12} + J'_{01} - J'_{03} = \mathrm{tr}(\tfrac{1}{2}J^2 + HJ'), \tag{4.21}$$

and

$$\begin{aligned} W_3 &= J_{21} - J_{01}J_{12} - J_{11}J_{03} + J_{01}J_{02}J_{03} + J'_{11} + J_{01}(J'_{01} - J'_{02}) + J''_{01} \\ &= \mathrm{tr}[\tfrac{1}{3}J^3 + P(J'J + J'')] + \mathrm{tr}(PJ)\,\mathrm{tr}(H^2J'), \end{aligned} \tag{4.22}$$

$H = \mathrm{diag}(1, 0, -1)$ being the reduction matrix and $P = \frac{1}{2}(H^2 + H)$ being the projection operator onto the first component of any vector. The gauge-invariant polynomials W_2 and W_3 are the second and third-order elements of the W-algebra of the $SL(3, R)$ Toda theory, the second-order polynomial W_2 being the Virasoro operator. The expression (4.22) is not homogeneous in the generators because the projection P is not homogeneous, but by subtracting $W'_2/2$ from (4.22) we obtain

$$\tilde{W}_3 = \mathrm{tr}[\tfrac{1}{3}J^3 + \tfrac{1}{2}(H(J'J + JJ') + H^2J'')] + \tfrac{1}{4}[\mathrm{tr}(HJ)\,\mathrm{tr}(H^2J') - \mathrm{tr}(H^2J)\,\mathrm{tr}(HJ')], \tag{4.23}$$

which is homogeneously cubic in the generators.

5. W-Bases: Generalized Toda Theories for Arbitrary G and B

We now turn to the generalized Toda theories corresponding to any of the subgroups B of any (maximally non-compact) WZNW group G. We first note that the number n of blocks in the reduction due to the reduction matrix $H = \mathbf{w}\cdot\mathbf{H}$ is $n = \mathbf{w}\cdot\psi + 1$, where ψ is the highest root of G. We then write the constrained currents (2.12) more explicitly as

$$J^{\text{constr.}} = M_{-1} + \sum_{d=0}^{n-1} J_d \cdot E_d, \tag{5.1}$$

where the grading is with respect to $H = \mathbf{w}\cdot\mathbf{H}$, and $J_d \cdot E_d$ means $\sum_r J^r_d E^r_d$, where the summation index r runs for all generators of grade d (whose range therefore may vary with d). The elements of the gauge group are of the form

$$g(\alpha) = \exp\left(\sum_{d=1}^{n-1} a_d \cdot E_d\right), \tag{5.2}$$

where the a_d's are the parameters. We define the nested subgroups G_h of gauge-transformations

$$g_h(\alpha_h) = \exp\left(\sum_{d=h}^{n-1} a_d \cdot E_d\right), \tag{5.3}$$

where α_h denotes all the parameters for $d \geqq h$, and the coset representatives

$$c_h(a_h) = \exp(a_h \cdot E_h), \tag{5.4}$$

for the cosets G_h/G_{h+1}. In particular we have

$$g_h(\alpha_h) = g_{h+1}(\tilde{\alpha})c_h(a_h), \tag{5.5}$$

where the $\tilde{\alpha}$ are some functions of the α_h.

As we did before we shall make an assumption about the non-degeneracy of the matrix M_{-1}. To see what assumption we should make we express the assumption for the $SL(N,R)$ case in a more general form. It is not difficult to see that the $SL(N,R)$ non-degeneracy assumption is that the adjoint action of the matrix M_{-1} on the Lie algebra of the gauge-group (see (4.6)) is not singular (has no kernel). Indeed this is why all of the a_h appear in (4.9) and can be compensated by linear combinations of the J_h. The natural extension of this assumption to any group G is that the adjoint action of M_{-1} on the Lie algebra of the gauge-group has no kernel, and this is the assumption that we shall make. If we denote the space of all generators of G of weight h by S_h, then since M_{-1} has a definite weight, this assumption can also be expressed by saying that the kernels of the maps $E^r_h \to \tilde{E}^r_{h-1} = \mathrm{adj}(M_{-1})E^r_h \equiv [M_{-1}, E^r_h]$ of S_h into S_{h-1} for $h \geqq 1$ are zero. Note that S_0 is just the Lie algebra of the subgroup B and that in general these maps are only into, i.e. the images $\tilde{S}_{h-1}$ of the maps are only subspaces of S_{h-1}. (For $SL(N,R)$ they are onto only if $h=1$ and the blocks are onedimensional.) Let F^r_{1-h} be linear combinations of the generators E^r_{1-h} which are trace orthogonal (dual) to the $\tilde{E}^s_{h-1}$:

$$\mathrm{tr}(F^r_{1-h}\tilde{E}^s_{h-1}) = \delta_{rs}, \tag{5.6}$$

where the non-degeneracy of M_{-1} guarantees that the indices r,s run from 1 to $\dim S_h$. Note that the F^r_{1-h} are not unique unless the map is onto, $\tilde{S}_{h-1} = S_{h-1}$. But this will not affect the results.

