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The Asymptotics of the Arakelov-Green's Function and Faltings' Delta Invariant

R. Wentworth*

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

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Abstract. We study the behavior of the Arakelov-Green's function and Faltings' delta invariant on degenerating Riemann surfaces.

1. Introduction

The analysis of the behavior of metrics, Green's functions, eigenvalues, and determinants of Laplace operators on spaces of degenerating Riemann surfaces has been carried out in many different contexts. Given a compact Riemann surface, one makes a particular choice from the conformal class of metrics compatible with the complex structure associated to the surface, and a fundamental problem is to relate the geometrical data arising from the metric to the complex analytic structure of the underlying surface. The metrics chosen, however, have nearly always been those with constant curvature. In this paper, we consider degeneration phenomena with respect to a different choice; namely, the Arakelov metric [2, 14, 21, 23].

One way to describe this metric is as follows: the Riemann surface M is embedded into its Jacobian variety $J(M)$ via the Abel map I. The canonical metric on M is the pullback by I of the flat metric on $J(M)$ induced by the polarization of $J(M)$ as an abelian variety, and the Arakelov metric can be defined by prescribing that its curvature be proportional to the Kähler form of the canonical metric. This only determines the metric up to a constant, but there is a second quantity associated to the canonical metric – the Green's function. We can fix the scaling of the Arakelov metric by requiring the logarithm of the distance in this metric to be exactly the singularity of the Green's function along the diagonal.

Spaces of degenerating surfaces correspond to paths in moduli space leading to the boundary points. These are obtained from compact surfaces by shrinking

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finitely many closed loops to points, called nodes. Conversely, the boundary points may be opened up by cutting out disks surrounding the nodes and attaching annuli. There is a distinction between separating and non-separating nodes, and we shall constantly refer to these two possibilities as Case I and Case II, respectively.

The behavior of the hyperbolic metric, which for genus ≥ 2 has constant curvature -1, has been extensively studied under this type of degeneration (see [24, 36] and the references therein). Lengths of closed geodesics serve as parameters for moduli space, and Selberg Trace Formula techniques can be used to study, for example, the asymptotics of regularized determinants [10, 18, 20, 22, 31, 34, 35].

In our case, the starting point for the analysis of the Arakelov-Green's function is an expression in terms of a theta function and theta divisor (see Proposition 5.2):

$$\log G(x, y) = \frac{1}{h!} \int\limits_{\Theta+x-y} \nu^{h-1} \log \|\vartheta\| + A .$$

This formula is due to Bost [6], and in Sect. 5 we give a proof based on the bosonization formula [1, 13, 14, 17, 35]. With the Green's function in this form, the problem is essentially transferred to the Jacobian variety, where explicit formulae for the degeneration are available.

We now outline the organization of this paper and our main results. In Sect. 2, we recall some well-known facts about Riemann surfaces and their Jacobian varieties. We introduce the canonical metric on the surface and the prime form, which is the fundamental tool for calculation. Finally, we define the Arakelov metric and the Arakelov-Green's function.

In Sects. 3 and 4, we discuss the degeneration of Riemann surfaces to singular surfaces with nodes. There are two cases, depending upon whether the node separates the surface. Our main goal is to establish formulae for the asymptotics of the theta function which will be used in analysing the Green's function. For this, we use certain properties of theta functions on Riemann surfaces, such as Fay's trisecant identity.

We then introduce the Faltings invariant [14] and proceed to determine its asymptotic behavior under the degeneration model described in Sects. 3 and 4. The analysis hinges on the formula for the Green's function above. In Sects. 6 and 7, we use this formula to derive pointwise asymptotics for the Green's function; the results are contained in Theorems 6.10 and 7.2. With these estimates and the bosonization formula, the asymptotics of Falting's invariant $\delta(M)$ are obtained (see Sect. 8):

Main Theorem. *For M_τ a family of compact Riemann surfaces of genus h, degenerating as $\tau \to 0$ to surfaces M_1, M_2 of genus, $h_1, h_2 > 0$ joined at a node,*

a) $$\lim_{\tau \to 0} \left[\delta(M_\tau) + 4 \frac{h_1 h_2}{h} \log|\tau| \right] = \delta(M_1) + \delta(M_2) .$$

For M_τ of genus $h+1$ degenerating to a surface M of genus h with two punctures *a and b identified at a node,*

b)
$$\lim_{\tau\to 0}\left[\delta(M_\tau)+\frac{4h+3}{3(h+1)}\log|\tau|+6\log(-\log|\tau|)\right]$$
$$=\delta(M)-\frac{2(2h-3)}{3(h+1)}\log G(a,b)-2\log 2\pi,$$

where $\log G(a, b)$ *is the Arakelov-Green's function of* M.

Finally, in Appendix B we use these results to exhibit the factorization of the bosonic string integrand for genus two.

2. Theta Functions and Jacobian Varieties

In this section, we gather definitions, notations, and certain elementary results which will facilitate the computations later on. Let M be a compact Riemann surface of genus h (we will assume all genera to be positive). A *symplectic homology basis* is a basis $\{A_j, B_j\}$, $1 \le j \le h$, for $H_1(M, \mathbb{Z})$ satisfying the intersection pairings

$$\#[A_i, A_j]=0, \qquad \#[B_i, B_j]=0, \qquad \#[A_i, B_j]=\delta_{ij}. \tag{2.1}$$

The dimension of the space of holomorphic 1-forms, or *abelian differentials of the first kind,* is given by the Riemann-Roch formula and is simply the genus h. The *normalized basis of abelian differentials* associated to the given symplectic homology basis $\{A_j, B_j\}$ is a basis $\omega_1, \ldots, \omega_h\}$ satisfying $\int_{A_j} \omega_k = \delta_{jk}$. The B-periods are then determined, and they define the *period matrix*: $\Omega_{ij} = \int_{B_i} \omega_j$. The period matrix is symmetric, has positive definite imaginary part, and satisfies [15]

$$\operatorname{Im}\Omega_{ij}=\frac{\sqrt{-1}}{2}\int_M \omega_i\wedge\bar{\omega}_j. \tag{2.2}$$

The *Jacobian variety* associated to M is defined by $J(M) = \mathbb{C}^h/\Gamma$, where Γ is the rank $2h$ lattice $\Gamma = \mathbb{Z}^h + \Omega\mathbb{Z}^h$. The Jacobian is a principally polarized abelian variety, and the Hodge metric of the polarization has the Kähler form

$$v=\frac{\sqrt{-1}}{2}\sum_{j,k=1}^{h}(\operatorname{Im}\Omega)^{-1}_{jk}\,dZ_j\wedge d\bar{Z}_k. \tag{2.3}$$

The *Abel map* embeds M into its Jacobian, $I : z \to \int_{z_0}^{z} \vec{\omega}$, where z_0 is a base point which will remain fixed throughout, and $\vec{\omega}$ is the vector in $\mathbb{C}^h$ whose i^{th} component is ω_i. We shall often abbreviate the notation and denote, for example, $I(x) - I(y)$ by $x - y$. The pullback of v by I is $I^*v = h\mu$, where μ is the *canonical metric*:

$$\mu=\frac{\sqrt{-1}}{2h}\sum_{j,k=1}^{h}(\operatorname{Im}\Omega)^{-1}_{jk}\,\omega_j\wedge\bar{\omega}_k. \tag{2.4}$$

Notice that by (2.2), $\int_M \mu = 1$.

The *theta function* associated to Ω is defined by

$$\vartheta(Z, \Omega) = \sum_{n \in \mathbb{Z}^h} \exp \pi i({}^t n \Omega n + 2{}^t n Z). \tag{2.5}$$

It is holomorphic in both variables (we will often omit the matrix argument) and quasi-periodic with respect to Γ. For example, if $m, n \in \mathbb{Z}^h$, then

$$\vartheta(Z + m + \Omega n, \Omega) = \exp(-\pi i^t n \Omega n - 2\pi i^t n Z)\vartheta(Z, \Omega).$$

The above implies that the *theta divisor* Θ, the zero set of ϑ, is a well-defined subvariety of $J(M)$. It also follows that the norm of theta,

$$\|\vartheta\|^2(Z) = \exp(-2\pi^t \operatorname{Im} Z (\operatorname{Im} \Omega)^{-1} \operatorname{Im} Z) \, |\vartheta|^2(Z), \tag{2.6}$$

is periodic with respect to Γ.

The image of the product M^d under the map $(p_1, \ldots, p_d) \xrightarrow{\phi^d} \sum_{i=1}^{d} I(p_i)$ will be denoted by W_d. The Jacobi Inversion and Riemann Vanishing theorems are essentially the statements $W_h = J(M)$ and $W_{h-1} - k^{z_0} = \Theta$, respectively [16, 27]. Here, k^{z_0} is related to the *Riemann class* Δ,

$$\Delta - (h-1)z_0 = k^{z_0},$$

and,

$$k_j^{z_0} = \frac{1 - \Omega_{jj}}{2} + \sum_{\substack{i=1 \\ i \neq j}}^{h} \int_{A_i} \omega_i(z) \int_{z_0}^{z} \omega_j. \tag{2.7}$$

Δ is a divisor, not necessarily positive, of degree $h - 1$. The Riemann class is independent of the choice of base point, as may be seen by direct computation.

The volumes of the subvarieties W_d with respect to the metric v are given by the following

Proposition 2.8. *For* $1 \leq d \leq h$, $\operatorname{Vol}(W_d) = \binom{h}{d}$.

Proof. This is essentially Poincaré's formula relating the homology class of W_d to an appropriate power of Θ [19, p. 350]. The proof, however, will be useful later, so we reproduce it here. We first restrict to the case where $\operatorname{Im} \Omega$ is the identity matrix. This amounts to a change in the $\mathbb{C}^h$ coordinates by a matrix B, $Z \mapsto BZ$, where $B^2 = (\operatorname{Im} \Omega)^{-1}$. In this situation, we have

$$\frac{\sqrt{-1}}{2} \int_M \omega_i \wedge \bar{\omega}_j = \delta_{ij}, \qquad v = \frac{\sqrt{-1}}{2} \sum_{j=1}^{h} dZ_j \wedge d\bar{Z}_j.$$

The volume of W_d for $1 \leq d \leq h$ is $\frac{1}{d!} \int_{W_d} v^d$ by Wirtinger's formula. We can write

$$v^d = \left(\frac{\sqrt{-1}}{2}\right)^d d! \sum_J \bigwedge_{j \in J} (dZ_j \wedge d\bar{Z}_j),$$

where J runs through the set $\{(j_1, \ldots, j_d) \mid 1 \leq j_1 < \cdots < j_d \leq h\}$. Now we pull back to the surface. For $1 \leq d \leq h$, the map ϕ^d has (generically) degree $d!$, so we have

$$\begin{aligned}\mathrm{Vol}(W_d) &= \sum_J \frac{1}{d!} \int_{M^d} \left(\frac{\sqrt{-1}}{2}\right)^d (\phi^d)^* \bigwedge_{j \in J} (dZ_j \wedge d\bar{Z}_j) \\ &= \sum_J \frac{1}{d!} \int_{M^d} \left(\frac{\sqrt{-1}}{2}\right)^d \bigwedge_{j\in J} \left(\sum_{k=1}^{d} \pi_k^* \phi^* dZ_j\right) \bigwedge_{j \in J} \left(\sum_{k=1}^{d} \pi_k^* \phi^* d\bar{Z}_j\right),\end{aligned}$$

where π_k is the projection of M^d onto the k^{th} factor. Denote by $S(d)$ the symmetric group on d letters, and define $\delta(\sigma, \sigma')$ for $\sigma, \sigma' \in S(d)$, $\delta(\sigma, \sigma') = 1 \Leftrightarrow \sigma = \sigma'$. Then the above is equal to

$$\begin{aligned}&\sum_J \frac{1}{d!} \left(\frac{\sqrt{-1}}{2}\right)^d \int_{M_1 \times \cdots \times M_d} \sum_{\sigma, \sigma'} \omega_{j_{\sigma(1)}}(p_1) \wedge \bar{\omega}_{j_{\sigma'(1)}}(p_1) \wedge \cdots \wedge \omega_{j_{\sigma(d)}}(p_d) \wedge \bar{\omega}_{j_{\sigma'(d)}}(p_d) \\ &= \sum_J \frac{1}{d!} \sum_{\sigma, \sigma'} \delta(\sigma, \sigma') = \sum_J 1 = \binom{h}{d}.\end{aligned}$$

In particular, we have $\mathrm{Vol}(J(M) = W_h) = 1$, and $\mathrm{Vol}(W_{h-1}) = \mathrm{Vol}(\Theta) = h$.

We also introduce the *prime form* $E(z, w)$ of M. It plays the role of $z - w$ in local coordinates, however it is multi-valued and transforms as a $-1/2$ form in each variable [27]. For us, what is important is that the prime form can be used to construct meromorphic objects on the surface. For example, one can express the *canonical differentials of the second and third kinds* as [16].

$$\omega(z, w) = \partial_z \partial_w \log E(z, w) dz dw, \qquad \omega_{b-a}(z) = \partial_z \log \frac{E(z, b)}{E(z, a)}. \tag{2.9}$$

$\omega(z, w)$ has zero A-periods in both variables and is meromorphic with a pole of order two only along the diagonal $z = w$. $\omega_{b-a}(z)$ has zero A-periods and is meromorphic with poles only at a and b with residues -1 and 1, respectively. We also have the *Riemann bilinear relations* [16]:

$$\int_a^b \omega(z, \cdot) = \omega_{b-a}(z), \tag{2.10}$$

$$\int_y^x \omega_{b-a}(\cdot) = \log \frac{E(x, b)E(y, a)}{E(x, a)E(y, b)}. \tag{2.11}$$

As in the case of the theta function we can construct a real-valued form

$$F(x, y) = \exp(-2\pi {}^t \mathrm{Im}(x - y) (\mathrm{Im}\, \Omega)^{-1} \mathrm{Im}(x - y)) \, |E(x, y)|^2. \tag{2.12}$$

$F(x, y)$ transforms as a $(-1/2, -1/2)$ form in each variable. The exponential factor removes the multi-valuedness of $E(x, y)$ at the expense of holomorphic factorization. A simple computation gives, for $z \neq w$,

$$\partial_z \partial_{\bar{z}} \log F(z, w) dz \wedge d\bar{z} = 2\pi\sqrt{-1} h \mu(z). \tag{2.13}$$

The *Arakelov-Green's function* on a compact Riemann surface M of genus $h > 0$ is characterized by the following [2]:
a) $G(x, y)$ has a zero of order 1 along the diagonal,
b) $G(x, y) = G(y, x)$,
c) for $z \neq w$, $\partial_z \partial_{\bar{z}} \log G(z, w) = \pi\sqrt{-1}\mu_{z\bar{z}}$,
d) $\int_M \log G(x, y)\mu(y) = 0$.

