Commun. Math. Phys. 137, 519–531 (1991)

Communications in Mathematical Physics
© Springer-Verlag 1991

An Example of a Generalized Brownian Motion

Marek Bożejko[1] **and Roland Speicher**[2]

[1] Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2/4, PL-50-384 Wrocław, Poland
[2] Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, W-6900 Heidelberg, Federal Republic of Germany

Received July 1, 1990; in revised form October 12, 1990

Abstract. We present an example of a generalized Brownian motion. It is given by creation and annihilation operators on a "twisted" Fock space of $L^2(\mathbb{R})$. These operators fulfill (for a fixed $-1 \leqq \mu \leqq 1$) the relations $c(f)c^*(g) - \mu c^*(g)c(f) = \langle f, g\rangle 1$ ($f, g \in L^2(\mathbb{R})$). We show that the distribution of these operators with respect to the vacuum expectation is a generalized Gaussian distribution, in the sense that all moments can be calculated from the second moments with the help of a combinatorial formula. We also indicate that our Brownian motion is one component of an n-dimensional Brownian motion which is invariant under the quantum group $S_\nu U(n)$ of Woronowicz (with $\mu = \nu^2$).

1. Introduction

We will present a representation of the relations

$$c(f)c^*(g) - \mu c^*(g)c(f) = \langle f, g\rangle 1 \qquad (f, g \in L^2(\mathbb{R}))$$

for a fixed μ with $-1 \leqq \mu \leqq 1$ on a "twisted" Fock space (not to be confused with the twisted Fock space of Pusz and Woronowicz [PWo]). There are at least three reasons for studying these relations:

i) They provide an interpolation between the bosonic and fermionic relations. Independently from our work, Greenberg [Gre] proposed the same relations as a first (non-relativistic) field theory that allows small violations of the exclusion principle (i.e. of Fermi statistics) or of Bose statistics.
ii) They give an example of a generalized Brownian motion.
iii) They exhibit a relation with the twisted $S_\nu U(n)$ of Woronowicz [Wor 1, Wor 2]: the Brownian motion of ii) can be considered as one component of an n-dimensional Brownian motion which is $S_\nu U(n)$-invariant. This also shows how the twisted creation and annihilation operators of [PWo] (appearing in the second quantization procedure based upon the twisted $S_\nu U(n)$) arise in a central limit theorem.

Concerning point i) we should mention the work of Lindsay and Parthasarathy [LiP], who also construct interpolations (Bose-Fermi bridges) between the Fock representations of the bosonic and fermionic relations. But whereas they do this by introducing relations in the full Fock space and thus getting inevitably additional relations between $c(f)c(g)$ and $c(g)c(f)$, we will produce our representation by changing the scalar product in the full Fock space. One of our motivations for this form of interpolation was to include for $\mu=0$ the Fock representation of the Cuntz algebra O_∞. This cannot be obtained within the framework of [LiP].

The rest of the introduction is devoted to explaining the points ii) and iii) more explicitly.

A (generalized) Brownian motion is a structure which is characterized by non-commutative analogues of the properties of classical Brownian motion. In non-commutative probability theory such objects (or more general white noises) are used for modelling non-commutative heat baths (cf. [Küm 1, Maa 1, Spe 1]). Kümmerer and Prin [KPr] have shown that a generalized Brownian motion allows the development of a stochastic integration theory which ensures that such a Brownian motion can be coupled to other systems with the help of stochastic differential (Langevin) equations.

We now give a short review of classical Brownian motion and then translate this to the non-commutative frame. Let $(\Omega, \Sigma, P, (Z_t^1, \dots, Z_t^n)_{t\in\mathbb{R}})$ be an n-dimensional classical complex Brownian motion. For $I=[t_1, t_2)$, let $Z_I^i := Z_{t_2}^i - Z_{t_1}^i$ be the increment of the i^{th} coordinate of this process. Let $\mathfrak{R}$ be the ring generated by all semiclosed intervals I of the above form. Then the definition of Z_I^i extends to $I\in\mathfrak{R}$ such that the mapping $I\mapsto(Z_I^1, \dots, Z_I^n)$ is finitely additive. The distribution of $(Z_I^1, \dots, Z_I^n)$ is an n-dimensional Gaussian distribution with covariance matrix depending only on $\lambda(I)$, the Lebesgue-measure of I.

It is clear that $I\mapsto(Z_I^1, \dots, Z_I^n)$ is characterized by the requirements that $Z_{I_1\cup I_2}^i = Z_{I_1}^i + Z_{I_2}^i$ $(i=1, \dots, n;\ I_1\cap I_2=\emptyset)$ and that all increments are independent and have a stationary distribution given by the limit law of a central limit theorem. It should also be noted that this limit law (Gaussian distribution) is characterized by the fact that all moments can be calculated in a combinatorial easy way with the help of the second moments. We are only interested in moments of our process, i.e. on the collection of all $\int \hat{Z}_{I_1}^{k(1)}(\omega)\dots\hat{Z}_{I_r}^{k(r)}(\omega)dP(\omega)$, $r\in\mathbb{N}$, $I_j\in\mathfrak{R}$ and $k(j)\in\{1, \dots, n\}$, where $\hat{Z}$ stands for Z or its complex conjugate $\bar{Z}$. Thus we consider two processes to be equivalent if all their corresponding moments are the same.