Now suppose that there exists a current $J^{(h)}$ of the form (5.1) for which the gauge transformation that is induced by the original gauge transformation of J is only with respect to $g_h(\alpha_h)$,

$$J^{(h)} \to \mathrm{Adj}(g_h(\alpha_h))J^{(h)}. \tag{5.7}$$

From (5.5) we then have

$$\begin{aligned} J^{(h)} &\to \mathrm{Adj}(g_{h+1}(\tilde{\alpha}))\,\mathrm{Adj}(c_h(a_h))J^{(h)} \\ &= \mathrm{Adj}(g_{h+1}(\tilde{\alpha}))\left(M_{-1} - a_h\cdot\tilde{E}_{h-1} + \sum_{d=0}^{h-1} J_d^{(h)}\cdot E_d + O(d \geqq h)\right) \\ &= M_{-1} - a_h\cdot\tilde{E}_{h-1} + \sum_{d=0}^{h-1} J_d^{(h)}\cdot E_d + O(d \geqq h). \end{aligned} \tag{5.8}$$

From (5.8) we see at once that if we define the quantities

$$j_h^r = -\operatorname{tr}(J^{(h)} F_{1-h}^r) \tag{5.9}$$

then they gauge-transform according to

$$j_h^r \to j_h^r + a_h^r. \tag{5.10}$$

In particular, the coset representatives $c_h(j_h)$ gauge-transform according to

$$c_h(j_h) \to c_h(j_h + a_h). \tag{5.11}$$

Hence, if we now define the currents

$$J^{(h+1)} = \mathrm{Adj}(c_h^{-1}(j_h))J^{(h)}, \tag{5.12}$$

then by exactly the same argument that led from (4.10) to (4.14) we conclude that the components of $J^{(n)}$ in the sequence of (4.13) are gauge-invariant polynomials. Also, as in the two preceding sections one sees that they form a complete set and that $J^{(n)}$ is a DS current,

$$J^{(n)} = J^{\mathrm{DS}}. \tag{5.13}$$

The procedure of the last three sections may be summarized in a more abstract way as follows: Suppose $J^{(h)}$ is a current that gauge transforms only with respect to the subgroup G_h. Then the components of $J^{(h)}$ of weight $k < h-1$ do not transform at all, and the components of weight $h-1$ transform according to

$$\begin{aligned} J_{h-1}^{(h)} &\to J_{h-1}^{(h)} + [c_h, M_{-1}] \\ &= J_{h-1}^{(h)} - \mathrm{adj}(M_{-1})c_h. \end{aligned} \tag{5.14}$$

Hence, if we assume that $\mathrm{adj}(M_{-1})$ is non-singular and define

$$J^{(h+1)} = \mathrm{Adj}(c_h^{-1}(j_h))J^{(h)}, \quad \text{where} \quad j_h = -(\mathrm{adj}(M_{-1}))^{-1} P_h J_{h-1}^{(h)}, \tag{5.15}$$

and P_h is the projection on the subspace $\tilde{S}_{h-1}$ of S_{h-1}, the j_h and $J^{(h+1)}$ transform according to

$$j_h \to j_h + a_h \quad \text{and} \quad J^{(h+1)} \to \mathrm{Adj}(g_{h+1}(\alpha_{h+1}))J^{(h+1)}, \tag{5.16}$$

respectively. It then follows by induction that the components of the n^{th} current $J^{(n)}$ in the sequence defined by (5.16) are gauge-invariant polynomials, and, because of the construction, form a complete set.

6. General Procedure for Computing W-Algebras

Let us consider now the construction of the W-algebras themselves. The general idea is the same as was used in ref. [1], namely to consider the (current-dependent) KM transformations that keep the constrained currents in the DS-gauge *form*-invariant and compute the changes in the non-zero components due to these transformations. Because the gauge-invariant polynomials W are linear in the DS current-components, and the KM transformations are canonical, these changes are just the changes in the W's that are induced by the (Poisson-bracket) W-algebra, and thus the structure functions for the Poisson-bracket W-algebra can be obtained from them by inspection. In other words we proceed as follows: First we determine the most general matrix K which leaves J^{DS} form-invariant, i.e. that satisfies

$$[K, J^{DS}] - K' = \delta J^{DS}, \tag{6.1}$$

where it is understood that δJ^{DS} satisfies the same conditions as J^{DS}. Then we parametrize K in some convenient way as $K = K(\alpha^a(z))$, where the α's are a set of $\dim B$ parameters, $a = 1, \ldots, \dim B$. Since the components of J^{DS} are the gauge-invariant polynomials W the canonical transformations

$$J^{DS} \to J^{DS} + \delta_\alpha J^{DS} \tag{6.2}$$

define the corresponding canonical transformations

$$W \to W + \delta_\alpha W \tag{6.3}$$

of the matrix of gauge-invariant polynomials corresponding to J^{DS}. Since these transformations are canonical we are guaranteed that the variations of the W's can be written in the form

$$\delta_\alpha W(w) = \int dz \alpha^a(z) [W_a(z), W(w)], \tag{6.4}$$

for some suitable choice of $W_a(z)$. Then, once the $W_a(z)$ are identified in terms of the $W(w)$, the W-algebra can be obtained from (6.4) by inspection.