Here, $\mu = \mu_{z\bar{z}} dz \wedge d\bar{z}$ is the $(1, 1)$ form defined in (2.4). Any metric on M compatible with the complex structure can be expressed in conformal coordinates $ds^2 = 2g_{z\bar{z}} dz d\bar{z}$. The *Arakelov metric* is defined by requiring its curvature to be proportional to μ.

$$\partial_z \partial_{\bar{z}} \log g^{z\bar{z}} = 4\sqrt{-1}(h - 1)\mu_{z\bar{z}}. \tag{2.14}$$

This determines $g_{z\bar{z}}$ up to a constant, which is fixed by the condition

$$\log g_{z\bar{z}} = \lim_{w \to z} [2 \log G(z, w) - \log |z - w|^2]. \tag{2.15}$$

We then have [11, 17, 33]:

$$2 \log G(z, w) = \log F(z, w) + \tfrac{1}{2} \log g_{z\bar{z}} + \tfrac{1}{2} \log g_{w\bar{w}}, \tag{2.16}$$

which will prove very useful in computations later.

3. Degeneration of Riemann Surfaces – I

We now discuss a model for the degeneration of compact Riemann surfaces to Riemann surfaces with nodes. These singular surfaces may be regarded as the union of finitely many compact surfaces with particular points, the *punctures*, identified by the local equation $zw = 0$. In the case of a single node, we have two cases depending upon whether the node separates the (degenerate) surface or not. Case I, the *separating node* model, will be discussed in this section, Case II, the *non-separating node* model, will be considered in Sect. 4.

The general reference for this section is [16] (see also, [37]). We consider the degeneration of M into two surfaces M_1, M_2 with genera $h_1, h_2 > 0$, which are joined at a node p. A model for this degeneration is constructed as follows: choose points $p_1, p_2 \in M_1, M_2$ and coordinates $z_i : U_i \to D$ centered at p_i, $i = 1, 2$. D is the unit disk in $\mathbb{C}$. Henceforth, we shall denote both p_1 and p_2 by p. Let $S = \{(x, y, t) \mid xy = t;\ x, y, t \in D\}$ and S_t be the fiber for fixed t. For $t \in D$, remove the disks $|z_i| < |t|$, and glue together the remaining surfaces by means of the annulus S_t according to the prescription

$$z_1 \to \left(z_1, \frac{t}{z_1}, t\right), \qquad z_2 \to \left(\frac{t}{z_2}, z_2, t\right).$$

The resulting surfaces then form an analytic family $\mathscr{M} \to D$ with fibers M_t, $t \neq 0$, each of genus $h = h_1 + h_2$ and M_0 a "stable curve", i.e. a boundary point in the Mumford-Deligne compactification of moduli space [26]. In a neighborhood of the double point, it is often useful to use normalizing coordinates,

$$\mathscr{X} = \tfrac{1}{2}(x + y), \qquad \mathscr{Y} = \tfrac{1}{2}(x - y). \tag{3.1}$$

For $\mathscr{X}$ near zero, the annuli are given by the double coverings $\mathscr{Y} = \pm\sqrt{\mathscr{X}^2 - t}$, ramified at $\pm\sqrt{t}$. For any point $z \in M_{1,2} - \{p\}$ and t sufficiently small, there is a natural section $z(t)$ of $\mathcal{M} \to D$ with $z(0) = z$ given by the above identification. In general, we shall indicate a smooth section $z(t)$ with $z(t) \in M_1 - \{|z_1| < |t|^{1/2}\}$ for all small t by saying $z \in M_1 \cap M_t$.

We can combine two symplectic homology bases for M_1, M_2 to get a basis for each fiber M_t. We shall assume this is ordered so that $\{A_j, B_j \mid j \leq h_1\}$ are closed curves in $M_1 \cap M_t$, and $\{A_j, B_j \mid j > h_1\}$ are closed curves in $M_2 \cap M_t$. We then have the fundamental

Proposition 3.2 [16, p. 38], [37, p. 129]. *For sufficiently small t, we can find a normalized basis of abelian differentials $\omega_1, \ldots, \omega_h$ for M_t, holomorphic in t, which have the following expansions:*

$$\omega_i(x, t) = \begin{cases} \omega_i^{(1)}(x) + O(t^2), & \text{for} \quad x \in M_1 - U_1, \\ -t\omega_i^{(1)}(p)\omega^{(2)}(x, p) + O(t^2), & \text{for} \quad x \in M_2 - U_2, \end{cases}$$

$$\omega_j(x, t) = \begin{cases} \omega_j^{(2)}(x) + O(t^2), & \text{for} \quad x \in M_2 - U_2, \\ -t\omega_j^{(2)}(p)\omega^{(1)}(x, p) + O(t^2), & \text{for} \quad x \in M_1 - U_1, \end{cases}$$

Here, $i \leq h_1$, $j > h_1$. The $\omega_i^{(1)}$ form a normalized basis for the abelian differentials on M_1, and likewise for $\omega_j^{(2)}$ on M_2. $\omega^{(1)}(x, y)$, $\omega^{(2)}(x, y)$ are the canonical differentials of the second kind on M_1, M_2 (see (2.9)). The terms $\lim_{t\to 0} \frac{O(t^2)}{t^2}$ are meromorphic differentials with poles of at most order four at p. The evaluations at p are carried out in the local coordinates z_i.

An easy consequence of the proposition is that for the period matrix associated to the homology basis described above,

$$\lim_{t\to 0} \Omega(t) = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}, \tag{3.3}$$

where Ω_1 is the period matrix of M_1 with respect to the original homology basis, and so on. The family $\mathcal{J} \to D$ with fiber $J(M_t)$, $t \neq 0$ is an analytic family of abelian varieties. From (3.3) we see that the fiber over zero is a product torus, $\mathcal{J}_0 = J(M_1) \times J(M_2)$.

As we vary the period matrix, the theta divisors, Θ_t, form an analytic family of subvarieties of $J(M_t)$. For $t \neq 0$, Θ_t is irreducible, since it is a translate of W_{h-1}. For $t = 0$, however, the theta divisor splits, $\Theta_0 = \Theta_1 \times J(M_2) \cup J(M_1) \times \Theta_2$; the two irreducible subvarieties in Θ_0 intersect in $\Theta_1 \times \Theta_2$.

We now consider the Abel map:

Corollary 3.4. *Let x, y be local smooth sections of $\mathcal{M} \to D$ for t near zero. For part* (c) *below, we further assume that in local coordinates, $\dot{x}(0) = \dot{y}(0) = 0$. Then for $j > h_1$,*

a) $\lim_{t\to 0} \int_y^x \vec{\omega}(\cdot, t) = (x - y, 0) \in \mathcal{J}_0$ *for* $x, y \in M_1 \cap M_t$,

b) $\lim_{t\to 0} \int_y^x \vec{\omega}(\cdot, t) = (x - p, p - y) \in \mathcal{J}_0$ *for* $x \in M_1 \cap M_t$, $y \in M_2 \cap M_t$,

c) $\lim_{t\to 0} \partial_t \int_y^x \omega(\cdot, t) = -\omega_j^{(2)}(p)\omega_{x-y}^{(1)}(p)$ *for* $x, y \in M_1 \cap M_t$.

In the above, the superscripts indicate to which surface the differentials belong. The form $\omega_{x-y}(z)$ is the canonical differential of the third kind defined in (2.9)

Proof. For the proof of part (c), we begin by noting that both sides are holomorphic in each variable for $x, y \in M_1 - \{p\}$ and are multi-valued with the same periods. Then if $x, y \in M_1 - U_1 \cap M_t$, we have from Proposition 3.2,

$$\frac{\partial}{\partial t}(x-y)_{t_j}|_{t=0} = \frac{\partial}{\partial t}\int_y^x \omega_j(\cdot, t)|_{t=0} = -\omega_j^{(2)}(p)\int_y^x \omega^{(1)}(\cdot, p)$$
$$= -\omega_j^{(2)}(p)\omega_{x-y}^{(1)}(p),$$

by (2.10). Since the two sides agree on an open set, they are equal everywhere. Parts (a) and (b) also follow easily from the expansions of the abelian differentials. For this and estimates of the remainders in these limits, see Appendix A.

Putting this together with the result for the theta divisor, we have

Corollary 3.5. *For local smooth sections x, y of the degeneration, the* $\lim_{t\to 0} \Theta_t + x - y$ *in* $\mathscr{J}_0$ *is:*
a) $\Theta_1 + x - y \times J(M_2) \cup J(M_1) \times \Theta_2$ *for* $x, y \in M_1 \cap M_t$,
b) $\Theta_1 + x - p \times J(M_2) \cup J(M_1) \times \Theta_2 + p - y$ *for* $x \in M_1 \cap M_t$, $y \in M_2 \cap M_t$.

The simple observation is that the translated theta divisor of part (a) of the corollary is, in the limit, no longer translated along the second factor. We shall see in Sect. 6 that this gives rise to a $\log|t|$ singularity for the Arakelov-Green's function.

We now examine the theta function $\vartheta(Z, \Omega(t))$; we shall write simply $\vartheta_t(Z)$. It is analytic in both arguments for t near the origin. We shall be particularly interested in the case where Z is a translate by $x - y$ of a point in the theta divisor. There are essentially two possibilities:

Proposition 3.6. *Let x, y be local smooth sections of the degeneration, $x \in M_1 \cap M_t$, $y \in M_2 \cap M_t$. Let $Z(t)$ be a local smooth section of $\Theta_t \to D$ with $Z(0) = (Z_1, Z_2)$. Then* $\lim_{t\to 0} \vartheta_t(Z + x - y) = \vartheta_1(Z_1 + x - p)\vartheta_2(Z_2 + p - y)$, *where* $\vartheta_1(\cdot) \equiv \vartheta(\cdot, \Omega_1)$.

Proof. The proposition follows directly from the definition (2.5) and part (b) of Corollary 3.4.

The second possibility is where x and y are on the same surface. Let $\Delta(t)$ denote the Riemann class of M_t associated to the homology basis described above. We first use the definition (2.7) of k to establish the following simple

Lemma 3.7. *Fix positive divisors $\mathscr{D}_1$, $\mathscr{D}_2$ in $M_1 - \{p\}$, $M_2 - \{p\}$ with degrees, d_1, d_2, respectively. Suppose $d_1 + d_2 = h - 1$. We consider local smooth extensions of the divisors to the degeneration $\mathscr{J} \to D$. Then*

$$\lim_{t\to 0} \mathscr{D}_1 + \mathscr{D}_2 - \Delta(t) = (\mathscr{D}_1 - (h_2 - d_2)p - \Delta_1, \mathscr{D}_2 - (h_1 - d_1)p - \Delta_2).$$

Proposition 3.8. *Fix positive divisors $\mathscr{D}_1 \subset M_1 - \{p\}$ of degree h_1 and $\mathscr{D}_2 \subset M_2 - \{p\}$ of degree $h_2 - 1$ such that $e = \mathscr{D}_1 - p - \Delta_1 \in J(M_1)$, and $f = \mathscr{D}_2 - \Delta_2 - \in \Theta_2$. Now extend them as local smooth sections of the degeneration. If x, y are local sections in $M_1 \cap M_t$, then we have the following expansion for $\vartheta_t(\mathscr{D}_1 + \mathscr{D}_2 - \Delta(t) + x - y)$:*

$$\vartheta_t = t\,\frac{\vartheta_1(x-p+e)\vartheta_1(y-p-e)E_1(x,y)}{\vartheta_1(e)E_1(x,p)E_1(y,p)}\sum_{k>h_1}\partial_{z_k}\partial_2(f)\omega_k^{(2)}(p) + O(t^2),$$

where $E_1(x, y)$ is the prime form for M_1 introduced in Sect. 2.

