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Classical W-Algebras*

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Abstract. We reconsider the relation between classical W-algebras and deformations of differential operators, emphasizing the consistency with diffeomorphisms. Generators of the W-algebra that are k-differentials are constructed by a systematic procedure. The method extends, following Drinfeld and Sokolov, to W-algebras based on arbitrary simple Lie algebras.

1. Introduction

Among the many relations between integrable systems and conformal field theories, it seems of interest to investigate the concept of W-algebras introduced by Zamolodchikov [1] which play an important role in both fields. In spite of a large number of works pertaining to this subject [2–16], we think that this object has not yet been completely defined. Our purpose here, in a classical context, is to contribute to its clarification.

The Virasoro algebra or its W-extensions appear naturally in the context of classical integrable systems of KdV types [4–15]. We wish to carry out this analysis in a covariant way with respect to diffeomorphisms. In a more complete presentation under preparation, we plan to exhibit these same algebras in a more geometric framework following the recent work of Sotkov and Stanishkov [16]. We shall briefly sketch here a natural construction of the simplest W-algebras pertaining to the fundamental representation of simple Lie groups treating in detail the $SL(n)$, $SO(2n)$ and G_2 cases. Our work relies heavily on an (admittedly incomplete) reading of a fundamental paper by Drinfeld and Sokolov [17].

We shall deal with differential (or more generally pseudo-differential) operators with regular coefficients acting on regular functions. By regular we mean according to the context, infinitely differentiable or analytic functions, either in a fixed neighbourhood or in a pointed neighbourhood of a point, or possibly even along

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the real axis (in the real case). One could even use formal series. The important point is that in any case the notion of derivative be well defined. We also need the notion of integral taken along a non-trivial cycle denoted generically $\mathscr{C}$, for instance the real axis for coefficients vanishing fast enough at infinity, or a period for periodic functions, or else a cycle around a deleted point in the complex plane. In some instances, the integral may extend over only part of the cycle. The above shows that the constructions can be carried in a purely algebraic way along the lines of [18].

Our discussion is carried in three steps. In the case of $sl(n)$ we first decompose a normalized n^{th} order differential operator D mapping $-\frac{n-1}{2}$ differentials into $\frac{n+1}{2}$ differentials into a canonical sum involving currents w_j of weight j ranging between 2 and n in correspondence with the generators of the A_{n-1} W-algebra. We then define associated deformations of D, each one depending on an arbitrary function, which generalize the KdV flows and in the simplest W_3-instance, amount to localizing the latter. In the last step, using the second Adler–Gelfand–Dikii Hamiltonian structure [18] one is able to compute the Poisson brackets between the w generators. This endows the polynomials in the w's and their derivatives with a Lie algebra structure containing the Virasoro algebra generated by w_2. In a last section we succinctly present the generalization to an arbitrary simple Lie algebra and discuss in more detail the slightly more involved case of the D_n Lie algebra.

2. Action of Changes of Variables

We consider linear differential operators of degree n, $D = d^n + \sum_{j=1}^{n} a_j d^{n-j}$ with $d = d/dx$, acting on functions $f(x)$. By a change of function, $f(x) \to \exp -\left(\frac{1}{n}\int^x a_1(u)du\right) f(x)$, one may dispose of the first coefficient a_1. Therefore, with no loss of generality, we shall only consider operators of the form

$$D = d^n + \sum_{j=2}^{n} a_j d^{n-j}. \tag{2.1}$$

We are interested to study how D and its coefficients $a_j(x), j = 2, \ldots, n$, transform under changes of variables $x \to t$. Let $\mathscr{F}_h$ denote the space of functions f that transform as h-differentials (conformal weight h):

$$f(x) = \left(\frac{dt}{dx}\right)^h f(t). \tag{2.2}$$

(Here and in the following, we make a slight abuse of notation, denoting with the same symbol the function before and after the change of variable.)

Proposition 1. *There exists a natural transformation of the functions $a_2, \ldots, a_n$ such that the operator D maps the space $\mathscr{F}_{-(n-1)/2}$ into the space $\mathscr{F}_{(n+1)/2}$.*

Proof. Let $f_1, f_2, \ldots, f_n$ be n linearly independent functions in the kernel of D. Since a_1, the logarithmic derivative of their wronskian W vanishes, W is a constant and by a change of normalization of the f's, may be set equal to 1,

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1^{(n-1)} & \cdots & f_n^{(n-1)} \\ f_1^{(n-2)} & \cdots & f_n^{(n-2)} \\ \vdots & \ddots & \vdots \\ f_1 & \cdots & f_n \end{vmatrix} = 1. \tag{2.3}$$

Let us then define the differential operator D by its action on the function f,

$$[Df] = \begin{vmatrix} f^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \\ f^{(n-1)} & f_1^{(n-1)} & \cdots & f_n^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f & f_1 & \cdots & f_n \end{vmatrix}. \tag{2.4}$$

It is readily seen that D is of the form (2.1), and it is a simple lemma [19] that if $f_1, f_2, \ldots, f_n$ and f belong to $\mathscr{F}_h$, then $W(f_1, f_2, \ldots, f_n)$ belongs to $\mathscr{F}_{nh+(n(n-1))/2}$ and $[Df]$ to $\mathscr{F}_{(n+1)h+(n(n+1))/2}$. The choice of $h = -\dfrac{n-1}{2}$ preserves the condition (2.3). By identification of the coefficients $a_2, a_3, \ldots, a_n$ with minors of the determinant (2.4), one finds their transformation law, and Proposition 1 follows. Under the change $x \to t$,

$$D_t = \phi^{(n+1)/2} D_x \phi^{(n-1)/2}. \tag{2.5}$$

Here and in the following we use the following notations: $\phi(t)$ denotes the jacobian dx/dt of the change of variable, $b(t) = \phi'(t)/\phi(t)$ its logarithmic derivative and the schwarzian derivative $s(t)$ reads

$$s(t) = \{\{x, t\}\} = b'(t) - \frac{1}{2} b^2(t) = \left(\frac{\dfrac{d^3x}{dt^3}}{\dfrac{dx}{dt}} \right) - \frac{3}{2} \left(\frac{\dfrac{d^2x}{dt^2}}{\dfrac{dx}{dt}} \right)^2. \tag{2.6}$$

It is denoted with double pairs of curly brackets to distinguish it from the Poisson brackets introduced later. We shall say that an operator transforms covariantly if it obeys (2.5). As a consequence of $f \in \mathscr{F}_{-(n-1)/2}$ the quantity $dx f[Df]$ is a 1-differential showing that its integral over a cycle is a well defined quantity.

Let us discuss more explicitly how the functions $a_2, \ldots, a_n$ transform. According to (2.5) a_1 vanishes in any coordinate system. On the other hand, a_2 does not transform as a 2-differential but has an "anomalous term" proportional to the schwarzian derivative of the change of variable

$$a_2(t) = a_2(x) \left(\frac{dx}{dt} \right)^2 + c_n \{\{x, t\}\}. \tag{2.7}$$

and this is reminiscent of the transformation law of the energy-momentum "tensor:" this is not an accident, see below. In (2.7), the "central charge" c_n reads

$$c_n = \frac{n(n^2-1)}{12}. \tag{2.8}$$

We recall that under composition of changes of variable, $u \to x \to t$, the schwarzian derivatives transform according to:

$$\{\{u,t\}\} = \{\{u,x\}\}\left(\frac{dx}{dt}\right)^2 + \{\{x,t\}\} \tag{2.9}$$

which implies the consistency of (2.7) and shows that $a_2(x)$ transforms as $c_n\{\{u,x\}\}$, with u a fixed coordinate for which a_2 vanishes. The other coefficients $a_3, \ldots, a_n$ of (2.1) have more complicated transformations involving higher and higher derivatives of $b(t)$. Their variations under *infinitesimal* change of variable, $x = t + \varepsilon(t)$, may be written in a closed form

$$\begin{aligned}\delta a_k &= \varepsilon a_k' + k\varepsilon' a_k + \frac{1}{2}\binom{n+1}{n-k}(k-1)\varepsilon^{(k+1)} \\ &\quad - \sum_{l=2}^{k-1}\left\{\binom{n-l}{k-l+1} - \frac{n-1}{2}\binom{n-l}{k-l}\right\}\varepsilon^{(k-l+1)}a_l.\end{aligned} \tag{2.10}$$

However, we have the following

Theorem 1. *There exist linear combinations w_k of $a_k, a_{k-1}, \ldots, a_2$ and their derivatives, with coefficients polynomials in a_2 and its derivatives that transform as k-differentials ($k \geqq 3$):*

$$\begin{aligned}w_k &= \sum_{l=2}^{k} B_{kl}a_l^{(k-l)} + \sum_{\substack{0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_r \\ \sum p_i + 2r = k}} C_{p_1\cdots p_r} a_2^{(p_1)} \cdots a_2^{(p_r)} \\ &\quad + \sum_{\substack{0 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_r \\ 3 \leqq l \leqq k - \sum p_i - 2r}} D_{p_1\cdots p_r, l}\, a_2^{(p_1)} \cdots a_2^{(p_r)} a_l^{(k-l-\sum p_i - 2r)},\end{aligned} \tag{2.11a}$$

$$\delta w_k = \varepsilon w_k' + k\varepsilon' w_k. \tag{2.11b}$$

Moreover these relations are invertible and one can express in a similar fashion a_k linearly in $w_k, \ldots, w_2 = a_2$ and their derivatives, with coefficients that are differential polynomials of a_2.