So, in practice, all one has to do is compute the most general K that keeps J^{DS} form-invariant, parametrize it in a suitable manner, and compute the variations of the components of J^{DS} for each parameter. Once this is done, and the components of W_a identified in terms of the W, the W-algebra can be read off from (6.4). In identifying the base-elements W_a it is useful to use the fact that the Poisson-brackets of any two elements must be anti-symmetric. Although this method of computing W-algebras is much more efficient than many others, it is still quite laborious for more than two blocks and for general WZNW groups G. Hence in this paper we shall restrict ourselves to the 2-block reductions of $SL(N, R)$.

7. W-Algebra for Generalized Liouville Theories

To illustrate the basic idea, and because this is an exceptional case that has to be treated separately anyway, let us first consider the case where the two blocks in the maximal Toda theory are equidimensional. In that case $N = 2n$, the reducing matrix H is just $H = \frac{1}{2}\mathrm{diag}(I_n, -I_n)$, where I_n is the n-dimensional unit matrix and from (3.16) we see that, in the canonical basis, the constrained DS-current is of the form

$$J^{\mathrm{DS}} = \begin{pmatrix} 0 & T \\ I & C \end{pmatrix} \quad \text{where} \quad \operatorname{tr} C = 0. \tag{7.1}$$

(Strictly speaking, the T and C should be written as T^{DS} and C^{DS}, but we drop the superscripts to simplify the notation.) We write the most general $SL(2n, R)$ matrix K in the form

$$K = \begin{pmatrix} x & y \\ \tau & \gamma \end{pmatrix} \quad \text{where} \quad \operatorname{tr} K = 0. \tag{7.2}$$

The Greek submatrices are the natural independent parameters because they are conjugate to the C and T submatrices in the current with respect to the KM centre, and the Latin submatrices are to be determined from the condition that with respect to a KM transformation by K the current J^{DS} remains form-invariant. The KM variation $\delta J^{\mathrm{DS}} = [K, J^{\mathrm{DS}}] - K'$ of J^{DS} generated by K is easily seen to be

$$\delta J^{\mathrm{DS}} = -\begin{pmatrix} T\tau - y + x' & T\gamma - xT - yC + y' \\ x + C\tau - \gamma + \tau' & y + [C, \gamma] - \tau T + \gamma' \end{pmatrix}, \tag{7.3}$$

and from this one sees at once that K will leave J^{DS} form-invariant if, and only if,

$$x = \gamma - C\tau - \tau' \quad \text{and} \quad y = T\tau + x'. \tag{7.4}$$

The general K matrices satisfying these conditions split naturally into

$$K_\gamma = \begin{pmatrix} \gamma & \gamma' \\ 0 & \gamma \end{pmatrix} \quad \text{and} \quad K_\tau = \begin{pmatrix} -\theta & T\tau - \theta' \\ \tau & \tau_0 I \end{pmatrix}, \tag{7.5}$$

where

$$\operatorname{tr} \gamma = 0, \quad \theta = C\tau + \tau' - \tau_0 I \quad \text{and} \quad N\tau_0 = \operatorname{tr}(C\tau + \tau'), \tag{7.6}$$

and the τ_0 is inserted in order to make K_τ traceless. From (7.3) one can read off the variations in the components of J^{DS} due to K_γ and K_τ, namely,

$$\begin{aligned} &\delta_\gamma C = [\gamma, C] - 2\gamma', \qquad \delta_\gamma T = [\gamma, T] + \gamma' C - \gamma'', \\ &\delta_\tau C = [\tau, T] + \theta' - \tau_0', \quad \delta_\tau T = T\tau C - (T\tau)' - (\theta + \tau_0) T - \theta' C + \theta''. \end{aligned} \tag{7.7}$$

The display in (7.7) defines the W-algebra for this case. (When allowance is made for partial integration the display is anti-symmetric.) Let us denote the elements of the W-algebra by the corresponding components $C_a = \operatorname{tr}(\sigma_a C)$ and $T_a = \operatorname{tr}(\sigma_a T)$ of C and T, where the σ's are the generators of $GL(n, R)$ in the fundamental (n-dimensional) representation (and thus include a multiple of the unit matrix as well as the usual $SL(n, R)$ generators). Then the W-algebra given by (7.7) is easily seen to take the explicit form

$$\begin{aligned} &[C_a(z), C_b(w)] = -f_{ab}^{e} C_e(w)\delta(z - w) + 2g_{ab}\delta'(z - w), \\ &[C_a(z), T_b(w)] = -f_{ab}^{e} T_e(w)\delta(z - w) - h_{ab}^{e} C_e(w)\delta'(z - w) - g_{ab}\delta''(z - w), \end{aligned} \tag{7.8a}$$

