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Fusion Rings and Geometry*

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Abstract. The algebraic structure of fusion rings in rational conformal field theories is analyzed in detail in this paper. A formalism which closely parallels classical tools in the study of the cohomology of homogeneous spaces is developed for fusion rings, in general, and for current algebra theories, in particular. It is shown that fusion rings lead to a natural orthogonal polynomial structure. The rings are expressed through generators and relations. The relations are then derived from some potentials leading to an identification of the fusion rings with deformations of affine varieties. In general, the fusion algebras are mapped to affine varieties which are the locus of the relations. The connection with modular transformations is investigated in this picture. It is explained how chiral algebras, arising in $N = 2$ superconformal field theory, can be derived from fusion rings. In particular, it is argued that theories of the type $SU(N)_k/SU(N-1)$ are the $N = 2$ counterparts of Grassmann manifolds and that there is a natural identification of the chiral fields with Schubert varieties, which is a graded algebra isomorphism.

1. Introduction

In recent years much interest has been focused on conformal field theory in two dimensions in connection with string theory and two dimensional critical phenomena. It was argued that string theories in four dimensions are described by some conformal field theories and that the properties of the string theories are a consequence of the characteristics of the two dimensional theory. A particularly fruitful set of ideas is the relation between algebraic properties of the conformal field theory and the emergence of space-time geometry. It was demonstrated, and conjectured to hold in general [1], that all $N = 2$ superconformal field theories are in one to one relation with complex manifolds. The interplay between algebraic

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** On leave of absence from the Weizmann Inst., Rehovot, Israel and geometrical pictures have been very fruitful in the exploration of string theory.

In this paper we will propose a different, through related, connection between non-supersymmetric conformal field theories and geometry. We will show that there is a remarkable analogy with tools and notions introduced in the context of $N = 2$ superconformal field theories. In particular, the chiral algebra will be shown to be closely related to fusion rings in rational conformal field theory, with the primary fields playing the role of the chiral fields. We shall then proceed to describe manifolds which are affine varieties whose moduli space is identical with the fusion rings.

Much of the examples discussed here are drawn from the class of $SU(N)$ theories which are analyzed here in detail. We develop here a Schubert-like calculus for the theories, based on Pieri and Giambelli formulas. The close similarity with the classical Schubert calculus is explained through the connection to $N = 2$ superconformal field theory, and in particular, it is explained how the theories of the type $SU(N+1/SU(N)$ are the $N = 2$ counterparts of Grassmann manifolds.

This paper is organized as following. In Sect. (2) the fusion ring of current algebra theories is investigated. A Schubert calculus is developed for theories of the type $SU(N)$. It is then used to derive a Borel-like picture which expresses the rings in terms of generators and relations. It is shown that the primary fields, when expressed in terms of the generators form a system of orthogonal polynomials. In the case of $SU(N)$ the measure is shown to be given by the discriminant of the N^{th} order polynomial equation. The rings are then expressed in terms of a potential which is obtained by integrating the relations. A connection with the theory of symmetric polynomials is heavily used for this purpose.

In Sect. (3) the connection with $N = 2$ superconformal field theory is described. In particular, we shown that the entire cohomology of Grassmannians can be recovered from the results of Sect. (2). The chiral algebra of the theories of the type $SU(N+1)/SU(N)$ is calculated and is shown to be described by the classical Schubert calculus. This also nicely demonstrates the connection with geometry, as well as enabling us to write down the precise manifolds that rational $N = 2$ theories correspond to.

The geometrical picture is further explored in Sect. (4). We use a classical construction in algebraic geometry to map every fusion ring to some affine variety in C^n. We then proceed to characterize the ring through the variety. In particular it is shown that the modular matrix can be throught of as the values of the primary fields on the points of the variety. A different geometrical construction is then offered, where the fusion ring is described as the moduli space of an affine variety.

2. Fusion Rings

The primary fields in a current algebra theory are labeled by representations of some finite dimensional Lie algebra G whose weights we denote by Λ. The highest weights obey $\Lambda\theta \leq k$, where k is the level which is equal to some integer. θ is the highest root. Each such primary field, denoted by G^Λ, gives rise to an infinite tower of fields in the corresponding integrable highest weight representation of current algebra. The fusion rules tell us which of the invariant couplings in the product,

$$[G^{\Lambda_1}] \times [G^{\Lambda_2}] = \sum_{\Lambda} N^{\Lambda}_{\Lambda_1,\Lambda_2} [G^{\Lambda}], \tag{2.1}$$

vanishes, where we denote by $[G^{\Lambda}]$ the tower of fields in the block whose highest weight is G^{Λ}.

Our basic tool for studying the fusion rules in current algebra is the depth criteria derived in [2]. Consider the correlation function of three primary fields,

$$\langle G^{\Lambda_1}_{\lambda_1}(z_1) G^{\Lambda_2}_{\lambda_2}(z_2) G^{\Lambda_3}_{\lambda_3}(z_3)\rangle = \sum_{i=1}^{N} f^{(i)}_{\lambda_1,\lambda_2,\lambda_3} C_{(i)} f(z_1,z_2,z_3), \tag{2.2}$$

where $f(z_1,z_2,z_3)$ is a constant function dependent on the dimensions; f^{ijk} are the Clebsch-Gordan coefficients of the Lie algebra, N is the number of singlets in the representation and the C_i are some constants. The fusion rules tell us which of the C_i is nonzero for a given k.

There are two sets of group theoretic equations obeyed by the correlation functions. The first is,

$$\sum_{j=1}^{n} t^a_j \langle \phi_1 \phi_2 \dots \phi_n \rangle = 0, \tag{2.3}$$

which is the usual finite algebra symmetry expressing the fact that the correlation function is invariant under the group. (Here t^a_j acts on the j^{th} field). The solutions to these equations are the invariant couplings or Clebsh-Gordan coefficients. The second set of equations is an algebraic set of equations,

$$\sum_{\substack{l_i \\ l_1+l_2+\dots+l_n=M+1}} \frac{(M+1)!}{l_1!l_2!\dots l_n!} \frac{(\tau^\theta_1)^{l_1}(\tau^\theta_2)^{l_2}\dots(\tau^\theta_n)^{l_n}}{(z-z_1)^{l_1}(z-z_2)^{l_2}\dots(z-z_n)^{l_n}}$$
$$\langle G_\lambda(z) G_1(z_1) G_2(z_2)\dots G_n(z_n)\rangle = 0, \tag{2.4}$$

where $M = k - \lambda\theta$. This equation follows froms the invariance under the affine algebra and the Ward identities.

In the case of the three point function Eq. (2.4) assumes a particularly simple form,

$$(\tau^\theta_1)^{n_1}(\tau^\theta_2)^{n_2}(\tau^\theta_3)^{n_3}\langle G^{\Lambda_1}_{\lambda_1}(z_1) G^{\Lambda_2}_{\lambda_2}(z_2) G^{\Lambda_3}_{\lambda_3}(z_3)\rangle = 0, \tag{2.5}$$

for any $n_1+n_2+n_3 = k+1$. This is the depth rule. The depth rule. Eq. (2.5) sets to zero some of the C_i at level k. Here τ^θ denotes the highest root component of the current. It is clear from the structure of the depth rule that any given coupling would be allowed for a sufficiently large k. Owing to the associativity of the operator product algebra, the fusion rules form an associative commutative algebra over the complex numbers. More precisely, since the structure constants of this algebra are integral, it is sometimes more useful to view it as a ring, and allow multiplication only by integers. This is the structure that will be emphasized in this note. Equation (2.5) is particularly easy to analyze in the case of $SU(2)$ and we find that [2]

$$[i] \times [j] = \sum_{\substack{l=|i-j| \\ l-i-j=0 \bmod 2}}^{\max(i+j,2k-i-j)} [l], \tag{2.6}$$

where $\Lambda = \frac{1}{2} i\alpha$ is the weight (i is twice the isospin).

It is clear that the highest root θ can be replaced in the depth rule by an arbitrary positive root, as the entire discussion goes through mutatis mutandi. This set of equations would be an algebraic consequence of the one above (as follows from the primitive vector theorem for Kac-Moode integrable modules), however, it will be convenient in application to have these additional equations.

We can recast the depth rule into an evidently equivalent form. Consider the product $L(\Lambda) \times \Lambda(\Lambda_1) \times \Lambda(\Lambda_2)$, where $L(\Lambda)$ denotes an integrable highest weight module of the finite algebra G. Let v be the highest weight vector of $L(\Lambda)$ and v_1, v_2 arbitrary vectors in the modules $L(\Lambda_1)$ and $L(\Lambda_2)$. If there exist integers l_1 and l_2 such that $l_1 + l_2 = k + \Lambda\alpha + 1$ and $(t^\alpha)^{l_1} v_1 \neq 0$ and $(t^\alpha)^{l_2} v_2 \neq 0$ for some negative root α, then the coupling $v \times v_1 \times v_2$ vanishes.

The two sets of equations, Eq. (2.3) and Eq. (2.5), determine precisely which invariant couplings can appear for every group and every k and, in particular, the number of such couplings. To demonstrate how this is done, let us first prove this theorem (stated in Ref. [3]).

Theorem. *Let σ be any external automorphism of the extended Dynkin diagram, $\sigma(\Lambda) = \sigma(0) + \omega_\sigma(\Lambda)$. Then the product $\Lambda_1 \times \sigma(\Lambda_2) = \sigma(\Lambda_1) \times \Lambda_2$, where Λ, Λ_1 and λ_2 are any three integrable highest weights for the algebra G at level k.*

Proof. From the associativity of the fusion rules it follows that the statement is equivalent to

$$\Lambda \times \sigma(0) = \sigma(\Lambda) . \tag{2.7}$$

In one direction simply set $\Lambda_2 = 0$. To prove the other direction multiplicity by Λ_2 and use the associativity.

So let as assume that the correlator

$$\langle G^{\sigma(0)}_{\sigma(0)} G^{\Lambda_1}_{\lambda_2} G^{\bar{\Lambda}_2}_{\lambda_2} \rangle \tag{2.8}$$

is nonvanishing. To prove the theorem it is enough to show that this implies that $\Lambda_2 = \sigma(\Lambda_1)$. Now, if the correlator Eq. (2.8) is nonzero, we can assume without loss of generality that λ_2 is a lowest weight, as we can apply Eq. (2.3), to push λ_2 down and λ_1 up, taking any positive root α in Eq. (2.3). The process will end when $G^{\bar{\Lambda}_2}_{\lambda_2}$ is annihilated by any τ^α, α a negative root, and is thus a lowest weight, $\lambda_2 = -\bar{\Lambda}_2$. Thus we can assume that $\lambda_2 = -\Lambda_2$ without any loss of generality.

We can further apply the Clebsch-Gordan rule, Eq. (2.3), with any negative root $\alpha < 0$ provided that $\alpha\sigma(0) = 0$. For such roots there will be only one surviving term in Eq. (2.3) proving that the correlation function Eq. (2.8) vanishes unless

$$t^\alpha \Phi^{\Lambda_1}_{\lambda_1} = 0 , \tag{2.9}$$

for any positive root $\alpha > 0$ such that $\alpha\sigma(0) = 0$.

The discussion so far was, in fact, general, and applies equally well to any product. Let us now use the depth rule, Eq. (2.5), with any root $\alpha > 0$ such that $\alpha\sigma(0) = k$. ($\sigma(0)$ is a minimal weight, i.e., a fundamental weight associated to a long simple root which appears with multiplicity one in the highest root θ, or $\theta\sigma(0) = 1$. Thus, the product $\alpha\sigma(0)$ is either 0 or k for any positive root α.) We find that $t^{-\alpha}\Phi^{\Lambda_1}_{\lambda_1} = 0$ for any such root.

It can be seen that for any $\alpha > 0$, $\alpha\sigma(0) = 0$ if and only if $\omega_\sigma^{-1}(\alpha) > 0$. Thus, the foregoing discussion implies

$$t^{\omega_\sigma} \Phi^{\Lambda_1}_{\lambda_1} = 0, \quad \text{for any } \alpha > 0 . \tag{2.10}$$

By standard properties of the Weyl group this is equivalent to $\lambda_1 = \omega_\sigma(\Lambda_1)$. Finally, since the weight of the correlator is zero we have, $\sigma(0) + \lambda_1 + \lambda_2 = 0$, and

$$\Lambda_2 = \sigma(0) + \omega_\sigma(\Lambda_1) = \sigma(\Lambda_1)\,. \tag{2.11}$$

proving the theorem.