It is the aim of the rest of this section to establish this result, to give explicit expressions for the matrix B_{kl} and its inverse and to show how the remaining coefficients may be obtained in a systematic way. For illustration, the formulae for the lowest values of k are displayed in Table I. As these k-differentials w_k will appear later as the generators of the w-algebra, we shall refer to them as "currents" and to k as their "spin." It is of course equally well their conformal weight.

More precisely, we are going to prove that any operator D may be written as a sum of differential operators

$$D = \Delta_2(a_2) + \sum_{k=3}^{n} \Delta_k(w_k, a_2), \tag{2.12}$$

each of which maps the space $\mathscr{F}_{-(n-1)/2}$ into the space $\mathscr{F}_{(n+1)/2}$. In (2.12), Δ_k is linear in w_k and its derivatives.

Let us first consider the term $\Delta_2(a_2)$ that depends solely on a_2 (and its derivatives). Given the function $a_2(x)$, let u denote a variable such that $a_2(u) = 0$.

Table I. w_k as functions of $a_l, SU(n)$ case

$$w_2 = a_2$$

$$w_3 = a_3 - \frac{n-2}{2} a_2'$$

$$w_4 = a_4 - \frac{n-3}{2} a_3' + \frac{(n-2)(n-3)}{10} a_2'' - \frac{(n-2)(n-3)(5n+7)}{10n(n^2-1)} a_2^2$$

$$w_5 = a_5 - \frac{n-4}{2} a_4' + \frac{3(n-3)(n-4)}{28} a_3'' - \frac{(n-2)(n-3)(n-4)}{84} a_2'''$$
$$+ \frac{(n-3)(n-4)(7n+13)}{14n(n^2-1)} ((n-2)a_2 a_2' - 2a_3 a_2)$$

This means that u is a solution of the equation

$$a_2(x) = c_n\{\{u, x\}\} \tag{2.13}$$

or that the jacobian $\phi(x) = du/dx$ and its logarithmic derivative $b(x)$ are such that

$$s(x) = b'(x) - \frac{1}{2} b^2(x) = \frac{a_2(x)}{c_n}. \tag{2.14}$$

The transformation from the function $a_2(x)$ to the function $b(x)$ is an example of a Miura transformation[1] and enables one to write $d^2 + \frac{1}{2}s = (d - \frac{1}{2}b)(d + \frac{1}{2}b)$. In the variable u where a_2 vanishes, the operator Δ_2 reduces to $(d/du)^n$. Therefore, since we want the operator to transform covariantly, in the variable x, it must read

$$\begin{aligned} \Delta_2(a_2) &= \phi^{(n+1)/2}(\phi^{-1}d)^n \phi^{(n-1)/2} \\ &= (d - jb)(d - (j-1)b) \cdots (d + jb), \end{aligned} \tag{2.15}$$

where we have set $n = 2j + 1$. For the consistency of this argument, we have to prove the

Proposition 2. *The expression (2.15) depends upon b only through the schwarzian derivative (2.13) and hence only on a_2 and its derivatives.*

Proof. Clearly, Δ_2 is a differential operator with coefficients that are polynomials in b and its derivatives and may be expressed through (2.14) as polynomials in b, s and derivatives of s. The proof amounts to showing that these polynomials reduce to their term independent of b. To see this, in the expression (2.15) we change b into $b + \delta b$, keeping $s = b' - \frac{1}{2}b^2$ fixed. This implies that δb satisfies the equation $\delta b' - b\delta b = 0$, or equivalently the commutation relation between differential operators

$$(d - (k+1)b)\delta b = \delta b(d - kb) \tag{2.16}$$

[1] For regular a_2, Eq. (2.14) always admits a solution defined, up to a constant $PSL(2)$ transformation, in a fixed neighbourhood of a point. As the following is only formal algebra, the introduction of this intermediate variable is clearly legitimate

for any k. The change of Δ_2 is thus

$$\begin{aligned}\delta\Delta_2 &= \sum_{k=-j}^{j} (d-jb)\cdots(d-(k+1)b)(-k\delta b)(d-(k-1)b)\cdots(d+jb)\\ &=\left(-\sum_{-j}^{j} k\right)\delta b(d-(j-1)b)\cdots(d+jb)\\ &=0. \end{aligned} \tag{2.17}$$

(See [14] and [4] for an alternative argument that the product (2.15) does not depend on the choice of a solution of (2.14).) Under a change of variable, the operator Δ_2 transforms covariantly, thanks to the transformation properties of the schwarzian derivative (2.6).

We now proceed to the construction of the operators $\Delta_k(w_k, a_2)$ along similar lines. Let $w_k(x)$ be a k-differential. In the variable u defined above, the operator has the general form

$$\Delta_k(w_k(u), 0) = \sum_{l=0}^{n-k} \alpha_{kl} w_k^{(l)} d^{n-k-l}, \tag{2.18}$$

and we seek coefficients α (with the normalization $\alpha_{k0} = 1$) such that after the change from u to x, it depends on b only through the schwarzian derivative (2.13),

$$\begin{aligned}\Delta_k(w_k(x), \alpha_2(x)) &= \phi^{(n+1)/2} \sum_{l=0}^{n-k} \alpha_{kl}[(\phi^{-1}d)^l \phi^{-k} w_k](\phi^{-1}d)^{n-k-l}\phi^{(n-1)/2}\\ &= \sum_{l=0}^{n-k} \alpha_{kl}[\mathscr{D}^l w_k]\mathscr{D}^{n-k-1}, \end{aligned} \tag{2.19}$$

where we have introduced the covariant derivative taking h-differentials to $h+1$-differentials:

$$\mathscr{D} f = (d - hb) f \tag{2.20}$$

thus, $\mathscr{D} w_k = (d-kb)w_k, \mathscr{D}^2 w_k = (d-(k+1)/b)(d-kb)w_k$, etc... and $\mathscr{D}^{n-k-l}$ in (2.19) maps $\mathscr{F}_{-(n-1)/2}$ into $\mathscr{F}_{((n+1)/2)-k-l}$. The square brackets in $[\mathscr{D}^l w_k]$ mean that $\mathscr{D}^l$ does not act further to the right.

Proposition 3. *The operator Δ_k in (2.19) depends upon b only through the schwarzian derivative (2.13) provided the coefficients α_{kl} are chosen as*

$$\alpha_{kl} = \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}}. \tag{2.21}$$

Proof. We proceed as before for Δ_2. Variation of b with s fixed produces a differential operator, and imposing that all its independent coefficients vanish yields recursion relations between the α's

$$l(l+2k-1)\alpha_{kl} = (k+l-1)(n+1-k-l)\alpha_{k,l-1}, \tag{2.22}$$

the solution of which (with the boundary condition $\alpha_{k0} = 1$) is given in (2.21).

We have thus shown how to construct $n-1$ independent differential operators transforming covariantly. Given an operator (2.1) depending on the functions $a_2, a_3, \ldots, a_n$, we can thus define the forms w_k by identifying w_3 as the coefficient of d^{n-3} in $D-\Delta_2(a_2)$, w_4 as the coefficient of d^{n-4} in $D-\Delta_2(a_2)-\Delta_3(\omega_3, a_2)$, etc.... This completes the proof of the existence of the decomposition (2.12). This identification is easy to read off for the linear terms

$$a_k = \sum_{l=2}^{k} A_{kl} w_l^{(k-l)} + \text{non-linear terms},$$

$$A_{kl} = \alpha_{l,k-1} = \frac{\binom{k-1}{k-l}\binom{n-l}{k-l}}{\binom{k+l-1}{k-l}}, \tag{2.23}$$

where by convention, we set $w_2 = a_2$. The inverse of the A matrix is the B matrix of (2.11), and reads (see Appendix):

$$B_{kl} = (-1)^{k-l} \frac{\binom{k-1}{k-l}\binom{n-l}{k-l}}{\binom{2k-2}{k-l}}. \tag{2.24}$$

Remark 1. The coefficients α of (2.21) make sense in (2.18) only for $k \geqq 3$. It turns out that for $k=2$ the expression (2.21) gives the terms linear in a_2 in the decomposition (2.12), namely

$$\Delta_2(a_2) - d^n = \sum_{l=0}^{n-2} \alpha_{2,l} a_2^{(l)} d^{n-2-1} + \text{non-linear terms}. \tag{2.25}$$

Indeed the linear terms in the right-hand side read

$$\begin{aligned}\sum_{m=-j}^{j} m d^{j+m} b d^{j-m} &= \frac{1}{2} \sum_{l=0}^{n-2} s^{(l)} d^{n-2-l} \sum_{l+1 \leqq j+m \leqq 2j} 2m \binom{j+m}{l+1} \\ &= \sum_{l=0}^{n-2} \alpha_{2,l} a_2^{(l)} d^{n-2-l}\end{aligned} \tag{2.26}$$

as a result of a simple identity (see Appendix). It follows that the terms linear in a_2 in (2.11a) are also given by the matrix B.