and

$$
\begin{aligned}
[T_a(z), T_b(w)] = \Big[& (h^{rs}_{ab} - h^{sr}_{ab}) T_r(w) C_s(w) - h^{rs}_{ab} C'_r(w) C_s(w) \\
& - h^e_{ab}(T'_e(w) - C''_e(w)) + \frac{1}{N}(C'_a(w)C_b(w) - C''_a(w)(\operatorname{tr}\sigma_b)) \Big] \delta(z-w) \\
& + \Big[h^{rs}_{ab} C_r(w) C_s(w) + (h^e_{ab} + h^e_{ba}) T_e(w) \\
& - 2h^e_{ab} C'_e(w) + \frac{1}{N}(2C'_a(w)(\operatorname{tr}\sigma_b) - C_a(w)C_b(w)) \Big] \delta'(z-w) \\
& + \Big[\frac{1}{N}(C_b(w)(\operatorname{tr}\sigma_a) - C_a(w)(\operatorname{tr}\sigma_b)) + f^e_{ab} C_e(w)) \Big] \delta''(z-w) \\
& + \Big[-g_{ab} + \frac{1}{N}(\operatorname{tr}\sigma_a)(\operatorname{tr}\sigma_b) \Big] \delta'''(z-w),
\end{aligned}
\tag{7.8b}
$$

where the primes on $\delta(z-w)$ mean differentiation with respect to z, the f^e_{ab} are the structure constants of $SL(n, R)$ and

$$
g_{ab} = \operatorname{tr}(\sigma_a \sigma_b), \quad h^e_{ab} = \operatorname{tr}(\sigma^e \sigma_a \sigma_b) \quad \text{and} \quad h^{rs}_{ab} = \operatorname{tr}(\sigma^r \sigma_a \sigma^s \sigma_b). \tag{7.9}
$$

Note that the $[C_a, C_b]$ part of the algebra is just a KM algebra (with the centre double that of the original KM algebra). In this sense the W-algebra (7.8) may be regarded as a polynomial extension of a KM algebra. The sense in which it is an extension of a Virasoro algebra will be discussed in Sect. 9. For the moment we note only that for the $SL(2, R)$ (Liouville) case $C_a = 0$, and T_a reduces to a single component and that, since $J^{\mathrm{DS}} = J^{(2)}$, this component is identical to the Virasoro operator Λ obtained in (3.17). Indeed, for this single component the W-algebra (7.8) reduces to

$$
[\Lambda(z), \Lambda(w)] = -\Lambda'(w)\delta(z-w) + 2\Lambda(w)\delta'(z-w) - \tfrac{1}{2}\delta'''(z-w), \tag{7.10}
$$

which is just the Virasoro algebra.

8. W-Algebra for Generalized Toda Theories with $G = SL(N, R)$ and B Maximal

Let us now consider the generic maximal case when the two blocks are not equi-dimensional, i.e., the reduction matrix H is of the form $H = \frac{1}{N}\operatorname{diag}[nI_{m+n}, -(m+n)I_n]$, where $m \geqq 1$, $n \geqq 1$ and $N = m + 2n$. In that case the DS-current in the canonical basis is of the form (see (3.15))

$$
J^{\mathrm{DS}} = \begin{pmatrix} A & 0 & S \\ B & 0 & T \\ 0 & I & C \end{pmatrix}, \quad \text{where} \quad \operatorname{tr} A + \operatorname{tr} C = 0, \tag{8.1}
$$

$\dim A = m$ and $\dim I = \dim T = \dim C = n$. (As in (7.1) the entries in (8.1) should have superscripts DS, but we have omitted them to simplify the notation.) We then write the most general $SL(N, R)$ matrix as

$$K = \begin{pmatrix} \alpha & \beta & x \\ u & v & w \\ \sigma & \tau & \gamma \end{pmatrix}, \quad \text{where} \quad \operatorname{tr} K = 0. \tag{8.2}$$

Here again the Greek submatrices are to be regarded as the independent parameters and the Latins are to be determined by the form-invariance condition. The KM variation $\delta J^{\mathrm{DS}} = [K, J^{\mathrm{DS}}] - K'$ of J^{DS} generated by K is easily seen to be

$$\begin{pmatrix} [\alpha, A] + \beta B - S\sigma - \alpha' & x - A\beta - S\tau - \beta' & \alpha S + \beta T + xC - Ax - S\gamma - x' \\ uA + vB - B\alpha - T\sigma - u' & w - B\beta - T\tau - v' & uS + vT + wC - Bx - T\gamma - w' \\ \sigma A + \tau B - u - C\sigma - \sigma' & \gamma - v - C\tau - \tau' & \sigma S + \tau T + [\gamma, C] - w - \gamma' \end{pmatrix} \tag{8.3}$$

from which one sees that J^{DS} remains form-invariant if, and only if,

$$\begin{aligned} x &= A\beta + S\tau + \beta', \\ u &= \sigma A + \tau B - C\sigma - \sigma', \\ v &= \gamma - C\tau - \tau', \\ w &= B\beta + T\tau + v'. \end{aligned} \tag{8.4}$$

The matrices K that satisfy this condition split naturally into the six sets

$$K_\alpha = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_\beta = \begin{pmatrix} 0 & \beta & A\beta + \beta' \\ 0 & 0 & B\beta \\ 0 & 0 & 0 \end{pmatrix}, \quad K_\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & \gamma' \\ 0 & 0 & \gamma \end{pmatrix}, \tag{8.5}$$

$$K_0 = \frac{1}{N} \begin{pmatrix} 2n\alpha_0 & 0 & 0 \\ 0 & -m\alpha_0 & -m\alpha_0' \\ 0 & 0 & -m\alpha_0 \end{pmatrix}, \quad K_\sigma = \begin{pmatrix} 0 & 0 & 0 \\ \Sigma & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix}, \tag{8.6}$$

and

$$K_\tau = \begin{pmatrix} \tau_0 I & 0 & S\tau \\ \tau B & -\theta & T\tau - \theta' \\ 0 & \tau & \tau_0 I \end{pmatrix}, \tag{8.7}$$

where

$$\Sigma = \sigma A - C\sigma - \sigma', \quad \theta = C\tau + \tau' - \tau_0 I_n \quad \text{and} \quad N\tau_0 = \operatorname{tr}(C\tau + \tau'). \tag{8.8}$$