Proof. We must find the $O(t)$ term, since in this case the constant term vanishes identically. Using the above lemma,

$$\begin{aligned}&\frac{\partial}{\partial t}\vartheta_t(\mathscr{D}_1+\mathscr{D}_2-\Delta(t)+x-y)|_{t=0}\\&=\sum_{\mu=1}^{h}\partial_{Z_\mu}\vartheta_0\frac{\partial}{\partial t}(\mathscr{D}_1+\mathscr{D}_2-\Delta(t)+x-y)_\mu|_{t=0}+\sum_{\mu,\nu=1}^{h}\frac{\partial}{\partial\Omega_{\mu\nu}}\vartheta_0\frac{\partial}{\partial t}\Omega_{\mu\nu}|_{t=0},\end{aligned}$$

where $\vartheta_0=\vartheta_1\vartheta_2$ is evaluated at the point $(e+x-y, f)$. Using part (c) of Corollary (3.4), and the fact that theta satisfies

$$\frac{\partial}{\partial\Omega_{\mu\nu}}\vartheta=\frac{1}{4\pi i}\partial_{Z_\mu}\partial_{Z_\nu}\vartheta,$$

we have for the first order term,

$$\begin{aligned}&\vartheta_1(x-y+e)\left\{\sum_{j>h_1}\partial_{Z_j}\vartheta_2(f)\frac{\partial}{\partial t}(\mathscr{D}_1+\mathscr{D}_2-\Delta(t))_j|_{t=0}\right.\\&\qquad\left.-\omega^{(1)}_{x-y}(p)\sum_{j>h_1}\partial_{Z_j}\vartheta_2(f)\omega^{(2)}_j(p)+\frac{1}{4\pi i}\sum_{j,k>h_1}\partial_{Z_j}\partial_{Z_k}\vartheta_2(f)\frac{\partial}{\partial t}\Omega_{jk}|_{t=0}\right\}\\&\qquad+\frac{1}{2\pi i}\sum_{\substack{j\le h_1\\k>h_1}}\partial_{Z_j}\vartheta_1(e+x-y)\partial_{Z_k}\vartheta_2(f)\frac{\partial}{\partial t}\Omega_{jk}|_{t=0}\\&=-\left\{\omega^{(1)}_{x-y}(p)\vartheta_1(e+x-y)+\frac{1}{2\pi i}\sum_{j\le h_1}\partial_{Z_j}\vartheta_1(x-y+e)\omega^{(1)}_j(p)\right.\\&\qquad\left.-\frac{\vartheta_1(x-y+e)}{\vartheta_1(e)}\frac{1}{2\pi i}\sum_{j\le h_1}\partial_{Z_j}\vartheta_1(e)\omega^{(1)}_j(p)\right\}\\&\qquad\times\sum_{k>h_1}\partial_{Z_k}\vartheta_2(f)\omega^{(2)}_k(p)+R\,\frac{\vartheta_1(x-y+e)}{\vartheta_1(e)},\end{aligned}$$

where R is equal to

$$\begin{aligned}&\vartheta_1\sum_{k>h_1}\partial_{Z_k}\vartheta_2\frac{\partial}{\partial t}(\mathscr{D}_1+\mathscr{D}_2-\Delta(t))_k|_{t=0}+\frac{1}{2\pi i}\sum_{\substack{j\le h_1\\k>h_1}}\partial_{Z_j}\vartheta_1\partial_{Z_k}\vartheta_2\frac{\partial}{\partial t}\Omega_{jk}|_{t=0}\\&\qquad+\frac{1}{4\pi i}\vartheta_1\sum_{j,k>h_1}\partial_{Z_j}\partial_{Z_k}\vartheta_2\frac{\partial}{\partial t}\Omega_{jk}|_{t=0}.\end{aligned}$$

In the expression above the theta functions are evaluated at e and f. It is clear, however, that R is identically zero, since it is the derivative of the theta function evaluated along the theta divisor. The result is obtained by rewriting the remaining term. For this, we refer to [16, p. 25].

The following proposition also follows from Lemma 3.7.

Proposition 3.9. *Fix positive divisors $\mathscr{D}_1$, $\mathscr{D}_2$ of degrees $h_1 - 1$, h_2 in $M_1 - \{p\}$, $M_2 - \{p\}$. Now extend them as local smooth sections of $\mathscr{J} \to D$. If x, y, are local sections in $M_1 \cap M_t$, then*

$$\lim_{t\to 0} \vartheta_t(\mathscr{D}_1 + \mathscr{D}_2 - \Delta(t) + x - y) = \vartheta_1(\mathscr{D}_1 - \Delta_1 + x - y)\vartheta_2(\mathscr{D}_2 - p - \Delta_2),$$

and for generic choices of $\mathscr{D}_1$ and $\mathscr{D}_2$ the constant term does not vanish.

For reference, we express the result of Proposition 3.8 in terms of $\|\vartheta\|$ and $F(x, y)$. Using (2.6) and (2.12) we have, after some algebra,

Corollary 3.10. *For the situation as in Proposition 3.8, we have the following expansion for $\|\vartheta_t\|^2(\mathscr{D}_1 + \mathscr{D}_2 - \Delta(t) + x - y)$:*

$$\|\vartheta_t\|^2 = |t|^2 \frac{\|\vartheta_1\|^2(x-p+e)\,\|\vartheta_1\|^2(y-p-e)}{\|\vartheta_1\|^2(e)} \frac{F_1(x,y)}{F_1(x,p)F_1(y,p)} \times \|h_f^{(2)}\|^2(p) + O(|t|^3).$$

Here we have defined

$$\|h_f\|^2(z) = \exp(-2\pi\, {}^t\operatorname{Im} f(\operatorname{Im}\Omega)^{-1}\operatorname{Im} f)\left|\sum_{k=1}^{h} \partial_{Z_k}\vartheta(f)\omega_k(z)\right|^2,$$

and, as before, the evaluation at p is in terms of the local coordinates z_1, z_2.

4. Degeneration of Riemann Surfaces – II

In this section, we consider the degeneration of a compact Riemann surface of genus $h+1$ to singular surface of genus h with a single, non-separating node. The construction is similar to that of Sect. 3, except that in this case we glue on the annulus via local coordinates z_a and z_b centered at points a, b on a compact Riemann surface M of genus h. The resulting surfaces form an analytic family $\mathscr{M} \to D$ with fibers M_t, $t \neq 0$, each compact of genus $h+1$, and M_0 a stable curve. Notice that the node, which is the identification of a with b in M_0 does *not* disconnect the surface when removed, as opposed to the situation analysed previously.

We now choose a homology basis. Let $\{A_j, B_j\}_{j=1,\dots,h}$ be a symplectic basis for M away from the points a, b. The surfaces M_t, $t \neq 0$, each have genus $h+1$, so we must provide two more loops A_{h+1}, B_{h+1} to fill out the basis. A_{h+1} may be taken to be the boundary of the disk U_b, and B_{h+1} will then run across the handle. But now it is easy to see that as t goes once around the origin in D^*, we twist the handle, and the resulting curve B'_{h+1} will differ from the original by $\pm A_{h+1}$. As a consequence, the period matrix $\Omega(t)$ will be multi-valued. If we subtract $\log t$ from the $(h+1, h+1)$ component, however, we obtain a matrix which is analytic near $t = 0$. Specifically, we have

Proposition 4.1 [16, p. 38]. *The period matrix has the expansion,*

$$\Omega(t) = \begin{pmatrix} \Omega_{ij} + t\sigma_{ij} & a_i + t\sigma_{ih} \\ a_j + t\sigma_{hj} & \dfrac{1}{2\pi i}\log t + c_0 + c_1 t \end{pmatrix} + O(t^2),$$

where Ω is the period matrix for M, $\lim\limits_{t\to 0} \frac{O(t^2)}{t^2}$ is a finite matrix, and $a_j = \int_a^b \omega_j$.

This proposition follows from

Proposition 4.2 [16, p. 51], [37, p. 135]. *For sufficiently small t, we can find a normalized basis of abelian differentials $\omega_1, \ldots, \omega_{h+1}$ for M_t, holomorphic in t, which have the following expansions for $i \le h$:*

$$\omega_i(x, t) = \omega_i(x) - t(\omega_i(b)\omega(x, a) + \omega_i(a)\omega(x, b)) + O(t^2), \quad x \in M - U_a - U_b\,;$$

$$\omega_{h+1}(x, t) = \frac{1}{2\pi i}\,\omega_{b-a}(x) - t(\gamma_1\omega(x, b) + \gamma_2\omega(x, a)) + O(t^2), \quad x \in M - U_a - U_b\,.$$

Here, the ω_i form a normalized basis for the abelian differentials on M, and the γ_i's are constants (see [37]). The term $\lim\limits_{t\to 0} \frac{O(t^2)}{t^2}$ is a meromorphic differential with poles only at a or b. The evaluations at a and b are carried out in the local coordinates z_a, z_b.

It can be shown that the associated Jacobians form an analytic family $\mathscr{J} \to D$ of abelian varieties with $\mathscr{J}_0$ non-compact. For each $t \ne 0$, set $\Theta_{\delta(t)} = \Theta_t + \delta(t)$, where $\delta_j(t) = \frac{1}{2}\,\Omega_{j,h+1}(t)$. The family Θ_δ becomes an analytic subvariety of $\mathscr{J}$ if we require the zero fiber $\Theta_{\delta(0)}$ to be defined by

$$\exp 2\pi i Z_{h+1} = -\frac{\vartheta(Z_0 - \frac{1}{2})\,(b-a))}{\vartheta(Z_0 + \frac{1}{2})\,(b-a))}\,, \tag{4.3}$$

for $Z = (Z_0, Z_{h+1}) \in \mathbb{C}^h \times \mathbb{C} = \mathbb{C}^{h+1}$ (henceforth, theta functions and prime forms appearing without the t subscript denote the objects on the original surface).

We briefly describe the origin of Eq. (4.3). Recall from the definition (2.5) that the theta function is a sum $\sum\limits_{n\in\mathbb{Z}^{h+1}}$. From Proposition 4.1, the important contributions for small t are those from n_{h+1}. Set $n = (n_0, m)$, $m = n_{h+1}$. Then $\vartheta_t(Z - \delta(t))$ has a factor of

$$\begin{aligned}
&\exp \pi i(i \operatorname{Im} \Omega_{h+1,h+1}(t)m^2 + i2m \operatorname{Im} Z_{h+1} - im \operatorname{Im} \Omega_{h+1,h+1}(t)) \\
&\quad = \exp(-\pi(m^2 - m) \operatorname{Im} \Omega_{h+1,h+1}(t) - 2\pi m \operatorname{Im} Z_{h+1})) \\
&\quad = |t|^{\frac{m}{2}(m-1)} \times \text{remaining terms}\,.
\end{aligned}$$

It can be shown that the remaining terms are uniformly bounded for Z in compact sets and t near the origin. The zero order term then corresponds to $m = 0, 1$. Hence

$$\lim_{t\to 0} \vartheta_t(Z - \delta(t)) = \vartheta(Z_0 - \tfrac{1}{2}\,(b-a)) + e^{2\pi i Z_{h+1}}\vartheta(Z_0 + \tfrac{1}{2}\,(b-a))\,. \tag{4.4}$$

Equation (4.3) follows from the above.

We now give the analog of Propositions 3.6, 3.7, and 3.9.

Proposition 4.5. *Fix $(Z_0, Z_{h+1}) = Z \in \Theta_{\delta(0)}$, $Z_0 \notin \Theta - \frac{1}{2}\,(b-a)$, and extend to a local section of $\Theta_{\delta(t)}$. Then for x, y local smooth sections of $\mathscr{M}$, we have*

$$\lim_{t\to 0} \vartheta_t(Z + x - y - \delta(t)) = \frac{\vartheta(Z_0 + x - \frac{1}{2}\,(a+b))\vartheta(Z_0 - y + \frac{1}{2}\,(a+b))}{\vartheta(Z_0 + \frac{1}{2}\,(b-a))} \times \frac{E(x, y)E(a, b)}{E(x, a)E(y, b)}\,.$$

Proof. By the expansions (4.2) and Eq. (2.11), we see that for local sections $x, y \in M - \{a, b\}$ of the degeneration, we have

$$\lim_{t\to 0} \int_y^x \vec{\omega}_t = \left(\int_y^x \vec{\omega},\ \frac{1}{2\pi i} \log \frac{E(x, b)E(y, a)}{E(x, a)E(y, b)} \right). \tag{4.6}$$

Then from (4.4),

$$\begin{aligned}
&\vartheta_t(Z + x - y - \delta(t)) \\
&\quad = \vartheta(Z_0 + x - y - \tfrac{1}{2}(b - a)) \\
&\quad + e^{2\pi i Z_{h+1}} \frac{E(x, b)E(y, a)}{E(x, a)E(y, b)} \vartheta(Z_0 + x - y + \tfrac{1}{2}(b - a)) + O(t) \\
&\quad = \{\vartheta(Z_0 + x - y - \tfrac{1}{2}(b - a))\vartheta(Z_0 + \tfrac{1}{2}(b - a))E(x, a)E(y, b) \\
&\quad - \vartheta(Z_0 + x - y + \tfrac{1}{2}(b - a))\vartheta(Z_0 - \tfrac{1}{2}(b - a))E(x, b)E(y, a)\} \\
&\quad \times \{\vartheta(Z_0 + \tfrac{1}{2}(b - a))E(x, a)E(y, b)\}^{-1} + O(t).
\end{aligned}$$

The proposition now follows from Fay's trisecant identity (see [16, 27]).

Finally, we note the behavior of the Riemann class:

Proposition 4.7. *Fix a base point $z_0 \in M - \{a, b\}$ and extend to a local section of the degeneration. Then for $1 \le j \le h$,*

$$\lim_{t\to 0} [k_j^{z_0}(t) + \delta_j(t)] = k_j^{z_0} + \frac{1}{2} \int_{2z_0}^{a+b} \omega_j\,;$$

$$\lim_{t\to 0} [k_{h+1}^{z_0}(t) + \delta_{h+1}(t)] = \frac{1}{2} + \frac{1}{2\pi i} \sum_{j=1}^{h} \int_{A_j} \omega_j(z) \log \frac{E(z, b)E(z_0, a)}{E(z, a)E(z_0, b)}\,.$$

Proof. See [16], p. 57.

5. Faltings' Invariant

We now come to the main topic of this paper. The Faltings invariant $\delta(M)$ was introduced in [14] as a proportionality constant reflecting the equivalence of two metrics on a certain determinant line bundle; the one defined by Faltings and the other the pullback of the natural metric on $\mathcal{O}(-\Theta)$. In the following sections, we will exhibit the behavior of this invariant under the degeneration model described above. The main results are contained in Theorems 6.10, 7.2, and the Main Theorem (see Introduction).