In quantum probability we replace the random variables $Z_I^1, \dots, Z_I^n$ – which may also be considered as n pairs of random variables $(Z_I^1, \bar{Z}_I^1), \dots, (Z_I^n, \bar{Z}_I^n)$ – and the probability measure P by n pairs of operators $(c_I^1, c_I^{1*}), \dots, (c_I^n, c_I^{n*})$ with $c_{I_1\cup I_2}^i = c_{I_1}^i + c_{I_2}^i$ $(i=1, \dots, n;\ I_1\cap I_2=\emptyset)$ and a state ϱ on the $*$-algebra $\mathscr{C}$ generated by all these operators. Of course, c_I^{j*} denotes the adjoint of c_I^j. We call such a process $(\mathscr{C}, \varrho, ((c_I^1, c_I^{1*}), \dots, (c_I^n, c_I^{n*}))_{I\in\mathfrak{R}})$ a (n-dimensional) Brownian motion if its increments for disjoint time intervals are independent and if the distribution of $(c_I^1, c_I^{1*}), \dots, (c_I^n, c_I^{n*})$ with respect to ϱ (i.e. all moments $\varrho(\hat{c}_I^{k(1)}\dots\hat{c}_I^{k(r)})$ for all $r\in\mathbb{N}$ and $k(j)\in\{1, \dots, n\}$, where $\hat{c}$ stands for c or c^*) is given by the limit distribution (generalized Gaussian) of the corresponding central limit theorem. The problem of this definition is the meaning of "independence." Whereas in classical probability theory there is only one possible definition of "independence," the situation in non-commutative probability theory is not so simple. Independence of the operators $c_{I_1}^i$ and $c_{I_2}^j$ $(I_1, I_2\in\mathfrak{R},\ I_1\cap I_2=\emptyset)$ shall be understood as usual to be the independence of the *-algebras $\mathscr{C}_{I_1}=\langle c^i_{I_1}\,|\,i=1,\dots,n\rangle$ and $\mathscr{C}_{I_2}=\langle c^i_{I_2}\,|\,i=1,\dots,n\rangle$ with respect to the states $\varrho_{I_1}=\varrho/\mathscr{C}_{I_1}$ and $\varrho_{I_2}=\varrho/\mathscr{C}_{I_2}$. Here $\langle R\rangle$ denotes the *-algebra generated by all operators $r\in R$. Thus we have to define the independence of subalgebras $\mathscr{C}_k$ of an algebra $\mathscr{C}$ with respect to a state ϱ. Of course we are led by the classical situation and demand some form of factorization.

We will use the general characterization of independence which was given by Kümmerer [Küm 2]. On an algebraic level it can be posed as the factorizing of pyramidally ordered products, i.e.

$$\varrho(a_1\dots a_m b_m\dots b_1)=\varrho_{I_1}(a_1b_1)\dots\varrho_{I_m}(a_mb_m),$$

if $a_i, b_i\in\mathscr{C}_{I_i}$ and $I_1<I_2<\dots<I_m$, where $I_1<I_2$ means: for all $t_1\in I_1$ and $t_2\in I_2$ we have $t_1<t_2$. For other products no rule is prescribed. This factorizing of pyramidally ordered products ensures that the coupling of such a generalized Brownian motion to some other system reproduces the right "pyramidal" correlation functions (cf. "quantum regression theorem" in [AFL, Küm 1, Maa 1]).

Example. Let $a\in\mathscr{C}_{I_1}$, $b\in\mathscr{C}_{I_2}$ with $I_1<I_2$. Then $aabb$ and $abba$ are pyramidally ordered and we demand

$$\varrho(aabb)=\varrho(abba)=\varrho_{I_1}(aa)\varrho_{I_2}(bb),$$

but no formula for $\varrho(abab)$ is given.

So we have different possibilities for adding rules to enable us to calculate products which are not pyramidally ordered. These different rules lead to different forms of "independence," which then also give different classes of concrete Brownian motions. Until now three main forms of independence have been used, namely the commuting, the anti-commuting and the free independence, leading to Brownian motions which are given by creation and annihilation operators of the CCR-algebra, CAR-algebra, and Cuntz-algebra O_∞ [Cun, Eva], respectively (cf. [CuH, CoH, GvW, vWa, Heg, Avi, Voi 1, Voi 2, HuP, ApH, Sma, Spe 1]). For these three cases there also exist stochastic calculi [BSW, HuP, ApH, Maa 2, KSp, Spe 2] which yield stronger results (including Ito's formulas) than the general theory of [KPr].

We will now give our favourite axiomatic definition of "generalized Brownian motion." We should, however, mention that this definition puts the main emphasis on algebraic aspects. For a more C^*-algebraic version of this subject we refer to [Küm 2].

Definition. An *n-dimensional generalized Brownian motion* is a triple $(\mathscr{C},\varrho,((c^1_I,c^{1*}_I),\dots,(c^n_I,c^{n*}_I))_{I\in\mathfrak{R}})$ consisting of a *-algebra $\mathscr{C}$, a state ϱ on $\mathscr{C}$ and a finitely additive mapping $\mathfrak{R}\to\mathscr{C}^n$, $I\mapsto(c^1_I,\dots,c^n_I)$ such that

i) pyramidally ordered moments factorize (independence)
ii) the moments $\varrho(\hat{c}^{k(1)}_{I_1+t}\dots\hat{c}^{k(r)}_{I_r+t})$ are independent of $t\in\mathbb{R}$ for all $r\in\mathbb{N}$, $k(j)\in\{1,\dots,n\}$ and $I_j\in\mathfrak{R}$, where $\hat{c}$ stands for c or c^*, and $I+t:=\{s+t\,|\,s\in I\}$ (stationarity)
iii) we have for all $r\in\mathbb{N}$, $k(j)\in\{1,\dots,n\}$, $I\in\mathfrak{R}$

$$\varrho(\hat{c}^{k(1)}_I\dots\hat{c}^{k(r)}_I)=\begin{cases}0, & r \text{ odd},\\ \lambda(I)^{r/2}\varrho(\hat{c}^{k(1)}_{[0,1]}\dots\hat{c}^{k(r)}_{[0,1]}), & r \text{ even}\end{cases}$$

(Gaussianity of the distribution).

Note that the concept of Gaussianity makes no sense if one considers only the state ϱ on the *-algebra generated by the operators c_I^i for a fixed I. The concept only gets a meaning if one considers the whole process. To put it another way, the statement that a special measure on $\mathbb{R}$ is a generalized Gaussian one does not make any sense without specifying the corresponding form of independence.

In the following we will also use $(\mathscr{C}, \varrho, (c_I^1, \ldots, c_I^n)_{I\in\mathfrak{R}})$ instead of $(\mathscr{C}, \varrho, ((c_I^1, c_I^{1*}), \ldots, (c_I^n, c_I^{n*}))_{I\in\mathfrak{R}})$.

In Sect. 2 we construct operators $c^*(f)$ for $f\in L^2(\mathbb{R})$ on a twisted Fock space. Let us denote by χ_I the indicator function of $I\in\mathfrak{R}$. If we now put $n=1$ and $c_I^1=c^*(\chi_I)$ and take for ϱ the vacuum expectation state, then c_I^1 is a new example of a generalized Brownian motion, which interpolates between the bosonic, free and fermionic cases. This gives our point ii).