For example, let us consider the product $8 \times 8 \times 8$. The highest weight is $\alpha_1 + \alpha_2$ (α_i are the two simple roots of $SU(3)$), and there are essentially two possibilities for the weight λ_1 and λ_2, corresponding to $(\alpha_1 + \alpha_2) + (-\alpha_1) + (-\alpha_2)$ and $\alpha_1 + \alpha_2) + (-\alpha - \alpha_2) + (0)$. For the first possibility we cannot act with $t^{-\theta}$ as this will take us out of the weight diagram. On the second one we can act once (on the (0)) if the coupling is the anti-symmetric one. For the symmetric coupling the (0) is a t^θ singlet. We conclude that for $k \geq 2$ the 8_s coupling appears, and that for $k \geq 3$ the anti-symmetric one does, as well. (Representing the 8 by 3×3 traceless hermitian matrices, the couplings 8_s and 8_a are given by $\mathrm{Tr}(ABC \pm ACB)$ and the statement above follows.)

We will now derive a Schubert like calculus for the fusion ring of $SU(N)_k$ theories. Subsequently we shall connect this calculus with the classical Schubert calculus for the cohomology of Grassmann manifolds. Due to the usefulness of this relation we will introduce geometrical terminology from the outset. Consider the limit $k \to \infty$. Then the depth rule, Eq. (2.5), becomes null and all the products allowed by ordinary group theory are non-vanishing. As is well known, the representations of $SU(N)$ may be described by Young tableaux. A given weight Λ, at level k, may be written as $\Lambda = \sum_{i=1}^{k} \Lambda^{(a_i)}$, where $\Lambda^{(0)} = \Lambda^{(N)} = 0$. $\Lambda^{(r)}$ stands for the r^{th} fundamental weight of $SU(N)$ and $0 < a_1 \leq a_2 \leq \ldots a_k \leq N - 1$. The a_i's correspond to the height of the i^{th} column in the Young tableaux. The adjoint (8) of $SU(3)$ is, for example, in this notation [1,2]. There are two types of special representations. The fully antisymmetric ones

$$\bar{c}_r = [0, 0, \ldots, 0, r]\,, \tag{2.12}$$

and the fully symmetric ones,

$$c_i = [1, 1, \ldots, 1]\,. \tag{2.13}$$

We shall call the representation c_i the i^{th} Chern class and $\bar{c}_i$ the i^{th} normal Chern class. Our first result concerns the multiplication of the special classes (namely, the fully antisymmetric representation) with any of the representations. The rule, in this case, is simple – there is no truncation of the operator algebra as a consequence of the depth rule Eq. (2.5) when multiplying one of the $\bar{c}_i$ with two other integrable representations at level k. Thus the product, in this case, is essentially given by the ordinary group theory one, which can be summarized in the following Pieri-like formula,

$$\bar{c}_i \cdot [a_1, a_2, \ldots, a_k] = \sum_{\substack{a_i \leq b_i \leq a_{i+1} \\ \sum b_i = \sum a_i + r}} [b_1, b_2, \ldots, b_k]\,, \tag{2.14}$$

with the convention that $0 \leq b_i \leq N$ and $b_i = N$ is the same as $b_i = 0$. We shall outline the proof of this Pieri-like formula. The key to the proof is writing the depth criteria in a Young tableaux form. The coupling the three representations to a singlet may be described by a square Young tableaux which is of height

N and width l, which is greater or equal to the maximal level of the three representations in question. Now, the depth rule translates in the Young tableaux language to the following statement: the maximal number of a certain type of quarks appearing in the product which can be flipped to a certain other kind is less or equal to the level k, or the product vanishes. The minimal level at which a certain coupling is non-vanishing will be termed the depth of the product. Clearly, the depth of a product does not exceed the width l but can be smaller. Now, it is not hard to see that the product of $\bar{c}_1$ with any of the integrable representations at level k can be fitted into a diagram of width k, and is thus non-vanishing. Simularly, the product of the other $\bar{c}_r$, $r > 1$, may be shown not to vanish, though the details are slightly more complicated.

From the Pieri-like formula for the multiplication of the special classes, we may derive a Giambelli-like formula which expresses any representation as a polynomial in the classes $\bar{c}_i$,

$$[a_1, a_2, \ldots, a_k] = \det_{1\le i,j\le k} \bar{c}_{a_i+i-j}\,, \tag{2.15}$$

where det stands for the determinant of this matrix, which is defined with the convention that $\bar{c}_N = \bar{c}_0 = 1$ and $\bar{c}_i = 1$ for $i > N$ or $i < 0$. The Giambelli-like formula, Eq. (2.15), is a consequence of the Pieri formula, Eqs. (2.14). To see it, expand the determinant Eq. (2.15) along the furst column ($j = 1$). It follows, by induction on k, that

$$\Delta = \sum_{i=1}^{k} (-1)^{i+1} \bar{c}_{a_i+i-1} [a_1 - 1, a_2 - 1, \ldots, a_{i-1} - 1, a_{i+1}, \ldots, a_k]\,. \tag{2.16}$$

Applying the Pieri to Eq. (2.16), all the terms cancel except for the desired one, proving Eq. (2.15). The pieri and Giambelli formulas, Eq. (2.14–2.15), form the basis of our Schubert-like calculus for the fusion ring, and enable one to compute any product. For example, consider the product 8×8 in $SU(3)$ at level $k = 2$. The representation 8 is written in our notation as $8 = [1,2]$. Using Giambelli, it may be written as

$$[1,2] = \det \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} = xy - 1\,, \tag{2.17}$$

where we denoted $x = \bar{c}_1$ and $y = \bar{c}_2$. Using Pieri, we find

$$x[1,2] = [2,2] + [1,3] = [2,2] + [0,1], \quad y([2,2] + [0,1]) = [0] + 2[1,2]\,.$$

Thus

$$[1,2]^2 = (xy - 1)[1,2] = [0] + [1,2]\,, \tag{2.18}$$

or in ordinary notation, $8 \times 8 = 1 + 8$. Repeating this for the case of $k = 3$ we find,

$$[0,1,2] \times [0,1,2] = [0,0,0] + [1,1,1] + [2,2,2] + 2[0,1,2]\,, \tag{2.19}$$

or in ordinary notation $8 \times 8 = 1 + 10 + \overline{10} + 8_s + 8_a$, where 8_s and 8_a correspond to the symmetric and antisymmetric couplings, agreeing with our earlier derivation. Next, we would like to describe the fusion ring as a ring of polynomials in the variables $\bar{c}_i$ with some relations among these variables ("syzigies" in Hilbert's terminology). This would correspond, as we shall later see, to the Borel picture of the cohomology of Grassmannians.

As for any associative algebra, there is a canonical map from the algebra of polynomials in n indeterminates $P[x_1, x_2, \dots, x_n]$ to the fusion ring R. This map, denoted by ϕ is given by $\phi(x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}) = \bar{c}_1^{a_1} \bar{c}_2^{a_2} \dots \bar{c}_n^{a_n}$, where $n = N - 1$. Clearly, this map is an algebra homomorphism and thus according to the first isomorphism theorem R is isomorphic to the quotient ring P/I, where the ideal I is the kernel of the map, $I = \ker \phi$, i.e., the polynomials which vanish when substituting the $\bar{c}_i$. A set of generators of this ideal correspond to the syzigies satisfied by the ring. We will show that the fusion ring is given by the polynomial ring $P[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n]$ and the ideal is generated by the Chern classes $c_{k+1}, c_{k+2}, \dots, c_{k+N-1}$, when these are expressed as polynomials in the normal Chern classes using Giambelli,

$$R \approx \frac{P[\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n]}{(c_{k+1}, c_{k+2}, \dots, c_{k+n})}. \tag{2.20}$$

To prove Eq. (2.20), it is enough to show that all the integrable representations at level $k + 1$ are contained in the ideal I defined above. This is a consequence of the Pieri formula, since all the terms which are missing there are integrable representations at level $k + 1$. Modulo the ideal generated by all integrable representations at level $k + 1$, the Pieri formula is identical to the ordinary group theory multiplication, or the limit $k \to \infty$ of the fusion ring. Using, for example, Young tableaux theory, it is not hard to see that for $k \to \infty$ the $\bar{c}_i$ are algebraically independent and thus generate the ring of polynomials. In addition, any representation at level $k+1$ appears in some product of lower level representation, unaccompanied by any other representation at level $k + 1$. So, it must vanish in order for the Pieri formula at level k to hold. It is enough to show, then, that the ideal I is identical to the ideal generated by all level $k + 1$ representations. This can be done by expressing the level $k+1$ representations in terms of the Chern classes, which is possible by some rather lengthy manipulations of determinants, which make use of the Giambelli formula. We omit the detail. An alternative proof will follow from the discussion of Sect. (4).

Next, we will show that the relations c_i can be integrated to a potential V, or that the fusion ring can be written as $P(\bar{c}_1)/(\partial_i V)$, where $(\partial_i V)$ stands for the ideal generated by the derivatives of the potential. This is due to

$$c_i = (-1)^{j-1} \frac{\partial V_{i+j}}{\partial \bar{c}_j}, \tag{2.21}$$

where V_i is some potential which will be described below. In order to integrate the Chern classes we will use a rather remarkable connection with the theory of invariant polynomials. The fusion ring for $k \to \infty$ becomes the usual multiplication in group theory. The Pieri formula may be viewed as a recursion relation for the polynomials of each of the irreducible representations of $SU(N)$. Now, the characters of the representations obey precisely the same Pieri formula, and thus are the solutions of this recursion relation. Define, as usual the character function of a representation with the highest weight Λ to be

$$\mathrm{ch}_\Lambda(\theta_i) = \sum_{\lambda \in L(\Lambda)} e^{i\lambda \alpha_n \theta_n}, \tag{2.22}$$

where α_n are any of the simple roots of the algebra, and we defined the character as a function of the angular variables $\theta_i \in [0, 2\pi]$, for $i = 1, 2, \dots, N-1$. The sum ranges over all the weights of the representation $L(\Lambda)$. Define also the variables

$$q_i = e^{i(\theta_i - \theta_{i-1})}, \tag{2.23}$$

for $i = 1, 2, \ldots, N$, with the convention that $\theta_0 = \theta_N = 0$. The variables q_i obey $\prod_{i=1}^{N} q_i = 1$.

Now, we can make a change of variables from the normal Chern classes $\bar{c}_i$ to the character basis, or the θ_i or q_i variables. The character of the representation $[r]$ is

$$\bar{c}_i = \sum_{1 \le i_1 < i_2 < \ldots < i_n \le N} q_{i_1} q_{i_2} \ldots q_{i_N}, \tag{2.24}$$

as it is the fully anti-symmetric representation. The q_i are the quark states in physicists language. Equation (2.24) may be interpreted as a change of variables. With this change of variables, the solution to the Pieri and Giambelli formulas is given by the character function of the representation, ch_Λ. Note that the $\bar{c}_i$ are the generators of symmetric functions in the q_i, with the constraint that $\bar{c}_N = \prod_i q_i = 1$. These generators are customarily denoted by S_i. The characters are the Schure functions expressed in terms of the q_i. These can be described by the Weyl character formula,

$$\mathrm{ch}_\Lambda(\theta_i) = D^{-1} \sum_{w \in W} (-1)^w e^{iw(\Lambda+\varrho)\alpha_n\theta_n}, \tag{2.25}$$

where W stands for the Weyl group (which is the symmetric group, S_N, for $SU(N)$) and ϱ is half the sum of positive roots. D is the denominator of the Weyl character formula

$$D = \sum_{w \in W} (-1)^w e^{iw(\varrho)\alpha_n\theta_n} = e^{i\varrho\alpha_n\theta_n} \prod_{\alpha > 0} (1 - e^{-i\alpha\alpha_n\theta_n}). \tag{2.26}$$

Since the positive roots of $SU(N)$ can be written as $\varepsilon_i - \varepsilon_j$, where $1 < i < j \le N$ and where ε_i is an orthonormal set of unit vectors, the denominator can be written as,

$$\begin{aligned} D &= e^{i \sum \theta_n} \prod_{1 \le i < j \le N} [1 - e^{-i\theta_n(\delta_{ni} - \delta_{nj} - \delta_{n+1,i} + \delta_{n+1,j})}] \\ &= e^{i \sum \theta_n} \prod_{i<j} (1 - q_j q_i^{-1}) = \prod_{i<j} (q_i - q_j). \end{aligned} \tag{2.27}$$

This shows that in terms of the q_i, the denominator D is the Vandermonde determinant. Now, the Chern classes are given by the so-called complete symmetric functions, as they are the totally symmetric tensor products of the fundamental representation,

$$c_r = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_r \le N} q_{i_1} q_{i_2} \ldots q_{i_r}. \tag{2.28}$$