Remark 2. One may prove that the differential operators $\Delta_k(w_k, 0)$ are self-adjoint, up to a sign:

$$\Delta_k(w_k, 0) = (-1)^{n-k} \Delta_k^*(w_k, 0), \tag{2.27}$$

which follows from the following identity (see Appendix):

$$\sum_{l=0}^{q} (-1)^l \frac{(k+l-1)!}{l!(q-l)!(2k+l-1)!} = \frac{(k+q-1)!}{q!(2k+q-1)!}. \tag{2.28}$$

This property of (anti-)self-adjointness carries over to the operators $\Delta_2(a_2)$ and

$\Delta_k(w_k, a_2)$ as is readily verified. Accordingly, one may rewrite (2.18) in a more compact and symmetric form:

$$\Delta_k(w_k, 0) = \sum_{0 \leqq 2l \leqq n-k} \hat{\alpha}_{kl}[w_k^{(2l)}, d^{n-k-2l}]_+ \tag{2.29}$$

with a similar expression for $\Delta_k(w_k, a_2)$, $k \geqq 3$.

Remark 3. One may wonder what is the general expression of the polynomials in the a_k's their derivatives that transform as r-differentials, with r integer larger than 2. Their form may be obtained following the same method as used above to construct the w's: write the expression in the coordinate where $a_2 = 0$ as a differential polynomial in the w's; return to the generic coordinate, transforming the derivatives into covariant derivatives; derive the conditions on the coefficients that enable one to reconstruct a r-differential depending only on a_2. For example, $4w_3' w_4 - 3w_3 w_4'$ is a 8-differential. This method leads to a generating function for the number $N(r)$ of linearly independent r-differentials:

$$\sum_{r \geqq 3} N(r) q^r = q + \frac{1-q}{\prod_{h=3}^{n} \prod_{l \geqq 0} (1 - q^{h+l})}.$$

3. W-Algebras as Generalized KdV Flows

3.1. Infinitesimal Deformations of the Differential Operators. In its infinitesimal form, the previous analysis is a particular case of the following problem: find two infinitesimal differential operators X and Y mapping $\mathscr{F}_{-(n-1)/2}$ respectively $\mathscr{F}_{(n+1)/2}$ onto themselves, such that after a change of functions: $G = (1 + Y)g$ and $F = (1 + X)f$, the equation $g = Df$ takes the form $G = (D + \delta D)F$, with $D + \delta D$ still of the form (2.1). The variation of D is thus given by:

$$\delta D = YD - DX. \tag{3.1}$$

The particular case $X = Y$ corresponds to the $SL(n)$-generalized KdV flows of [17], for which the variations δ all commute. In the general situation considered here, they do not.

Let us recall a few definitions and results from (pseudo)differential calculus [18]. A pseudo-differential operator is a formal series in d with smooth function coefficients, involving negative integer powers of d as well, d^{-1} being defined as the formal inverse of d. Its commutation with functions is taken to be:

$$d^{-1} f = \sum_{i=0}^{\infty} (-1)^i f^{(i)} d^{-i-1} \tag{3.2}$$

so that iterating it

$$d^{-k} f = \sum_{i=0}^{\infty} (-1)^i \binom{k+i-1}{i} f^{(i)} d^{-i-k}. \tag{3.3}$$

We call valuation the smallest power of d appearing in the operator, if it is finite. One denotes by $(R)_+$ the differential part of any pseudo-differential operator R, i.e. its parts with no negative power of d, and $(R)_- = R - (R)_+$. The coefficient of d^{-1}

in R is called the residue of R and denoted $\mathrm{Res}(R)$. One shows [20] that pseudo-differential operators commute under the symbol Res, up to total derivatives:

$$\mathrm{Res}([R_1, R_2]) = \text{total derivative.} \tag{3.4}$$

This leads to the definition of the trace of a pseudo-differential operator R:

$$\mathrm{Tr}(R) = \int_{\mathscr{C}} dx\, \mathrm{Res}(R). \tag{3.5}$$

Any pseudo-differential operator R has a well defined formal inverse denoted R^{-1}. Finally the natural $\mathbf{Z}_2$ involution $*$, which leaves the functions invariant, and such that $d^* = -d$, extends to pseudo-differential operators: for any A and B, $(AB)^* = B^*A^*$.

Let us proceed now to the detailed study of (3.1). First of all, it is clear that the infinitesimal changes of the variable $x \to x + \varepsilon(x)$ are generated on $\mathscr{F}_{-(n-1)/2}$ respectively $\mathscr{F}_{(n+1)/2}$ by:

$$X_1 = \varepsilon d - \frac{n-1}{2}\varepsilon' \tag{3.6a}$$

$$Y_1 = \varepsilon d + \frac{n+1}{2}\varepsilon' = -X_1^*. \tag{3.6b}$$

This entails:

$$\delta_1 D = Y_1 D - D X_1 \tag{3.7}$$

which summarizes the transformations (2.10) of the coefficients of D under a change of variable. More generally we look for deformations (3.1) generated by higher degree differential operators X and Y. By inspection of the powers of d in (3.1), we find constraints relating X and Y. We have the following:

Proposition. *The most general variation of the form (3.1) is built from an arbitrary differential operator $\tilde{X}$ of valuation* 1, *and X and Y read*:

$$X = \tilde{X} - \frac{1}{n}\int^x \mathrm{Res}(D\tilde{X}D^{-1}), \tag{3.8a}$$

$$Y = (DXD^{-1})_+. \tag{3.8b}$$

Proof. Suppose X is of degree k, then the left-hand side of (3.1) is of degree $n-2$, which imposes that Y be of the same degree k and gives $k+2$ constraints obtained by setting to zero the coefficients of all powers of d between d^{n+k} and d^{n-1}. It is easy to see that the $k+1$ first constraints express the Y coefficients in terms of the X ones in a triangular fashion. Multiplying (3.1) by D^{-1} from the right, one finds that $Y - DXD^{-1} = \delta D D^{-1}$. The right-hand side is of degree -2, thus taking the differential part leads to (3.8b). Taking the part of degree -1 (the residue) and writing $X = \tilde{X} + \varepsilon_0$, we have

$$\begin{aligned}\mathrm{Res}\, D\tilde{X}D^{-1} &= -\mathrm{Res}\, D\varepsilon_0 D^{-1}\\ &= -\mathrm{Res}\,[D, \varepsilon_0]D^{-1}\\ &= -n\varepsilon_0'.\end{aligned} \tag{3.9}$$

Since the residue of a commutator is a total derivative, it makes sense to integrate

(3.9) and drop the constant of integration (which does not affect the definition of δ in (3.1) anyway), which completes the proof of the proposition. As an example the degree one operators X_1, Y_1 defined in (3.6) obviously satisfy (3.8) with $\tilde{X} = \varepsilon d$. By abuse of notation we will denote by $\delta_X = \delta_{\tilde{X}}$ the variation (3.1) acting on D hence on its coefficients.

These pairs (X, Y) can be safely restricted to be at most of order $n-1$ for the following reason. Applying the Euclidean division algorithm to a general X of order $k \geqq n$, one may write

$$\begin{aligned} X &= ZD + \hat{X}, \\ Y &= DZ + \hat{Y} \end{aligned} \tag{3.10}$$

with $(\hat{X}, \hat{Y})$ of order at most $n-1$, and related by (3.8b). One sees immediately that Z does not contribute to δD.

We are interested in computing commutators of δ's. From the definition (3.1), we get:

$$[\delta_{X'}, \delta_X] = \delta_{[X',X] + \delta_X X' - \delta_{X'} X}. \tag{3.11}$$

We now want to define a basis $\delta_k(\eta) = \delta_{X_{k(\eta)}}$ such that:

$$[\delta_1(\varepsilon), \delta_k(\eta)] = \delta_k(\varepsilon\eta' - k\eta\varepsilon'), \tag{3.12}$$

which amounts to saying that η transforms as a $-k$-differential under changes of variable. The corresponding X_k and Y_k are built of covariant pieces mapping $\mathscr{F}_{-(n-1)/2}$, respectively $\mathscr{F}_{(n+1)/2}$ into themselves, and could be constructed by a method similar to that of Sect. 2. Alternatively, we shall determine them by using a Hamiltonian language. The variations δ_k will be generated by Hamiltonians of the form $H_k = \int dx \eta(x) w_{k+1}(x)$, through some appropriate Poisson brackets. (Notice that the fact that η is a $-k$-differential guarantees the invariance of the former integral). Before doing so, we have to recall some facts about Poisson brackets on the manifold of differential operators D [21, 17].