To discuss iii) we consider an n-dimensional Brownian motion $(\mathscr{C}, \varrho, (c_I)_{I\in\mathfrak{R}})$, with $c_I=(c_I^1, \ldots, c_I^n)$ and look for symmetries. First, we consider classical symmetries. Let $U=(u_{ij})_{i,j=1,\ldots,n}\in SU(n)$. Then define a new Brownian motion $d_I := Uc_I$ by $(i=1,\ldots,n;\ I\in\mathfrak{R})$,

$$d_I^i := \sum_{j=1}^n u_{ji}c_I^j, \quad \text{thus} \quad d_I^{i*} = \sum_{j=1}^n \bar{u}_{ji}c_I^{j*}.$$

If d_I is equivalent to c_I, i.e. if all moments of $(d_I^1, d_I^{1*}), \ldots, (d_I^n, d_I^{n*})$ and $(c_I^1, c_I^{1*}), \ldots, (c_I^n, c_I^{n*})$ with respect to ϱ are the same, then we say that the Brownian motion c_I is invariant under $SU(n)$. For example, this is true for a classical n-dimensional complex Brownian motion if the covariance matrix is a multiple of the unit matrix.

For a non-commutative Brownian motion we can also consider more general invariance properties, namely invariance under the action of quantum groups. Let us take the twisted $S_\nu U(n)$ of Woronowicz [Wor 1, Wor 2]. For convenience we will only consider the case $n=2$. Then $S_\nu U(2)$ is given by

$$u=\begin{pmatrix} \alpha & -\nu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2 \otimes A,$$

where α and γ fulfill the relations

$$\begin{matrix} \alpha^*\alpha+\gamma^*\gamma=1 \\ \alpha\alpha^*+\nu^2\gamma^*\gamma=1 \end{matrix} \qquad \begin{matrix} \gamma^*\gamma=\gamma\gamma^* \\ \alpha\gamma=\nu\gamma\alpha \\ \alpha\gamma^*=\nu\gamma^*\alpha \end{matrix} .$$

Given a 2-dimensional Brownian motion $(\mathscr{C}, \varrho, (c_I^1, c_I^2)_{I\in\mathfrak{R}})$ we define $d_I=uc_I$ by

$$\begin{matrix} d_I^1 := \alpha\otimes c_I^1+\gamma\otimes c_I^2 \\ d_I^2 := -\nu\gamma^*\otimes c_I^1+\alpha^*\otimes c_I^2 \end{matrix}, \quad \text{thus} \quad \begin{matrix} d_I^{1*} := \alpha^*\otimes c_I^{1*}+\gamma^*\otimes c_I^{2*} \\ d_I^{2*} := -\nu\gamma\otimes c_I^{1*}+\alpha\otimes c_I^{2*} \end{matrix}.$$

Thus $d_I^i\in A\otimes\mathscr{C}$. To enable us to compare d_I and c_I we identify c_I^i with $1\otimes c_I^i$ and so embed $\mathscr{C}$ in $A\otimes\mathscr{C}$. We denote the conditional expectation $A\otimes\mathscr{C}\to A$, $a\otimes c\mapsto a\varrho(c)$ by $1\otimes\varrho$. Now we can say that c_I and d_I are equivalent if all "moments" of the conditional expectation $1\otimes\varrho$ are the same. If this is the case we say that the given Brownian motion $c_I=(c_I^1, c_I^2)$ is invariant under $S_\nu U(2)$.

Of course, this invariance property is only a statement about the Gaussian distribution of the increments for each fixed $I\in\mathfrak{R}$. But it is also related to the corresponding Brownian motion by the fact that all increments have to be put together in such a way that the (quantum) symmetry is preserved by this procedure. This means there is a connection between the chosen form of independence and the possible symmetries occurring for the Gaussian distribution of the corresponding central limit theorem.

We now want to indicate that our 1-dimensional Brownian motion $c^*(\chi_I)$ may be considered as one component of a 2-dimensional $S_\nu U(2)$-invariant Brownian motion. Consider a 2-dimensional Brownian motion with $c_I^i = c_i^*(\chi_I)$ $(i=1,2)$, where the operators $c_1(f)$ and $c_2(g)$ fulfill the following relations ($f, g \in L^2(\mathbb{R})$):

$$c_1(f)c_2^*(g) = \nu c_2^*(g)c_1(f),$$

$$c_2(f)c_1(g) = \nu c_1(g)c_2(f),$$

$$c_1(f)c_1^*(g) = \langle f, g\rangle 1 + \nu^2 c_1^*(g)c_1(f) - (1-\nu^2)c_2^*(g)c_2(f),$$

$$c_2(f)c_2^*(g) = \langle f, g\rangle 1 + \nu^2 c_2^*(g)c_2(f).$$

Then (c_I^1, c_I^2) is $S_\nu U(2)$-invariant: indeed, the process is written down in such a way as to satisfy for fixed $f=g$ (with $\|f\|=1$) the relations of the creation and annihilation operators of [PWo], which are according to their construction $S_\nu U(2)$-invariant. We can also phrase this by saying that the above relations are, for fixed $f=g$, invariant under the action of $S_\nu U(2)$.

If we choose f, g with $fg=0$, then the above process reveals to us the appropriate form of independence which gives, via a central limit theorem, an $S_\nu U(2)$-invariant Gaussian distribution.

We see that the 1-dimensional Brownian motion constructed in this paper can be regarded as the second component $c_2^*(f)$ of the above 2-dimensional Brownian motion (with $\mu=\nu^2$).

A more complete treatment of the statements of this introduction will be given elsewhere [Spe 3].

2. Realization of the Brownian Motion

We want to define Brownian motions which are interpolations between fermionic, free and bosonic Brownian motion. Hence we consider for $\mathscr{H} = L^2(\mathbb{R})$ the relations

$$c(f)c^*(g) - \mu c^*(g)c(f) = \langle f, g\rangle 1 \qquad (f, g \in \mathscr{H}),$$

with $-1 \leq \mu \leq 1$. For $\mu = -1$ they correspond to the CAR-algebra, for $\mu = 0$ to the Cuntz-algebra O_∞ and for $\mu = +1$ to the CCR-algebra. We will give a representation of the relations which generalizes the Fock space representations of the CAR-, Cuntz-, and CCR-algebras.

The following considerations will be for an arbitrary complex separable Hilbert space $\mathscr{H}$, in the end we will specialize to $\mathscr{H} = L^2(\mathbb{R})$.