The generating function for the symmetric functions S_i is

$$\sum_{i=1}^{N} (-1)^i \bar{c}_i t^i = \prod_{i=1}^{N} (1 - q_i t), \tag{2.29}$$

as can be seen by expanding the product. The generating function for the complete symmetric functions is

$$\sum_{r=0}^{\infty} c_r t^r = \prod_{i=1}^{N} (1 - q_i t)^{-1}, \tag{2.30}$$

as can again be seen by expanding each of the geometrical series; the terms on the right-hand-side are of the form $\sum_{m_i \geq 0} q_1^{m_1} q_2^{m_2} \ldots q_n^{m_n} t^{m_1+m_2+\ldots+m_n}$, which is equivalent to Eq. (2.28). Define the potential V_m by

$$V_m = \frac{1}{m} \sum_{i=1}^{N} q_i^m. \tag{2.31}$$

A generating function for the potentials is

$$\begin{aligned} V(t) &= \sum_{m=1}^{\infty} (-1)^{m-1} V_m t^m = \sum_{i=1}^{N} \log(1 - q_i t) = \log \prod_{i=1}^{N} (1 - q_i t) \\ &= \log \left(\sum_{i=0}^{N} (-1)^i \bar{c}_i t^i \right). \end{aligned} \tag{2.32}$$

It follows that

$$\begin{aligned} \sum_{m=1}^{\infty} (-1)^{m-1} t^m q_n^{m-1} &= \frac{\partial V(t)}{\partial q_n} = \prod_{i=1}^{N} (1 - q_i t)^{-1} \frac{\partial}{\partial q_n} \left(\sum_{p=0}^{N} (-1)^p S_p t^p \right) \\ &= \left[\sum_{r=0}^{\infty} c_r t^r \right] \left[\sum_{p=0}^{N} (-1)^p t^p \frac{\partial S_p}{\partial q_n} \right]. \end{aligned} \tag{2.33}$$

Comparing sides we find that

$$(-1)^{m-1} q_n^{m-1} = \sum_{r+p=m} (-1)^p c_r \frac{\partial S_p}{\partial q_n}. \tag{2.34}$$

Multiplying both sides by $\partial q_n / \partial S_q$, summing over n and using $\sum_n \frac{\partial q_n}{\partial s_q} \frac{\partial S_p}{\partial q_n} = \delta_{pq}$, it follows that

$$(-1)^{m-1} \sum_n \frac{\partial q_n}{\partial s_q} \frac{\partial}{\partial q_n} V_m (-1)^{m-1} \frac{\partial V_m}{\partial S_q} = (-1)^q c_{m-q}, \tag{2.35}$$

and we have proved that the integral of the Chern classes is given by V_m, Eq. (2.21).

To summarize, the fusion ring of $SU(N)_k$ is the quotient ring

$$R_k = \frac{P[x_1, x_2, \ldots, x_n]}{(\partial_i V)}, \tag{2.36}$$

where the variables x_i are identified with the fully anti-symmetric representations, $\bar{c}_i = [i]$, for $i = 0, 1, \ldots, n$, where $n = N - 1$. The potential V is given by the symmetric function $V = \frac{1}{N+k} \sum_{i=1}^{N} q_i^{N+k}$ expressed in terms of the generators of symmetric functions which are $x_n = \sum_{1 \le i_1 < i_2 \ldots \le i_r \le i_n} q_{i_1} q_{i_2} \cdots q_{i_n}$.

The ring R_k does not contain all the information of the conformal field theory. Specifically, we miss the invariant bilinear form on the conformal blocks. We can define the scalar product of two primary fields through the two point function $\langle \phi_\lambda(z_1)\phi_\mu(z_2)\rangle$. The bilinear form is defined through

$$(\phi_\lambda, \phi_\mu) = \delta_{\lambda,\bar{\mu}}, \tag{2.37}$$

where $\bar{\mu}$ is the complex conjugate representation (or the time reversal of it). In other words, the bilinear product (ϕ_λ, ϕ_μ) counts the number of times the unit operator appears in the product $\phi_\lambda \phi_\mu$ according to the fusion rules. It follows that this bilinear product is non-degenerate over C.

We wish to express this bilinear product as an integral over the generators x_i with some suitably chosen contour of integration. This, in particular, will define a structure of orthogonal polynomials for the primary fields of the theory. In other words, we would like to find a measure M such that

$$(\phi_\lambda, \phi_\mu^*) = \int_C \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\, M(x_1, x_2, \ldots, x_n)\phi_\lambda(x_i)\phi_\mu(x_i)^*, \tag{2.38}$$

where C is some contour of integration and $\phi_\mu^* = \phi_{\bar{\mu}}$, where $\bar{\mu}$ is the complex conjugate representation. We shall first describe a general method for deriving M, using the theory of orthogonal polynomials, and then give a more direct route for doing so based on the Weyl character formula.

We can rewrite $M = e^{-U}$, where U is some potential, assuming that M vanishes on the boundary of the contour of integration. With the definition Eq. (2.37), the polynomials ϕ_λ, expressed in terms of the generators $\bar{c}_i$, become a system of orthogonal polynomials, i.e., they form an orthonormal basis for the algebra of polynomials $P[x_1, x_2, \ldots, x_n]$. Since (ϕ_λ, ϕ_μ) is the number of singlets in the product, Eq. (2.37) would follow if,

$$\delta_{\lambda,0} = (\phi_\lambda, 1) = \int_C \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n\, \phi_\lambda(x_i) e^{-U}, \tag{2.39}$$

owing to the fusion rules algebra. By partial integration, using the assumption that M vanishes on the boundary of C, we find

$$\left(\frac{\partial \phi_\lambda}{\partial x_i}, 1\right) = \int_C \frac{\partial \phi_\lambda}{\partial x_i} e^{-U} = \int_C \phi_\lambda e^{-U} \frac{\partial U}{\partial x_i} = \left(\phi_\lambda, \frac{\partial U}{\partial x_i}\right), \tag{2.40}$$

where the integral, as before, is over all the x_i. Since the ϕ_λ form an orthonormal system, we may expand,

$$\frac{\partial U}{\partial x_i} = \sum_\mu a_\mu^i \phi_\mu, \tag{2.41}$$

where the a_μ^i are some complex constants, and the sum is over all the weights μ of the algebra. Substituting into Eq. (2.40) we find, using the orthonormality, that

$$a_\mu \delta_{\bar{\mu}\lambda} = \left(\frac{\partial \phi_\lambda}{\partial x_i}, 1\right), \tag{2.42}$$

and so

$$\frac{\partial U}{\partial x_i} = \sum_\lambda \phi_\lambda \left(\frac{\delta \phi_\lambda}{\partial x_i}, 1 \right). \tag{2.43}$$

Since $\left(\frac{\delta \phi_\lambda}{\partial x_i}, 1 \right)$ is the number of times the singlet representation appears in the derivative of ϕ_λ, it is not hard to compute this quantity directly, getting an expansion for the derivatives of U, which can then be summed and integrated to find M.

Let us illustrate this orthogonal polynomial method by considering the case of $SU(2)$. By deriving the Giambelli formula, Eq. (2.15), it follows that

$$\frac{\partial P_n}{\partial x} = \sum_{r+s=n-1} P_r(x) P_s(x), \tag{2.44}$$

where $P_r(x)$ is the polynomial representing $[1, 1, \dots, 1]$ (r times), or the representation with $\Lambda = \frac{1}{2} r\alpha$ and $x = \bar{c}_1 = [1]$. Thus, the number of singlets is

$$(P_r'(x), 1) = \begin{cases} 1 & \text{for } r = \text{odd}, \\ 0 & \text{for } r = \text{even}, \end{cases} \tag{2.45}$$

and so

$$U' = \sum_{n=0}^{\infty} P_{2n+1}(x). \tag{2.46}$$

Using the Pieri formula, Eq. (2.14), it is easy to see that $(x^2 - 4)U' = -x$, and so $U' = -x/(x^2 - 4)$. It follows that $M = \sqrt{4 - x^2}$, up to a constant, and the contour of integration is $x \in [-2, 2]$.

In principle, we can use this method to compute the measure in more complicated cases. However, the calculations become very involved, and so we shall use instead a Lie algebraic method. Recall that we can express the polynomials in terms of the Weyl character formula,

$$[a_1, a_2, \dots, a_n] = \frac{1}{D} \sum_{w \in W} (-1)^w e^{iw(\Lambda+\varrho)\alpha_n \theta_n}, \tag{2.47}$$

where D is the denominator Eq. (2.26), and $\Lambda = \sum \Lambda^{(a_i)}$ is the heighest weight of the representation $[a_1, a_2, \dots, a_k]$. In this notation $\bar{\Lambda} = [N - a_k, N - a_{k-1}, \dots, N - a_1]$. Consider now the integral

$$I = \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \dots \mathrm{d}\theta_n \, D^2 \, \mathrm{ch}_\lambda(\theta_i) \, \mathrm{ch}_\mu(\theta_i)^*. \tag{2.48}$$

From the Weyl character formula Eq. (2.25) we find

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{2\pi} \dots \int_0^{2\pi} \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \dots \mathrm{d}\theta_n \sum_{w,\hat{w}} (-1)^w (-1)^{\hat{w}} e^{i[w(\Lambda+\varrho) - \hat{w}(\mu+\varrho)]\alpha_n \theta_n} \\ &= \sum_{w,\hat{w}} (2\pi)^n (-1)^w (-1)^{\hat{w}} \delta\left[w(\Lambda + \varrho) - \hat{w}(\mu + \varrho)\right] \\ &= (2\pi)^n (n+1)! \, \delta_{\Lambda,\mu}, \end{aligned} \tag{2.49}$$

where we used the well known fact that $\omega(\Lambda+\varrho) = \hat{w}(\mu+\varrho)$ if and only if $w = \hat{w}$ and $\Lambda = \mu$, where Λ and μ are two highest weight vectors (see, for example [4]).

Thus Eq. (2.48) is our desired orthonormal integral. Note also, in this respect, that $\mathrm{ch}_\mu(\theta_i)^* = \mathrm{ch}_{\bar{\mu}}(\theta_i)$, where $\bar{\mu}$ is the complex conjugate representation and where the θ_i are taken to be real, and so we can indeed write Eq. (2.48) in the form Eq. (2.37).

Let us now express this integral in terms of the generators $S_i = \bar{c}_i$. We need to compute the Jacobian for the change of variables from θ_i to $\bar{c}_i$. To do so, it is convenient to relax the constraint $\prod_i q_i = \bar{c}_N = 1$ and allow any values for q_i and $\bar{c}_N$. Also, we can introduce a radical variable for the θ_i, $q_i = re^{i(\theta_i-\theta_{i-1})}$. We can then integrate the measure with respect to the redundant variable, inserting a delta function to insure the constraint. Now, the Jacobian for the change of variables from q_i to $\bar{c}_i$ is,

$$J = \frac{\partial \bar{c}_i}{\partial q_i} = \det_{r,s}\left(\sum_{\substack{i_1<i_2<\ldots<i_r \\ i_j \neq s}} q_{i_1}q_{i_2}\ldots q_{i_r}\right) = \prod_{i<j}(q_i - q_j) = D\,, \tag{2.50}$$

i.e., it is given by the Vandermonde determinant. This is easy to see by noting that the Jacobian is a totally anti-symmetric function of the same degree as D, and thus must be equal to it up to a constant.