We begin by proving a formula which relates the Arakelov-Green's function to an integral over the theta divisor. This expression is originally due to Bost [6] – we give a proof based on the *spin-1 bosonization formula* [14], which may also be taken as the definition of $\delta(M)$:

$$\begin{aligned}
&\|\vartheta\|\,(p_1 + \cdots + p_{h-1} + x - y - \Delta) \\
&\qquad = e^{-\delta(M)/8} (\det \operatorname{Im} \Omega)^{-3/4} \|\det \omega_j(p_k)\| \frac{\prod_{j=1}^{h} G(p_i, y)}{\prod_{j<k} G(p_j, p_k)}\,.
\end{aligned} \tag{5.1}$$

The left-hand side is defined by (2.6) and (2.7). On the right-hand side, we have set $p_h = x$, and

$$\|\det \omega_j(p_k)\|^2 = \left(\prod_{i=1}^{h} g^{z_i \bar{z}_i}\right) \det \omega_j(z_k)\, \overline{\det \omega_j(z_k)},$$

where z_k is a local coordinate around p_k, and $g^{z\bar{z}} = (g_{z\bar{z}})^{-1}$ is the Arakelov metric.

As suggested by the name, (5.1) also arises in physics, where it is but one of many identities obtained by equating the correlation functions of a certain conformal field theory of free fermions with a scalar theory coupled to a background charge. In this setting, the δ-invariant can be related to the zeta regularized determinant of the Laplace operator with respect to the Arakelov metric [1, 17, 32]:

$$\delta(M) = c_h - 6 \log \frac{\det' \Delta_g}{\operatorname{Area}(M, g)},$$

where c_h is a constant depending only upon the genus.

Proposition 5.2 [6, Proposition 1]. *There exists a constant A such that*

$$\log G(x, y) = \frac{1}{h!} \int_{\Theta + x - y} \nu^{h-1} \log \|\vartheta\| + A.$$

Proof. Taking the log of both sides of (5.1) we have

$$\log \|\vartheta\| = -\frac{\delta(M)}{8} - \frac{3}{4} \log \det \operatorname{Im} \Omega + \log \| \det \omega_j(p_k) \|$$
$$+ \sum_{j=1}^{h-1} \log G(p_j, y) - \sum_{j<k} \log G(p_j, p_k) + \log G(x, y).$$

To carry out the integral over Θ, we pull back to M^{h-1} as in Sect. 2. We are primarily interested in terms like $\log G(p_i, x)$ and $\log G(p_i, y)$. From the proof of Proposition 2.8, such an integral becomes

$$\frac{1}{h} \sum_J \frac{1}{(h-1)!} \left(\frac{\sqrt{-1}}{2}\right)^{h-1} \int_{M_1 \times \cdots \times M_{h-1}} \log G(p_i, x)$$
$$\times \sum_{\sigma, \sigma'} \omega_{j_{\sigma(1)}}(p_1) \wedge \bar{\omega}_{j_{\sigma'(1)}}(p_1) \wedge \cdots \wedge \omega_{j_{\sigma(h-1)}}(p_{h-1}) \wedge \bar{\omega}_{j_{\sigma'(h-1)}}(p_{h-1}).$$

Integrating over $p_j \neq p_i$, we see that all terms vanish other than those with $\sigma(j) = \sigma'(j)$. We have

$$\frac{1}{h} \sum_J \frac{1}{(h-1)!} \frac{\sqrt{-1}}{2} \int_{M_i} \log G(p_i, x) \sum_{\sigma, \sigma'} \delta(\sigma, \sigma') \omega_{j_{\sigma(i)}}(p_i) \wedge \bar{\omega}_{j_{\sigma'(i)}}(p_i),$$

$$= \frac{1}{h} \sum_J \frac{1}{(h-1)} \sum_{j \in J} \frac{\sqrt{-1}}{2} \int_{M_i} \log G(p_i, x) \omega_j(p_i) \wedge \bar{\omega}_j(p_i),$$

$$= \int_{M_i} \log G(p_i, x) \frac{\sqrt{-1}}{2h} \sum_{j=1}^{h} \omega_j(p_i) \wedge \bar{\omega}_j(p_i) = \int_{M_i} \mu(p_i) \log G(p_i, x) = 0.$$

Furthermore, note that $\frac{1}{(h-1)!} v^{h-1}$ restricted to $\Theta + x - y$ is the induced volume form, and h is its volume, so $\frac{1}{h!} \int_{\Theta+x-y} v^{h-1} = 1$. After integration, the only terms remaining are $\log G(x, y)$ and another term which manifestly depends on the point x

$$A = -\frac{1}{h!} \int_M \mu(y) \int_{\Theta+x-y} v^{h-1} \log \|\vartheta\| .$$

However, by symmetry of the Green's function and of the theta divisor in x and y, it is seen that A is in fact constant.

To simplify notation in the sequel, we make the following definitions:

$$\begin{aligned} \log K(x, y) &= \frac{1}{h!} \int_{\Theta+x-y} v^{h-1} \log \|\vartheta\| , \\ \log \|H\| &= \frac{1}{h!} \int_{J(M)} v^h \log \|\vartheta\| , \\ A &= -\int_M \mu(y) \log K(x, y) . \end{aligned} \tag{5.3}$$

6. Asymptotics of the Green's function – Case I

We now let M degenerate to the surfaces M_1, M joined at a double point p. First, however, we would like to briefly comment on the parameterization.[1] The degeneration model described above depends on the choice of coordinates z_1, z_2. A different choice would have the effect of rescaling the dgeneration parameter t. More precisely, suppose we consider the sets

$$S = \{(z, w, t) \mid zw = t;\ z, w, t \in D\} , \qquad \tilde{S} = \{(\tilde{z}, w, \tilde{t}) \mid \tilde{z}w = \tilde{t};\ \tilde{z}, w, \tilde{t} \in D\} .$$

Here we are thinking of $\tilde{z}$ as an analytic function of z near the origin with $\tilde{z}(0) = 0$. For fixed w, $\tilde{t}$ is a function of t alone, and $d\tilde{t}/dt(0) = d\tilde{z}/dz(0)$. Now suppose both z $\tilde{z}$ are local coordinates for M_1 centered at the point p. Define the parameters $\tau = t\sqrt{g_{z\bar{z}}(0)}$, and similarly for $\tilde{\tau}$. We compute

$$\left| \frac{d\tilde{\tau}}{d\tau}(0) \right| = \left| \frac{d\tilde{t}}{d\tau}(0) \sqrt{g_{\tilde{z}\bar{\tilde{z}}}(0)} \right| = \left| \frac{1}{\sqrt{g_{z\bar{z}}(0)}} \frac{d\tilde{z}}{dz}(0) \left| \frac{d\tilde{z}}{dz}(0) \right|^{-1} \sqrt{g_{z\bar{z}}(0)} \right| = 1 .$$

In particular, we have $|\tilde{\tau}| = |\tau| + O(|\tau|^2)$. This local computation easily extends to the family of degenerating surfaces. Thus, if for the local coordinates z_1, z_2 on the surfaces M_1, M_2 we define the parameter

$$\tau = t\sqrt{g_{z_1\bar{z}_1}(0)}\, \sqrt{g_{z_2\bar{z}_2}(0)} , \tag{6.1}$$

then expansions to the first order in τ are independent of the initial choice of coordinates.

[1] Thanks to Scott Wolpert for clarification of this point

Now consider the expression for the Green's function in Proposition 5.2. For the theta divisor translated by $x - y$, $x, y \in M_1 \cap M_t$, we have by Corollary 3.5 that $\Theta_t + x - y \to \Theta_1 + x - y \times J(M_2) \cup J(M_1) \times \vartheta_2$, and $\|\vartheta_t\|$ vanishes on the second subvariety, the volume of which is h_2. Thus, we expect a divergence like $\frac{h_2}{h} \log|t|$ for $\log K_t(x, y)$. This is the idea behind

Proposition 6.2. *Let M_t of genus $h = h_1 + h_2$ degenerate into two surfaces M_1 and M_2 of genus h_1 and h_2, respectively, as described in Sect. 3, and define the parameter τ as in (6.1). Then for local smooth sections $x, y \in M_1 \cap M_t$, $x \neq y$,*

$$\text{a)}\quad \lim_{t\to 0}\left[\log K_t(x, y) - \frac{h_2}{h}\log|\tau|\right] = \log G_1(x, y) - \frac{h_2}{h}\log G_1(x, p)G_1(y, p) + \frac{h_1}{h}(\log\|H_2\| - A_1) + \frac{h_2}{h}(\log\|H_1\| - A_2)\,;$$

For smooth sections $x \in M_1 \cap M_t$, $y \in M_2 \cap M_t$,

$$\text{b)}\quad \lim_{t\to 0}\log K_t(x, y) = \frac{h_1}{h}(\log K_1(x, p) + \log\|H_2\|) + \frac{h_2}{h}(\log K_2(y, p) + \log\|H_1\|)\,,$$

where the subscripts 1 and 2 indicate the functions associated to M_1 and M_2, respectively.

Proof. Consider first part (a), so we have local sections $x, y \in M_1 \cap M_t$. For each $t \neq 0$, we define an open set in $J(M_t)$ whose limiting value at zero is essentially $J(M_1) \times \Theta_2$. Let

$$\mathscr{E}_t = \left\{\sum_{i=1}^{h_1} I(p_i) + \sum_{j=1}^{h_2-1} I(q_j) - \Delta(t)\mid p_i \in M_1 \cap M_t,\ q_j \in M_2 \cap M_t\right\}.$$

Lemma 6.3. $\lim_{t\to 0}\frac{1}{|t|^r}(\mathrm{Vol}(\mathscr{E}_t) - h_2) = 0,$ *for all* $r < 1$.

Proof. We refer to our calculation of the volumes in Proposition 2.8: the map $(M_1 \cap M_t)^{h_1} \times (M_2 \cap M_t)^{h_2-1} \xrightarrow{\psi} \mathscr{E}_t$ has degree (generically) $h_1!(h_2-1)!$. As before, we calculate the volume of $\mathscr{E}_t$ by pulling back by ψ. The result is

$$\sum_J \frac{1}{h_1!(h_2-1)!}\left(\frac{\sqrt{-1}}{2}\right)^{h-1} \times \sum_{\sigma,\sigma' \in S(h-1)}\left(\int_{M_1\cap M_t}\omega_{j_{\sigma(1)}}\wedge\bar\omega_{j_{\sigma'(1)}}\right)\cdots\left(\int_{M_2\cap M_t}\omega_{j_{\sigma(h-1)}}\wedge\bar\omega_{j_{\sigma'(h-1)}}\right).$$

Now we claim that

$$\int_{M_1\cap M_t}\frac{\sqrt{-1}}{2}\omega_{j_{\sigma(k)}}\wedge\bar\omega_{j_{\sigma'(k)}} = \delta_1(\sigma, \sigma'; J, k) + O(|t|\log|t|)\,,$$

where

$$\delta_1(\sigma, \sigma'; J, k) = \begin{cases} 1, & \text{if}\quad \sigma(k) = \sigma'(k)\quad \text{and}\quad j_{\sigma(k)} \le h_1\,;\\ 0, & \text{otherwise}.\end{cases}$$

There is a similar expression for the integral over $M_2 \cap M_t$, with the condition for $\delta_2(\sigma, \sigma';\ J, k) = 1$ being that $\sigma(k) = \sigma'(k)$ and $j_{\sigma(k)} > k$. The claim follows from the expansions of the abelian differentials which we defer to Appendix A. From the claim, we have for the volume of $\mathscr{E}_t$,

$$\sum_J \frac{1}{h_1!(h_2-1)!} \sum_{\sigma,\sigma'} \prod_{k=1}^{h_1} \delta_1(\sigma, \sigma';\ J, k) \prod_{l=h_1+1}^{h-1} \delta_2(\sigma, \sigma';\ J, l) + O(|t| \log|t|).$$

Now,

$$\prod_{k=1}^{h_1} \delta_1(\sigma, \sigma';\ J, k) \prod_{l=h_1+1}^{h-1} \delta_2(\sigma, \sigma';\ J, l) = 1 \Leftrightarrow \sigma = \sigma'$$
$$\text{and} \quad \sigma \in S(h_1) \times S(h_2 - 1).$$

This also forces h_1 of the j_k's to be $\leq h_1$, so J must have the form

$$J = (1, 2, \ldots, h_1, j_{h_1+1}, \ldots, j_{h-1}), \qquad h_1 + 1 \leq j_{h_1+1} < \cdots < j_{h-1} \leq h.$$

There are exactly h_2 distinct J's of this form, and so we have the lemma.

We now estimate

$$\begin{aligned} \log K_t(x, y) - \frac{h_2}{h} \log|t| &= \frac{1}{h!} \int\limits_{\Theta_t - \mathscr{E}_t} v_t^{h-1} \log \|\vartheta_t\| (\cdot + x - y) \\ &\quad + \frac{1}{h!} \int\limits_{\mathscr{E}_t} v_t^{h-1} \{\log \|\vartheta_t\| (\cdot + x - y) - \log|t|\} \qquad (6.4) \\ &\quad + \frac{1}{h} (\mathrm{Vol}(\mathscr{E}_t) - h_2) \log|t|. \end{aligned}$$

By Lemma 6.3, the last term vanishes as $t \to 0$. The other terms will also be continuous, the point being that the integrands are now well behaved as $t \to 0$.