All our operators are defined on a dense subset $\mathscr{F}$ of the full Fock space $\bigoplus_{n\geq 0} \mathscr{H}^{\otimes n}$, where $\mathscr{H}^0 \cong \mathbb{C}$. By $\Omega = (1, 0, 0 \ldots)$ we denote the vacuum and by $\mathscr{F}$ the set of finite linear combinations of product vectors. Later we will introduce a scalar product $\langle\ ,\ \rangle_\mu$ and take the completion $\mathscr{F}_\mu$ of $\mathscr{F}$ with respect to this scalar product.

We define $c^*(f)$ and $c(f)$ on $\mathscr{F}$ by linear extension of $(f, h_i \in \mathscr{H})$,

$$c^*(f)\Omega = f,$$
$$c^*(f)h_1 \otimes \ldots \otimes h_n = f \otimes h_1 \otimes \ldots \otimes h_n,$$

and

$$c(f)\Omega = 0,$$
$$c(f)h_1 \otimes \ldots \otimes h_n = \sum_{k=1}^{n} \mu^{k-1} \langle f, h_k \rangle h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n.$$

The symbol $\check{h}_k$ means that h_k has to be deleted in the product.

We thus have defined $c^*(f)$ as the (left) creation operator [Eva] and $c(f)$ as a "twisted" (left) annihilation operator.

Remark. For a better understanding of the asymmetry between $c(f)$ and $c^*(f)$ it is advantageous to think of $\mathscr{F}$ as polynomials in $\dim \mathscr{H}$ many non-commuting indeterminants z_i (corresponding to an orthonormal basis $\{f_i\}$ of $\mathscr{H}$). Then $c^*(f_i)$ and $c(f_i)$ correspond to the multiplication operator z_i and the partial derivative ∂_{z_i}, respectively. The sum appearing in the definition of $c(f_i)$ is then nothing but the product rule for differentiating.

Lemma 1. *The operators $c(f)$ on $\mathscr{F}$ fulfill, for all $f, g \in \mathscr{H}$, the relation*

$$c(f)c^*(g) - \mu c^*(g)c(f) = \langle f, g \rangle 1.$$

Proof.

$$\begin{aligned}[c(f)c^*(g)]h_1 \otimes \ldots \otimes h_n &= c(f)g \otimes h_1 \otimes \ldots \otimes h_n \\ &= \langle f, g \rangle h_1 \otimes \ldots \otimes h_n + \mu g \otimes [c(f)h_1 \otimes \ldots \otimes h_n] \\ &= [\langle f, g \rangle 1 + \mu c^*(g)c(f)]h_1 \otimes \ldots \otimes h_n. \quad \square\end{aligned}$$

It remains to present a scalar product which makes $c(f)$ and $c^*(f)$ adjoint. First, we give a formal definition of this scalar product $\langle\, , \rangle_\mu$. The main problem will be to prove its positive definiteness.

We define the symmetric bilinear form $\langle\, , \rangle_\mu$ on $\mathscr{F}$ by

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_m \rangle_\mu := 0 \quad \text{for} \quad n \neq m$$

and otherwise recursively by

$$\begin{aligned}\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_\mu &:= \langle g_2 \otimes \ldots \otimes g_n, c(g_1)h_1 \otimes \ldots \otimes h_n \rangle_\mu \\ &= \sum_{k=1}^{n} \mu^{k-1} \langle g_1, h_k \rangle \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_\mu.\end{aligned}$$

Lemma 2. *We have, for all $f \in \mathscr{H}$ and $\xi, \eta \in \mathscr{F}$:*

$$\langle c^*(f)\xi, \eta \rangle_\mu = \langle \xi, c(f)\eta \rangle_\mu.$$

Proof. It is sufficient to prove, for all $g_i, h_j \in \mathscr{H}$, that

$$\langle c^*(f)g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_{n+1} \rangle_\mu = \langle g_1 \otimes \ldots \otimes g_n, c(f)h_1 \otimes \ldots \otimes h_{n+1} \rangle_\mu.$$

But this is clear from the definition of $\langle\, , \rangle_\mu$. $\square$

It follows immediately that

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_\mu = \langle \Omega, c(g_n) \ldots c(g_1) h_1 \otimes \ldots \otimes h_n \rangle .$$

Now we want to write $\langle\,,\rangle_\mu$ as $\langle \xi, \eta \rangle_\mu = \langle \xi, P_\mu \eta \rangle$ with $P_\mu : \mathscr{F} \to \mathscr{F}$, where $\langle\,,\rangle := \langle\,,\rangle_0$ is the usual scalar product on the full Fock space. We write

$$P_\mu = \bigoplus_{n=0}^{\infty} P_\mu^{(n)} \quad \text{with} \quad P_\mu^{(n)} : \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n} .$$

To define $P_\mu^{(n)}$ we use the following unitary representation $\pi \mapsto U_\pi$ of the symmetric group S_n on $\mathscr{H}^{\otimes n}$:

$$U_\pi h_1 \otimes \ldots \otimes h_n = h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)} .$$

By $i(\pi)$ we denote the number of inversions of $\pi \in S_n$, i.e.

$$i(\pi) := \# \{(i,j) \in \{1, \ldots, n\}^2 \mid i < j,\ \pi(i) > \pi(j)\} .$$

Then we put

$$P_\mu^{(n)} := \sum_{\pi \in S_n} \mu^{i(\pi)} U_\pi .$$

In particular, $P_\mu^{(0)} = P_\mu^{(1)} = 1$, i.e. we have not changed the scalar product on the vacuum and the one particle space.

Remark. It is clear, that $P_\mu^{(n)}$ is, for each n, a bounded operator on $\mathscr{H}^{\otimes n}$. Later, in the proof of Lemma 4, we will see that it has the norm $\prod_{i=0}^{n-1} (1 + \mu + \ldots + \mu^i)$. Hence P_μ is in the case $\mu > 0$ an unbounded operator on the full Fock space.