The Jabobian of the change of variables from q_i to θ_i is not hard to compute directly and we omit the details. The result is that it is equal to $i^{N-1}N$. Inserting a delta function and integrating over the constraints we find that

$$i^{N-1}\prod_{i=1}^{n} d\theta_i = \frac{1}{D}\prod_{i=1}^{n} d\bar{c}_i\,. \tag{2.51}$$

It follows that the orthogonal integral, Eq. (2.49), assumes the form[1]

$$\begin{aligned} I &= \int\limits_C \frac{D(\bar{c}_i)}{(2\pi i)^n}\prod_{i=1}^{n} d\bar{c}_i[a_1, a_2, \ldots, a_k]\,[N-b_k, N-b_{k-1}, \ldots, N-b_1] \\ &= \prod_{i=1}^{k}\delta(a_i - b_i)\,. \end{aligned} \tag{2.52}$$

The measure M is thus seen to be identical to the Vandermonde determinant, expressed in terms of the $\bar{c}_i$ variables. Denote by $R = D^2$. Then $D = \sqrt{R}$. The q_i are the solutions of the polynomial equation,

$$0 = \prod_{i=1}^{N}(q - q_i) = q^N - \left(\sum q_i\right)q^{N-1} + \ldots + (-1)^N\prod q_i\,, \tag{2.53}$$

and so this equation is

$$q^N - \bar{c}_1 q^{N-1} + \ldots + (-1)^{N-1}\bar{c}_{N-1}q + (-1)^N\bar{c}_N = 0\,, \tag{2.54}$$

with the convention that $\bar{c}_N = 1$. Thus we see that R is precisely the discriminant of the most general N^{th} order polynomial equation, Eq. (2.54), with the constant

[1] The change of variables c_i to q_i is $(n+1)! = |W(G)|$ fold, and so the contour of integration, C, in Eq. (2.52) is the maximal torus modulo the action of W

term $\bar{c}_N$ set to one. In the case of $SU(2)$, we have found earlier, using the orthogonal polynomial method, that $M = \sqrt{R}$, where $R = 4 - \bar{c}_1^2$. This is indeed the discriminant of the second order polynomial equation, $q^2 - \bar{c}_1 q + 1 = 0$. The calculation of the discriminants of polynomial equations is described in the appendix.

We conclude this section with some examples. The simplest case is $SU(2)$. The fields of $SU(2)$ at level k are labeled by the highest weights $\lambda = \frac{1}{2}\,\alpha m$, where $0 \le m \le k$ is any integer (which is twice the isospin). We shall denote this field by (m). In our previous notation $\lambda = [1, 1, \dots, 1]$ (m times). The fusion rules can be written, in this case, Eq. (2.6),

$$(m) \times (n) = \sum_{\substack{l=|m-n| \\ l-m-n=\text{even}}}^{\max(2k-m-n,\,m+n)} (l)\,. \tag{2.55}$$

It is not hard to see that for $n = 1$, this product rule is identical to the Pieri formula Eq. (2.14).

The Giambelli formula, Eq. (2.15), expresses the field (m) as a polynomial in $x = (1)$. Explicitly, we find

$$(m) = P_m(x) = \det \begin{pmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x \end{pmatrix}.$$

A closed form for $P_m(x)$ follows from the Weyl character formula, Eq. (2.25). The relevant change of variables, Eq. (2.24), assumes the form $x = \exp(i\theta) + \exp(-i\theta) = 2\cos\theta$. Then Eq. (2.25) becomes

$$P_m(2\cos\theta) = \mathrm{ch}_{\frac{1}{2}m\alpha}(\theta) = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(m+1)\theta}{\sin\theta}\,. \tag{2.56}$$

The denominator is

$$D = e^{i\theta} - e^{-i\theta} = 2i\sin\theta\,. \tag{2.57}$$

Thus, $P_m(x) = U_m(x/2)$, where $U_m(x)$ are the Chebyshev polynomials of the second kind, defined as

$$U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}\,. \tag{2.58}$$

The Pieri formula is simply the well known recursion relation for Chebyshev polynomials of the second kind, whereas the Giambelli formula appears to be new. The fusion ring is given, according to our earlier discussion, by $P[x]/(U_{k+1}(x/2))$, where $P[x]$ denotes the ring of polynomials in the indeterminate x over the integers.

The measure, as we computed earlier, is $M = \sqrt{4 - x^2}$, which translates into the well known orthogonality relation for Chebyshev polynomials,

$$\int_{-2}^{2} dx \sqrt{4 - x^2}\, P_m(x) P_n(x) = 2\pi\delta_{n,m}\,. \tag{2.59}$$

The potential V_{k+1} is, Eq. (2.31),

$$(k+1)V_{k+1}(2\cos\theta) = \sum q_i^{k+1} = e^{i(k+1)\theta} + e^{-i(k+1)\theta} = 2\cos(k+1)\theta\,, \qquad (2.60)$$

i.e., the Potentials are Chebyshev polynomials of the first kind. The fact that $P_{k+1}(x) = \frac{\mathrm{d}V_{k+1}}{\mathrm{d}x}$, Eq. (2.21), is the usual relation between Chebyshev polynomials of the first and second kinds. The first few polynomials are $P_m(x) = 1, x, x^2-1, x^3-2x, \ldots$, where as the first few potentials are $V_m(x) = 2, x, x^2/2, x^3/3 - x, x^4/-x^2, \ldots$. As an example of the fusion ring consider $k = 2$. Then, according to Eq. (2.36), $R = P[x]/(x^3-2x)$. The elements of this ring are $a + bx + cx^2$, where a, b and c are arbitrary integers. The addition of elements is term by term, while the product rule is

$$\begin{aligned}&(a+bx+cx^2)(\alpha+\beta x+\gamma x^2)\\ &\quad = a\alpha + x(b\alpha + a\beta + 2b\gamma + 2c\beta) + x^2(c\alpha + b\beta + a\gamma + 2c\gamma)\,,\end{aligned} \qquad (2.61)$$

i.e., R is the additive group Z^3 embedded with the above product structure. The scalar product in R is defined through $(1,1) = (x,x) = (x^2,1) = (x^2,x^2) = 1$ and $(1,x) = (x,x^2) = 0$.

Let us as a second example consider the case of $SU(3)$. In this case the polynomials are $P_{n,m}(x,y) = [1,1,\ldots,1,2,2,\ldots 2]$, where there are n one's and m two's. $P_{n,m}(x,y) = x^n y^m + \text{l.o.t.}$, where l.o.t. stands for lower order terms and the order of x and y is one (this is not hard to see from the Giambelli formula). Giambelli gives the polynomials as a determinant. The polynomials up to level two are, $P_{0,0} = 1$, $P_{1,0} = x$, $P_{0,1} = y$, $P_{0,2} = \det\begin{pmatrix} x & 1 \\ y & x\end{pmatrix} = x^2 - y$, $P_{1,1} = xy - 1$, $P_{2,2} = y^2 - 1$. The polynomials at level k are all the polynomials whose leading coefficient is of order k. These span the k^{th} order subspace of the algebra of polynomials (this is general for all $SU(N)$). The orthogonality relation of the polynomials is

$$\int_C \mathrm{d}x\,\mathrm{d}y\, M P_{n,m}(x,y) P_{l,s}(x,y) = \delta_{n,s}\delta_{m,l}\,, \qquad (2.62)$$

where C is the contour $\theta_i \in [0, 2\pi]$ parametrized by the variables,

$$x = e^{i\theta_1} + e^{-i\theta_1 + i\theta_2} + e^{-i\theta_2}, \qquad (2.63)$$

and $y = x^*$ (the complex conjugate of x). The measure M, as discussed earlier, is the square root of the discriminant, $D = \sqrt{R}$. From the appendix, the discriminant of the third order polynomial, $q^3 - xq^2 + yq - 1$ is $-R = 27 - x^2y^2 - 4x^3 4y^3 + 18xy$ and so the measure is (up to a factor)

$$M = \sqrt{27 - x^2y^2 - 4x^3 - 4y^3 + 18xy}\,. \qquad (2.64)$$

In terms of the variables $q_i = e^{i(\theta_i - \theta_{i-1})}$ the polynomials are the Schure functions, specialized to these values of the variables q_i. The potentials are given by

$$V_m = \frac{1}{m}\sum_i q_i^k = \frac{1}{m}\left[e^{im\theta_1} + e^{im(\theta_2-\theta_1)} + e^{-im\theta_2}\right], \qquad (2.65)$$

expressed in terms of the variables x and y. The explicit form of the potentials can be calculated by integrating the Giambelli formula expressions for the Chern classes, or alternatively by computing directly the first three potentials and then using the recursion relation,

$$(k+3)V_{k+3} - (k+2)xv_{k+2} + (k+1)yV_{k+1} - kV_k = 0\,, \tag{2.66}$$

which follows by multiplying by q^k the equation obeyed by all the q_i,

$$q^3 - xq^2 + yq - 1 = 0\,, \tag{2.67}$$

(see Appendix (A)) and summing over i. Clearly, $\sum q_i^0 = 3$, $V_1 = \sum q_i = x$, $2V_2 = \sum q_i^2 = x^2 - 2y$. We can now compute recursively using Eq. (2.66), $3V_3 = x^3 - 3xy + 3$, $4V_4 = x^4 - 4x^2y + 4x + 2y^2$, $5V_5 = x^5 - 5x^3y + 5xy^2 + 5x^2 - 5y$, etc. Alternatively, we can use the generating function, Eq. (2.32),

$$V_m = \frac{(-1)^m}{m!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \log(1 - tx + t^2y - t^3)|_{t=0}\,. \tag{2.68}$$

The fusion ring at level k is the polynomial ring $P[x, y]/(\partial_i V_{k+3})$. For $k = 2$, for example, the relations are generated by the derivatives of V_5 which are the Chern classes,

$$c_4 = [1111] = \frac{\partial V_5}{\partial x} = x^4 - 3x^2y + y^2 + 2x\,, \tag{2.69}$$

$$c_3 = [111] = -\frac{\partial V_5}{\partial y} = x^3 - 2xy + 1\,. \tag{2.70}$$

It can be checked that the same answer follows from Giambelli. Hence, the fusion ring is generated by the two variables x and y modulo the relations, $x^3 = 2xy - 1$ and $x^4 = 3x^2y - y^2 - 2x$. The fields can be represented by 1, x, y, xy, x^2, y^2 and the products can be computed from the relations, e.g., $x^2y = x + y^2$, $y^3 = x^3 = 2xy - 1$, and so forth.

3. The Connection with Grassmannians

The formalism that we have developed in the previous section of the fusion ring of $SU(n+1)_k$ is remarkably similar to the cohomology ring of Grassmann manifolds, i.e., the manifolds

$$G_k(C^{n+k}) = \frac{U(n+k)}{U(n) \times U(k)}\,, \tag{3.1}$$

whose points are k dimensional vector subspaces in C^{n+k}. There is an obvious symmetry in the Grassmann manifold of exchanging n with k. In the geometrical language, this corresponds to replacing a k dimensional hyperplane with its normal plane.

The relation with Grassmann manifolds can be made precise through the connection with $N = 2$ superconformal field theory (see [5] for a review). This connection is interesting for its own sake.

As was established in [1], every $N = 2$ superconformal field theory (with a central charge that is a multiple of three, corresponds to some complex manifold. On the other hand, the G current algebras are the building blocks of the rational

$N = 2$ theories. Consider the quotient theory G/H, where (G, H) is some reductive pair, i.e., H is a subgroup of G obtained by the removal of some nodes from the Dynkin diagram of G and is of the same rank as G. As was shown in [6] this theory has an $N = 2$ superconformal invariance. An important subclass of these theories is described by the quotient, $SU(n+1)/SU(n) \times U(1)$. We shall denote this theory by $SU(n+1)_k/SU(n)$.

In each $N = 2$ theory there is a special class of fields, called chiral primary fields, which obey the relation $\Delta = Q/2$, where Δ is the dimension and Q is the $U(1)$ charge, and a similar relation holds for the right moving algebra. These fields are chiral in the two-dimensional supersymmetry sense, as well as being highest weight vectors of the Virasoro algebra. As was established in [7] the operator products among the chiral fields are associative, i.e., we can define an associative algebra structure on the chiral fields via their operator products,

$$C_i C_j = f_{ij}^k C_k , \tag{3.2}$$

where f_{ij}^k are the operator products. This algebra is an associative, commutative, finite dimensional, graded algebra. It is graded by the $U(1)$ charges, Strictly speaking, since there are left and right moving $U(1)$ charges, this algebra is actually bigraded. However, we shall limit ourselves here to theories in which the right and left moving charges are identical, and so there will be a unique grading.

In the framework of the general connection between $N = 2$ theories and complex manifolds, we shall now establish that the theories $SU(n+1)_k/SU(n)$ are the conformal field theory counterparts of Grassmann manifolds. This will also explain the striking similarity of the formalism developed in Sect. (2) for the fusion ring, with classical tools in the study of such manifolds: the Schubert and Borel pictures of the cohomology, along with the connection to the theory of invariant polynomials.

The calculation of the chiral algebra of theories of the type $SU(n+1)_k/SU(n)$ was described in [7] and, for the general reductive pair in [8,9]. The result is simple – the chiral fields of the theory are in one to one correspondence with the primary fields of $SU(n+1)_k$. Each such field, denoted by C^Λ, where Λ is a highest weight of $SU(n+1)$ at level k, is obtained through the decomposition

$$G^\Lambda = H^\Lambda C^\Lambda , \tag{3.3}$$

where G^Λ is the highest weight component of a primary field and H^Λ is the primary field in the H current algebra to which it decomposes. Thus, we can compute the chiral algebra as a ratio of the structure constants among the primary fields in the current algebra theories. Unfortunately, however, though much is known about these structure constants [10,2], they have not been computed in general, at the present. Thus, instead of using the actual structure constants, we will substitute the fusion coefficients. We shall then argue that this procedure leads to the correct answer.