As before, the τ_0 has been inserted in order to make K_τ traceless. Note that $\operatorname{tr} \theta = (m + n)\tau_0$. From (8.3) one can now read off the variations in the components of J^{DS} due to the K's and one finds the following table:

	$\hat{A}$	$\hat{C}$	$\operatorname{tr} A$	B	S	T
δ_α	$[\alpha, A] - \alpha'$	0	0	$-B\alpha$	αS	0
δ_γ	0	$[\gamma, C] - 2\gamma'$	0	γB	$-S\gamma$	$[\gamma, T] + \gamma' C - \gamma''$
δ_0	0	0	$\frac{-2mn}{N}\alpha_0'$	$-B\alpha_0$	$\alpha_0 S$	$\frac{m}{N}(-\alpha_0' C + \alpha_0'')$
δ_β	$(\widehat{\beta B})$	$-(\widehat{B\beta})$	$\operatorname{tr}(\beta B)$	0	$*$	$*$
δ_σ	$-(\widehat{S\sigma})$	$(\widehat{\sigma S})$	$-\operatorname{tr}(S\sigma)$	$*$	0	$*$
δ_τ	0	$[\tau, T] + \hat{\theta}'$	$-m\tau_0'$	$*$	$*$	$*$

(8.9)

where hat means that the trace part is to be removed, e.g., $\hat{A} = A - \frac{1}{m}\operatorname{tr} A$, and the lower right-hand 3×3 subtable is

	B	S	T
δ_β	0	$\beta(T - C') + AX + X'$	$BX - (B\beta)'$
δ_σ	$-T\sigma + \Sigma A - \Sigma'$	0	ΣS
δ_τ	$(\tau B)A - C(\tau B) - 2\tau' B - \tau B'$	$(S\tau)C - A(S\tau) - (S\tau)'$	Z

(8.10)

where

$$X = \beta C - A\beta - \beta' \quad \text{and} \quad Z = [\tau, BS] + T\tau C - (\theta + \tau_0)T - (T\tau)' - \theta' C + \theta''. \tag{8.11}$$

The array (8.9) defines the W-algebra for the general maximal $SL(N, R)$ Toda theory. Note that the first three rows and columns in (8.9) define an $S(L(m) \times L(n))$ KM algebra, and the first four rows and columns an $S(L(m) \times L(n)) \wedge A(n)$ KM algebra, where $A(n)$ is the real abelian Lie group of dimension n^2. Thus the W-algebra defined by (8.9) is a polynomial extension of KM algebra, and the KM subalgebra is quite large.

To write out the W-algebra defined by (8.9) would be quite laborious on account of the parameters being block-matrices so we shall content ourselves with writing it out for the $S(L(2) \times L(1))$ reduction of $SL(3, R)$. In this case $\hat{A} = \hat{C} = 0$ and if we write $\operatorname{tr} A = -\operatorname{tr} C = a$ one obtains from the last four rows and columns of (8.9) the four-dimensional array:

	a	B	S	T
δ_0	$-\frac{2}{3}\alpha_0'$	$-B\alpha_0$	$S\alpha_0$	$\frac{1}{3}(a\alpha_0' + \alpha_0'')$
δ_β	$B\beta$	0	$\bar{T}\beta - 3a\beta' - \beta''$	$-\bar{B}\beta - 2B\beta'$
δ_σ	$-S\sigma$	$-\bar{T}\sigma - 3(a\sigma)' + \sigma''$	0	$2aS\sigma - S\sigma'$
δ_τ	$\frac{1}{3}[(a\tau)' - \tau'']$	$\bar{B}\tau - 2(B\tau)'$	$-(2Sa + S')\tau - S\tau'$	Z

(8.12)

where $\bar{T} = T - 2a^2 - a'$ and $\bar{B} = 2aB + B'$, the last entry Z is given by

$$Z = -T\tau' - (T\tau)' - \tfrac{2}{3}[(aa' + a'')\tau + (a^2 + 2a')\tau' - \tau'''], \tag{8.13}$$

and B, S and T are no longer matrices but simple functions. From (8.12) one can read off the W-algebra in an obvious notation as