Lemma 3. *We have for all* $\xi, \eta \in \mathscr{F}$,

$$\langle \xi, \eta \rangle_\mu = \langle \xi, P_\mu \eta \rangle .$$

Proof. It is sufficient to prove, for all $n \in \mathbb{N}$ and all $g_i, h_j \in \mathscr{H}$:

$$\begin{aligned} \langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_\mu &= \langle g_1 \otimes \ldots \otimes g_n, P_\mu^{(n)} h_1 \otimes \ldots \otimes h_n \rangle \\ &= \sum_{\pi \in S_n} \mu^{i(\pi)} \langle g_1, h_{\pi(1)} \rangle \ldots \langle g_n, h_{\pi(n)} \rangle . \end{aligned}$$

We do this by induction. For $n = 1$ the assertion is clear. Assume the assertion is true for $n-1$. Denote by $S_{n-1}^{(k)}$ the set of all bijections from $\{2, \ldots, n\}$ to $\{1, \ldots, \check{k}, \ldots, n\}$. Then we can write each $\pi \in S_n$ as $\pi = \begin{pmatrix} 1 \to k \\ \sigma \end{pmatrix}$, where $\pi(1) = k$ and $\sigma \in S_{n-1}^{(k)}$ with $\sigma(l) = \pi(l)$ for $l = 2, \ldots, n$. The number of inversions $i(\sigma)$ for $\sigma \in S_{n-1}^{(k)}$ is defined in the same way as for $\pi \in S_n$. It is easy to see that $i(\pi) = k - 1 + i(\sigma)$ if $\pi = \begin{pmatrix} 1 \to k \\ \sigma \end{pmatrix}$. By using $\sum_{k=1}^{n} \sum_{\sigma \in S_{n-1}^{(k)}} = \sum_{\pi \in S_n}$ we get

$$\begin{aligned} &\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_\mu \\ &= \sum_{k=1}^{n} \mu^{k-1} \langle g_1, h_k \rangle \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_\mu \\ &= \sum_{k=1}^{n} \mu^{k-1} \langle g_1, h_k \rangle \sum_{\sigma \in S_{n-1}^{(k)}} \mu^{i(\sigma)} \langle g_2, h_{\sigma(2)} \rangle \ldots \langle g_n, h_{\sigma(n)} \rangle \\ &= \sum_{\pi \in S_n} \mu^{i(\pi)} \langle g_1, h_{\pi(1)} \rangle \ldots \langle g_n, h_{\pi(n)} \rangle . \quad \square \end{aligned}$$

Now we tackle the problem of the positive definiteness of $\langle\, , \rangle_\mu$. This can be inferred from more general theorems in [BJS]. But we prefer to give an elementary proof.

Proposition 1. a) *The operator P_μ is positive for all $\mu \in [-1,1]$.*
b) *The operator P_μ is strictly positive for all $\mu \in (-1,1)$.*

Proof. It is sufficient to consider $P_\mu^{(n)}$ for all $n \in \mathbb{N}$.
a) We will first prove that $\varphi_\mu : \pi \mapsto \mu^{i(\pi)}$ is a positive definite function on S_n, i.e.

$$\sum_{\pi,\sigma \in S_n} \mu^{i(\pi^{-1}\sigma)} r(\sigma)\overline{r(\pi)} \geqq 0$$

for arbitrary $r : S_n \to \mathbb{C}$.
We define

$$\Phi := \{(i,j) \mid i \neq j,\ 1 \leqq i,\ j \leqq n\},$$
$$\Phi^+ := \{(i,j) \in \Phi \mid i < j\},$$

and for $\pi \in S_n$ and $A \subset \Phi$,

$$\pi(A) := \{(\pi(i), \pi(j)) \mid (i,j) \in A\} \subset \Phi .$$

By $|A|$ we denote the number of elements of the set A. Note in the following that we have $|\pi(A)| = |A|$. Then we have $i(\pi) = |\pi(\Phi^+)\backslash \Phi^+|$ and $i(\pi) = i(\pi^{-1}) = |\pi^{-1}(\Phi^+)\backslash \Phi^+| = |\Phi^+ \backslash \pi(\Phi^+)|$. Denoting by $A \Delta B$ the symmetric difference $(A\backslash B) \cup (B \backslash A)$ of A and B we get $2i(\pi) = i(\pi) + i(\pi^{-1}) = |\pi(\Phi^+) \Delta \Phi^+|$, which yields for $\pi, \sigma \in S_n$,

$$2i(\pi^{-1}\sigma) = |\sigma(\Phi^+) \Delta \pi(\Phi^+)| .$$

By using the characteristic function χ_A and χ_B for $A, B \subset \Phi$ we can write

$$|A \Delta B| = \sum_{x \in \Phi} |\chi_A(x) - \chi_B(x)| = \sum_{x \in \Phi} |\chi_A(x) - \chi_B(x)|^2 .$$

Now we consider first the case $0 < \mu \leqq 1$ and put $\mu = e^{-\lambda}$ $(\lambda \geqq 0)$. We have

$$\begin{aligned} \mu^{i(\pi^{-1}\sigma)} &= \exp(-\lambda i(\pi^{-1}\sigma)) \\ &= \exp\left(-\frac{\lambda}{2} |\sigma(\Phi^+) \Delta \pi(\Phi^+)|\right) \\ &= \prod_{x \in \Phi} \exp\left(-\frac{\lambda}{2} |\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)|^2\right). \end{aligned}$$

Remembering the fact that the pointwise product of two positive definite functions is again positive definite [Sch] we see that it is sufficient to prove for all $x \in \Phi$,

$$\sum_{\pi,\sigma \in S_n} \exp\left(-\frac{\lambda}{2} |\chi_{\sigma(\Phi^+)}(x) - \chi_{\pi(\Phi^+)}(x)|^2\right) r(\sigma)\overline{r(\pi)} \geqq 0 .$$

Putting $y_0 := 0$, $y_1 := 1$ and

$$r(y_0) := \sum_{\substack{\sigma \text{ with} \\ x \notin \sigma(\Phi^+)}} r(\sigma), \qquad r(y_1) := \sum_{\substack{\sigma \text{ with} \\ x \in \sigma(\Phi^+)}} r(\sigma)$$

we have

$$\begin{aligned}
&\sum_{\pi,\sigma\in S_n} \exp\left(-\frac{\lambda}{2}|\chi_{\sigma(\Phi^+)}(x)-\chi_{\pi(\Phi^+)}(x)|^2\right) r(\sigma)\overline{r(\pi)}\\
&=\sum_{i,j=0}^{1} \exp\left(-\frac{\lambda}{2}|y_i-y_j|^2\right) r(y_i)\overline{r(y_j)}\\
&\geqq 0,
\end{aligned}$$

since $x\mapsto \exp\left(-\frac{\lambda}{2}|x|^2\right)$ is a positive definite function on the additive group $\mathbb{R}$. Thus for $0<\mu\leqq 1$ the function $\varphi_\mu:\pi\mapsto\mu^{i(\pi)}$ is positive definite.