As in Sect. (2), we shall be representing each integrable highest weight Λ as $\Lambda = \sum \Lambda^{(a_i)}$ or $\Lambda = [a_1, a_2, \ldots , a_k]$, where the integers a_i are such that $0 \le a_1 < a_2 < \ldots a_k \le n$. The $U(1)$ charge of the chiral field $C^\Lambda = \Phi^\Lambda_{\Lambda,Q}$ is

$$(k+g)Q = \sum_{i=1}^{k} a_i . \tag{3.4}$$

Consider now the fusion ring of this theory, restricted to the chiral fields. These fields fuse according to

$$C^{\Lambda_1} \times C^{\Lambda_2} = f^{\Lambda(G)}_{\Lambda_1,\Lambda_2} f^{\Lambda(H)}_{\Lambda_1,\Lambda_2} \delta(Q - Q_1 - Q_2) C^{\Lambda}, \tag{3.5}$$

where $f^{\Lambda(G)}_{\Lambda_1,\Lambda_2}$ are the fusion coefficients of $G = SU(n+1)$ and similarly $f(H)$ are the fusion coefficients of $H = SU(n)$. Let us, as in Sect. (2), define first the special classes C^{Λ}, where Λ is one of the fundamental weights, which will be again referred to as the normal Chern classes, and denoted accordingly by $\bar{c}_r = C^{\Lambda_r}$, where Λ_r is the r^{th} fundamental weight. Consider the product of one of the special classes with some field G^{Λ}. The fusion ring is a subset of the Pieri-like formula, Eq. (2.14), with some of the coefficients set to zero according to Eq. (3.5). The delta function in Eq. (3.5) is equal to $\delta\left(r + \sum a_i - \sum b_i\right)$. Thus, ignoring the H algebra, the fusion rules assume the form,

$$[r] \times [a_1, a_2, \dots, a_k] = \sum_{\substack{a_i \le b_i \le a_{i+1} \\ r + \sum a_i = \sum b_i}} [b_1, b_2, \dots, b_k], \tag{3.6}$$

with the convention that $0 \le b_i \le n$. In other words, the fusion rule Eq. (3.6) is identical to the Pieri-like formula, Eq. (2.14), except that now $b_i = N$ is no longer allowed to appear in the product.

Next, we shall argue that H does not change this product, and that the full fusion ring of the chiral fields is given by Eq. (3.6). If $\Lambda = [a_1, a_2, \dots, a_k]$ is some integrable representation of $SU(n+1)$ at level k, then the corresponding weight of $SU(n)$, also denoted by Λ (in an abuse of notation, is the same weight, with each of the a_i which is equal to n is changed to zero. Thus, the products of H that we need to consider are those where each of the $b_i = n$ in Eq. (3.6) is replaced by $b_i = 0$. Note, now, that this is, in fact, precisely, the description of the Pieri-like formula of $SU(n)$ (which is at lavel $k+1$). Thus, we conclude that Eq. (3.6), indeed, represents the fusion of the chiral fields in the theory $SU(n+1)_k/SU(n)$.

We would wish to argue now that Eq. (3.6) is in fact the correct form for the chiral algebra, and that the structure constants are indeed all equal to zero or one, with the appropriate normalizations for the fields. In other words, our argument so far implies that the chiral algebra is of the form

$$\bar{c}_r[a_i] = \sum_{\substack{a_i \le b_i \le a_{i+1} \\ \sum b_i = \sum a_i + r}} f^r_{a_i} [b_1, b_2, \dots, b_k],$$

where $f^r_{a_i} \neq 0$ are some unknown constants. We can now use the associativity of the algebra to find relations among these constants. A useful tool for doing so is the Hasse diagram defined as follows [11]. Define the partial ordering among the Schubert classes, $[a_i] \le [b_i]$ iff $a_i \le b_i$ for all i. This is called the Bruhat order (of a Coxeter group) and is identical to the inclusion order of Schubert varieties. This poset (partially ordered set) has a length function which is $\sum a_i$. In the Hasse diagram every point stands for a Schubert variety, and two varieties whose length differ by one are connected by a line iff one is greater than the other. Now, it is easy to see that $\bar{c}_1 \times [a_i]$ contains exactly all the terms in the Hasse diagram greater than $[a_i]$ and connected to it. It follows that the constants $f^1_{a_i}$ can all be assumed to be equal to one (up to normalizations of the fields) except for a set of lines in the Hasse diagram, whose removal would make it into a tree, i.e., the unknown constants are in one-to-one correspondence with the cycle basis of the Hasse diagram.

We can now use the associativity of the algebra to compute all other constants. By computing the product $\bar{c}_1\bar{c}_r[a_i]$ in two different ways, we find all the constants $f^r_{a_i}$, with $r > 1$, as well as getting some relations among the constants $f^1_{a_i}$ which represent the cycle basis of the Hasse diagram. It can be seen that, up to these relations, the cycle basis indeed parametrizes different algebras with the generic form described above. If we supplement the algebra with the assumption that the inner product is[2] $(a_i, b_i) = \delta(a_i + b_{k+1-i} - n)$, these constants can be seen to be equal to one, and there is indeed a unique algebra which is identical to the restriction of the fusion algebra.

As in Sect. (2) we can derive a Giambelli formula which expresses the chiral fields as polynomials in the generators $\bar{c}_i$,

$$[a_1, a_2, \dots, a_k] = \det_{i,j} \bar{c}_{a_i+i-j}\,, \tag{3.7}$$

with the convention that $c_i = 0$ for $i < 0$ or $i \geq N$. Note that the only difference between this Giambelli formula and the one describing the fusion ring, Eq. (3.7), is in the convention that $\bar{c}_N = 0$ rather than $\bar{c}_N = 1$.

Amazingly, the Pieri and Giambelli formulas that describe the chiral algebra are identical to the classical Pieri and Giambelli formulas describing the wedge product in the cohomology ring of the Grassmann manifolds $G_k(C^{n+k})$ [12, 11]. The basis of this cohomology is given by the Schubert varieties $[a_1, a_2, \dots, a_k]$ which are defined as follows. Let $0 \subset V_1 \subset V_2 \dots \subset V_{n+k}$ be a fixed flag in $G_k(C^{n+k})$, i.e., an ascending series of vector subspaces in C^{n+k}, such that $\dim(V_i) = i$. Then, the Schubert variety $[a_1, a_2, \dots, a_k]$ is defined as

$$[a_1, a_2, \dots, a_k] = \{v \in G_k(C^{n+k}) \mid \dim(v \cap V_{a_i+i} \geq i)\}\,. \tag{3.8}$$

We conclude that the cup algebra of the Schubert varieties (or alternatively, via de-Rahm isomorphism, the wedge algebra in the cohomology ring of Grassmannians) is identical to the chiral algebra of the conformal field theories $SU(n+1)_k/SU(n)$. More precisely

Theorem (3.1). *There is a graded algebra isomorphism from the chiral algebra of $SU(n+1)_k/SU(n)$ to the cohomology ring of the Grassmann manifold $G_k(C^{n+k})$. The isomorphism map identifies the field C^Λ, where $\Lambda = \sum \Lambda^{(a_i)}$ and $0 \leq a_1 \leq a_2 \leq \dots \leq a_k \leq n$ with the Schubert variety $[a_1, a_2, \dots, a_k]$.*

Proof. Outlined above (with the assumption on the bilinear product). It remains only to show that the grading is preserved. The $U(1)$ charge is $Q = \sum a_i$. It is well known that the complex codimension of this Schubert variety is also $\sum a_i$. As this dimension is also half the degree of the form, the theorem follows.

The intersection pairing of forms in the cohomology ring of some manifold V is the bilinear product,

$$(\alpha, \beta) = \int_V \alpha \wedge \beta\,.$$

[2] The inner product is defined through the pairing $[a_i]\,[b_i] = \alpha C_{\max}$, where α is the inner product. It might be possible to justify this assumption from a conformal field theory calculation

For the Schubert varieties this form is equal to [12, 11]

$$([a_i],[b_i]) = \prod_i \delta(a_i + b_{k+1-i} - n)\,.$$

Note that this is indentical to the result obtained from conformal field theory. In fact, this pairing is identical to the one introduced for the fusion rings in Sect. (2); the complex conjugate of the representation $[a_1, a_2, \ldots, a_k]$ is easily seen to be $[N - a_k,\ N - a_{k-1}, \ldots, N - a_1]$. The transpose of a chiral field is defined by [7] $C^t = C_{\max}C^\dagger$ where $C_{\max} = C^{kA^{(n)}}$ is the chiral field with maximal $U(1)$ charge. It is easy to see that $(C^A)^t = C^{A'}$ where $A' = (n - a_k, n - a_{k-1} \pm n - a_1)$. It follows that the pairing of the chiral fields in conformal field theory is identical with the intersection pairing of Schubert varieties (up to normalizations), and the Poincaré dual of a Schubert variety corresponds to the transpose of a chiral field.

The structure of the chiral algebra is immediate from this theorem and from the results of Sect. (2). The fusion ring is given by

$$P[\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n]/(c_{k+1}, c_{k+2}, \ldots, c_{k+n})\,. \tag{3.9}$$

The chiral algebra is given, according to the foregoing discussion, by the homogeneous part of the fusion ring. That is, making the change of variables $\bar{c}_i \to \lambda^i \bar{c}_i$, where λ is some complex constant, and taking the limit of $|\lambda| \to \infty$.

It follows that the chiral algebra is given by the same formula, Eq. (3.9), where now the Chern classes are evaluated using the classical Giambelli formula Eq. (3.7). An equivalent way to write this is as follows. It is not hard to see that the Giambelli formula and the ideal of relations imply that

$$(1 - c_1 + c_2 - \ldots + (-1)^k c_k)\,(1 + \bar{c}_1 + \bar{c}_2 + \ldots \bar{c}_n) = 1\,, \tag{3.10}$$

where this equation is regarded as a set of equations for each of the homogeneous components. For example, the first equation is $c_1 - \bar{c}_1 = 0$, the second is $c_2 - c_1\bar{c}_1 + \bar{c}_2 = 0$, etc. The first k equations give the Giambelli formula expression for the Chern classes, whereas, the $k+1$ up to $k+n$ give the relations of the algebra, $c_i = 0$ for $k+1 \le i \le k+n$. It follows that Eq. (3.10) is a complete description of the chiral algebra. This equation, which is due to Borel, is a very natural geometrical description of the cohomology of Grassmann manifolds: it expresses the fact that the complete Chern class times the complete normal Chern class is equal to one, or that the direct sum of the tangent bundle and the normal bundle is a trivial vector bundle, and that there are no additional relations in the algebra. This is the Borel picture of the cohomology. We conclude that the description of fusion rings by generators and relations is the analog of the Borel picture. The symmetry of exchanging n and k is also evident from Eq. (3.10) through the exchange of the symmetric representations with the anti-symmetric one[3].

Finally, integrating the relations to get the potential, is the same as in Sect. (2). The potential V_k is given by the highest degree components of the potential of Sect. (2). It is immediate, thus, that it is equal to

$$V_k = \frac{1}{k+n+1} \sum_{i=1}^{n} q_i^{k+n+1}\,, \tag{3.11}$$

[3] Our definition of the Chern classes differs by a sign from the conventional one, $c_r = (-1)^r [r]$

expressed in terms of the generators $\bar{c}_r = \sum_{i_1<i_2<\ldots<i_r} q_{i_1}, i_2, \ldots, i_r$. The only difference from the fusion ring is that now $\bar{c}_N = 0$ rather than $\bar{c}_N = 1$. It follows that,

$$c_i = (-1)^p \frac{\partial V_{i+p}}{\partial \bar{c}_p}, \tag{3.12}$$

as before, and that the chiral algebra is $P[\bar{c}_i]/\left(\frac{\partial V_{k+N}}{\partial \bar{c}_i}\right)$.

The description of the theories $SU(n+1)_k/SU(n)$ as scalar field theories with a potential, of the deformation type of V was first described in [7][4]. It follows from the result above, that the theories may be written by the lagrangian

$$\mathscr{L} = \int \mathrm{d}^4\theta K(\Phi_i, \bar{\Phi}_i) + \left[\int \mathrm{d}^2\theta V_{k+n+1}(\Phi_i) + \text{c.c.}\right], \tag{3.13}$$

where Φ_i is a scalar superfield multiplet associated with each of the generators, and with the precise potentials described above. This shows that the superpotential is not only a deformation of the homogeneous space cohomology, but is precisely identical to it, with the totally anti-symmetric representations playing the role of the normal Chern classes, while the symmetric ones play the role of the Chern classes.