3.2. Poisson Brackets. Following [21, 17], Hamiltonian structures (or Poisson brackets) are defined first on linear functionals of D, (i.e. of the a_i and their derivatives), and then extended by differentiation to arbitrary polynomial functionals. Let $l_U(D)$ be a linear functional,

$$\begin{aligned} l_U(D) &= \int dx \sum_{i=2}^{n} u_i(x) a_i(x) \\ &= \mathrm{Tr}\,(d^n + a_2 d^{n-2} + \cdots + a_n)(d^{1-n}u_2 + d^{2-n}u_3 + \cdots + d^{-1}u_n) \\ &= \mathrm{Tr}\, DU. \end{aligned} \tag{3.13}$$

Since a_1 vanishes one can freely add to U a term of the form $d^{-n}u_1$. The two Hamiltonian structures discussed in [21, 17] read

$$\begin{aligned} \{l_U(D), l_V(D)\}_1 &= \mathrm{Tr}\,(D[U,V]) = l_V(DU - UD) \\ \{l_U(D), l_V(D)\}_2 &= \mathrm{Tr}\,((DU)_+(DV) - (VD)(UD)_+) \\ &= l_V((DU)_+ D - D(UD)_+). \end{aligned} \tag{3.14}$$

The first one will not concern us here, whereas the second one seems to fit our goal: the expression in the right-hand side of the last Eq. (3.14) has the desired form (3.1), with $X=(UD)_+$, $Y=(DU)_+=(DXD^{-1})_+$. Some care has to be exercised, however, when using the second Poisson bracket on differential operators D with a vanishing coefficient a_1. It is not generally true that the expression $((DU)_+D-D(UD)_+)$ respects this property. In other words, the X just mentioned does not satisfy (3.8a). One may thus decide to add a further term to U [21]:

$$\hat{U}=U+d^{-n}u_1 \tag{3.15}$$

which does not affect $l_U(D)$ but does modify the second Hamiltonian structure. Adjusting the value of u_1 to

$$u_1=\frac{1}{n}\int^x \operatorname{Res}[U,D] \tag{3.16}$$

is essential to remove the unwanted term of order $n-1$ in $((DU)_+D-D(UD)_+)$. (The resulting Poisson bracket turns out to identify with the one inherited from the Miura transformation [17].) An alternative way consists in introducing a third Hamiltonian structure:

$$\begin{aligned}\{l_U(D),l_V(D)\}_3&=\operatorname{Tr}\left\{\left(\int^x \operatorname{Res}[D,U]\right)[D,V]\right\}\\&=\int_C dx\left(\int^x \operatorname{Res}[D,U]\right)\operatorname{Res}[D,V].\end{aligned} \tag{3.17}$$

It is easily seen that it has the right antisymmetry. As for the Jacobi identity, it follows simply from the fact that the new Poisson bracket is "coordinated"[2] to the second one and moreover, that adding it to the second bracket just amounts to the change of $U\to\hat{U}$ in the latter:

$$\begin{aligned}&\{l_U(D),l_V(D)\}_2+\lambda\{l_U(D),l_V(D)\}_3\\&=\operatorname{Tr}(((DU)_++\lambda\textstyle\int\operatorname{Res}[D,U])D-D((UD)_++\lambda\int\operatorname{Res}[D,U]))V\\&=\{l_{\hat{U}}(D),l_V(D)\}_2\end{aligned} \tag{3.18}$$

with $\hat{U}=U+\lambda d^{-n}\int\operatorname{Res}[U,D]$. Ultimately, we choose $\lambda=1/n$.

3.3. Explicit Formulae for the X_k and Y_k. We can now use this formalism to write explicit formulae for the X's and Y's. We apply the well known fact [17, 14] that the Poisson brackets between a linear functional $l_U(D)$ and a general differential polynomial functional $\Psi(D)$, is entirely determined by the former and the gradient of the latter, namely:

$$\{\Psi(D),l_U(D)\}=\{l_{V_\Psi}(D),l_U(D)\}, \tag{3.19}$$

where

$$V_\Psi=\sum_{i=1}^{n-1} d^{-i}\frac{\delta\Psi(D)}{\delta a_{n-i+1}}, \tag{3.20}$$

[2] Two Poisson brackets are said to be coordinated if an arbitrary linear combination of them also satisfies the axiom of antisymmetry and the Jacobi identity [17]

and $\frac{\delta}{\delta a_j}$ is a short-hand notation for $\sum_k (-d)^k \frac{\delta}{\delta a_j^{(k)}}$. Note that we are extending the application of (3.14) to the functional $l_{V_\Psi}(D) = \mathrm{Tr}(V_\Psi D)$ which is not linear. Taking $\Psi(D) = H_k = \int dx \varepsilon w_{k+1}$, we get:

$$\begin{aligned}\delta_k l_U(D) &= \{\textstyle\int dx \varepsilon w_{k+1}, l_U(D)\}_{2+\lambda 3} \\ &= l_U(\delta_k D) \\ \delta_k D &= (DV_k)_+ D - D(V_k D)_+ ,\end{aligned} \tag{3.21}$$

where $V_k \equiv \hat{V}_{H_k}$ with the notations of (3.15)–(3.16). It is now easy to identify $X_k = (V_k D)_+$ and $Y_k = (DV_k)_+$. Knowing w_{k+1}, this gives compact expressions for the parts $\tilde{X}_k$ and $\tilde{Y}_k$ of valuation one of X_k and Y_k,

$$\tilde{X}_k = \left(\left(\sum_{i=1}^{n-1} d^{-i} \frac{\delta H_k}{\delta a_{n-i+1}} \right) \left(d^n + \sum_{j=2}^{n} a_j d^{n-j} \right) \right)_{++}, \tag{3.22a}$$

$$\tilde{Y}_k = \left(\left(d^n + \sum_{j=2}^{n} a_j d^{n-j} \right) \left(\sum_{i=1}^{n-1} d^{-i} \frac{\delta H_k}{\delta a_{n-i+1}} \right) \right)_{++}. \tag{3.22b}$$

The $++$ subscript means that we keep only the contribution of valuation one. From (3.22) the full expression of X_k and Y_k may be reconstructed as explained in (3.8). The above expressions become much simpler if one sets a_2 and all w_j to zero. In that case, one has:

$$\frac{\delta H_k}{\delta a_i} = (-1)^{k+1-i} B_{k+1,i} \varepsilon^{(k+1-i)} \tag{3.23}$$

so that the terms independent of w in $\tilde{X}_k$ and $\tilde{Y}_k$ take a particularly simple form:

$$\tilde{X}_k(a_2 = 0, w_3 = 0, \ldots, w_n = 0) = \sum_{s=1}^{k} B_{k+1,s+1} \varepsilon^{(k-s)} d^s, \tag{3.24a}$$

$$\begin{aligned}\tilde{Y}_k(a_2 = 0, w_3 = 0, \ldots, w_n = 0) &= \sum_{s=1}^{k} C_{k+1,s+1} \varepsilon^{(k-s)} d^s \\ &= (-1)^k \tilde{X}_k^*(a_2 = 0, w_3 = 0, \ldots, w_n = 0),\end{aligned} \tag{3.24b}$$

where the B coefficients have been introduced in (2.24), the C coefficients are:

$$C_{k+1,s+1} = (-1)^k \sum_{l=0}^{k-s} (-1)^{l+s} \binom{l+s}{l} B_{k+1,l+s+1} = B_{k+1,s+1}(n \to -n) \tag{3.25}$$

and use has been made of the identity (see Appendix):

$$B_{k+1,s+1} = (-1)^{k-s} \sum_{l=0}^{k-s} \binom{n-s-1}{l} B_{k+1,l+s+1}. \tag{3.26}$$

A sample of the first X_k and Y_k is displayed in Table II.

3.4. W-Algebra. The local, i.e. x-dependent flows δ_k that act on the differential operator D do not in general reduce to the KdV flows when the infinitesimal parameter is taken to be x-independent. The generators of the latter are the $u_{k+1} = \mathrm{Res}(D^{k/n})$ which for $k = 1, 2$ coincide with w_2, w_3, [16], but do not in general

Table II. The generators X_k and Y_k, $SU(n)$ case

$$X_1 = \varepsilon d - \frac{n-1}{2}\varepsilon'$$

$$Y_1 = \varepsilon d + \frac{n+1}{2}\varepsilon'$$

$$X_2 = \varepsilon d^2 - \frac{n-2}{2}\varepsilon' d + \left\{\frac{2}{n}\varepsilon a_2 + \frac{1}{12}(n-1)(n-2)\varepsilon''\right\}$$

$$Y_2 = \varepsilon d^2 + \frac{n+2}{2}\varepsilon' d + \left\{\frac{2}{n}\varepsilon a_2 + \frac{1}{12}(n+1)(n+2)\varepsilon''\right\}$$

$$X_3 = \varepsilon d^3 - \frac{n-3}{2}\varepsilon' d^2 + \left\{\frac{(n-2)(n-3)}{10}\varepsilon'' + \frac{6}{5}\frac{3n^2-7}{n(n^2-1)}a_2\varepsilon\right\}d$$
$$+ \left\{\frac{3}{n}w_3\varepsilon - \frac{3(n+2)(n-7)}{10n(n+1)}a_2'\varepsilon - \frac{(n-3)(4n+7)}{5n(n+1)}a_2\varepsilon' - \frac{(n-1)(n-2)(n-3)}{5!}\varepsilon'''\right\}$$

$$Y_3 = \varepsilon d^3 + \frac{n+3}{2}\varepsilon' d^2 + \left\{\frac{(n+2)(n+3)}{10}\varepsilon'' + \frac{6}{5}\frac{3n^2-7}{n(n^2-1)}a_2\varepsilon\right\}d$$
$$+ \left\{\frac{3}{n}w_3\varepsilon + \frac{3(n-2)(n+7)}{10n(n-1)}a_2'\varepsilon + \frac{(n+3)(4n-7)}{5n(n-1)}a_2\varepsilon' + \frac{(n+1)(n+2)(n+3)}{5!}\varepsilon'''\right\}$$

coincide with w_{k+1}. Accordingly, $X_k \neq Y_k$, $k \geqq 3$, even for ε a constant (see Table II). The KdV flows are isospectral in a fixed coordinate frame, a property which is not invariant under diffeomorphisms. The relationship between the two families of flows remains to be clarified.