For $-1\leqq\mu<0$ we notice

$$\varphi_\mu(\pi)=(-1)^{i(\pi)}\varphi_{-\mu}(\pi).$$

Since $\pi\mapsto(-1)^{i(\pi)}$ is a character on S_n and hence positive definite, we get the assertion for $-1\leqq\mu<0$ by the fact that the pointwise product of two positive definite functions is again positive definite.

Since for $\mu=0$ we have $\varphi_0(1)=1$, $\varphi_0(\pi)=0$ $(\pi\neq 1)$, which is clearly positive definite, we have the positive definiteness of φ_μ for all $\mu\in[-1,1]$.

Now we show that $P_\mu^{(n)}$ is a positive operator, i.e. that $\langle\eta,P_\mu^{(n)}\eta\rangle\geqq 0$ for all $\eta\in\mathscr{H}^{\otimes n}$. We have (with $\{\xi_i\}$ denoting a CONS of the Hilbert space $\mathscr{H}^{\otimes n}$),

$$\begin{aligned}
\langle\eta,P_\mu^{(n)}\eta\rangle&=\sum_{\pi\in S_n}\mu^{i(\pi)}\langle\eta,U_\pi\eta\rangle\\
&=\frac{1}{n!}\sum_{\pi,\sigma\in S_n}\mu^{i(\pi^{-1}\sigma)}\langle\eta,U_{\pi^{-1}\sigma}\eta\rangle\\
&=\frac{1}{n!}\sum_{\pi,\sigma\in S_n}\mu^{i(\pi^{-1}\sigma)}\langle U_\pi\eta,U_\sigma\eta\rangle\\
&=\frac{1}{n!}\sum_{\pi,\sigma\in S_n}\sum_{\{\xi_i\}}\mu^{i(\pi^{-1}\sigma)}\langle U_\pi\eta,\xi_i\rangle\langle\xi_i,U_\sigma\eta\rangle\\
&=\frac{1}{n!}\sum_{\{\xi_i\}}\left\{\sum_{\pi,\sigma\in S_n}\mu^{i(\pi^{-1}\sigma)}\overline{\langle\xi_i,U_\pi\eta\rangle}\langle\xi_i,U_\sigma\eta\rangle\right\}\\
&\geqq 0,
\end{aligned}$$

since φ_μ is positive definite.

b) It is sufficient to show that φ_μ is a strictly positive definite function on S_n for $\mu\in(-1,1)$. We only consider $\mu\in(0,1)$; the case $\mu\in(-1,0)$ is analogous and $\mu=0$ is trivial.

We call φ_μ degenerate if it is not strictly positive definite. Assume that there exists a $\mu\in(0,1)$ with φ_μ degenerate. Since the pointwise product of two strictly positive definite functions is again strictly positive definite [Sch] we can conclude that also $\varphi_{\mu'}$ with $\mu':=\sqrt{\mu}$ is degenerate. Since $\sqrt{\mu}\neq\mu$ we get in this way infinitely many different μ_i with φ_{μ_i} degenerate. But the fact that φ_μ is degenerate can be stated as the vanishing of $\det A$, where $A:=(\mu^{i(\pi^{-1}\sigma)})_{\pi,\sigma\in S_n}$. Since $\det A$ is a non-constant polynomial in μ, it has only a finite number of zeros, and we obtain a contradiction. Thus there exists no $\mu\in(0,1)$ with φ_μ degenerate. □

We now denote by $\mathscr{F}_\mu$ the completion of $\mathscr{F}$ with respect to the scalar product $\langle\,,\,\rangle_\mu$. Note that in the cases $\mu=-1$ and $\mu=+1$ we first have to divide out the kernel of P_μ before taking the completion.

Remarks. 1) Note the following special cases:

i) $\mu=0$: $P_0=\mathrm{id}$. For $k:=\dim\mathscr{H}<\infty$ we get the full Fock space representation of an extension of the Cuntz algebra O_k by the algebra of compact operators, for $k=\infty$ we get the full Fock space representation of the Cuntz algebra O_∞ itself (cf. [Eva]).
ii) $\mu=1$: $P_1^{(n)2}=n!\,P_1^{(n)}$; $1/n!\,P_1^{(n)}$ projects onto the space of symmetric functions, thus our scalar product implements automatically the additional relations $c(f)c(g)=c(g)c(f)$. We get a representation of the CCR-algebra over $\mathscr{H}$.
iii) $\mu=-1$: $P_{-1}^{(n)2}=n!\,P_{-1}^{(n)}$; $1/n!\,P_{-1}^{(n)}$ projects onto the space of antisymmetric functions, thus our scalar product implements automatically the additional relations $c(f)c(g)=-c(g)c(f)$. We get a representation of the CAR-algebra over $\mathscr{H}$.

2) The strict positivity of P_μ in the interval $\mu\in(-1,1)$ shows that, apart from the above three cases, P_μ is not a multiple of a projection.

Lemma 4. *The operator $c(f)$ on $\mathscr{F}_\mu$ is bounded for $-1\leqq\mu<1$ and has the norm*

$$\|c(f)\|_\mu=\frac{1}{\sqrt{1-\mu}}\|f\|_{\mathscr{H}}\qquad(0\leqq\mu<1),$$

$$\|c(f)\|_\mu=\|f\|_{\mathscr{H}}\qquad(-1\leqq\mu\leqq 0).$$

Proof. First, we consider $-1\leqq\mu\leqq 0$. Then we have for $\xi\in\mathscr{F}$,

$$\begin{aligned}\langle c^*(f)\xi, c^*(f)\xi\rangle_\mu&=\langle f,f\rangle\langle\xi,\xi\rangle_\mu+\mu\langle c(f)\xi,c(f)\xi\rangle_\mu\\&\leqq\langle f,f\rangle\langle\xi,\xi\rangle_\mu.\end{aligned}$$

Thus we have $\|c^*(f)\|\leqq\|f\|$ and equality holds since $c^*(f)\Omega=f$.

Now we treat the case $0\leqq\mu<1$. We will show

$$\langle f\otimes\xi, f\otimes\xi\rangle_\mu\leqq\frac{1}{1-\mu}\langle\xi,\xi\rangle_\mu$$

for all $\xi\in\mathscr{F}$. It is sufficient to deal with $\xi\in\mathscr{H}^{\otimes n}$ for arbitrary $n\in\mathbb{N}_0$.