$$\begin{aligned}
&[W_a(z), W_a(w)] = \tfrac{2}{3}\delta'(z-w),\\
&[W_a(z), W_b(w)] = -W_b(w)\delta(z-w),\\
&[W_a(z), W_s(w)] = W_s(w)\delta(z-w),\\
&[W_a(z), W_t(w)] = \tfrac{1}{3}[-W_a(w)\delta'(z-w) + \delta''(z-w)],\\
&[W_b(z), W_b(w)] = 0,\\
&[W_b(z), W_s(w)] = [W_t(w) - 2W_a^2(w) - W_a'(w)]\delta(z-w)\\
&\qquad\qquad\qquad + 3W_a(w)\delta'(z-w) - \delta''(z-w),\\
&[W_b(z), W_t(w)] = [-2W_a(w)W_b(w) - W_b'(w)]\delta(z-w) + 2W_b(w)\delta'(z-w),\\
&[W_s(z), W_s(w)] = 0,\\
&[W_s(z), W_t(w)] = 2W_s(w)W_a(w)\delta(z-w) + W_s(w)\delta'(z-w),\\
&[W_t(z), W_t(w)] = -[W_t'(w) + \tfrac{2}{3}W_a(w)W_a'(w) + \tfrac{2}{3}W_a''(w)]\delta(z-w)\\
&\qquad\qquad\qquad + [2W_t(w) + \tfrac{2}{3}W_a^2(w) + \tfrac{4}{3}W_a'(w)]\delta'(z-w) - \tfrac{2}{3}\delta'''(z-w).
\end{aligned} \tag{8.14}$$

9. Primary Fields for Generalized Liouville Theories

The reduced WZNW theories are conformally-invariant and thus the W-algebras associated with them should be expressible in terms of a Virasoro operator and a set of primary fields. In other words, they should be Zamolodchikov algebras. The base-elements of the W-algebra (gauge-invariant polynomials) constructed so far are not automatically primary fields, because they contain the gravitational component $H(z) = h(z)\,\mathrm{tr}\,H^2$ of the constrained KM current J, and, as discussed in Sect. 2, this component transforms as a spin-1 *connection*. That is to say, under infinitesimal conformal transformations h acquires, in addition to the usual tensorial terms, the inhomogeneous term (see (2.8) with $k = 1$)

$$\Delta h(w) \equiv [\Lambda(z), h(w)]_{\text{inhom.}} = \delta''(z-w). \tag{9.1}$$

In this and the next section we determine where h occurs in the gauge-invariant polynomials and hence identify the Virasoro operator Λ and the primary fields. For clarity, and because of some special features, we treat only the generalized Liouville case in this section, leaving the general maximal Toda theories to Sect. 10.

In order to locate the gravitational component $h(z)$ in the gauge-invariant polynomials we decompose the original constrained current $J^{\text{constr.}}$ into its primary field and its $h(z)$ parts, i.e., we write

$$J^{\text{constr.}} = \tilde{J} + hH \leftrightarrow \begin{pmatrix} Y & T \\ I & C \end{pmatrix} = \begin{pmatrix} \tilde{Y} & T \\ I & \tilde{C} \end{pmatrix} + h(z)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{9.2}$$

where $\tilde{Y}, \tilde{C}$ and T contain only primary fields. From (3.16) the matrix $J^{(2)}$ of gauge-invariant polynomials is

$$J^{(2)} = \begin{pmatrix} 0 & T^{(2)} \\ I & C^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & T - YC + Y' \\ I & C + Y \end{pmatrix}, \tag{9.3}$$

and if we now use (9.2) and the covariant derivative $\mathscr{D} = \partial + h$ for the spin-one field Y to extract h explicitly we obtain

$$J^{(2)} = \begin{pmatrix} 0 & T - \tilde{Y}\tilde{C} + \mathscr{D}\tilde{Y} \\ I & \tilde{C} + \tilde{Y} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & (\tilde{Y} + \tilde{C})h + \frac{1}{2}h^2 - \mathscr{D}h \\ 0 & 0 \end{pmatrix}. \tag{9.4}$$

From (9.4) it is easy to see that with respect to conformal transformations the components of $J^{(2)}$ will acquire, in addition to the usual covariant variations, the inhomogeneous terms

$$\begin{aligned} \Delta C^{(2)} &= 0 \\ \Delta T^{(2)} &= -\tfrac{1}{2}[(\tilde{Y} + \tilde{C})\Delta h - \mathscr{D}_w \Delta h + h\Delta h], \\ &= -\tfrac{1}{2}[(Y + C)\Delta h - \partial_w \Delta h] = -\tfrac{1}{2}[C^{(2)}\Delta h - \partial_w \Delta h], \end{aligned} \tag{9.5}$$

where we have used (9.3) to reconvert $(Y + C)$ into $C^{(2)}$. Converting this result into W-language and using (9.1) we obtain

$$\Delta W_c = 0 \quad \text{and} \quad \Delta W_t(w) = -\tfrac{1}{2}[W_c(w)\delta''(z - w) + \delta'''(z - w)]. \tag{9.6}$$

Thus the W_c's are primary fields but the W_t's are not.

To identify the Virasoro operator Λ one now uses the fact that Λ must be that combination of the W's whose Poisson-bracket with the W's produces the usual tensorial conformal transformation terms (spin 1 and 2 for the W_c's and W_t's respectively) plus the inhomogeneous terms shown in (9.6), and it is easy to check from the array (7.8) that the combination

$$\Lambda = \operatorname{tr}(\tfrac{1}{2}W_c^2 + W_t), \tag{9.7}$$

has this property. That is to say,

$$\begin{aligned} [\Lambda(z), W_c(w)] &= -W_c'(w)\delta(z - w) + W_c(w)\delta'(z - w), \\ [\Lambda(z), W_t(w)] &= -W_t'(w)\delta(z - w) + 2W_t(w)\delta'(z - w) \\ &\quad - \tfrac{1}{2}W_c(w)\delta''(z - w) - \tfrac{1}{2}\delta'''(z - w). \end{aligned} \tag{9.8}$$

Thus the Λ defined in (9.7) is the required Virasoro operator. For $n = 1$, it coincides with the expression (3.17) obtained directly as a gauge-invariant polynomial. Its central coefficient c is seen from (7.8) to be $c = 6n$, which, for $k = 1$ is in agreement with the general result $c = 12k \operatorname{tr} H^2$ of Sect. 2.