With the explicit expressions of the w's, the X's and the Y's at our disposal, we can now form the Poisson brackets of the w's among themselves. In general $\{w_k(x), w_l(y)\}$ is by construction a sum of monomials in the w's and their derivatives times a derivative of $\delta(x-y)$. The set of Poisson brackets $\{w_k, w_l\}, k, l = 2, \ldots, n$ defines the W-algebra (more precisely the A_{n-1} W-algebra, see below). It always contains the (classical) Virasoro algebra generated by a_2 and the relations expressing that the $w_k, k \geqq 3$, transform as k-differentials:

$$\{a_2(y), a_2(x)\} = (a_2'(x) + 2a_2(x)d + c_n d^3)\delta(x-y), \tag{3.27a}$$

$$\{a_2(y), w_k(x)\} = (w_k'(x) + kw_k(x)d)\delta(x-y). \tag{3.27b}$$

As for the other brackets $\{w_k, w_l\}, k, l \geqq 3$, it is easier to compute and tabulate them again in the coordinate u where a_2 vanishes. According to an argument used repeatedly in this paper, if

$$\{w_k(u), w_l(v)\}|_{a_2=0} = \Delta(w_j, d_u)\delta(u-v) \tag{3.28}$$

with Δ some differential operator, then in the generic coordinate

$$\{w_k(x), w_l(y)\} = \phi^k \Delta(\phi^{-j}w_j, \phi^{-1}d)\phi^{l-1}\delta(x-y). \tag{3.29}$$

(The δ-function has contributed an extra ϕ^{-1}). The operator Δ must satisfy certain constraints in order that the right-hand side of (3.29) depends only on the schwarzian derivative of the change of coordinate. We illustrate these considerations on the set of Poisson brackets $\{w_k, w_l\}, k, l = 3, 4$, for generic n,

$$\{w_3(v), w_3(u)\}|_{a_2=0} = \left(2[w_4, d]_+ - \frac{(n-2)(n-1)n(n+1)(n+2)}{6!} d^5\right)\delta(u-v), \tag{3.30}$$

$$\{w_3(v), w_4(u)\}|_{a_2=0} = \left(5w_5 d + 2w_5' - \frac{(n-3)(n+3)}{70}(14w_3 d^3 + 14w_3' d^2 + 6w_3'' d + w_3''')\right)\delta(u-v),$$

$$\{w_4(v), w_4(u)\}|_{a_2=0} = \left(3[w_6, d]_+ - \frac{n^2-19}{30}(3[w_4, d^3]_+ - 2[w_4'', d]_+) - 3\frac{n-3}{n} w_3 d w_3 + \frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)}{20.7!} d^7\right)\delta(u-v),$$

where all the w's on the right-hand side are evaluated at u and d stands for d/du. Notice that even for $a_2 = 0$, non-linearities in the w's appear. In general, one can see that the Poisson brackets $\{w_k, w_l\}$ are at most cubic in the w_j, $j \geqq 3$. By restoring the dependence on a_2 and truncating to $n=3$, with $w_l = 0$, $l > 3$, or to $n=4$, with $w_l = 0$, $l > 4$, one finds respectively the formulae defining the W_3 and W_4 algebras. The latter had been explicitly described in [7].

One general feature of the W-algebra which is provided by our approach is the form of the central term. It is easy to see that the only contribution independent of the a's in $\delta_k a_l$ for $l \leqq k+1$ is

$$\begin{aligned}\delta_k a_l|_{a_i=0} &= -\sum_{q=0}^{k} \binom{n}{l+q} B_{k+1,q+1}\varepsilon^{(k+l)} \\ &= -(n-k)(n-k+1)\cdots(n+k)\frac{(-1)^k (k!)^2}{(2k)!(2k+1)!}\delta_{k+1,l}\varepsilon^{(k+l)}\end{aligned} \tag{3.31}$$

as a result of a simple identity (see Appendix). It follows that the central term of the W-algebra reads

$$\{w_k(y), w_l(x)\}|_{\text{central term}} = (n-k+1)(n-k+2)\cdots(n+k-1)\frac{(-1)^k((k-1)!)^2}{(2k-2)!(2k-1)!}\delta_{k,l} d^{k+l-1}\delta(x-y). \tag{3.32}$$

4. Generalization to Other Lie Groups

Let us start from (2.1) again. We consider instead an equivalent family of first order matrix operators acting on a n-vector of functions

$$\mathscr{D}(Q) = \mathbf{I}d - J + Q, \tag{4.1}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $J_{i,j} = \delta_{i,j+1}$, and Q is an upper triangular $n \times n$ matrix, i.e. elements below the diagonal vanish (we shall call strictly upper triangular, matrices with vanishing elements below and on the diagonal). For any such Q, there exists $S \in SL(n)$, with $S - \mathbf{I}$ strictly upper triangular such that $S\mathscr{D}(Q)S^{-1} = \mathscr{D}(Q_{\mathrm{can}})$, where Q_{can} can be chosen as a $n \times n$ matrix filled with zeros, except for the first line which reads $(0, a_2, a_3, \ldots, a_n)$. Alternatively, Q_{can} can be chosen to have as its only non-vanishing column the last one $((-1)^n a_n^*, \ldots, a_2^*, 0)^T$, where $(-1)^n D^* = \sum_p a_p^* d^{n-p}$. Of course this S has in general x-dependent matrix elements. The operator D of (2.1) is the "solved" version of $\mathscr{D}(Q_{\mathrm{can}})$ in the sense that if one looks at the kernel equation $\mathscr{D}\Psi = 0$, $\Psi^T = (\psi_1, \ldots, \psi_n)$, it is equivalent to the kernel equation $D\psi_n = 0$, all the other components ψ being expressed in terms of ψ_n. Note that the kernels of D and D^* are related: if one forms the wronskian determinant of n independent solutions of one of the operators, the $n-1 \times n-1$ minors of the first row are n independent solutions of the adjoint operator.

A natural framework to generalize the above construction to other Lie groups than $SL(n)$ (or Lie algebra A_{n-1}) has been found by Drinfeld and Sokolov [17]. In general, given a matrix Lie group G acting in a $d(G)$ dimensional complex vector space, one considers a first order $d(G) \times d(G)$ matrix differential operator of the form (4.1) with $J = \sum_{\text{positive simple roots } i} X_i$, where X_i denote the representatives of the corresponding generators in the representation of dimension $d(G)$ of the Cartan–Chevalley basis of the Lie group G. The matrix Q is upper triangular and belongs to the Borel subalgebra generated by the Cartan algebra and the generators Y_i, both represented as upper triangular matrices. In fact, one considers equivalence classes of operators of the form (4.1) under the adjoint action of elements S belonging to a subgroup with nilpotent Lie algebra generated by the Y_i. Note that again, $S - \mathbf{I}$ is strictly upper triangular. The gauge reduction of $\mathscr{D}$ enables one in general to restrict Q to some canonical form Q_{can} or alternatively to consider a solved equivalent form for a (pseudo)-differential operator related to $\mathscr{D}$. The number of independent coefficients is equal to the rank of the Lie algebra. More precisely, for an appropriate gradation, these coefficients will have as degrees the Coxeter exponents augmented by one unit. (In particular the largest one is the Coxeter number.)

In the following we study the case of $G = SO(2n)$ with the simply laced Lie algebra D_n in some detail. In [17] it was shown that thanks to gauge equivalence $\mathscr{D}$ can be "solved" through a *pseudo*-differential operator D of degree $2n-1$ satisfying the properties:

$$D^* = -D, \tag{4.2a}$$

$$(D)_- = (-1)^n u_n d^{-1} u_n. \tag{4.2b}$$

The operator D maps $\mathscr{F}_{-(n-1)}$ into $\mathscr{F}_n$. The appearance of a pseudo-differential operator results from an elimination between a pair of coupled ordinary differential operators; see below an illustration. Using (4.2) one can write:

$$D = d^{2n-1} + \sum_{k=1}^{n-1} \frac{1}{2}[a_{2k}, d^{2n-2k-1}]_+ + (-1)^n u_n d^{-1} u_n. \tag{4.3}$$

The reader will notice that the indices of the coefficients of the a's and u_n run over one plus the exponents of D_n: $1, 3, \ldots, 2n-3, n-1$. In particular when n is even, the exponent $n-1$ has multiplicity two.

The structure of D implies that $f[Df] = [d(R(f,f))]$, where R is a quadratic scalar form in f and its derivatives. This follows from

$$f[a, d^{2r-1}]_+ f = d(f d^{2r-2} a f - [df] d^{2r-3} a f + \cdots + [d^{2r-2} f] a f) \tag{4.4}$$

and $(f u_n)[d^{-1}(u_n f)] = \frac{1}{2}[d(d^{-1} u_n f)^2]$. Thus $R(f,f)$ is x-independent on the kernel of D and provides an invariant pseudo-differential quadratic form as expected in a theory related to $SO(2n)$. At the price of introducing a coupled function, one could get rid of the pseudo-differential term.