Take, for example $\frac{1}{h!} \int_{\mathscr{E}_t} v_t^{h-1} \{\log \|\vartheta_t\| (\cdot + x - y) - \log|t|\}$. Fix a fundamental domain for $\mathscr{J}_0 \subset \mathbb{C}^h = \mathbb{C}^{h_1} \times \mathbb{C}^{h_2}$. Let

$$\mathscr{E}_0' = \left\{ (e, f) \,\middle|\, \vartheta_1(x - p + e) \vartheta_1(y - p - e) \sum_{k > h_1} \partial_{Z_k} \vartheta_2(f) \omega_k^{(2)}(p) = 0 \right\} \bigcup \Theta_1 \times \Theta_2.$$

Let V_ε be an ε-neighborhood of $\mathscr{E}_0' \cup K$, where K is some compact set containing the fundamental domain. Now we divide up the integral:

$$\frac{1}{h!} \int\limits_{\mathscr{E}_t} v_t^{h-1} \{\log \|\vartheta_t\| (\cdot + x - y) - \log|t|\} = \frac{1}{h!} \int\limits_{\mathscr{E}_t - V_\varepsilon} \{\ldots\} + \frac{1}{h!} \int\limits_{V_\varepsilon \cap \mathscr{E}_t} \{\ldots\}. \qquad (6.5)$$

Using the compactness of $J(M_t)$ for all small t, and reducing ε still further if necessary, we can contain $\{V_\varepsilon \cap \mathscr{E}_t \,|\, t \text{ small}\}$ in the union of finitely many balls $D_1, \ldots, D_m$. We choose ε and the D_j's small enough so that $\log \|\vartheta_t\| (\cdot + x - y) - \log|t| < 0$ on D_j for small t and all j. Therefore,

$$-\frac{1}{h!} \int\limits_{V_\varepsilon \cap \mathscr{E}_t} v^{h-1} \{\log \|\vartheta\| (\cdot + x - y) - \log|t|\} \leq \sum_{k=1}^{m} -\frac{1}{h!} \int\limits_{D_k \cap \mathscr{E}_t} \{\ldots\},$$

and the expression on the right is bounded as $t \to 0$. This follows from the following lemma, which is easily proved by using the Weierstrass Preparation Theorem:

Lemma 6.6. *Suppose $f(z, t) \not\equiv 0$ is analytic for z in the ball D^n and t near zero. Let dv denote the Euclidean volume form. Then $\int_{D_\varepsilon} dv(z) \log |f(z, t)|$ is continuous at $t = 0$, and $\lim_{\varepsilon \to 0} \int_{D_\varepsilon} dv(z) \log |f(z, 0)| = 0$.*

For the first term in (6.5), the integrand as $t \to 0$ is bounded on the entire domain, so we may apply the dominated convergence theorem and Corollary 3.10

$$\begin{aligned}
&\lim_{t \to 0} \frac{1}{h!} \int_{\mathscr{E}_t - V_\varepsilon} v_t^{h-1} \{\log \|\vartheta_t\| (\cdot + x - y) - \log |t|\} \\
&= \frac{1}{h} \int_{J(M_1) \times \Theta_2 - V_\varepsilon} \frac{1}{h_1!} v_1^{h_1}(e) \times \frac{1}{(h_2 - 1)!} v_2^{h_2 - 1}(f) \\
&\times \left\{ \log \frac{\|\vartheta_1\| (x - p + e) \|\vartheta_1\| (y - p - e)}{\|\vartheta_1\| (e)} \right. \\
&\left. + \frac{1}{2} \log \frac{F_1(x, y)}{F_1(x, p) F_1(y, p)} + \log \|h_f^{(2)}\| (p) \right\}.
\end{aligned}$$

Since ε is arbitrary, we may deduce the continuity of the left-hand side of (6.5). Letting $\varepsilon \to 0$, the second term in (6.5) vanishes, and we are left with the above, integrated over all of $J(M_1) \times \Theta_2$,

$$\begin{aligned}
&\lim_{t \to 0} \frac{1}{h!} \int_{\mathscr{E}_t} v_t^{h-1} \{\log \|\vartheta_t\| (\cdot + x - y) - \log |t|\} \\
&= \frac{h_2}{h} \log \|H_1\| + \frac{h_2}{2h} \log \frac{F_1(x, y)}{F_1(x, p) F_1(y, p)} + \frac{1}{h \cdot (h_2 - 1)!} \int_{\Theta_2} v_2^{h_2 - 1}(f) \log \|h_f^{(2)}\| (z).
\end{aligned}$$

The last term above is given by

$$\frac{1}{h!} \int_{\Theta} v^{h-1}(f) \log \|h_f\| (z) = \frac{1}{2} \log g_{z\bar{z}} - A \tag{6.7}$$

which follows from (2.15) and Proposition 5.2. A similar argument holds for the other term in (6.4). We just state the result:

$$\lim_{t \to 0} \frac{1}{h!} \int_{\Theta_t - \mathscr{E}_t} v_t^{h-1} \log \|\vartheta_t\| (\cdot + x - y) = \frac{h_1}{h} (\log K_1(x, y) + \log \|H_2\|).$$

Substituting for the F's via (2.16), we obtain part (a) of the proposition. Part (b) is a consequence of Proposition 3.6 and part (b) of Corollary 3.5.

Corollary 6.8. *For the situation in Proposition 6.2 and A as in 5.3,*

$$\lim_{t \to 0} \left[A_t + \frac{h_1 h_2}{h^2} \log |\tau| \right] = \frac{h_1}{h} (A_1 - \log \|H_2\|) + \frac{h_2}{h} (A_2 - \log \|H_1\|).$$

Proof. Roughly speaking, we should simply integrate the asymptotics of μ_t with those of K_t. Some consideration of the higher order terms is necessary:

Lemma 6.9. *For μ as defined by 2.4, we have for $z \in M_i \cap M_t$, $i = 1, 2$,*

$$\mu_t(z) = \frac{h_i}{h}\,\mu_1(z) + O(|t|)\,.$$

Moreover, in the local coordinates z_i about the node, there is a constant C such that $|O(|t|)| \le C|t|\,|dz|^2/|z|^2$.

Proof. This follows directly from Proposition 3.2, Eq. (3.3) and the bounds obtained from expanding the abelian differentials as in Appendix A.

The lemma assures that integration of the $O(|t|)$ part of the measure in the annulus against terms of the type $\log|z|$ arising from the Green's functions in the limits of Proposition 6.2 will contribute only to order

$$|t| \int\limits_{|t|^{1/2}<|z|} \frac{|dz|^2}{|z|^2} \log|z| \sim \text{const.}\,|t|\,(\log|t|)^2\,,$$

and so in the limit we may ignore them. We shall see in the next section that this is not the case for degeneration to a non-separating node.

To prove Corollary 6.8, we may therefore simply integrate the zero order terms in Proposition 6.2 and Lemma 6.9. Choose $x \in M_1 \cap M_t$. Then

$$\begin{aligned}
A_t &= -\int\limits_{M_t} \mu_t(y)\log K_t(x, y) \\
&\sim -\int\limits_{M_1\cap M_t} \frac{h_1}{h}\,\mu_1(y)\log K_t(x, y) - \int\limits_{M_2\cap M_t} \frac{h_2}{h}\,\mu_2(y)\log K_t(x, y) \\
&\sim -\frac{h_1h_2}{h^2}\log|\tau| + \frac{h_1h_2}{h^2}\log G_1(x, p) - \frac{h_1^2}{h^2}(\log\|H_2\| - A_1) \\
&\quad - \frac{h_1h_2}{h^2}(\log\|H_1\| - A_2) - \frac{h_1h_2}{h^2}(\log K_1(x, p) + \log\|H_2\|) \\
&\quad - \left(\frac{h_2}{h}\right)^2 \left(\int\limits_{M_2\cap M_t} \mu_2(y)\log K_2(y, p) + \log\|H_1\|\right).
\end{aligned}$$

Note that we have used the normalization of the Green's function, property (d) of Sect. 2. The corollary now follows by applying the definition (5.3).

We now have the main result of this section:

Theorem 6.10. *Let M_t of genus $h = h_1 + h_2$ degenerate into two surfaces M_1 and M_2 of genus h_1 and h_2, respectively, as described in Sect. 3, and define the parameter τ as in (6.1). Then for local smooth sections $x, y \in M_1 \cap M_t$, $x \neq y$,*

a)
$$\lim_{t\to 0}\left[\log G_t(x, y) - \left(\frac{h_2}{h}\right)^2 \log|\tau|\right] = \log G_1(x, y) - \frac{h_2}{h}\log G_1(x, p)G_1(y, p)\,;$$

For smooth sections $x \in M_1 \cap M_t$, $y \in M_2 \cap M_t$,

b)
$$\lim_{t\to 0}\left[\log G_t(x, y) + \frac{h_1h_2}{h^2}\log|\tau|\right] = \frac{h_1}{h}\log G_1(x, p) + \frac{h_2}{h}\log G_2(y, p)\,.$$

Proof. Combine Proposition 6.2 with Corollary 6.8 and (5.3).

7. Asymptotics of the Green's Function – Case II

In this section, we study the behavior of the Arakelov-Green's function under the degeneration of surfaces described in Sect. 4. The situation is quite different from that of degeneration to a separating node, since we no longer have a nice geometric argument, i.e. the reducibility of the theta divisor over the zero fiber, to indicate how $\log|t|$ singularities might arise.

Nevertheless, the main tool is again the expression from Proposition 5.2. Referring to the degeneration model of Sect. 4, we define, by analogy with (6.1), the parameter

$$\tau = t\sqrt{g_{z_a\bar{z}_a}(0)}\,\sqrt{g_{z_b\bar{z}_b}(0)}\,. \tag{7.1}$$

Theorem 7.2. *For the family M_t of compact Riemann surfaces of genus $h+1$ degenerating to a surface M of genus h with a non-separating node as described in Sect. 4, τ as defined in (7.1), and local smooth sections $x \neq y$, $x, y \in M - \{a, b\}$, we have,*

$$\begin{aligned}\lim_{t\to 0}&\left[\log G_t(x, y) - \frac{1}{12(h+1)^2}\log|\tau|\right]\\ &= \log G(x, y) + \frac{5}{6(h+1)^2}\log G(a, b)\\ &\quad - \frac{1}{2(h+1)}\log G(x, a)G(x, b)G(y, a)G(y, b)\,.\end{aligned}$$

Proof. The basic idea behind the proof is as follows: for fixed $Z \in \mathbb{C}^{h+1}$, we saw in Sect. 4 that $\Omega(t)$, and therefore $\vartheta_t(Z)$ were multi-valued for $t \in D^*$, the punctured disk. Let $\delta(t)$ be the vector defined by $\delta_j = \frac{1}{2}\Omega_{j,h+1}$. Then from (4.4) we know that $\vartheta_t(Z-\delta(t))$ is single-valued and analytic for t near the origin. The integral of the translate of $\log|\vartheta_t|$ may be evaluated using Proposition 4.5, and the divergence arising from the norm $\|\cdot\|$ may be treated separately.

To separate the norm from theta requires that the calculations take place in the universal cover of $J(M)$. To make things precise, we will fix a particular choice of fundamental domain. Denote by e^j the vector in $\mathbb{C}^{h+1}$ with all components zero except the j^{th} which is 1. Then for a lattice $\Gamma = \mathbb{Z}^{h+1} + \Omega\mathbb{Z}^{h+1}$, a fundamental domain for $\mathbb{C}^{h+1}/\Gamma$ may taken as

$$\mathscr{F} = \left\{\sum_{j=1}^{h+1}\alpha_j e^{(j)} + \sum_{j=1}^{h+1}\beta_j\Omega\cdot e^{(j)}\,\middle|\,\alpha_j, \beta_j \in \mathbb{R}; -\frac{1}{2} < \alpha_j, \beta_j < \frac{1}{2}\right\}.$$

For any set $U \subset \mathbb{C}^{h+1}$, we shall denote $[U] = U \cap \mathscr{F}$.

In addition, we will have to integrate terms such as $\operatorname{Im}(x-y)$ over M_t with respect to y. This, of course, requires that we specify the path of integration. To do this, we cut the surface along the cycles $\{A_j, B_j\}_{1\le j\le h}$, $B_{h+1,h+1}$, and the *waist* of the annulus $\{|z_{a,b}| = |t|^{1/2}\}$. This specifies the path, and we denote this choice by $[M_t]$.

We begin by recalling Proposition 4.1:

Lemma 7.3. *Define the constant* $c = \lim_{t\to 0}\left[\operatorname{Im}\Omega_{h+1,h+1}(t) + \frac{1}{2\pi}\log|\tau|\right]$. *Then,*

a) $$(\operatorname{Im}\Omega)^{-1}(t) = \begin{pmatrix} (\operatorname{Im}\Omega)^{-1}_{ij} & (*) \\ (*) & \dfrac{1}{-\frac{1}{2\pi}\log|\tau| + c} \end{pmatrix} + O(1/\log|\tau|),$$

where $\lim_{t\to 0}\log|\tau|\, O(1/\log|\zeta|)$ *is a finite matrix, and*

$$(*)_j = -\sum_{k=1}^{h}(\operatorname{Im}\Omega)^{-1}_{jk}\operatorname{Im}(b-a)_k\left(-\frac{1}{2\pi}\log|\tau| + c\right)^{-1};$$

b) $$(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}(t) = \frac{1}{-\frac{1}{2\pi}\log|\tau| + c} + \frac{{}^t\operatorname{Im}(b-a)\,(\operatorname{Im}\Omega)^{-1}\operatorname{Im}(b-a)}{\left(-\frac{1}{2\pi}\log|\tau| + c\right)^2} + O(1/(\log|\tau|)^3),$$

Proof. Both parts (a) and (b) are easily obtained from Proposition 4.1.

Lemma 7.4. *Up to* $O(1/\log|\tau|)$ *terms, we have*

a) $$v_t^h(Z) \sim v^h(Z) + \frac{\sqrt{-1}}{2}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}(t)\cdot h v^{h-1}\wedge dZ_{h+1}\wedge d\bar{Z}_{h+1};$$

b) $$\mu_t(z) \sim \frac{h}{h+1}\mu(z) + \frac{\sqrt{-1}}{2}\frac{1}{h+1}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}(t)\omega_{h+1}\wedge\bar{\omega}_{h+1}(z,t).$$

Moreover, in part (b) *there exist constants* C_1, C_2, C_3 *such that for* z *the local coordinate around the punctures* a *or* b,

$$|O(1/\log|\tau|)| \le \left(C_1 + \frac{C_2}{|z|} + C_3\frac{|\tau|}{|z|^2}\right)\frac{|dz|^2}{-\log|\tau|},$$

and for part (a), $\lim_{\tau\to 0}\log|\tau|\,O(1/\log|\tau|)$ *contains no* $dZ_{h+1}\wedge d\bar{Z}_{h+1}$ *factor.*

Proof. The lemma follows from Proposition 4.1 and the bounds obtained by expanding the abelian differentials (see Appendix A).