To identify the primary fields one notes from (9.8) and the conformal transformation properties of the derivative that the combinations $W_\tau = \hat{W}_t - \frac{1}{2}W_c'$ are primary fields. Since the remaining base-element $\operatorname{tr} W_t$ can be replaced by Λ we then see that a Virasoro-primary-field basis of the W-algebras for the generalized Liouville theories is

$$\Lambda, \; W_c \quad \text{and} \quad W_\tau = \hat{W}_t - \tfrac{1}{2}W_c'. \tag{9.9}$$

Note that, because $\partial + h$ is the covariant derivative for spin-one fields, the combinations $\hat{W}_t + \frac{1}{2}hW_c$ are also primary fields. But they are not gauge-invariant on account of the factor h.

10. Primary Fields for Maximal Generalized Toda Theories

As in the generalized Liouville case we first decompose the original constrained current $J^{\text{constr.}}$ into its primary-field part $\tilde{J}$ and gravitational part $H(z) = h(z)\,\mathrm{tr}\,H^2$:

$$J^{\text{constr.}} = \begin{pmatrix} A & X & S \\ B & Y & T \\ 0 & I & C \end{pmatrix} = \begin{pmatrix} \tilde{A} & X & S \\ B & \tilde{Y} & T \\ 0 & I & \tilde{C} \end{pmatrix} + h(z) \begin{pmatrix} \frac{n}{N} & 0 & 0 \\ 0 & \frac{n}{N} & 0 \\ 0 & 0 & -\frac{m+n}{N} \end{pmatrix}, \tag{10.1}$$

where, from the tracelessness and trace-orthogonality to H of $\tilde{J}$ (first matrix on the right-hand side) we have

$$\mathrm{tr}(\tilde{A} + \tilde{Y}) = 0 \quad \text{and} \quad \mathrm{tr}\,\tilde{C} = 0. \tag{10.2}$$

From (3.15) the matrix $J^{(2)}$ of gauge-invariant polynomials is

$$J^{(2)} = \begin{pmatrix} A^{(2)} & 0 & S^{(2)} \\ B^{(2)} & 0 & T^{(2)} \\ 0 & I & C^{(2)} \end{pmatrix} = \begin{pmatrix} A & 0 & S + AX - XC + \partial X \\ B & 0 & T + BX - YC + \partial Y \\ 0 & I & C + Y \end{pmatrix}, \tag{10.3}$$

and if we now use (10.1) and the covariant derivative $\mathscr{D} = \partial + h$ for the spin-one fields X and $\tilde{Y}$ to extract h explicitly we obtain

$$J^{(2)} = \begin{pmatrix} \tilde{A} & 0 & S + \tilde{A}X - X\tilde{C} + \mathscr{D}X \\ B & 0 & T + BX - \tilde{Y}\tilde{C} + \mathscr{D}\tilde{Y} \\ 0 & I & \tilde{C} + \tilde{Y} \end{pmatrix} + \begin{pmatrix} \frac{n}{N}h & 0 & 0 \\ 0 & 0 & Z \\ 0 & 0 & -\frac{m}{N}h \end{pmatrix}, \tag{10.4}$$

where

$$Z = \frac{-n}{N}\left[(\tilde{Y} + \tilde{C})h + \frac{n}{N}h^2 - \mathscr{D}h \right]. \tag{10.5}$$

From (10.4) and (10.5) it is easy to see that with respect to conformal transformations the components of $J^{(2)}$ will acquire, in addition to the usual tensorial variations, the inhomogeneous pieces

$$\begin{pmatrix} \Delta A^{(2)} & 0 & \Delta S^{(2)} \\ \Delta B^{(2)} & 0 & \Delta T^{(2)} \\ 0 & 0 & \Delta C^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{n}{N}\Delta h & 0 & 0 \\ 0 & 0 & \frac{-n}{N}\left[(\tilde{Y}+\tilde{C})\Delta h + \frac{2n}{N} h\Delta h - \mathscr{D}_w \Delta h\right] \\ 0 & 0 & -\frac{m}{N}\Delta h \end{pmatrix}. \tag{10.6}$$

Using (10.1), (10.3) and $\mathscr{D}$ to reconvert all the quantities in (10.6) into components of $J^{(2)}$ we obtain

$$\begin{pmatrix} \Delta A^{(2)} & 0 & \Delta S^{(2)} \\ \Delta B^{(2)} & 0 & \Delta T^{(2)} \\ 0 & 0 & \Delta C^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{n}{N}\Delta h & 0 & 0 \\ 0 & 0 & \frac{-n}{N}[C^{(2)}\Delta h - \partial_w \Delta h] \\ 0 & 0 & -\frac{m}{N}\Delta h \end{pmatrix}. \tag{10.7}$$

Translating this result into W-algebra language, we see that from (9.1) all the W's are primary fields except the W_0 associated with the gravitational field $h(z)$ and defined as $W_0 = \operatorname{tr} A^{(2)}$, and the W_t's, which, under infinitesimal conformal transformations, acquire the inhomogeneous pieces

$$\Delta W_0 = \frac{mn}{N}\delta'' \quad \text{and} \quad \Delta W_t = -\frac{n}{N}[W_c\delta'' + \delta'''], \tag{10.8}$$

respectively.