The generalization of the covariance condition reads in infinitesimal form:

$$\delta D = YD - DX, \tag{4.5}$$

where we now allow pseudo-differential operators X, Y mapping respectively $\mathscr{F}_{-(n-1)}$ and $\mathscr{F}_n$ to themselves. The reparametrizations are now generated by:

$$X_1 = \varepsilon d - (n-1)\varepsilon', \tag{4.6a}$$

$$Y_1 = \varepsilon d + n\varepsilon' = -X_1^*, \tag{4.6b}$$

and lead to an anomalous transformation of a_2:

$$\delta_1 a_2 = \varepsilon a_2' + 2\varepsilon' a_2 + \frac{n(n-1)(2n-1)}{3}\varepsilon'''. \tag{4.7}$$

Example. It is instructive to consider explicitly the equivalence between the cases of the algebras A_3 and D_3 in order to match the corresponding coefficients and to expose the origin of the pseudo-differential term. Starting from a matrix of the form (4.1) with $n=4$ relative to A_3,

$$\begin{pmatrix} d & b_2 & b_3 & b_4 \\ -1 & d & 0 & 0 \\ 0 & -1 & d & 0 \\ 0 & 0 & -1 & d \end{pmatrix} \tag{4.8}$$

acting on column vectors $\mathbf{f} = (f_4, f_3, f_2, f_1)^T$ and $\mathbf{g} = (g_4, g_3, g_2, g_1)^T$, one forms the combinations

$$\begin{aligned} v_1 &= f_1 g_2 - f_2 g_1, & u_1 &= 2(f_3 g_4 - f_4 g_3), \\ v_2 &= f_1 g_3 - f_3 g_1, & u_2 &= 2(f_2 g_4 - f_4 g_2), \\ v_3 &= f_1 g_4 - f_4 g_1, & u_3 &= 2(f_2 g_3 - f_3 g_2), \end{aligned}$$

corresponding to the 6-dimensional antisymmetric tensor representation of A_3 which exhibits its isomorphism with D_3. Assuming that $\mathbf{f}$ and $\mathbf{g}$ belong to the kernel, one finds that the u's and v's satisfy

$$\begin{pmatrix} d & & -b_3 & & -2b_4 & \\ -1 & d & b_2 & & & -2b_4 \\ & -1 & d & & & \\ & -\frac{1}{2} & 0 & d & b_2 & b_3 \\ & & -\frac{1}{2} & -1 & d & \\ & & & & -1 & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_3 \\ v_2 \\ v_1 \end{pmatrix} = 0. \tag{4.9}$$

Solving for $u = u_3$ and $v = v_1$ one obtains

$$\begin{aligned} &(d^3 + b_2 d + b_3)v = u', \\ &(d^3 + d b_2 - b_3)u = 2[b_4, d]_+ v, \end{aligned} \tag{4.10}$$

or equivalently

$$\begin{aligned} D_{D_3} v &= (d^5 + [b_2, d^3]_+ + [b_3' - b_2'' - 2b_4 + \tfrac{1}{2}b_2^2, d]_+ - (b_3 - b_2')d^{-1}(b_3 - b_2'))v \\ &= 0. \end{aligned} \tag{4.11}$$

Therefore the operators

$$\begin{aligned} D_{A_3} &= d^4 + b_2 d^2 + b_3 d + b_4, \\ D_{D_3} &= d^5 + \tfrac{1}{2}[a_2, d^3]_+ + \tfrac{1}{2}[a_4, d]_+ - u_3 d^{-1} u_3 \end{aligned} \tag{4.12}$$

are equivalent provided one takes:

$$\begin{aligned} a_2 &= 2b_2, \\ u_3 &= \pm(b_3 - b_2'), \\ a_4 &= -4b_4 + 2b_3' - 2b_2'' + b_2^2. \end{aligned} \tag{4.13}$$

in agreement with [17].

The analysis of Eq. (4.5) for X and Y differential is exactly the same as in Sect. 3.1. It is entirely determined by the data of a valuation one differential operator $\tilde{X}$, and:

$$X = \tilde{X} - \frac{1}{2n-1} \int \mathrm{Res}(D\tilde{X}D^{-1}), \tag{4.14a}$$

$$Y = (DXD^{-1})_+. \tag{4.14b}$$

Of course the $\mathbf{Z}_2$ covariance of D under the involution $*$ (4.2a) must be preserved by the variations δ: $-\delta D^* = \delta D = -X^* D + DY^* = YD - DX$, so that $(Y + X^*)D = D(Y^* + X)$. This implies that $Y = -X^*$. But X and Y have the same degree, hence it is necessarily odd. The number of possible variations δ_k as well as the number of "currents" turns out to be equal to the rank of the algebra.

We still have to check that the variation of the "tail" of D: $\delta(u_n d^{-1} u_n) = \delta(u_n) d^{-1} u_n + u_n d^{-1} \delta(u_n)$ is compatible with the form of (4.5). The only contributions to $(YD - DX)_-$ come from $\Delta = (Y u_n d^{-1} u_n - u_n d^{-1} u_n X)_- = [Y u_n] d^{-1} u_n - (u_n d^{-1} u_n X)_-$, where $[Y u_n]$ is the function obtained by letting the differential operator Y act on u_n. But we know that this expression is odd under $\mathbf{Z}_2$, so: $\Delta = -\Delta^* = u_n d^{-1}[Y u_n] - (X^* u_n d^{-1} u_n)_-$, and $X^* = -Y$, so that we finally get: $\Delta = [Y u_n] d^{-1} u_n + u_n d^{-1} [Y u_n]$, which shows the compatibility between the negative parts in (4.5), and gives the variation of u_n:

$$\delta_X u_n = [Yu_n]. \tag{4.15}$$

Applying this to the first variation δ_1, we get:

$$\delta_1(u_n) = \varepsilon u_n' + n\varepsilon' u_n \tag{4.16}$$

which shows that u_n is a n-differential.

We can now proceed in two steps as before. 1) Write D as a sum of weight $2k$ contributions at $a_2 = 0$, and find their form by requiring that after a change of coordinates which restores the a_2 dependence, they only depend on the schwarzian derivative of the jacobian. 2) "Diagonalize" the variations δ so that they are compatible with Poisson brackets with $2k$-differentials.

Both steps can be carried out completely, we only quote the results:

1) Writing:

$$D = \Delta_2(a_2) + \Delta_0(u_n, a_2) + \sum_{k=2}^{n-1} \Delta_{2k}(w_{2k}, a_2), \tag{4.17}$$

where $\Delta_0(u_n, a_2 = 0) = u_n d^{-1} u_n = \Delta_0(u_n, a_2)$ is automatically covariant, due to the fact that u_n is a n-differential, and

$$\Delta_{2k}(w_{2k}, a_2 = 0) = \sum_{l=0}^{2n-2k-1} \alpha_{2k,l} w_{2k}^{(l)} d^{2n-2k-l-1} \tag{4.18}$$

for $k \geqq 2$, where w_{2k} are $2k$-differentials, we finally get:

$$\alpha_{2k,l} = \frac{\binom{2k+l-1}{l}\binom{2n-2k-1}{l}}{\binom{4k+l-1}{l}}. \tag{4.19}$$

One restores the a_2 dependence after a change of coordinate, by replacing the schwarzian derivative of the jacobian by $3a_2/n(n-1)(2n-1)$.

There is another more convenient method to obtain the $w_{2k}, k = 1, 2, \ldots, n-1$ of the $SO(2n)$ (D_n) case, from the w's of the $SL(2n-1)$ (A_{2n-2}) case. It relies on the remark that the covariance condition of the D operator (4.3) splits into two independent covariance conditions for $D_- = (-1)^n u_n d^{-1} u_n$ and D_+, expressing that both operators map $-(n-1)$-differentials onto n-differentials. The former means that u_n is a n-differential. The latter involves the differential operator D_+ of degree $2n-1$, which can be seen as the $\mathbf{Z}_2$-odd projection (antiself-dual with respect to $*$) of the degree $2n-1$ differential operator (2.1) of the A_{2n-2} case, which precisely maps $-(n-1)$-differentials onto n-differentials. Such a projection is easily performed using the form (2.29), in which we set $w_{2l+1} = 0$, for $1 \leqq l \leqq n-1$, leaving only anti-commutators of functions and odd powers of d, that are manifestly antiself-dual. We proceed by eliminating $a_{2l+1}(A_{2n-2})$ from $w_{2l+1}(A_{2n-2}) = 0$ and substituting in $w_{2l}(A_{2n-2})$, $1 \leqq l \leqq n-1$. We get as a result $w_{2l}(D_n)$ as a function of the $a_{2j}(A_{2n-2})$. The final answer must be expressed in terms of the $a_{2j}(D_n)$ defined by (4.3), which are related to $a_{2l}(A_{2n-2})$ through:

$$D_+ = d^{2n-1} + \sum_{i=1}^{n-1} \frac{1}{2}[d^{2n-2i-1}, a_{2i}(D_n)]_+$$
$$= d^{2n-1} + \sum_{j=2}^{2n-1} a_j(A_{2n-2}) d^{2n-j-1} \tag{4.20}$$

so that:

$$a_{2i}(A_{2n-2}) = a_{2i}(D_n) + \frac{1}{2}\sum_{j=1}^{i-1} \binom{2n-2i+2j}{2j} a_{2i-2j}^{(2j)}(D_n) \tag{4.21a}$$

$$a_{2i+1}(A_{2n-2}) = \frac{1}{2}\sum_{j=0}^{i} \binom{2n-2i+2j}{2j+1} a_{2i-2j}^{(2j+1)}(D_n). \tag{4.21b}$$

Substituting (4.21a) in w_{2l} above yields the desired result. Note that the relation (4.21b) is equivalent to the above elimination from $w_{2i+1}(A_{2n-2}) = 0$. The first few w's are displayed in Table III. As an example, $w_4(D_n)$ (Table III) can be obtained from $w_4(A_{2n-2})$ (Table I, with $n \to 2n-1$), by substituting $a_3(A_{2n-2}) = \frac{2n-3}{2} a_2'(A_{2n-2})$ $(w_3(A_{2n-2}) = 0)$, and going back to the $a_{2i}(D_n)$ coefficients, using $a_2(A_{2n-2}) = a_2(D_n)$ and $a_4(A_{2n-2}) = a_4(D_n) + \frac{1}{2}(2n-3)(n-2)a_2''(D_n)$.