We also need an expression for the constant term in the expansion of the period matrix[2] :

Lemma 7.5. *Let* c *be defined as in Lemma 7.3. Then*

$$\pi(c - {}^t\operatorname{Im}(b-a)\,(\operatorname{Im}\Omega)^{-1}\operatorname{Im}(b-a)) = \log G(a,b).$$

Proof. Consider $\int_{B_{h+1}}\omega_{h+1}(\cdot,t) = \int_y^x \omega_{h+1}(\cdot,t)$ for x near b and y near a with local coordinates $z = t^{1/2}$. By the estimates in Appendix A, we need only look at the

[2] The author thanks John Fay for suggesting this lemma

zero order term. From (4.6),

$$\operatorname{Im}\Omega_{h+1,h+1}(t)=\operatorname{Im}\int_y^x \omega_{h+1}(\cdot,t)\sim\operatorname{Im}\frac{1}{2\pi i}\log\frac{E(x,b)E(y,a)}{E(x,a)E(y,b)}$$

$$\sim-\frac{1}{2\pi}\log|t|+\frac{1}{\pi}\log|E(a,b)|$$

(see also [37]). The lemma now follows from (2.12) and (2.16).

We now continue with the proof of Theorem 7.2. We separate the terms into

$$\log\|\vartheta_t\|(Z+x-y-\delta(t))=\{\text{norm}\}+\{\text{theta}\},$$

where,

$$\begin{aligned}\{\text{norm}\}&=-\pi\,{}^t\operatorname{Im}(Z+x-y-\delta(t))\,(\operatorname{Im}\Omega)^{-1}(t)\operatorname{Im}(Z+x-y-\delta(t))\\ \{\text{theta}\}&=\log|\vartheta_t|(Z+x-y-\delta(t)).\end{aligned}$$

We must integrate the above with respect to

$$\frac{1}{(h+1)!}\int_{\Theta_{\delta(t)}} v_t^h(Z)-\frac{1}{(h+1)!}\int_{M_t}\mu_t(y)\int_{\Theta_{\delta(t)}} v_t^h(Z)$$

Expanding {norm}, we see that for terms not involving y, the $\mu_t(y)$ integral is unity, and such terms will simply cancel out. Now the second terms in parts (a) and (b) of Lemma 7.4 vanish pointwise, but we keep them for the reason stated after the proof of Lemma 6.9; namely, the integrands will have log singularities as the points near the punctures, and when integrated against the second terms they will contribute to order $\log|t|$.

Lemma 7.6. *Fix the base point z_0 for the Abel map away from the punctures. Then for $1\le j\le h$, we have the following expansions (up to $o(1)$ terms, unless otherwise specified):*

a)
$$\int_{[\Theta_{\delta(t)}]} v_t^h\operatorname{Im}Z_{h+1}\sim\frac{1}{2\pi}\int_{[J(M)]} v^h\log\left|\frac{\vartheta(Z+\frac12(b-a))}{\vartheta(Z-\frac12(b-a))}\right|$$
$$+\frac{h}{2\pi}\int_{[\Theta]} v^{h-1}\log\left|\frac{E(z_0,b)\vartheta(Z+\frac12(b-a))}{E(z_0,a)\vartheta(Z-\frac12(b-a))}\right|,$$

b)
$$\int_{[M_t]}\mu_t(y)\operatorname{Im}(x-y)_{h+1}\sim\frac{1}{2\pi}\log\left|\frac{E(x,a)}{E(x,b)}\right|+\frac{h}{2\pi(h+1)}$$
$$\times\int_{[M]}\mu(y)\log\left|\frac{E(y,a)}{E(y,b)}\right|,$$

c)
$$\int_{[M_t]} \mu_t(y)\,\mathrm{Im}(x-y)_{h+1}^2$$
$$= -\frac{1}{12(h+1)}\frac{1}{(2\pi)^3}(\mathrm{Im}\,\Omega)^{-1}_{h+1,h+1}(\log|t|)^3$$
$$+\frac{1}{2(h+1)}\frac{1}{(2\pi)^3}(\mathrm{Im}\,\Omega)^{-1}_{h+1,h+1}(\log|t|)^2\log|E(a,b)|+O(1),$$

d)
$$\int_{[\Theta_{\delta(t)}]} v_t^h\,\mathrm{Im}\,Z_j \sim \int_{[J(M)]} v^h\,\mathrm{Im}\,Z_j + h\int_{[\Theta]} v^{h-1}\,\mathrm{Im}\,Z_j,$$

e)
$$\int_{[M_t]} \mu_t(y)\,\mathrm{Im}(x-y)_j \sim \frac{h}{h+1}\int_{[M]} \mu(y)\,\mathrm{Im}(x-y)_j$$
$$+\frac{1}{h+1}\,\mathrm{Im}\left(x-\frac{1}{2}(a+b)\right)_j.$$

Proof. Consider, for example, part (a). By Lemma 7.4, we have,

$$\frac{1}{(h+1)!}\int_{[\Theta_{\delta(t)}]} v_t^h\,\mathrm{Im}\,Z_{h+1}$$
$$=\frac{1}{(h+1)!}\int_{[\Theta_{\delta(t)}]} v^h\,\mathrm{Im}\,Z_{h+1}+\frac{\sqrt{-1}}{2}\frac{h}{(h+1)!}(\mathrm{Im}\,\Omega)^{-1}_{h+1,h+1}$$
$$\times\int_{[\Theta_{\delta(t)}]} v^{h-1}\wedge dZ_{h+1}\wedge d\bar{Z}_{h+1}\,\mathrm{Im}\,Z_{h+1}+o(1).$$

By (4.3), the first term on the right-hand side is continuous, and

$$\to\frac{1}{2\pi(h+1)}\frac{1}{h!}\int_{[J(M)]} v^h\log\left|\frac{\vartheta(Z+\frac{1}{2}(b-a))}{\vartheta(Z-\frac{1}{2}(b-a))}\right|.$$

For the second term, we realize $\Theta_{\delta(t)}$ as a translate of W_h. As in Sect. 2, we define a map

$$\phi^h:(p_1,\dots,p_h)\in M_t^h\mapsto p_1+\dots+p_h-\Delta(t)-\delta(t)\in\Theta_{\delta(t)}.$$

ϕ^h has degree $h!$ (generically), so

$$\int_{\Theta_{\delta(t)}} v^{h-1}\wedge dZ_{h+1}\wedge d\bar{Z}_{h+1}\,\mathrm{Im}\,Z_{h+1}$$
$$=\frac{1}{h!}\sum_{j,k=1}^{h}\int_{M_t^h}(\phi^h)^*v^{h-1}\wedge\omega_{h+1}(p_j,t)\wedge\bar{\omega}_{h+1}(\bar{p}_k,\bar{t})\,\mathrm{Im}\,Z_{h+1}.$$

If $j \neq k$, we will have an $O(1)$ term which will be annihilated by $(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}$, so the above is equal (up to a finite term) to

$$\frac{1}{(h-1)!}\int\limits_{M^{h-1}}(\phi^{h-1})^{*}v^{h-1}\int\limits_{[M_t]}\omega_{h+1}(p,t)\wedge\bar{\omega}_{h+1}(\bar{p},\bar{t})\operatorname{Im}\int\limits_{z_0}^{p}\omega_{h+1}$$
$$+\frac{1}{(h-1)!}\int\limits_{M^{h-1}}(\phi^{h-1})^{*}v^{h-1}\int\limits_{[M_t]}\omega_{h+1}(p,t)\wedge\bar{\omega}_{h+1}(\bar{p},\bar{t})$$
$$\times\left\{\operatorname{Im}\sum_{j=1}^{h-1}\int\limits_{z_0}^{p_j}\omega_{h+1}(\cdot,t)-(k^{z_0}(t)+\delta(t))_{h+1}\right\}.$$

The outside integral in the first line is just $h!$ (see Sect. 2). In the second line, we have from (2.2),

$$\int\limits_{M_t}\omega_{h+1}\wedge\bar{\omega}_{h+1}=-2\sqrt{-1}\operatorname{Im}\Omega_{h+1,h+1}, \tag{7.7}$$

so we are left with

$$=h!\int\limits_{[M_t]}\omega_{h+1}(p,t)\wedge\bar{\omega}_{h+1}(\bar{p},\bar{t})\operatorname{Im}\int\limits_{z_0}^{p}\omega_{h+1}(\cdot,t)-2\sqrt{-1}\operatorname{Im}\Omega_{h+1,h+1}\int\limits_{[W_{h-1}]}v^{h-1}$$
$$\times\operatorname{Im}\left\{\sum_{j=1}^{h-1}\int\limits_{z_0}^{p_j}\omega_{h+1}(\cdot,t)-(k^{z_0}(t)+\delta(t))_{h+1}\right\}. \tag{7.8}$$

In order to evaluate the second line, define

$$\tilde{Z}(t)=\sum_{j=1}^{h-1}\int\limits_{z_0}^{p_j}\vec{\omega}(\cdot,t)-(k^{z_0}(t)+\delta(t)).$$

Then $\tilde{Z}(t)=p_1+\dots+p_{h-1}+z_0-(\Delta(t)+\delta(t))$, so $\tilde{Z}(t)\in\Theta_{\delta(t)}$. Now by Proposition 4.7, we have for $1\le j\le h$,

$$\lim_{t\to0}\tilde{Z}_j(t)=p_1+\dots+p_{h-1}+z_0-\tfrac{1}{2}(a+b)-\Delta.$$

Using (4.3),

$$\lim_{t\to0}\operatorname{Im}\tilde{Z}_{h+1}(t)=-\frac{1}{2\pi}\log\left|\frac{\vartheta(Z+z_0-b)}{\vartheta(Z+z_0-a)}\right|,$$

where $Z=p_1+\dots+p_{h-1}-\Delta$. Therefore, the second line in (7.8) is asymptotic to

$$\sim\frac{\sqrt{-1}}{\pi}\operatorname{Im}\Omega_{h+1,h+1}\int\limits_{[\Theta]}v^{h-1}\log\left|\frac{\vartheta(Z+z_0-b)}{\vartheta(Z+z_0-a)}\right|.$$

Now we use (4.6) and the estimates from Appendix A to show that, up to a finite term, the top line of (7.8)

$$\sim \frac{1}{2\pi} h! \int\limits_{|t|^{1/2}}^{1} \frac{dr}{r} (-2\sqrt{-1}) \frac{1}{2\pi} \left\{ \log r - \log|E(a,b)| + \log \left|\frac{E(z_0,b)}{E(z_0,a)}\right| \right.$$
$$\left. - \log r + \log|E(a,b)| + \log \left|\frac{E(z_0,b)}{E(z_0,a)}\right| \right\}$$
$$= \frac{\sqrt{-1}}{\pi} h! \frac{1}{2\pi} \log|t| \log \left|\frac{E(z_0,b)}{E(z_0,a)}\right| .$$

Part (a) of the proposition now follows by multiplying by $(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}$ and using the expansion of Lemma 7.3. The proofs of the other parts of the lemma are similar.

We now notice that the only terms in the norm which survive as $t \to 0$ are

$$\{\text{norm}\} \sim -2\pi \sum_{j,k=1}^{h} (\operatorname{Im}\Omega)^{-1}_{j,k}(t) \operatorname{Im} Z_j \operatorname{Im}(x-y)_k$$
$$- \pi \sum_{j,k=1}^{h} (\operatorname{Im}\Omega)^{-1}_{j,k}(t) \operatorname{Im}(x-y)_j \operatorname{Im}(x-y)_k$$
$$- 2\pi \sum_{j=1}^{h} (\operatorname{Im}\Omega)^{-1}_{j,h+1}(t) \operatorname{Im}(x-y)_j \operatorname{Im}(x-y)_{h+1}$$
$$- \pi (\operatorname{Im}\Omega)^{-1}_{h+1,h+1}(t) (\operatorname{Im} x - y)^2_{h+1} + \pi \operatorname{Im}(x-y)_{h+1} .$$

The contributions from these terms may be readily evaluated by applying Lemma 7.6. We omit the computations.

We now turn to the theta terms. Referring to the proof of Lemma 7.6 and Proposition 4.5, we have

$$\frac{1}{(h+1)!} \int\limits_{[\Theta_{\delta(t)}]} v_t^h \log|\vartheta_t| (Z + x - y - \delta(t))$$
$$\sim + \frac{1}{h+1} \frac{1}{h!} \int\limits_{[J(M)]} v^h \log \left| \frac{\vartheta(Z + x - \frac{1}{2}(a+b))\vartheta(Z - y + \frac{1}{2}(a+b))}{\vartheta(Z + \frac{1}{2}(b-a))} \right|$$
$$+ \frac{1}{h+1} \log \left|\frac{E(x,y)E(a,b)}{E(x,a)E(y,b)}\right| + \frac{\sqrt{-1}}{2} \frac{h}{(h+1)!} (\operatorname{Im}\Omega)^{-1}_{h+1,h+1}$$
$$\times \frac{1}{h!} \sum_{j=1}^{h} \int\limits_{[M]^{h-1}} (\phi^{h-1})^* v^{h-1} \int\limits_{[M_t]} \omega_{h+1}(p,t) \wedge \bar{\omega}_{h+1}(\bar{p},\bar{t})$$
$$\times \{\log|\vartheta| (Z_0 + p - \tfrac{1}{2}(a+b)) + \log|\vartheta| (Z_0 - y + \tfrac{1}{2}(a+b))$$
$$- \log|\vartheta| (Z_0 + \tfrac{1}{2}(b-a)) + \log|E(p,y)| - \log|E(p,a)|$$
$$+ \log|E(a,b)| - \log|E(y,b)|\} .$$

Here, we have expressed the expansion in Proposition 4.5 in terms of $(p\,y)$ instead of (x, y). We therefore have set $Z_0 = p_1 + \cdots + p_{h-1} + x - \Delta(t) - \delta(t)$, and from Proposition 4.7, it is seen that $Z_0 \to \tilde{Z} + x - \frac{1}{2}(a+b)$ as $t \to 0$, where $\tilde{Z} \in \Theta$.