To see the connection between the chiral algebra and the equation of motion, Eq. (3.13) let us recapitulate the arguments of [7]. The equation of motion which follows from the supersymmetric lagrangian Eq. (3.13) is the usual one for a Wess-Zumino model,

$$K_{\phi_i,\bar{\phi}_j} D_+\bar{D}_+\phi_j^* = \frac{\partial V}{\partial \phi_i}, \tag{3.14}$$

where ϕ_i is the lowest component of the superfield Φ_i. Now, note that the right-hand-side of Eq. (3.14) is a polynomial in the chiral fields ϕ_i while the left-hand-side is a non-chiral field. It follows that the polynomials $\frac{\partial V}{\partial \phi_i}$ vanish when computed in the chiral algebra (using the structure constants to define the algebra). Thus the chiral algebra of the theory Eq. (3.13) assumes the form $\frac{P[\phi_1, \phi_2, \ldots, \phi_n]}{(\partial V/\partial \phi_i)}$. Hence also, the potential appearing in the lagrangian, is identical to the one computed from the chiral algebra.

This set of arguments applies, in fact, equally well to non-conformal scalar field theories. Consider the perturbations of the conformal lagrangian Eq. (3.13). These are described by adding to the potential V arbitrary elements of the chiral algebra. Thus, the new theory will be described by the potential $\hat{V} = V + C$, where C is any chiral field, expressed as a polynomial in the generators. The potential no longer needs to be quasi-homogeneous, and the theory is not necessarily conformal. However, the equation of motion is still of the form Eq. (3.14), and thus the chiral algebra (again, defined through the structure constants among the chiral fields) is given by the quotient

$$\mathscr{C} = \frac{P[\phi_1, \phi_2, \ldots, \phi_n]}{(\partial_i \hat{V})}. \tag{3.15}$$

[4] The fact that the Poincaré polynomial of this theory is identical with the Poincaré polynomial of the cohomology of Grassmann manifolds was subsequently noted in [8, 13]

We thus see that this relatively simple derivation enables us to calculate exactly the structure constants among the chiral fields in any two dimensional scalar field theory[5].

As explained in [7] this also allows us to write down the manifold which corresponds to an $N = 2$ string theory built with these conformal field theories. The manifold, as was shown there, is given by

$$\sum_i V_i = 0, \tag{3.16}$$

where V_i are the potentials for each of the theories, and the variables ϕ_i are considered now to be complex numbers in weighted projective space. This generalizes the correspondence with manifolds of [1] from the minimal series $SU(2)$ to the more general $SU(n+1)/SU(n)$. Owing to the calculation of the chiral algebra in non-conformal field theories, described above, we can map $N = 2$ supersymmetric theories to manifolds, even in the absence of conformal invariance. The corresponding manifold is again given by Eq. (3.16) with V replaced by $\hat{V}$ and with the variables ϕ_i now considered to be complex numbers in C^n, i.e., a complex affine variety rather then a complex projective one. The elements of the chiral algebra can now be identified with the moduli space of the affine variety. We shall see more of this type of correspondence in the next section.

4. The Geometrical Picture

In this section we will apply some algebraic geometric notions to fusion rings. Let us begin with some elements of ring theory and algebraic geometry [15, 12]. Let A be some commutative algebra over the pair of fields (k, K), where K is the algebraic closure of k)[6]. For any ring, R, two types of radicals can be defined. First the Jacobson radical, $J(R)$, which is the intersection of all the maximal ideals in R. Equivalently, $J(R)$ is the union of all the ideals, I, for which $(1+x)$ is invertible for any $x \in I$. A ring for which the Jacobson radical vanishes is called semisimple. The quotient ring $R/J(R)$ is semisimple. Another radical one can define is the nilradical, $N(R)$, which is equal to (for a commutative ring) the set of all nilpotent elements in R, i.e., elements x such that $x^n = 0$ for some n. Clearly, the nilradical is contained in J since if x is nilpotent, $(1+x)^{-1} = 1 - x + x^2 + \dots (-1)^n x^n$ and so $1 + x$ is invertible, implying that $x \in J(R)$.

Now, assume that the nilradical of the algebra R vanishes, or that there are no nilpotent elements. Let $x_1, x_2, \dots, x_n$ be a set of generators of R. There is a natural map from the polynomial algebra $P[x_1, x_2, \dots, x_n] = P[x]$ to R given by substituting the generators. The kernel of this map is an ideal, I, in $P[x]$. Clearly $R = P[x]/I$. A set of generators of I is called the relations (or syzigies) of the algebra R. Consider the zero set of I, or the root locus, M, which consists of all the points $x \in K^n$ for which $p(x) = 0$ for any $p \in I$. M is some affine variety in K^n. Consider the ideal J which consists of all the polynomials which

[5] The chiral algebra of perturbed minimal models (of types A, D, E_6) was recently calculated directly using perturbation theory in [14]. The result described there agrees with ours

[6] Actually, we will allow k to be a ring, provided it is an integral domain, and then take K to be a field containing the algebraic closure of the field of fractions of k. For fusion rings, k will be taken as the ring of integers which is not a field but an integral domain. K will then be taken to be the complex numbers

vanish on M. Hilbert's Nullstelenzats theorem asserts that any elements of $p \in J$, raised to a sufficiently high power, lies in I, $p^m \in I$. It follows that if $N(R) = 0$ then J must be identical to I, since $p^n \in I$ implies that $p \in I$. Thus every such algebra can be mapped uniquely to an affine variety M. The converse of this is also true; any variety M would give rise to the algebra $P[x]/J$, where J is the ideal of polynomials which vanish on M. The algebra R itself can be interpreted as the algebra of polynomial maps (with coefficients in k) from the variety M to the field K or, in other words, the algebra of functions on the variety M. One introduces the notion of regular maps between the varieties M and N as the polynomial maps with coefficients in k which take M to N. Clearly, a regular map induces a homomorphism on the corresponding algebras and vice-versa. The composition of regular maps is the same as the composition of homomorphisms. More precisely the category of finitly generated commutative algebras with vanishing nilradical is equivalent to the category of affine varieties, with the functorial map defined above. It follows that any algebraic question can be phrased geometrically and vice versa. The algebraic R is called the coordinate ring of the affine variety M.

Now, suppose that R is finite dimensional over k. Then, since R can be identified as the algebra of polynomial functions, $p : M \to K$, it must be that M consists of a finite number of points, whose number is $\dim(R)$. The following basis can be chosen for R (over K). Let x_i be the i^{th} point of M (in some arbitrary order). Define the functions,

$$\zeta_i(x_j) = \delta_{ij} , \tag{4.1}$$

namely, ζ_i vanished in all the points except one. Clearly, the product of two such functions is

$$\zeta_i \zeta_j = \zeta_i \delta_{ij} , \tag{4.2}$$

as is seen by substituting any of the points of M. Equation (4.2) shows that over K any two such algebras are isomorphic if they have the same dimension. Equivalently any n points in C^m can be mapped to any other n points by a regular map (provided, of course, that $k = K$). The detailed structure of the algebra (or alternatively, the variety M) reveals itself only when k is not algebraically closed. In the case of the fusion rings, the most suitable choice is $k = Z$, the integers, and $K = C$, the complex numbers.

Now, fusion rings in a rational conformal field theory are semisimple. It follows that they can be cast in the form Eq. (4.2) along the general lines described above. The ζ_i are closely related to the basis introduced in [16] for the fusion rules. To see it, note that if $\beta_i = \sum_j S_{ij}\,[j]$, where S is the matrix of modular transformations, and $[j]$ is any primary field block, then

$$\beta_i \beta_j = \lambda_i \delta_{ij} \beta_j , \qquad \text{where } \lambda_i = S_{i0}^{-1}. \tag{4.3}$$

Now, the β_i are polynomials in x. Substituting any of the points of M we find that

$$\beta_i(x)\beta_j(x) = \lambda_i \beta_i(x) \delta_{ij} . \tag{4.4}$$

It follows that $\beta_i(x) = 0$ or $\beta_i(x) = \lambda_i$ and it is non-zero for a unique point. Thus $\beta_i = \lambda_i \zeta_i$, and

$$[j] = S^{\dagger}_{ij} \beta_i = \frac{S^{\dagger}_{ij}}{S_{i,0}} \zeta_i . \tag{4.5}$$

Substituting any of the points $x \in M$ and denoting by $p_j(x)$ the polynomial which represents the field $[j]$ we find that,

$$p_j(x_k) = \frac{S^\dagger_{jl}}{S_{i,0}} \zeta_l(x_k) = \frac{S^\dagger_{jk}}{S_{i,0}}. \tag{4.6}$$

In particular, since the generators are identified with some of the fields, $p_\alpha(x) = x^{(\alpha)}$ for $\alpha \in S$ where S is some subset of the fields, it follows that the points of M are given by

$$x_i^{(\alpha)} = \frac{S^\dagger_i \alpha}{S_{i,0}}. \tag{4.7}$$

Thus we have expressed the variety M in terms of the modular matrix S. Further, S can be described as the values of the polynomials of R on the variety M, or the functions that each primary field corresponds to. To summarize,

Theorem (4.1). *Any fusion ring can be written in the form $P[x_1, x_2, \dots, x_n]/I$, where x_i are the generators and $I = (p_1, p_2, \dots, p_n)$ is the ideal of relations. The syzigies p_i are algebraically independent and the solution of the system and polynomial equations $p_i(x) = 0$ is given by the points $x^{(\alpha)} = S^\dagger_{i\alpha}/S_{i0}$, where S is the modular matrix and $[\alpha]$ for $\alpha \in S$ are the generators. Equivalently, It can be described as the ideal consisting of all polynomials vanishing at these points.*

Consider a fusion ring generated by one primary field, which will be denoted by $x = [1]$. Then M is an affine variety in C, or the points $x_i \in C$, where

$$x_i = \frac{S^\dagger_{i1}}{S_{i,0}}, \qquad \text{for } i = 0, 1, 2, \dots n. \tag{4.8}$$

The ideal I of polynomials vanishing on M is (since $P[x]$ is a principal ideal domain) generated by some polynomial $p \in P[x]$. Clearly, $p(x)$ is proportional to $\prod_{i=0}^{n} (x - x_i)$. Hence

$$R = \frac{P[x]}{(p)}, \qquad \text{where } p(x) = \prod_{i=0}^{n} \left(x - \frac{S^\dagger_{i,1}}{S_{i,0}} \right). \tag{4.9}$$

Note that this is the structure of R as a ring over Z and that, in particular, the coeficients of $p(x)$ are all integral. Now, it is easy to see that ζ_i is given by

$$\zeta_i = p_i(x)/p_i(x_i), \qquad \text{where } p_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - x_j), \tag{4.10}$$

since $\zeta_i(x_j) = \delta_{ij}$. From this we find an explicit expression for the primary fields as polynomials in x,

$$p_j(x) = \sum_l S^\dagger_{jl} \beta_l = \sum_l S^\dagger_{jl}/S_{j,0} \zeta_l = \sum_{l=0}^{n} \frac{S^\dagger_{jl}}{S_{j0}} \prod_{\substack{r=0 \\ r \neq i}}^{n} \frac{x - S^\dagger_{r1}/S_{i0}}{S^\dagger_{l1}/S_{l0} - S^\dagger_{r1}/S_{r0}}. \tag{4.11}$$

Let us now describe the variety M which corresponds to the fusion rings of Sect. (2). There, we found that the fusion ring of $SU(N)_k$ can be described as the quotient $P[x_1, x_2, \ldots, x_n]/(\partial_i V)$, where x_i are the anti-symmetric representations, and the potential is $V = \frac{1}{N+k}\sum_{i=1}^{N} q_i^{k+N}$ with the x_i identified as the symmetric functions in the q_i's, $n = N-1$ and with the constraint $x_N = q_1 q_2 \ldots q_N = 1$. The manifold $\frac{\partial V}{\partial x_i} = 0$ is identical to the manifold $\frac{\partial V}{\partial q_i} = 0$ subject to the constraint, except at the points where the Jacobian, which is equal to the Vandermonde determinant, vanishes. Introducing a lagrange multiplier λ, we see that the manifold M is given by the extremum with respect to q_i and λ of the potential

$$U = \frac{1}{k+N}\sum_{i=1}^{N} q_i^{N+k} - k\left(\prod_{i=1}^{N} q_i - 1\right). \tag{4.12}$$