We denote by $\pi_i\in S_n$ $(n>i)$ the transpositions of the symmetric group, i.e. π_i interchanges i and $i+1$. One sees easily that each element π of the symmetric group can be written uniquely in the form $\pi_{k(1)}\pi_{k(1)+1}\pi_{k(1)+2}\ldots\pi_{k(1)+r(1)}\pi_{k(2)}\pi_{k(2)+1}\pi_{k(2)+2}\ldots\pi_{k(2)+r(2)}\ldots\pi_{k(i)}\pi_{k(i)+1}\pi_{k(i)+2}\ldots\pi_{k(i)+r(i)}$ with $i\geqq 0$ ($i=0$ gives the unit element $1\in S_n$), $r(i)\geqq 0$ and $k(1)>k(2)>\ldots>k(i)$. Furthermore, the length of this product, i.e. $(r(1)+1)+(r(2)+1)+\ldots+(r(i)+1)$ is equal to $i(\pi)$. This shows that we can write $P_\mu^{(n+1)}$ in the following form (by identifying U_π with π):

$$P_\mu^{(n+1)}=1\otimes P_\mu^{(n)}(1+\mu\pi_1+\mu^2\pi_1\pi_2+\ldots+\mu^n\pi_1\pi_2\ldots\pi_n).$$

Using this equation and its adjoint version we get

$$\begin{aligned}P_\mu^{(n+1)}P_\mu^{(n+1)}&=1\otimes P_\mu^{(n)}(1+\ldots+\mu^n\pi_1\ldots\pi_n)(1+\ldots+\mu^n\pi_n\ldots\pi_1)1\otimes P_\mu^{(n)}\\&\leqq 1\otimes P_\mu^{(n)}(1+\mu+\ldots+\mu^n)(1+\mu+\ldots+\mu^n)1\otimes P_\mu^{(n)}.\end{aligned}$$

This implies

$$P_\mu^{(n+1)}\leqq(1+\mu+\ldots+\mu^n)1\otimes P_\mu^{(n)}<\frac{1}{1-\mu}1\otimes P_\mu^{(n)}.$$

Now we obtain

$$\begin{aligned}\langle f\otimes\xi, f\otimes\xi\rangle_\mu &= \langle f\otimes\xi, P_\mu^{(n+1)} f\otimes\xi\rangle \\ &< \frac{1}{1-\mu}\langle f\otimes\xi, 1\otimes P_\mu^{(n)}(f\otimes\xi)\rangle \\ &= \frac{1}{1-\mu}\langle f,f\rangle\langle\xi, P_\mu^{(n)}\xi\rangle .\end{aligned}$$

This gives the boundedness of $c^*(f)$. That the norm of $c^*(f)$ is equal to $1/\sqrt{1-\mu}\|f\|$ can be seen from $c^*(f)f^{\otimes n}=f^{\otimes(n+1)}$ and

$$\langle f^{\otimes(n+1)}, f^{\otimes(n+1)}\rangle_\mu=(1+\mu+\ldots+\mu^n)\langle f,f\rangle\langle f^{\otimes n}, f^{\otimes n}\rangle_\mu . \quad \square$$

This lemma implies that Lemmata 1 and 2, which were proved only on the dense domain $\mathscr{F}$, remain valid for $\mu\in[-1,1)$ also on $\mathscr{F}_\mu$. The case $\mu=+1$ is an exceptional one. In this case our operators are unbounded ones defined only on the dense domain $\mathscr{F}$.

Now we can consider our Brownian motion. In the following we choose $\mathscr{H}=L^2(\mathbb{R})$ and fix a $\mu\in[-1,1]$. By $\mathscr{C}_\mu$ we denote the *-algebra generated by all $c(\chi_I)$ for $I\in\mathfrak{R}$ and by ϱ the vacuum expectation state on $\mathscr{C}_\mu$. Then one only has to use the definition of $c(f)$ and $c^*(f)$ to see the stationarity of the distribution and, by induction, the factorizing of pyramidally ordered moments. To justify the name "Brownian motion" for the object $(\mathscr{C}_{\mu,\xi}, ((c(\chi_I), c^*(\chi_I)))_{I\in\mathfrak{R}})$, it only remains to show the Gaussianity of the corresponding distribution. This is a consequence of the fact that all moments of our operators are determined in a special way by the second moments. This is, of course, an implication of a central limit theorem, which will be considered elsewhere [Spe 3]. To describe this connection between the moments we have to define the number of inversions $i(\mathscr{V})$ of the special partitions $\mathscr{V}=\{(e_1,z_1),\ldots,(e_{r/2},z_{r/2})\}$ of the set $\{1,2,\ldots,r\}$, where $e_i<e_j$ for $i<j$ and $e_i<z_i$ for all $i=1,\ldots,r/2$ (r even). It is defined as

$$i(\mathscr{V}):=\#\{(i,j)\mid e_i<e_j<z_i<z_j\}.$$

The set of all such partitions will be denoted by $\mathscr{P}_2(1,\ldots,r)$.

Examples. $i((1,2),(3,4))=0$, $i((1,3),(2,4))=1$, $i((1,4),(2,5),(3,6))=3$.

Remark. If we visualize our partitions as in [Spe 1] by building bridges (connecting e_i with z_i), then $i(\mathscr{V})$ is the minimal number of crossing points of these lines. In particular, for an admissible partition $\mathscr{V}$ (cf. [Spe 1]) we have $i(\mathscr{V})=0$.

Now we have

Proposition 2. *Let $c_I:=c(\chi_I)$ and $c_I^1:=c_I$, $c_I^2:=c_I^*$. Then we have for all $r\in\mathbb{N}$ and $k(1),\ldots,k(r)\in\{1,2\}$,*

$$\varrho(c_I^{k(1)}\ldots c_I^{k(r)})=\begin{cases}0, & r \text{ odd} \\ \sum\limits_{\mathscr{V}\in\mathscr{P}_2(1,\ldots,r)} \varrho(c_I^{k(e_1)}c_I^{k(z_1)})\ldots\varrho(c_I^{k(e_{r/2})}c_I^{k(z_{r/2})})\mu^{i(\mathscr{V})}, & r \text{ even}\end{cases},$$

where the sum runs over all partitions $\mathscr{V}=\{(e_1,z_1),\ldots,(e_{r/2},z_{r/2})\}$. The matrix $(Q_I)_{ij}=Q_I(i,j)=\varrho(c_I^ic_I^j)$ is given by

$$Q_I=\begin{pmatrix}0 & \lambda(I)\\ 0 & 0\end{pmatrix}.$$

Example.

$$\varrho(c_I^{k(1)}c_I^{k(2)}c_I^{k(3)}c_I^{k(4)}) = Q_I(k(1),k(2))Q_I(k(3),k(4)) + Q_I(k(1),k(4))Q_I(k(2),k(3)) + \mu Q_I(k(1),k(3))Q_I(k(2),k(4)).$$

In particular, $\varrho(c_I c_I c_I^* c_I^*) = \lambda(I)(1+\mu)$ and $\varrho(c_I c_I^* c_I c_I^*) = \lambda(I)$.