We have to evaluate the above expression for the integral of theta and subtract from it its integral with respect to $\mu_t(y)$. As before, the terms independent of y will cancel, so we may ignore them. By (7.7), the terms dependent on y are

$$\begin{aligned}
&\frac{1}{h+1}\frac{1}{h!}\int\limits_{[J(M)]} v^h \log|\vartheta|\left(Z - y + \tfrac{1}{2}(a+b)\right)\\
&+\frac{1}{h+1}\log|E(x,y)| - \log|E(y,b)|\\
&+\frac{\sqrt{-1}}{2}\frac{h}{h+1}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}\int\limits_{[M_t]}\omega_{h+1}(p,t)\\
&\wedge\bar{\omega}_{h+1}(\bar{p},\bar{t})\log|E(p,y)|\\
&+\frac{h}{h+1}\frac{1}{h!}\int\limits_{[W_{h-1}]} v^{h-1}\log|\vartheta|\left(Z_0 - y + \tfrac{1}{2}(a+b)\right).
\end{aligned}\tag{7.10}$$

As $t \to 0$,

$$\begin{aligned}
\{7.10\} \sim\ &\frac{1}{h+1}\frac{1}{h!}\int\limits_{[J(M)]} v^h\log|\vartheta|\left(Z - y + \tfrac{1}{2}(a+b)\right)\\
&+\frac{1}{h+1}\log|E(x,y)| - \log|E(y,b)|\\
&+\frac{h}{2(h+1)}\log|E(y,a)E(y,b)|\\
&+\frac{h}{h+1}\frac{1}{h!}\int\limits_{[\Theta]} v^{h-1}\log|\vartheta|(Z + x - y).
\end{aligned}\tag{7.11}$$

To (7.11), we have to add

$$\begin{aligned}
&-\int\limits_{[M_t]}\mu_t(y)\frac{1}{(h+1)!}\int\limits_{[\Theta_{\delta(t)}]} v_t^h\log|\vartheta_t|(Z + x - y - \delta(t))\\
&\sim -\frac{h}{h+1}\int\limits_{[M]}\mu(y)\ \{7.10\}\\
&\quad-\frac{\sqrt{-1}}{2}\frac{1}{h+1}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}\int\limits_{[M_t]}\omega_{h+1}(y,t)\\
&\wedge\bar{\omega}_{h+1}(\bar{y},\bar{t})\ \{7.10\},
\end{aligned}\tag{7.12}$$

by Lemma 7.4. The second term may be written

$$
\begin{aligned}
&-\frac{1}{2(h+1)}\left\{\frac{1}{h+1}\frac{1}{h!}\int_{[J(M)]} v^h \log|\vartheta|\left(Z+\tfrac{1}{2}(b-a)\right)|\vartheta|\left(Z-\tfrac{1}{2}(b-a)\right)\right.\\
&+\frac{1}{h+1}\log|E(x,a)E(x,b)|\\
&\left.+\frac{h}{h+1}\frac{1}{h!}\int_{[\Theta]} v^{h-1}\log|\vartheta|(Z+x-a)|\vartheta|(Z+x-b)\right\}\\
&+\frac{1}{h+1}\frac{1}{2\pi}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}\int_{|t|^{1/2}}^{1}\frac{dr}{r}\{\log|E(a,b)|+\log r\}\\
&-\frac{h}{(h+1)^2}\frac{1}{(2\pi)^4}\left((\operatorname{Im}\Omega)^{-1}_{h+1,h+1}\right)^2\int_{|t|^{1/2}<|z|<1}\frac{|dz|^2}{|z|^2}\int_{|t|^{1/2}<|w|<1}\frac{|dw|^2}{|w|^2}\\
&\times\{2\log|z-w|+2\log|E(a,b)|\}\,.
\end{aligned}
$$

At this point, we need

Lemma 7.13 *For real* $\varepsilon>0$,

$$
\frac{1}{(2\pi)^2}\int_{\varepsilon<|z|<1}\frac{|dz|^2}{|z|^2}\int_{\varepsilon<|w|<1}\frac{|dw|^2}{|w|^2}\log|z-w|=\frac{1}{3}(\log\varepsilon)^3+O(\log\varepsilon)\,.
$$

Proof. The proof follows by dividing the w-integral into domains $|w|<|z|$ and $|w|>|z|$ and expanding the logarithm. We omit the details.

By the lemma above and Lemma 7.3, we have

$$
\begin{aligned}
\{7.12\}=&-\frac{h}{h+1}\int_{[M]}\mu(y)\{7.11\}\\
&-\frac{1}{8(h+1)}\frac{1}{2\pi}(\operatorname{Im}\Omega)^{-1}_{h+1,h+1}(\log|t|^2)\\
&-\frac{h}{12(h+1)^2}\frac{1}{(2\pi)^2}\left((\operatorname{Im}\Omega)^{-1}_{h+1,h+1}\right)^2(\log|t|)^3\\
&+\frac{1}{2(h+1)^2}\log|E(a,b)|-\frac{1}{2(h+1)^2}\log|E(x,a)E(x,b)|\\
&-\frac{1}{2(h+1)^2}\frac{1}{h!}\int_{[J(M)]}v^h\log|\vartheta|\left(Z+\tfrac{1}{2}(b-a)\right)|\vartheta|\left(Z-\tfrac{1}{2}(b-a)\right)\\
&-\frac{h}{2(h+1)^2}\frac{1}{h!}\int_{[\Theta]}v^{h-1}\log|\vartheta|(Z+x-a)|\vartheta|(Z+x-b)\,.
\end{aligned}
$$

Theorem (7.2) now follows by adding the contributions from (7.9) to (7.11) and the above expression: all of the prime forms are replaced via (2.2) by F's, which are in turn related to the Green's function by (2.16). Through (2.6) and the expression (5.2) (and Lemma 7.5), we obtain Green's functions from $\log|\vartheta|$'s. The procedure requires an enormous amount of tedious and uninteresting algebra which we shall omit.

8. Asymptotics of the Faltings Invariant

As mentioned previously, the Faltings invariant, $\delta(M)$, was introduced in [14], and there it was indicated that the singularity of δ at the stable curves with separating nodes should be $\sim \sqrt{-\log|t|}$, but this is erroneous. In fact, δ should be regarded as a Weil function on moduli up to a loglog term coming from the period matrix ([23], p. 144), and hence the singularity should be $\sim \log|t|$. Using the results of Sects. 6 and 7, we show that this is indeed the case and that the next order term is simply related to the δ's associated to the stable curves.

Proof (of Main Theorem, see Introduction). Consider first part (a). We shall use the defining equation (2.15) for the Arakelov metric to find its asymptotics. Interchanging the limits and using the result of Proposition 6.10, we have

$$\log g^t_{z\bar{z}} = 2\left(\frac{h_2}{h}\right)^2 \log|\tau| + \log g^{(1)}_{z\bar{z}} - 4\,\frac{h_2}{h}\log G_1(z,p) + o(1)\,, \tag{8.1}$$

for $z \in M_1 \cap M_t$.

For $\delta(M_t)$, we use the bosonization formula (5.1), and the asymptotics for the Green's function and the metric. We have to evaluate

$$\exp\delta(M) = \frac{\|\det\omega_i(p_j)\|^8}{\|\vartheta\|^8}\,\frac{\prod\limits_{j=1}^{h} G(p_j,y)^8}{\prod\limits_{j<k} G(p_j,p_k)^8}\,(\det\operatorname{Im}\Omega)^{-6}\,. \tag{8.2}$$

Choose h_1 of the p_i's and y to be in $M_1 \cap M_t$, and h_2 of the p_i's in $M_2 \cap M_t$. Then $|\det\omega_i(p_j)| \to |\det\omega_i^{(1)}(p_j)|\,|\det\omega_i^{(2)}(p_j)|$, and $\|\vartheta_t\|^2 \to \|\vartheta_1\|^2\,\|\vartheta_2\|^2$. By counting the number of Green's functions and metrics, we find

$$\exp\delta(M_t) \sim \frac{|\tau|^{-8h_1\left(\frac{h_2}{h}\right)^2}\,|\tau|^{-8h_2\left(\frac{h_1}{h}\right)^2}}{|\tau|^{8\left(\frac{h_2}{h}\right)^2\frac{h_1(h_1-1)}{2}}\,|\tau|^{8\left(\frac{h_1}{h}\right)^2\frac{h_2(h_2-1)}{2}}\,|\tau|^{-8\frac{h_1^2h_2^2}{h^2}}}\,,$$

so $\delta(M_t) = -4\,\dfrac{h_1h_2}{h}\log|\tau| + O(1)$, as in the proposition. If one carefully manipulates the $O(1)$ terms from Sect. 6, the splitting of δ is easily obtained.

The proof of part (b) is slightly more involved. The asymptotics of the metric are obtained from (2.15) and Proposition 7.2. For z away from the punctures,

$$\begin{aligned}\log g^t_{z\bar{z}} \sim\ & \frac{1}{6(h+1)^2}\log\tau + \log g_{z\bar{z}} + \frac{5}{3(h+1)^2}\log G(a,b)\\ & - \frac{2}{h+1}\log G(z,a)G(z,b)\,.\end{aligned} \tag{8.3}$$

Next, from Proposition (4.2), we have up to $O(|t|)$,

$$|\det \omega_i(p_j, t)| \sim \left| \det \begin{pmatrix} \omega_1(p_1) & \dots & \omega_h(p_1) & \frac{1}{2\pi i}\,\omega_{b-a}(p_1) \\ \vdots & \ddots & \vdots & \vdots \\ \omega_1(p_h) & \dots & \omega_h(p_h) & \frac{1}{2\pi i}\,\omega_{b-a}(p_h) \\ \omega_1(x) & \dots & \omega_h(x) & \frac{1}{2\pi i}\,\omega_{b-a}(x) \end{pmatrix} \right| .$$

The zero order term can be evaluated ([16], p. 24); again to $O(|t|)$,

$$|\det \omega_i(p_j, t)| = (2\pi)^{-1}\,|\det \omega_i(p_j)| \left| \frac{\vartheta(b-x-e)\vartheta(x-a-e)E(a,b)}{\vartheta(b-a-e)\vartheta(e)E(x,a)E(x,b)} \right| ,$$

where $e = p_1 + \dots + p_h - a - \Delta$. For the theta function, we write the argument as $Z(t) + x - y - \delta(t)$, where $Z(t)$ is the local section of $\Theta_{\delta(t)}$ defined by

$$Z(t) = p_1 + \dots + p_h - \Delta(t) + \delta(t) .$$

Splitting away the norm, we have

$$\begin{aligned} &\log \|\vartheta_t\|\,(Z(t) + x - y - \delta(t)) \\ &= -\pi{}^t\operatorname{Im}(Z(t) + x - y)\,(\operatorname{Im}\Omega)^{-1}(t)\operatorname{Im}(Z(t) + x - y) \\ &\quad + \pi \operatorname{Im}(Z(t) + x - y)_{h+1} - \frac{\pi}{4}\operatorname{Im}\Omega_{h+1,h+1}(t) \\ &\quad + \log|\vartheta|\,(Z(t) + x - y - \delta(t)) \\ &\sim -\frac{\pi}{4}\operatorname{Im}\Omega_{h+1,h+1}(t) - \pi{}^t\operatorname{Im}(Z_0 + x - y)\,(\operatorname{Im}\Omega)^{-1}\operatorname{Im}(Z_0 + x - y) \\ &\quad - \frac{1}{2}\log\left|\frac{\vartheta(Z_0 - \frac{1}{2}(b-a))}{\vartheta(Z_0 + \frac{1}{2}(b-a))}\right| + \frac{1}{2}\log\left|\frac{E(x,a)E(y,b)}{E(y,a)E(x,b)}\right| \\ &\quad + \log\left|\frac{\vartheta(Z_0 + x - \frac{1}{2}(a+b))\vartheta(Z_0 - y + \frac{1}{2}(a+b))}{\vartheta(Z_0 + \frac{1}{2}(b-a))}\,\frac{E(x,y)E(a,b)}{E(x,a)E(y,b)}\right| . \end{aligned}$$

Here, we have used (4.3), Proposition 4.5, and Lemma 7.3. Now, by Proposition 4.7,

$$Z_0 = p_1 + \dots + p_h - \tfrac{1}{2}(a+b) - \Delta = e - \tfrac{1}{2}(b-a) .$$

Using this expression, we may simplify the above. Combined with the result for the determinant, we have, in terms of the parameter τ (note also Lemma 7.5).

$$\begin{aligned} \log|\det \omega_i(p_j, t) - \log\|\vartheta_t\| \sim &-\tfrac{1}{8}\log|\tau| - \log 2\pi + \tfrac{1}{4}\log G(a,b) \\ &- \tfrac{1}{2}\log\|\vartheta\|\,(p_1 + \dots + p_h - a - \Delta) \\ &- \tfrac{1}{2}\log\|\vartheta\|\,(p_1 + \dots + p_h - b - \Delta) \\ &+ \log|\det \omega_i(p_j)| - \log|x-y| + O(|x-y|) . \end{aligned} \tag{8.4}$$

We now use Proposition 4.1, (8.3), and Proposition 7.2 to determine the asymptotics of the remaining factors. Combining these with (8.4), we see that those involving a and b each combine to contribute $\frac{1}{2}\delta(M)$. Since the choice of points was arbitrary, we may let $x \to y$ and note that the remaining terms vanish by (2.15). This completes the proof of the theorem.