By deriving U with respect to q_i and λ we obtain the conditions for an extremum,

$$q_i^{N+k} = \lambda, \qquad \prod_{i=1}^{N+k} q_i = 1\,. \tag{4.13}$$

Making the change of variables, $q_i = \exp(i\theta_i - i\theta_{i-1})$ with $\theta_0 = \theta_N = 0$, we find that,

$$\theta_i = \frac{s_i}{N+k} + \frac{z\Lambda_1 \cdot \Lambda_i}{N+k} + 2\pi n_i\,, \tag{4.14}$$

where s_i, n_i and z are any integers. Defining the variables

$$\phi = \frac{N+k}{2\pi}\sum_{i=1}^{n} \theta_i \alpha_i\,, \tag{4.15}$$

where α_i are the simple roots, it follows that U has an extremum if and only if

$$\phi \in M^* \operatorname{mod}(k+N)M\,, \tag{4.16}$$

where M is the root lattice and M^* is its dual (which is the weight lattice). From these solutions we need to exclude the points at which the Jacobian vanishes, which do not correspond to an extremum with respect to the x_i. These are the solutions where $q_i = q_j$, for some i and j. From the denominator formula, Eq. (2.26–2.27), it follows that these correspond to ϕ's such that $w(\phi) - \phi \in (k+N)M$, where w is some reflection in the Weyl group. Now, the variables x_i are expressed in terms of the ϕ's through the Weyl character formula, Eq. (2.25), and it follows that

$$x_\lambda(\theta_i) = \frac{\sum_{x\in W} (-1)^w e^{2\pi i\omega(\lambda+\varrho)\phi/(k+N)}}{\sum_{x\in W} (-1)^w e^{2\pi i w(\varrho)\phi/(k+N)}}\,, \tag{4.17}$$

where λ correspond to the anti-symmetric representations, $\lambda = \Lambda^{(i)}$. The same formula holds for the other representations, with λ taken as the highest weight. From this equation it is easy to see that $\mathrm{ch}_\lambda(\phi) = \mathrm{ch}_\lambda[\omega(\phi)]$ implying that ϕ and $\omega(\phi)$ lead to the same solution. Consequently, the manifold M consists of the vectors in $M^* \operatorname{mod}(k+N)M$ modulo the anti-invariant action of W. As is well known, e.g. [4], these vectors can be described as $\Lambda + \varrho$, where Λ is an integrable highest weight at level k. Thus, the points of M are given by the integrable highest weight vectors at level k, Λ, whose number is, indeed, equal to the dimension of the fusion ring R, in accordance with our general theory. The points are $\phi_\Lambda = 2\pi(\Lambda + \varrho)/(k+N)$. Substituting these points into the character formula, we find that the values of the primary fields at these points are

$$\mathrm{ch}_\mu(\lambda) = \frac{S^\dagger_{\lambda,\mu}}{S_{0,\mu}}, \tag{4.18}$$

where S is the unitary matrix

$$S_{\lambda,\mu} = \left|\frac{M^*}{(k+g)M}\right|^{1/2} \sum_{\omega\in W} (-1)^w e^{-2\pi i w(\lambda+\varrho)\,(\mu+\varrho)/(k+g)}. \tag{4.19}$$

(We used $S_{0,\mu} = S^\dagger_{0,\mu}$.) The partition functions of the current algebra transform according to the matrix of modular transformations, which is identical to Eq. (4.19) [2]. Thus we have verified directly Theorem (4.1) for the $SU(N)_k$ theories.

Consider, for example, the case of $SU(2)$ [7]. We have found that (r) is given by Chebyshev polynomials,

$$U_r(2\cos\phi) = \frac{\sin(r+1)\phi}{\sin\phi}. \tag{4.20}$$

Thus, quite clearly, $U_{k+1}(x) = 0$ if and only if $\phi_n = \pi(n+1)/(k+2)$, for $n = 0, 1, \ldots k$, or $x_n = 2\cos(\pi(n+1)/(k+2))$. We can now calculate the value of $U_r(x_n)$,

$$U_r(x_n) = \frac{\sin\left[\dfrac{\pi(r+1)\,(n+1)}{k+2}\right]}{\sin\left[\dfrac{\pi(r+1)}{k+2}\right]}, \tag{4.21}$$

which is indeed indentical (up to a factor which is fixed by the unitarity) to the matrix of modular transformations, Eq. (4.19), which reads for $SU(2)$,

$$S_{i,j} = \frac{2}{\sqrt{k+2}} \sin\left[\frac{\pi(i+1)\,(j+1)}{k+2}\right]. \tag{4.22}$$

From Eqs. (4.10–4.11) one finds product formulae for Chebyshev polynomials of the second kind. It is left as an exercise to check these relations by explicitly substituting the matrix S.

Let us briefly discuss the application of the formalism described above in the investigation of automorphisms of the fusion rules. As already remarked, from the basis of functions on the variety M, it is evident that any two such algebras over C are isomorphic if, and only if, their dimensions are equal. The isomorphism simply sends ζ_i to ζ'_i, where ζ and ζ' are the two basis of the algebras. Clearly, also the automorphism group of the algebras over C is S_d, the permutation group, where $d = \dim(R)$. Any automorphism is of the from $\zeta_i \to \zeta_{p(i)}$, where p is some permutation. This can be seen also geometrically: the automorphisms correspond to any permutation of the points of the variety M. The question becomes much more complicated when considering R as a ring, allowing only integral coefficients. The automorphisms are then polynomials in

[7] The description of the fusion rules as a truncation of character multiplication, was already noted in [16] in the case of $SU(2)$

R which have integral coefficients. Geometrically, this translates to the question which regular maps (i.e., polynomial maps with coefficients in Z) take the points of M into themselves. Clearly, only a subgroup of the permutation group S_d survives, as most of these maps do not have integral coefficients (but rather some algebraic integers). Thus, the question of finding the automorphisms under Z translates into a difficult arithmetic question which will not be addressed here.

The discussion in much of this paper centered around theories of the type $SU(N)$. However, much of the machinery developed here applies equally well to any rational conformal field theory and, in particular, to current algebras based on other types of groups.

Consider the Schubert-like calculus. We have based it on Young tableaux and thus as it stands it is specific to $SU(N)$ type groups. However, it is quite likely that one can develop a similarly elegant calculus for any group. The basic steps are finding the set of generators and writing a Pieri-like formula for their products. Such a set of generators for the general group are the primary fields associated with the fundamental weights. The solution of the Pieri-like formula once written down is given, again, by the characters of the finite algebras, for large enough k. In particular, the characters expressed in terms of the generators again form a system of orthonormal polynomials, just as in the $SU(N)$ case, with a measure which is D^2/J, where D is the denominator of the Weyl character formula and J is the Jacobian for the change of variables, from the θ_i which parametrize the maximal torus to the fundamental weights. (In the case of $SU(N)$ we have found $D = J$. This might also hold for the other groups.) Now, the Pieri-like formula differs from the character products by the fact that some representations are missing. The missing representations form an ideal, I_k, specific to the level, Thus the fusion ring is given by

$$R_k = P[x_1, x_2, \ldots, x_n]/I_k, \tag{4.23}$$

where the x_i correspond to the i^{th} fundamental weight. At $k \to \infty$ the algebra is a free polynomial algebra. Now according to the discussion of Sect. (3), the points where I_k vanishes correspond to a variety M which consists of d points in C^n, where d is the number of integrable highest weights at level k (i.e., the dimension of the algebra). In the case of $SU(N)$ we found that these points correspond to the elements $2\pi(\Lambda + \varrho)/(k+g)$ in the Cartan subalgebra, where g is the dual Coxeter number. It is reasonable to guess that these would be the points in general, and thus,

$$x_i = \mathrm{ch}_\mu \left(\frac{2\pi i(\Lambda + \varrho)}{k+g} \right), \qquad \text{where } \mu = \Lambda^i, \tag{4.24}$$

and Λ^i is the i^{th} fundamental weight. As explained earlier, the values of the primary fields when expressed as polynomials in the generators evaluated at these points, correspond to the modular matrix. This is indeed consistent with Eq. (4.24) since

$$\mathrm{ch}_\mu \left(\frac{2\pi(\Lambda + \varrho)}{k+g} \right) = \frac{S^\dagger_{\mu,\Lambda}}{S_{0,\Lambda}}, \tag{4.25}$$

where S is the matrix of modular transformations, Eq. (4.19), as is easy to see from the Weyl character formula Eq. (2.25). We have thus formed a complete picture of the fusion rules: It is the algebra of polynomials in n variables x_i modulo the ideal of polynomials vanishing at the points x_Λ given by Eq. (4.24).

The unitarity of S implies that the primary fields are linearly independent in this algebras and thus form a basis of it (as the dimension is equal to the number of primary fields). It follows that Eq. (4.23) indeed represents an algebra which is identical to the usual group multiplication, modulo some identifications, which is closed among the integrable representations and is consistent with the relation to the modular transformations.

It is not hard to describe more explicitly the ideal I_k. Consider the expression for the primary fields, Eq. (4.25). Clearly, if $w_\alpha(\Lambda+\varrho)-(\Lambda+\varrho) \in (k+g)M_l$, where M_l is the lattice generated by the long roots, then ch_Λ vanishes for all the points in M. It follows that the ideal I_k contains all the representations with highest weight Λ such that,

$$\alpha(\Lambda+\varrho) = 0 \bmod (k+g)\,, \tag{4.26}$$

for some root of α of G. Choosing, in particular, $\alpha = \theta$ (the highest root), and using the relation $\varrho\theta = g-1$, it follows that I_k contain all the representations at level $k+1$ (i.e., $\theta\Lambda = k+1$). This, gives also, another proof for Eq. (2.20).

The mapping of an associative algebra to an affine variety in C^n, described above, is general to all finite dimensional semi-simple algebras. It is a somewhat lacking description since it translates algebraic questions into hard arithmetical ones. The fusion rings described here, afford however a different geometrical description which is more natural from the viewpoint of quantum field theory. As we found here, all the fusion rings can be described by potentials V which are polynomials in the generators $V(x_{1,2},\ldots,x_n)$ with integral coefficients. The fusion ring itself is the quotient $P[x]/(\partial_i V)$. Now, consider the manifold,

$$V(x_1,x_2,\ldots,x_n) = 0\,, \tag{4.27}$$

where the x_i are considered to be complex variables in C. This is some affine variety, which is a hypersurface of codimension one in C^n, where n is equal to the number of generators in the ring. Now, what is the relation between R and the variety M? Consider the perturbations of the complex structure of M. These can be described as polynomials which can be added to the defining equation, Eq. (4.27),

$$\hat{V} = V + \varrho = 0\,, \tag{4.28}$$

which do not change the degrees of V. However, linear redefinitions of the x_i lead to the same complex structure. Such redefinitions give polynomials which are proportional to derivatives of V. It follows that the complex structures of M (i.e., its moduli space) are described by the ring $P[x_1,x_2,\ldots,x_n]/(\partial_i V)$. Namely, this is precisely the primary fields of the conformal field theory, and we recovered the fusion ring as the deformations (or moduli space) of the hypersurface M.

5. Discussion

In this paper we have focused on two apparently unconnected finite dimensional associative algebras arising in conformal field theory. One is the fusion rule algebra expressing the way the primary fields fuse in the operator product algebra. The other is the operator product algebra of the chiral fields in an $N=2$ superconformal field theory. Both physically and mathematically these algebras are quite different. Physically, the fusion rules express the truncation of the operator products owing to the existence of an extended algebra in a rational conformal field theory, while the chiral algebra describes actual correlation functions of some special fields, in a theory which is not necessarily rational. Mathematically, the two algebras are also as different as one can get. While the fusion algebra is semi-simple, the chiral algebra is a local algebra, i.e., an algebra with a unique maximal ideal, which is hence identical with the Jacobson radical. Thus the close relation and complete analogy between the two structures, described here, is somewhat surprising.

The first level of this relation is the direct one. Namely, as we have shown, the fusion ring of the the theories $SU(N)_k$ is almost identical to the chiral algebras of the $N = 2$ theories $SU(N)_k/SU(N-1)$. The connection is two-fold. First, the chiral algebra of this theory is the homogeneous part of the fusion ring of $SU(N)_k$, i.e., taking the limit $\lambda \to \infty$ in the change of variables $x_i \to \lambda^i x_i$. The second is that the fusion ring of $SU(N)_k$ can be described, precisely, as the chiral algebra of the theory $SU(N+1)_{k-1}/SU(N)$ specialized to the value $C^{A^N} = q_1 q_2 \dots q_n = 1$. [This is most easily seen from the explicit expressions of the algebras, Eq. (3.9).] Thus, we see that these two mathematically and physically very different objects are identical, in essence. It follows that there is a close analogy between the two algebras and one can write a dictionary for translating notions from one to the other: chiral field $\to$ primary field, structure constant $\to$ fusion coefficient, weighted projective variety $\to$ affine variety.