Proof. We only have to note that the formula is valid by definition of the scalar product $\langle\ ,\ \rangle_\mu$ for products of the form $c_I \ldots c_I c_I^* \ldots c_I^*$ (i.e. $e_i < z_j$ for all i and j) and that both sides of the formula change in the same way if we replace in $c_I^{k(1)} \ldots c_I^{k(r)}$ a factor $c_I c_I^*$ by $c_I^* c_I$. $\square$

Now we can summarize our result.

Theorem. *The triple* $(\mathscr{C}_\mu, \varrho, ((c(\chi_I), c^*(\chi_I)))_{I\in\mathfrak{R}})$ *is, for* $\mu \in [-1,1]$, *a generalized Brownian motion fulfilling the relations*

$$c(f)c^*(g) - \mu c^*(g)c(f) = \langle f, g\rangle 1.$$

References

[ApH] Applebaum, D.B., Hudson, R.L.: Fermion Ito's formula and stochastic evolutions. Commun. Math. Phys. **96**, 473–496 (1984)

[Avi] Avitzour, D.: Free products of C^*-algebras. Trans. Am. Math. Soc. **271**, 423–435 (1982)

[BJS] Bożejko, M., Januszkiewics, T., Spatzier, R.J.: Infinite Coxeter groups do not have Kazhdan's property. J. Operator Theory **19**, 63–68 (1988)

[BSW] Barnett, C., Streater, R.F., Wilde, I.F.: The Ito-Clifford-integral. J. Funct. Anal. **48**, 172–212 (1982)

[CoH] Cockroft, A.M., Hudson, R.L.: Quantum mechanical Wiener process. J. Multivariate Anal. **7**, 107–124 (1977)

[CuH] Cushen, C.D., Hudson, R.L.: A quantum-mechanical central limit theorem. J. Appl. Prob. **8**, 454–469 (1971)

[Cun] Cuntz, J.: Simple C^*-algebras generated by isometries. Commun. Math. Phys. **57**, 173–185 (1977)

[Eva] Evans, D.E.: On O_n. Publ. RIMS, Kyoto Univ. **16**, 915–927 (1980)

[GvW] Giri, N., von Waldenfels, W.: An algebraic version of the central limit theorem. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **42**, 129–134 (1978)

[Gre] Greenberg, O.W.: Q-mutators and violations of statistics. University of Maryland Preprint 91-034, 1990

[Heg] Hegerfeld, G.C.: Noncommutative analogs of probabilistic notions and results. J. Funct. Anal. **64**, 436–456 (1985)

[HuP] Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolution. Commun. Math. Phys. **93**, 301–323 (1984)

[Küm 1] Kümmerer, B.: Survey on a theory of non-commutative stationary Markov processes. In: Quantum probability and applications. III. Oberwolfach 1987, Lecture Notes in Mathematics, vol. 1303. Berlin, Heidelberg, New York: Springer 1988, pp. 154–182

[Küm 2] Kümmerer, B.: Markov dilations and non-commutative Poisson processes. Preprint

[KPr] Kümmerer, B., Prin, J.: Generalized white noise and non-commutative stochastic integration. Preprint

[KSp] Kümmerer, B., Speicher, R.: Stochastic integration on the Cuntz-algebra O_∞. Preprint

[Maa 1] Maassen, H.: Theoretical concepts in quantum probability; quantum Markov processes. Preprint, Nijmegen 1988

[Maa 2] Maassen, H.: Quantum Markov processes on Fock space described by integral kernels. In: Quantum probability and applications. II. Heidelberg 1984, Lecture Notes in Mathematics, vol. 1136, Berlin, Heidelberg, New York: Springer 1985, pp. 361–374

[LiP] Lindsay, J.M., Parthasarathy, K.R.: Cohomology of power sets with applications in quantum probability. Commun. Math. Phys. **124**, 337–364 (1989)
[PWo] Pusz, W., Woronowicz, S.L.: Twisted second quantization. Preprint
[Sch] Schur, I.: Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. J. Reine Angew. Math. **140**, 1–29 (1911)
[Sma] Schürmann, M.: Quantum stochastic processes with independent and additive increments. Preprint, Heidelberg 1989
[Spe 1] Speicher, R.: A new example of "Independence" and "White noise." Probab. Th. Rel. Fields **84**, 141–159 (1990)
[Spe 2] Speicher, R.: Stochastic integration on the full Fock space with the help of a kernel calculus. To appear in Publ. RIMS **27** (1991)
[Spe 3] Speicher, R.: In preparation
[Voi 1] Voiculescu, D.: Symmetries of some reduced free product C^*-algebras. In: Operator algebras and their connection with topology and ergodic theory. Busteni, Romania, 1983, Lecture Notes in Mathematics, vol. 1132. Berlin, Heidelberg, New York: Springer 1985
[Voi 2] Voiculescu, D.: Addition of certain non-commuting random variables. J. Funct. Anal. **66**, 323–346 (1986)
[vWa] von Waldenfels, W.: An algebraic central limit theorem in the anti-commuting case. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **42**, 135–140 (1978)
[Wor 1] Woronowicz, S.L.: Twisted $SU(2)$ group. An example of a non-commutative differential calculus. Publ. RIMS **23**, 117–181 (1987)
[Wor 2] Woronowicz, S.L.: Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups. Invent. Math. **93**, 35–76 (1988)

Communicated by A. Connes

Note added in proof. Philip Feinsilver called our attention to the following article: Arik, M., Coon, D.D., Lam, Y.: Operator algebra of dual resonance models. J. Math. Phys. **16**, 1776–1779 (1975), where the same relations were considered on a formal level; but none of our main results is contained in this article.