Appendix A

In this appendix, we collect some elementary estimates for integrals of abelian differentials.

Estimates outside the annulus are trivial, since poles in the expansions for the differentials develop only at the node. We therefore concentrate on the region $\{z \mid |t|^{1/2} < |z| < \varepsilon\}$, for a local coordinate z ($= z_1$ or z_2 in Case I – z_a or z_b in Case II) and ε sufficiently small.

Proposition A.1. *Let $\omega(z, t)$ be a linear combination of the abelian differentials $\omega_j(z, t)$ for small $|t|$. Then there is a positive constant C such that for all smooth sections x, y of the degeneration,*

$$\left| \int_y^x \omega(\cdot, t) - \int_y^x \omega(\cdot, 0) \right| \leq -C|t|^{1/2} \log |t| \,.$$

Proof. Following reference [16], we expand the differential in terms of the normalizing coordinate $\mathscr{X}$ for $|\mathscr{X}|$ small:

$$\omega(z, t) = \sum_{n=0}^{\infty} a_n(t)\mathscr{X}^n d\mathscr{X} + b_n(t)\mathscr{X}^n d\mathscr{Y} \,,$$

where a_n, b_n are holomorphic near the origin. Choose ε sufficiently small such that $\sum_{n=0}^{\infty} a_n(t)\varepsilon^n$ and $\sum_{n=0}^{\infty} b_n(t)\varepsilon^n$ converge for $|t| < \varepsilon^2$. In terms of the local coordinate z, we have (see (3.1))

$$\omega(z, t) = \sum_{n=0}^{\infty} 2^{-(n+1)} a_n(t)\,(z + t/z)^n (1 - t/z^2) dz \pm 2^{-n} b_n(t)\,(z + t/z)^n \,\frac{dz}{z} \,, \quad \text{(A.2)}$$

the $\pm$ indicating on which half of the annulus z lies (note also that we have shifted the b_n index). Consider the second term in (A.2), the analogous estimate for the first term being similar. We estimate

$$|z^{n-1}((1 + t/z^2)^n - 1)| \leq \sum_{p=1}^{n} \binom{n}{p} |z|^{n-2p-1}|t|^p \quad \text{(A.3)}$$

for $n \geq 1$ (the case $n = 0$ can be treated separately). Assume for the moment that n is odd. Then

$$\begin{aligned} \int_{z^{-1}(|t|^{1/2})}^{z^{-1}(\varepsilon)} |dz|\,|z^{n-1}((1 + t/z^2)^n - 1)| &\leq \sum_{p=1}^{n} \binom{n}{p} |t|^p \,\frac{1}{n - 2p}\,(\varepsilon^{n-2p} - |t|^{\frac{n-2p}{2}}) \\ &\leq \text{const.}\, |t|^{1/2} 2^n \varepsilon^{n-1} \,, \end{aligned}$$

since we are assuming $|t| < \varepsilon^2$. If n is even, we have to consider the case where $p = n/2$:

$$\begin{aligned}\int_{z^{-1}(|t|^{1/2})}^{z^{-1}(\varepsilon)} |dz|\,|z^{n-1}((1+t/z^2)^n - 1)| &\leq \binom{n}{n/2} \int_{z^{-1}(|t|^{1/2})}^{z^{-1}\varepsilon} |dz|\,\frac{|t|^{n/2}}{|z|} + \sum_{p \neq n/2} \cdots \\ &\leq -\frac{1}{2}\binom{n}{n/2} |t|^{1/2} \log|t| \varepsilon^{n-1} + \sum_{p\neq n/2} \cdots \\ &\leq -\text{const.}\,|t|^{1/2} \log|t| 2^n \varepsilon^{n-1}\,.\end{aligned}$$

The proposition follows from these bounds and the convergence of $\sum a_n(t)\varepsilon^n$ and $\sum b_n(t)\varepsilon^n$.

Proposition A.4. *Let ω_1, ω_2 be differentials as in Proposition A.1, and D a domain in M_1, M_2, or M. Then there is a positive constant C such that for all small $|t|$,*

$$\left|\int_{M_t \cap D} \omega_1 \wedge \bar{\omega}_2(z, t) - \int_{M_t \cap D} \omega_1 \wedge \bar{\omega}_2(z, 0)\right| \leq \begin{cases} -C|t| \log|t|, & \textit{for Case I}, \\ -C|t|^{1/2} \log|t|, & \textit{for Case II}. \end{cases}$$

Proof. We split the form as follows:

$$\begin{aligned}\omega_1 \wedge \bar{\omega}_2(z, t) - \omega_1 \wedge \bar{\omega}_2(z, 0) = {} & [\omega_1(z, t) - \omega_1(z, 0)] \wedge \bar{\omega}_2(\bar{z}, t) \\ & + \omega_1(z, 0) \wedge [\bar{\omega}_2(\bar{z}, t) - \bar{\omega}_2(\bar{z}, t)]\,.\end{aligned}$$

Concentrating on the second term of (A.2) combined with (A.3), we must estimate

$$\begin{aligned}&\int_{|t|^{1/2}<|z|<\varepsilon} |dz|^2\,|z^{n-1}(1+t/z^2)^n z^{m-1}((1+t/z^2)^m - 1)| \\ &\leq 2\pi \sum_{p=0}^{n} \sum_{q=1}^{m} \binom{n}{p}\binom{m}{q} \int_{|t|^{1/2}}^{\varepsilon} dr\, r^{m+n-2p-2q-1} |t|^{p+q} \\ &\leq -\text{const.}\,|t|^{\frac{n+1}{2}} \log|t| 2^{m+n} \varepsilon^{m-1} + \text{const.}\,|t| 2^{m+n} \varepsilon^{m+n-2}\,.\end{aligned}$$

If $n \geq 1$ we have the desired estimate by the convergence of $\sum a_n(t)\varepsilon^n$ and $\sum b_n(t)\varepsilon^n$. One can also show that the first term in (A.2) combined with (A.3) gives the desired estimate (essentially the case $n = 1$ above). We therefore have a bound

$$\left|\int_{M_t \cap D} \omega_1 \wedge \bar{\omega}_2(z, t) - \omega_1 \wedge \bar{\omega}_2(z, 0)\right| \leq -C' b_0(t)\,|t|^{1/2} \log|t| - C''|t| \log|t|$$

for some positive constants C', C''. Now from the expansions of the differentials (3.2) and (4.2), it follows that $b_0(0) = 0$ for Case I and $\neq 0$ for the expansion of ω_{h+1} in Case II. The proposition follows immediately.

Appendix B

We would like to present an application of the previous results in the special case of a surface of genus two. The boundary of the moduli space consists of singular curves, the components of which are elliptic. For elliptic curves, explicit formulae for the Arakelov-Green's function and $\delta(M)$ are known (for genus two, see references [6, 8]). Set $h_1 = h_2 = 1$; then for a torus of modulus, τ $M = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ (see [14]; note the difference in the definition of $\|\cdot\|$),

$$\begin{aligned}
&\text{a) } \log G(z, w) = \log G(z-w, 0) = \log \|\vartheta\|\,(z - w + \Delta) - \log|\eta|\,(\tau)\,,\\
&\text{b) } g_{z\bar{z}} = 4\pi^2 |\eta(\tau)|^4\,, \qquad\qquad \text{(B.1)}\\
&\text{c) } \delta(M) = -24 \log \|\eta\|\,(\tau) - 8\log 2\pi\,,
\end{aligned}$$

where $\Delta = \dfrac{1+\tau}{2}$, η is the Dedekind eta function, and $\|\eta\|\,(\tau) = (\operatorname{Im}\tau)^{1/4}\,|\eta(\tau)|$. We would like to consider the bosonic ($d = 26$) string integrand (for a review, see [12]). In terms of $\delta(M)$, this can be expressed ($h > 1$) [32],

$$|\phi_1 \wedge \cdots \wedge \phi_{3h-3}|^2 \exp(\tfrac{9}{4}\,\delta(M)) \times \frac{\prod_{i\neq j} G(p_i, p_j)}{\|\det \phi_i(p_j)\|^2} (\det \operatorname{Im}\Omega)^{1/2}\, \|\vartheta\|^2 \left(\sum_{j=1}^{3h-3} I(p_j) - \Delta\right).$$

Here, the ϕ_i's are the quadratic differentials, and the p_j's are $3h-3$ arbitrarily chosen points. In the genus two case, we may take

$$\phi_1 = \omega_1^2\,, \qquad \phi_2 = \omega_1\omega_2\,, \qquad \phi_3 = \omega_2^2\,.$$

Then one can show that for the measure,

$$|\phi_1 \wedge \phi_2 \wedge \phi_3|^2 = \text{const.}\, |d\Omega_{11} \wedge d\Omega_{12} \wedge d\Omega_{22}|^2\,.$$

We fix the constant of proportionality equal to one. Suppose we choose p_1, $p_3 \in M_1$, $p_2 \in M_2$. Then by (B.1) and Propositions 3.2, 6.10, and 8.1,

$$\begin{aligned}
\prod_{i\neq j} G(p_i, p_j) &\sim |t|^{-1/2} (2\pi)^{-1} G_1(p_1, p_3)^2 G_2(p_2, 0)^2\, |\eta(\tau_1)|^{-1}\, |\eta(\tau_2)|^{-1}\\
\|\det \phi_i(p_j)\|^2 &\sim |t|^{-} (2\pi)^{-18} \frac{G_1(p_1, 0)^4 G_1(p_3, 0)^4 G_2(p_2, 0)^4}{|\eta(\tau_1)|^{22}\, |\eta(\tau_2)|^{14}}\\
&\quad \times |\omega^{(1)}(p_1, 0) - \omega^{(1)}(p_3, 0)|^2\\
\exp \tfrac{9}{4}\,\delta &\sim |t|^{-9/2} (2\pi)^{-45} (\operatorname{Im}\tau_1)^{-27/2} (\operatorname{Im}\tau_2)^{-27/2}\, |\eta(\tau_1)|^{-63}\, |\eta(\tau_2)|^{-63}\,,
\end{aligned}$$

and the remaining factors are continuous and split as $t \to 0$. Thus, we have

$$\begin{aligned}
&\exp(\tfrac{9}{4}\,\delta)\, \frac{\prod_{i\neq j} G(p_i, p_j)}{\|\det \phi_i(p_j)\|^2} (\det \operatorname{Im}\Omega)^{1/2}\, \|\vartheta\|^2 \left(\sum_{j=1}^{3} I(p_j) - \Delta\right)\\
&\to |t|^{-4} (2\pi)^{-28} (\operatorname{Im}\tau_1)^{-13} (\operatorname{Im}\tau_2)^{-13}\, |\eta(\tau_1)|^{-42}\, |\eta(\tau_2)|^{-50}\\
&\quad \times \frac{G_1(p_1, p_3)^2}{G_1(p_1, 0)^4 G_1(p_3, 0)^4}\, \frac{\|\vartheta_1\|^2\,(p_1 + p_2 - \Delta_1)}{|\omega_1(p_1, 0) - \omega_1(p_3, 0)|^2}\, \frac{\|\vartheta_2\|^2 (p_3 - \Delta_2)}{G_2(p_2, 0)^2}\,.
\end{aligned}$$

Now we specialize to the case where $p_1 = 1/2$, $p_3 = \tau_1/2$. We also use (B.1) to rewrite G_1, G_2, and the fact that [27]

$$\|\vartheta_1\|^4 (\tau_1/2)\, \|\vartheta_1\|^4 (1/2) = 2^4 |\eta(\tau_1)|^{12}\, |\vartheta_1(0)|^{-4}\,,$$

to obtain for the above expression

$$|t|^{-4}(2\pi)^{-28}(\operatorname{Im}\tau_1)^{-13}(\operatorname{Im}\tau_2)^{-13}\,|\eta(\tau_1)|^{-48}|\eta(\tau_2)|^{-48}\,\frac{2^{-4}|\vartheta_1(0)|^8}{|\omega_1(1/2,0)-\omega_1(\tau_1/2,0)|^2}\,.$$

Now for the canonical differentials of the second kind on a torus, we have ([16], p. 35),

$$\omega_1(1/2,0)-\omega_1(\tau_1/2,0)=\wp(1/2)-\wp(\tau_1/2)\,,$$

where $\wp$ is the Weierstrass function. Furthermore, at these particular points, we have ([30], p. 286),

$$|\wp(1/2)-\wp(\tau_1/2)||^2=\pi^4|\vartheta_1(0)|^8\,.$$

Putting these results together, we see that the two-loop string integrand factorizes as $\Omega_{12}\sim 2\pi it\to 0$,

$$\begin{aligned}\text{string integrand}\to{}&(2\pi)^{-28}\frac{|d\Omega_{12}|^2}{|\Omega_{12}|^4}(\operatorname{Im}\tau_1)(\operatorname{Im}\tau_2)\\&\times\frac{|d\tau_1|^2}{|\eta(\tau_1)|^{48}}(\operatorname{Im}\tau_1)^{-14}\frac{|d\tau_2|^2}{|\eta(\tau_2)|^{48}}(\operatorname{Im}\tau_2)^{-14}\\&\times\text{lower order terms}\,.\end{aligned}$$

We see that the middle line is the product of the two one-loop integrands [29], and the top line may be interpreted as the insertion of a tachyon. This result has previously been obtained by realizing the genus two integrand as [5, 25] (see also [3, 7, 9]),

$$|d\Omega_{11}\wedge d\Omega_{12}\wedge d\Omega_{22}|^2\,|\chi_{10}|^{-2}(\det\operatorname{Im}\Omega)^{-13}\,,$$

where χ_{10} is a modular form of weight 10 defined in terms of theta characteristics.

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Note. We would like to point out that the coefficient $4h_1h_2/h$ appearing in part (a) of the Main Theorem has been previously announced by Bost in lectures as the IAS and by Jorgenson in [21].

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