In the case of $N = 2$ theories we have encountered two quite different geometrical interpretations. The first, is the original map of $N = 2$ theories to manifolds with vanishing first Chern class, conjectured to be general in [1]. A different type of geometrical correspondence follows from the analysis of Sect. (3). There we have seen that the chiral fields are identical to the Schubert varieties which form the natural basis of the cohomology of Grassmann manifolds. Thus, the chiral algebra is isomorphic to the cohomology of the corresponding Grassmann manifolds, with the two natural bases being completely identical. So, in one geometrical description the chiral algebra is the ring of deformations of the manifold, while in the other it is the entire cohomology. The connection between these two geometrical pictures is as yet a little mysterious. Somehow, one should be able to get from one geometry to the other in a procedure that closely imitates the gluing of subtheories and the $U(1)$ projection needed to derive the conformal field theory which corresponds to a manifold of vanishing first Chern class. This is an entirely geometrical question. Namely, for each complex manifold M (with the correct dimension) one can define a manifold of lower dimension, N, such that the deformations of N are the cohomology of M. Thus, the complete construction of $N = 2$ string theory can be phrased and studied in purely geometrical terms.

The geometrical correspondence is particularly well understood in the case that the theory admits a scalar field theory realization. As was shown in [7] this reflects itself by having an integrable chiral algebra, i.e., an algebra of the form $P[x_1, x_2, \dots, x_n]/(\partial_i V)$, where V is some potential. The potential is used, in turn, to construct the manifold.

As we have seen in Sect. (4), this line of thought has an analog in the study of fusion rings. It turns out that the fusion rings studied here are all integrable. A study of various other examples leads us of the following:

Conjecture. All fusion rings encountered in rational conformal field theory are integrable, with a potential V which is a polynomial with integral coefficients.

Thus, the fusion rings are the close analog of the scalar field theories and hence, bearing the correctness of this conjecture, all rational conformal field theories can be classified through their corresponding potentials. These potentials lead, in turn, as discussed in Sect. (4) to a similar geometrical interpretation for the fusion ring. The latter can be described as the deformations of the affine variety, $V(x_1, x_2, \dots, x_n) = 0$, where $x_i \in C$. Thus, one can map a rational conformal field theory to some affine variety and study it geometrically. A related question is what is the quantum field theory interpretation of this construction. Namely, can one think of the conformal field theory as some sort of a sigma model on the corresponding manifold? Is there, for example, a scalar field theory construction which explains the relation found here between the fusion ring and the deformations of its corresponding affine variety? In relation to string theory: does this imply that the corresponding string theories are geometrical in nature? Clearly, a deeper understanding of these questions is called for.

Appendix A. Discriminants of Polynomial Equations

Consider the most general n^{th} order polynomial equation,

$$\prod_{i=1}^{n} (x - q_i) = x^n - S_1 x^{n-1} + S_2 x^{n-2} - \dots + (-1)^n S_n = 0\,, \tag{A.1}$$

where the coefficients S_i are expressed in terms of the roots as

$$S_i = \prod_{i_1 < i_2 < \dots < i_r} q_{i_1} q_{i_2} \cdots q_{i_r}\,. \tag{A.2}$$

The discriminant of this equation, R, is defined as the square of the Vandermonde determinant of the roots,

$$R = \prod_{i<j} (q_i - q_j)^2. \tag{A.3}$$

The function R is symmetric in the q_i and thus can be expressed (according to the symmetric function theorem) as a polynomial in the generators of symmetric function, which are the S_r. This polynomial is the discriminant. It vanishes if and only if Eq. (A.1) has degenerate solutions.

Now, we can calculate the discriminant directly by noting that $R = (\det M)^2$, where $M_{i,j}$ is the matrix $M_{i,j} = q_i^{j-1}$, and so $R = \det(M^t M)$,

$$R = \left(\det_{i,j} q_i^{j-1}\right)^2 = \det_{i,j} \sum_k q_k^{i-1} q_k^{j-1} = \det_{1 \le i,j \le N} V_{i+j-2}\,, \tag{A.4}$$

where $V_m = \sum_k q_k^m$ are the potentials defined in Sect. (2). These potentials can be calculated in various ways. For example, by calculating the Chern classes c_i using the Pieri formula, and then integrating these to get the potentials, using Eq. (2.21). Another way is to use the recursion relation which follows by multiplying Eq. (A.1) by q^s and summing over k,

$$\sum_{i=0}^{n} (-1)^i V_{s+n-i} S_i = 0\,, \tag{A.5}$$

with the convention $S_0 = 1$. The first few terms can be calculated explicitly, and the rest from the recursion relation Eq. (A.5).

A less direct way to calculate the discriminant is the following. Let us first set $q_n = 0$. Then $S_n = 0$ and

$$R_n = \prod_{i<j} (q_i - q_i)^2 = (q_1 q_2 \dots q_n)^2 \prod_{i<j\le n-1} (q_i - q_j)^2 = S_{n-1}^2 R_{n-1}\,, \tag{A.6}$$

where R_n is the discriminant of the n^{th} order equation, expressed in terms of S_i, $i = 1, 2, \dots n$. It follows that $R_n = R_{n-1}S_{n-1}^2 + O(S_n)$.

Let us now derive a differential equation for the discriminant. Substituting $q_i' = q_i + \varepsilon$, where ε is a small constant, the discriminant R_n does not change. The change in the symmetric functions is,

$$S_l' = \sum_{i_1<i_2<\dots<i_l} (q_{i_1} + \varepsilon)\,(q_{i_2} + \varepsilon)\dots(q_{i_l} + \varepsilon) = S_l + (n - l + 1)\varepsilon S_{l-1} + O(\varepsilon^2)\,. \tag{A.7}$$

Thus we find the differential equation,

$$\Delta R_n = \sum_{l=1}^{n} (n - l + 1) S_{l-1} \frac{\partial R_n}{\partial S_l} = 0\,. \tag{A.8}$$

Setting $R_n = \sum_{j=0}^{m} b_j S_n^j$ for some m (by degree counting $m = n-1$) and substituting into the differential equation Eq. (A.8), we obtain a recursion relation for the b_j's,

$$b_j = -\frac{1}{jS_{n-1}} \sum_{l=1}^{n-1} (n - l + 1) S_{l-1} \frac{\partial b_{j-1}}{\partial S_l}\,. \tag{A.9}$$

This equation, together with $b_0 = S_{n-1}^2 R_{n-1}$, Eq. (A.6), can be used to compute b_j recursively.

This method is considerably simpler, calculation wise, than the first one described, and is also particularly suitable for a mechanical evaluation of the discriminant. The formal computer language Mathematica was used to this end. The program which is simply a translation into Mathematica of the recursive definition of the discriminant, Eqs. (A.6, A.9), is listed below along with the output: the discriminants R_n for $2 \le n \le 5$. In the output $R(n)$ stands for R_n and $S(l)$ for S_l.

```
B/ : B[n_, 0] := Expand[S[n − 1]^2 * R[n − 1]]
S/ : S[0] = 1
B/ : B[n_, m_] := B[n, m] = Expand[−((1 * Sum[(n − 1[n, m] + 1) * S[l[n, m] − 1] *
     D[B[n, m − 1], S[l[n, m]]], {l[n, m], 1, n − 1}])/(m * S[n − 1]))]
R/ : R[1] = 1
R/ : R[n_] := R[n] = Expand[Sum[B[n, q[n]] * S[n]^q[n], {q[n], 0, n − 1}]]
Do[TeXForm[R[k]] >>> disc2.a, {k, 2, 5}]
```

The program took a second or two to execute, and can be easily used to compute the discriminants for larger k. The output of this program, printed in tex format into a file, is as follows,

$$
\begin{aligned}
R(2) &= S(1)^2 - 4S(2)\\
R(3) &= S(1)^2S(2)^2 - 4S(2)^3 - 4S(1)^3S(3) + 18S(1)S(2)S(3) - 27S(3)^2\\
R(4) &= S(1)^2S(2)^2S(3)^2 - 4S(2)^3S(3)^2 - 4S(1)^3S(3)^3 + 18S(1)S(2)S(3)^3\\
&\quad - 27S(3)^4 - 4S(1)^2S(2)^3S(4) + 16S(2)^4S(4) + 18S(1)^3S(2)S(3)S(4)\\
&\quad - 80S(1)S(2)^2S(3)S(4) - 6S(1)^2S(3)^2S(4) + 144S(2)S(3)^2S(4) - 27S(1)^4S(4)^2\\
&\quad + 144S(1)^2S(2)S(4)^2 - 128S(2)^2S(4)^2 - 192S(1)S(3)S(4)^2 + 256S(4)^3
\end{aligned}
$$

$$
\begin{aligned}
R(5) &= S(1)^2S(2)^2S(3)^2S(4)^2 - 4S(2)^3S(3)^2S(4)^2 - 4S(1)^3S(3)^3S(4)^2\\
&\quad + 18S(1)S(2)S(3)^3S(4)^2 - 27S(3)^4S(4)^2 - 4S(1)^2S(2)^3S(4)^3 + 16S(2)^4S(4)^3\\
&\quad + 18S(1)^3S(2)S(3)S(4)^3 - 80S(1)S(2)^2S(3)S(4)^3 - 6S(1)^2S(3)^2S(4)^3\\
&\quad + 144S(2)S(3)^2S(4)^3 - 27S(1)^4S(4)^4 + 144S(1)^2S(2)S(4)^4 - 128S(2)^2S(4)^4\\
&\quad - 192S(1)S(3)S(4)^4 + 256S(4)^5 - 4S(1)^2S(2)^2S(3)^3S(5) + 16S(2)^3S(3)^3S(5)\\
&\quad + 16S(1)^3S(3)^4S(5) - 72S(1)S(2)S(3)^4S(5) + 108S(3)^5S(5)\\
&\quad + 18S(1)^2S(2)^3S(3)S(4)S(5) - 72S(2)^4S(3)S(4)S(5)\\
&\quad - 80S(1)^3S(2)S(3)^2S(4)S(5) + 356S(1)S(2)^2S(3)^2S(4)S(5)\\
&\quad + 24S(1)^2S(3)^3S(4)S(5) - 630S(2)S(3)^3S(4)S(5)\\
&\quad - 6S(1)^3S(2)^2S(4)^2S(5) + 24S(1)S(2)^3S(4)^2S(5) + 144S(1)^4S(3)S(4)^2S(5)\\
&\quad - 746S(1)^2S(2)S(3)S(4)^2S(5) + 560S(2)^2S(3)S(4)^2S(5)\\
&\quad + 1020S(1)S(3)^2S(4)^2S(5) - 36S(1)^3S(4)^3S(5) + 160S(1)S(2)S(4)^3S(5)\\
&\quad - 1600S(3)S(4)^3S(5) - 27S(1)^2S(2)^4S(5)^2 + 108S(2)^5S(5)^2\\
&\quad + 144S(1)^3S(2)^2S(3)S(5)^2 - 630S(1)S(2)^3S(3)S(5)^2 - 128S(1)^4S(3)^2S(5)^2\\
&\quad + 560S(1)^2S(2)S(3)^2S(5)^2 + 825S(2)^2S(3)^2S(5)^2 - 900S(1)S(3)^3S(5)^2\\
&\quad - 192S(1)^4S(2)S(4)S(5)^2 + 1020S(1)^2S(2)^2S(4)S(5)^2\\
&\quad - 900S(2)^3S(4)S(5)^2 + 160S(1)^3S(3)S(4)S(5)^2\\
&\quad - 2050S(1)S(2)S(3)S(4)S(5)^2 + 2250S(3)^2S(4)S(5)^2 - 50S(1)^2S(4)^2S(5)^2\\
&\quad + 2000S(2)S(4)^2S(5)^2 + 256S(1)^5S(5)^3 - 1600S(1)^3S(2)S(5)^3\\
&\quad + 2250S(1)S(2)^2S(5)^3 + 2000S(1)^2S(3)S(5)^3 - 3750S(2)S(3)S(5)^3\\
&\quad - 2500S(1)S(4)S(5)^3 + 3125S(5)^4
\end{aligned}
$$

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