2) Using the Poisson brackets of Sect. 3.2, we can obtain the infinitesimal variations:

$$\delta_{2k+1}(\varepsilon)D = \{\textstyle\int dx\, \varepsilon w_{2k}, D\} = \delta_{X_{2k+1}} D \tag{4.22a}$$

$$\delta_0(\varepsilon)D = \{\textstyle\int dx\, \varepsilon u_n, D\} = \delta_{X_0} D \tag{4.22b}$$

and the differential operators X_{2k+1}, Y_{2k+1} are calculated from $X_{2k+1}(A_{2n-2})$ and

Table III. w_{2k} in terms of a_{2l} for the $SO(2n)$ case

$$w_0 = u_n$$

$$w_2 = a_2$$

$$w_4 = a_4 + \frac{(2n-3)(n-2)}{5} a_2'' - \frac{(2n-3)(n-2)(5n+1)}{10n(n-1)(2n-1)} a_2^2$$

$$w_6 = a_6 + \frac{2(2n-5)(n-3)}{9} a_4'' + \frac{(2n-3)(2n-5)(n-2)(n-3)}{252} a_2^{(4)}$$
$$- \frac{(2n-5)(2n-3)(n-2)(n-3)(14n+1)}{126n(n-1)(2n-1)} \left(\frac{a_2 a_2''}{5} - \frac{a_2'^2}{4} \right)$$
$$- \frac{(2n-5)(2n-3)(n-2)(n-3)(35n^2+21n+4)}{210(n(n-1)(2n-1))^2} a_2^3$$
$$- \frac{(2n-5)(n-3)(3n+2)}{3n(n-1)(2n-1)} a_2 w_4$$

Table IV. The X_{2k+1} operators and the Poisson brackets (in the coordinate where $a_2 = 0$) in the $D_4 = so(8)$ case

$$X_0 = -\tfrac{1}{2}\varepsilon d^{-1}u_4$$
$$X_1 = \varepsilon d - 3\varepsilon'$$
$$X_3 = \varepsilon d^3 - 2\varepsilon' d^2 + 2\varepsilon'' d - \varepsilon'''$$
$$X_5 = \varepsilon d^5 - \varepsilon' d^4 + \tfrac{2}{3}\varepsilon'' d^3 - \tfrac{1}{3}\varepsilon''' d^2 + \tfrac{5}{42}\varepsilon^{(4)} d - \tfrac{1}{42}\varepsilon^{(5)} + \tfrac{5}{6} w_4 \varepsilon d - \tfrac{1}{2} w_4 \varepsilon'$$
$$\{u_4(y), u_4(x)\} = \tfrac{1}{12}(6d^7 + 3[w_6, d]_+ - 2[w_4'', d]_+ + 3[w_4, d^3]_+)\delta(x-y)$$
$$\{u_4(y), w_4(x)\} = (-2[u_4'', d]_+ + 3[u_4, d^3]_+)\delta(x-y)$$
$$\begin{aligned}\{u_4(y), w_6(x)\} = {} & (\tfrac{11}{7}(2u_4 d^5 + \tfrac{5}{2}u_4' d^4 + \tfrac{5}{3}u_4'' d^3 + \tfrac{2}{3}u_4''' d^2) \\ & + \tfrac{1}{21}(5u_4^{(4)} d + \tfrac{1}{2}u_4^{(5)}) + \tfrac{4}{3}w_4 u_4 d + \tfrac{1}{2}(u_4 w_4)')\delta(x-y)\end{aligned}$$
$$\{w_4(y), w_4(x)\} = (6d^7 + 3[w_6, d]_+ + 2[w_4'', d]_+ - 3[w_4, d^3]_+)\delta(x-y)$$
$$\begin{aligned}\{w_4(y), w_6(x)\} = {} & (\tfrac{11}{7}(2w_4 d^5 + \tfrac{5}{2}w_4' d^4 + \tfrac{5}{3}w_4'' d^3 + \tfrac{2}{3}w_4''' d^2) + \tfrac{1}{21}(5w_4^{(4)} d + \tfrac{1}{2}w_4^{(5)}) \\ & + 6(\tfrac{4}{3}u_4^2 d + u_4 u_4') - \tfrac{1}{2}(\tfrac{4}{3}w_4^2 d + w_4 w_4'))\delta(x-y)\end{aligned}$$
$$\begin{aligned}\{w_6(y), w_6(x)\} = {} & (\tfrac{1}{21}d^{11} - \tfrac{1}{42}(26[w_6, d^5]_+ - 60[w_6'', d^3]_+ + 37[w_6^{(4)}, d]_+) \\ & + \tfrac{1}{36}(22[w_4^2, d^3]_+ - [24w_4 w_4'' + 39w_4'^2, d]_+) \\ & + \tfrac{1}{3}(22[u_4^2, d^3]_+ - [24u_4 u_4'' + 39u_4'^2, d]_+)\delta(x-y).\end{aligned}$$

$Y_{2k+1}(A_{2n-2})$ using the above procedure: one simply has to substitute (4.21a–b) in the expressions (3.22). As an example, $X_3(D_n)$ is obtained by setting $w_3 = 0$ in $X_3(A_{2n-2})$ (Table II, $n \to 2n-1$), and one reads directly the correct result, due to $a_2(A_{2n-2}) = a_2(D_n)$. X_0, Y_0 are two pseudo-differential operators, easily computed as:

$$\begin{aligned} X_0 &= -\tfrac{1}{2}\varepsilon d^{-1}u_n, \\ Y_0 &= -\tfrac{1}{2}u_n d^{-1}\varepsilon = -X_0^*. \end{aligned} \tag{4.23}$$

We still have to check the compatibility between the variation of the tail of D, $(-1)^n\delta_0(u_n d^{-1}u_n)$, and $(Y_0 D - DX_0)_-$. A straightforward calculation shows that they are compatible, provided:

$$\delta_0 u_n = \tfrac{1}{2}(-1)^n[D_+\varepsilon]. \tag{4.24}$$

This completes the description of the D classical W-algebra in its infinitesimal form. For definiteness, we have listed in Table IV the form of the X generators and the Poisson brackets for the D_4 algebra in the coordinate where $a_2 = 0$. One observes a certain symmetry between the roles played by the two currents u_4 (rescaled by $\sqrt{12}$) and w_4 of weight 4, and the fact that the central terms of the Poisson brackets are still diagonalized. To the best of our knowledge, this algebra had never been made explicit.

To summarize, we have obtained n currents $w_2 = a_2, w_4, \ldots, w_{2n-2}, u_n$, of weight $j+1$, j running over the set of Coxeter exponents of $D = SO(2n)$. The extension to the C_n and B_n Lie algebras is straightforward following [17]. The D operator becomes a self-dual differential operator of degree $2n$, respectively an anti self-dual differential operator of degree $2n-1$ (which amounts to taking $u_n = 0$ in the D case). One gets in all these cases, rank(G) currents of weight $j+1$, j running over the Coxeter exponents of G, the first current a_2 being anomalously transformed under reparametrizations and the other currents being $j+1$-differentials.

A further example is provided by the G_2 Lie algebra. It is a rank 2 algebra of Coxeter number 6 and exponents 1 and 5. One thus expects two currents, a_2 and w_6, and in the coordinate where $a_2 = 0$, the differential operator is simply

$$D = d^7 + \tfrac{1}{2}[w_6, d]_+ . \tag{4.25}$$

The operator D is covariant, mapping $\mathscr{F}_{-3}$ into $\mathscr{F}_4$ and returning to the generic coordinate, one recovers the form written in [17].

The case of G_2 may also be obtained from the one of D_4 in the same way as D_n was obtained from A_{2n-2}. Here one has to impose the vanishing of w_4 and u_4. Thus from the formulae of Table III taken for $n = 4$, one eliminates a_4 by imposing that $w_4 = 0$, and one finds

$$w_6(G_2) = a_6 - \frac{17}{14}a_2^2 + \frac{177}{588}a_2 a_2'' + \frac{293}{784}a_2'^2 - \frac{9}{686}a_2^3 \tag{4.26}$$

and

$$D_{G_2} = d^7 + \tfrac{1}{2}[a_2, d^5]_+ + \tfrac{1}{2}[\tfrac{1}{4}a_2^2 - 2a_2'', d^3]_+ + \tfrac{1}{2}[a_6, d]_+ . \tag{4.27}$$

The Poisson bracket of $w_6(G_2)$ with itself may be obtained from the one of D_4 (last line of Table IV) by restricting it to $w_4 = u_4 = 0$.