As in the generalized Liouville case the Virasoro operator is identified as that combination of the W's whose Poisson bracket with all the W's produces their usual tensor transformation properties (spin 1 for $\tilde{W}_a$, $\tilde{W}_c$, W_b and W_0, and spin 2 for W_s and W_t) plus the inhomogeneous terms (10.8). It is easy to check from (8.9) that the operator

$$\begin{aligned} \Lambda &= \operatorname{tr}[\tfrac{1}{2}(W_a^2 + W_c^2) + W_t] - W_0' \\ &= \operatorname{tr}[\tfrac{1}{2}(\tilde{W}_a^2 + \tilde{W}_c^2) + W_t] + \frac{n+m}{2mn} W_0^2 - W_0', \end{aligned} \tag{10.9}$$

has this property and is thus the required Virasoro operator. Note that in the DS gauge the matrices $\tilde{W}_a$ and $\tilde{W}_c$ are traceless and could therefore equally well be written as $\hat{W}_a$ and $\hat{W}_c$ as in Eq. (8.9). The centre c of the Virasoro algebra for the operator in (10.9) is seen from (8.11) to be $c = 12\frac{n(n+m)}{N}$, which for $k=1$ is in agreement with the general result $c = 12k \operatorname{tr} H^2$ of Sect. 2.

To identify the primary fields one notes from (10.8) and the conformal transformation properties of the derivative that the combinations

$$W_\tau = W_t - \frac{n}{N} W_c' - \frac{(m+n)}{nm^2}\left[W_0^2 + \frac{2mn}{N} W_0' \right]$$

are primary and thus a Virasoro-primary-field basis for the W-algebra of the general maximal Toda theories is

$$\tilde{W}_a, \quad W_b, \quad \tilde{W}_c, \quad W_s, \quad \Lambda, \quad \text{and} \quad W_\tau = W_t - \frac{n}{N} W_c' - \frac{(m+n)}{nm^2}\left[W_0^2 + \frac{2mn}{N} W_0' \right]. \tag{10.10}$$

This is the generalization of the result (9.9) for the generalized Liouville case. On account of the element W_0 it differs from the Liouville result not only in the existence of the extra W_0 terms in (10.10) but also in the fact that tr W_τ is a primary field and that the fields

$$W_t + \frac{1}{m^2}[mW_0 W_c - W_0^2 - mW_0'], \tag{10.11}$$

which are obtained from the W_τ by using the covariant derivative $\partial + h \to \partial + \frac{N}{mn} W_0$, to substitute $\frac{-N}{mn} W_0 \tilde{W}_c$ for $\tilde{W}_c'$, are both primary and gauge-invariant.

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References

1. Balog, J., Fehér, L., Forgacs, P., O'Raifeartaigh, L., Wipf, A.: Phys. Lett. **227B**, 214 (1990); **B244**, 435 (1990); Ann. Phys. **203**, 76 (1990)
2. Leznov, A. N., Savaliev, M. V.: Commun. Math. Phys. **74**, 111 (1980)
3. Goddard, P., Olive, D.: Int. J. Mod. Phys. **A1**, 303 (1986)
4. Bilal, A., Gervais, J.-L.: Phys. Lett. **206B**, 412 (1988); Nucl. Phys. **B314**, 646 (1989); **B318**, 579 (1989); Babelon, O.: Phys. Lett **215B**, 523 (1988)
5. Fateev, V. A., Lukyanov, S. L.: Int. Mod. Phys. **A3**, 507 (1988); Lukyanov, S. L.: Funct. Anal. Appl. **22**, 255 (1989); Yamagishi, K.: Phys. Lett. **205B**, 466 (1988); Mathieu, P.: Phys. Lett. **208B**, 101 (1988); Bakas, I.: Phys. Lett. **213B**, 313 (1988)
6. O'Raifeartaigh, L., Wipf, A.: Phys. Lett. **251B**, 361 (1990)
7. Zamolodchikov, A. B.: Theor. Math. Phys. **65**, 1205 (1986); Fateev, V. A., Zamolodchikov, A. B.: Nucl. Phys. **B280**, 644 (1987); Di Francesco, P., Itzykson, C., Zuber, J.-B.: Classical W-Algebras. Saclay Preprint SPh-T/90-149
8. O'Raifeartaigh, L.: Group Structure of Gaûge Theories. Cambridge: Cambridge University Press 1988, Chap. 11
9. Polyakov, A. M.: Mod. Phys. Lett. **A2**, 893 (1987)
10. Drinfeld, V., Sokolov, V.: J. Sov. Math. **30**, 1975 (1984)

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