It is well known that G_2 may be embedded in $SO(7)$. As discussed above (4.4), one may construct a differential quadratic form of weight 0 conserved on the kernel of D:

$$d(ff^{(6)} - f'f^{(5)} + f''f^{(4)} - \tfrac{1}{2}f'''^2 + w_6 f^2) = 0 \quad \text{if} \quad Df = 0. \tag{4.28}$$

The group G_2 is the automorphism group of octonions and leaves invariant an antisymmetric *ternary* form. Accordingly, we find that given three functions f, g and h in the kernel of D, the following scalar form is conserved

$$\mathscr{T}(f,g,h) = (4,3,2) - (5,3,1) + (5,4,0) + 2(6,2,1) - (6,3,0) + (7,2,0), \tag{4.29}$$

where the symbol (j,k,l) denotes the determinant

$$(j,k,l) = \begin{vmatrix} f^{(j)} & g^{(j)} & h^{(j)} \\ f^{(k)} & g^{(k)} & h^{(k)} \\ f^{(l)} & g^{(l)} & h^{(l)} \end{vmatrix} . \tag{4.30}$$

On the two cases of the D and G_2 algebras, we thus see the remarkable consistency between the form of the differential operator and the geometry of the underlying group.

After completion of this work, some recent papers that overlap with the present one have come to our attention [22] and [23]. For earlier references and an alternative treatment of the problem of Sect. 2, see [24].

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Appendix

1. Proof that $BA = \mathbf{I}$ (Eqs. (2.23), (2.24).)
Both A_{kl} and B_{kl} vanish for $l > k$. Thus for $k \geqq m$

$$\sum_{l=m}^{k} B_{kl} A_{lm} = \frac{(k-1)!(n-m)!(2m-1)!}{(n-k)!(2k-2)!(m-1)!} \sum_{l=m}^{k} (-1)^{k-l} \frac{(k+l-2)!}{(k-l)!(l-m)!(l+m-1)!}. \tag{A.1}$$

For $k > m$, the last sum reads

$$\begin{aligned} &\frac{1}{(k-m)!} \sum_{l=0}^{k-m} (-1)^{k-m-l} \binom{k-m}{l} \frac{(k+m+l-2)!}{(l+2m-1)!} \\ &= \frac{1}{(k-m)!} \left(\frac{d}{dx}\right)^{k-m-1} x^{k+m-2} \sum_{l=0}^{k-m} (-1)^{k-m-l} \binom{k-m}{l} x^{l} \Big|_{x=1} \\ &= \frac{1}{(k-m)!} \left(\frac{d}{dx}\right)^{k-m-1} x^{k+m-2} (x-1)^{k-m} \Big|_{x=1} \\ &= 0 \end{aligned}$$

whereas the expression (A.1) for $k = m$ is equal to 1. This completes the proof that the A and B matrices are inverse to one another.
2. Proof of the identity (2.26)
One may write

$$\begin{aligned} \sum_{l+1 \leqq j+m \leqq 2j} 2m \binom{j+m}{l+1} &= \sum_{l \leqq s \leqq n-2} (2s+3-n) \binom{s+1}{l+1} \\ &= \frac{d}{dx} \sum_{l \leqq s \leqq n-2} \oint \frac{dz}{2i\pi z^{l+2}} (1+z)^{s+1} x^{2s+3-n} \Big|_{x=1} \\ &= \frac{d}{dx} \oint \frac{dz}{2i\pi z^{l+2}} x^{1-n} \frac{((1+z)x^2)^n - ((1+z)x^2)^{l+1}}{(1+z)x^2 - 1} \Big|_{x=1} \\ &= (n+1) \binom{n}{l+2} - 2 \binom{n+1}{l+3} = \frac{(n+1)!(l+1)!}{(n-l-2)!(l+3)!} \end{aligned} \tag{A.2}$$

which is the desired identity.
3. Proof of the identity (2.28).
After multiplication of (2.28) by $\dfrac{(2k-1)!}{(k-1)!} q!$ one has to show that

$$\sum_{l=0}^{q} (-1)^l \frac{k(k+1)\cdots(k+l-1)}{2k(2k+1)\cdots(2k+l-1)} \frac{q!}{l!(q-l)!} = \frac{k(k+1)\cdots(k+q-1)}{2k(2k+1)\cdots(2k+q-1)}. \tag{A.3}$$

The generating function of the right-hand side is the hypergeometric function

$$F(k,1,2k;z)=\sum_{q=0}^{\infty}\frac{k(k+1)\cdots(k+q-1)}{2k(2k+1)\cdots(2k+q-1)}z^q \tag{A.4}$$

whereas the same sum on the left-hand side yields

$$\begin{aligned}&\sum_{q=0}^{\infty}z^q\sum_{l=0}^{q}(-1)^l\frac{q!}{l!(q-l)!}\frac{k(k+1)\cdots(k+l-1)}{2k(2k+1)\cdots(2k+l-1)}\\&=\sum_{l=0}^{\infty}(-1)^l\frac{k(k+1)\cdots(k+l-1)}{2k(2k+1)\cdots(2k+l-1)}\sum_{q=l}^{\infty}\frac{q!}{l!(q-l)!}z^q\\&=\frac{1}{1-z}\sum_{l=0}^{\infty}\frac{k(k+1)\cdots(k+l-1)}{2k(2k+1)\cdots(2k+l-1)}\left(\frac{z}{z-1}\right)^l\\&=\frac{1}{1-z}F\left(k,1,2k;\frac{z}{z-1}\right)\end{aligned} \tag{A.5}$$

and it is a classical identity that $F(k,1,2k;z)=\frac{1}{1-z}F\left(k,1,2k;\frac{z}{z-1}\right)$.

4. Proof of the identity (3.26).
Using the expression (2.24) the identity to be proven amounts to

$$\sum_{l=0}^{k-s}(-1)^{k-s-l}\frac{(k+l+s)!}{(k-l-s)!(l+s)!l!s!}=\frac{(k+s)!}{(k-s)!(s!)^2}. \tag{A.6}$$

Once again one introduces a generating function of both sides. Multiplied by z^s and summed over s from 0 to ∞, the right-hand side yields in fact the degree k Legendre polynomial

$$\begin{aligned}\sum_{s=0}^{k}\frac{(k+1)\cdots(k+s)k\cdots(k-s+1)}{s!s!}z^s&=F(k+1,-k,1;-z)\\&=(-1)^kP_k(-1-2z)\end{aligned} \tag{A.7}$$

whereas the left-hand side leads to

$$\begin{aligned}\sum_{s=0}^{k}\sum_{l=0}^{k-s}(-1)^{k-s-l}\frac{(k+l+s)!}{(k-l-s)!(l+s)!l!s!}&=\sum_{m=0}^{k}(-1)^{k-m}\frac{(k+m)!}{(k-m)!(m!)^2}\sum_{s=0}^{m}\binom{m}{s}z^s\\&=(-1)^kF(k+1,-k,1;1+z)\\&=P_k(1+2z).\end{aligned} \tag{A.8}$$

The desired identity follows from the parity properties of the Legendre polynomials.

5. Proof of the identity (3.31)
For $l\leq k$ we have to compute

$$\begin{aligned}&\frac{(n-k-1)!(2k)!}{n!}\sum_{q=0}^{k}\binom{n}{l+q}B_{k+1,q+1}\\&=\sum_{q=0}^{k}(-1)^{k-q}\frac{(n-q-1)!(k+q)!}{(n-q-l)!(l+q)!}\binom{k}{q}\end{aligned}$$

$$= \sum_{q=0}^{k} (-1)^{k-q}(n-q-1)\cdots(n-q-l+1)(k+q)\cdots(l+1+q)\binom{k}{q}x^q\bigg|_{x=1}$$
$$= \left(\frac{d}{dx}\right)^{l-1} x^{-n-1}\left(\frac{d}{dx}\right)^{k-l} x^k(x-1)^k\bigg|_{x=1}$$
$$= 0. \tag{A.9}$$

For $l = k+1$, the same method shows that, as a function of n, the result must vanish for $n = -1, \ldots, -k$,

$$\begin{aligned}\mathcal{N}_k(n) &= \frac{1}{n(n-1)\cdots(n-k)}\sum_{q=0}^{k}\binom{n}{k+q+1}B_{k+1,q+1} \\ &= \sum_{q=0}^{k}(-1)^{k-q}\frac{(n-q-1)\cdots(n-q-k)}{(2k)!(k+1+q)}\binom{k}{q}.\end{aligned} \tag{A.10}$$

For $n = -l$, $l = 1, \ldots, k$, the factor $k+1+q$ appears in the numerator and

$$\begin{aligned}(2k)!\mathcal{N}_k(-l) &= \sum_{q=0}^{k}(-1)^q(q+l+1)\cdots(q+k)(q+k+2)\cdots(q+l+k)\binom{k}{q} \\ &= \left(\frac{d}{dx}\right)^{k-1}\frac{1}{x}\left(\frac{d}{dx}\right)^{l-1}(1-x)^k x^{k+l}\Big|_{x=1} \\ &= 0\end{aligned} \tag{A.11}$$

and the coefficient of $(n+1)\cdots(n+k)$ is computed to be:

$$\begin{aligned}\frac{\mathcal{N}_k(n)}{(n+1)\cdots(n+k)} &= \sum_{q=0}^{k}(-1)^{k-q}\frac{1}{(k+q+1)(2k)!}\binom{k}{q} \\ &= \frac{(-1)^k}{(2k)!}\int_0^1 dx\, x^k(1-x)^k \\ &= (-1)^k\frac{(k!)^2}{(2k)!(2k+1)!},\end{aligned} \tag{A.12}$$

which is the announced result (3.31).

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