Commun. Math. Phys. 136, 1–14 (1991)

Communications in Mathematical Physics

The Higher Rank Virasoro Algebras*

J. Patera[1] and H. Zassenhaus[2]

[1] Centre de Recherches mathématiques, Université de Montréal, CP 6128-A, Montréal (Québec) H3C 3J7, Canada
[2] Department of Mathematics, Ohio State University, Columbus, Ohio 43210, USA

Received October 26, 1989

Abstract. Higher rank Virasoro algebras are defined and their properties studied: triangular decompositions, automorphism groups and finite dimensional subalgebras.

In the following paper we generalize what has recently been called the Virasoro algebra. We reluctantly continue to call that algebra the Virasoro algebra, since in [1] the algebra appears (only implicitly) without the central extension. Such an algebra was well known to E. Cartan [2] and was extensively studied [3, 4, 5] before World War II, on a suggestion of E. Witt, as an example of a simple Lie algebra (of infinite dimension over $\mathbb{C}$ and of finite dimension over $\mathbb{F}$ of char > 0). With the central extension added, the algebra is not simple but its representation theory is much richer. The central term apparently appeared for the first time in [6] with a footnote reference to J. Weis.

The purpose of this article is to generalize the notion of Virasoro algebras of rank 1 to higher rank Virasoro algebras and explore its algebraic properties like triangular decompositions, the automorphism groups and the finite dimensional subalgebras. Six representative examples are described in the last section. A comprehensive exposition of the rank one Virasoro algebra and its representations is found in [7].

1. Definitions

In [2] the infinite simple Lie algebra $L(\mathbb{Z}/\mathbb{C})$ with basis elements $e_j (j \in \mathbb{Z})$ and multiplication rule

$$[e_j, e_k] = (k - j)e_{j+k} \tag{1}$$

* Work supported in part by the Natural Sciences and Engineering Research Council of Canada and by the Fonds FCAR du Québec

over the complex number field $\mathbb{C}$ was defined as an example of a simple infinite dimensional Lie algebra. Using the same multiplication rule, but indices j running over the prime field of a field of characteristic p, a finite Lie algebra $L(\mathbb{Z}_p/\mathbb{C})$ was defined by E. Witt (see [3, 4]) providing examples of simple finite dimensional Lie algebras of prime characteristic p in case $p > 2$.

In [4] the finite dimensional simple Lie algebras $L(M/\mathbb{F})$ with basis elements $e_\alpha(\alpha \in M)$ and multiplication

$$[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta} \quad (\alpha, \beta \in M), \tag{2}$$

generalizing (1) were defined for any finite submodule M of a field of prime characteristic. By the same calculation for any submodule M of a field $\mathbb{F}$ the algebra $L(M/\mathbb{F})$ with basis elements $e_\alpha(\alpha \in M)$ and multiplication rule (2) turns out as a Lie algebra of dimension equal to the cardinality $|M|$ of M over the field $\mathbb{F}$ that is simple in case M is noncyclic or in case M is cyclic of order > 2.

The central extension $V(\mathbb{Z}/\mathbb{C})$ of $L(\mathbb{Z}/\mathbb{C})$ was introduced by J. Weis (see [6]). It plays an important role in many applications. What role is played by the universal central extension $V(M/\mathbb{C})$ of the Lie algebra $L(M/\mathbb{C})$ corresponding to the free submodules M of $\mathbb{C}$ of higher rank?

We remark that there holds the module decomposition

$$L(M/\mathbb{F}) = \bigoplus_{\alpha \in M} \mathbb{F}e_\alpha \tag{3}$$

for any submodule M of any field $\mathbb{F}$ which is both a finest grading in the sense defined in [8] and a Cartan decomposition relative to the Cartan subalgebra $\mathbb{F}e_0$. Therefore for any central extension $\mathfrak{L}$ given by the exact sequence

$$0 \rightarrow N = \mathbb{F}\mathscr{C} \xrightarrow{\iota} \mathfrak{L} \xrightarrow{\varepsilon} L(M/\mathbb{F}) \rightarrow 0 \tag{4}$$

and the condition

$$[\mathfrak{L}, \iota N] = 0 \tag{4b}$$

it follows that the ε-inverse image

$$H = \iota N + \mathbb{F}f_0 \quad (f_0 \in \mathfrak{L}, \varepsilon f_0 = e_0) \tag{5}$$

of $\mathbb{F}e_0$ is the Cartan subalgebra of $\mathfrak{L}$ satisfying

$$[H, \mathbb{F}\varepsilon^{-1}e_\alpha) = \mathbb{F}f_\alpha \quad (0 \neq \alpha \in M) \tag{6}$$

so that the one dimensional linear spaces $\mathbb{F}f_\alpha$ are the root spaces (see [9] 7.1 for the Virasoro case). Hence we can extend any $\mathbb{F}$-basis of the center N of $\mathfrak{L}$ to an $\mathbb{F}$-basis of $\mathfrak{L}$ by means of basis elements f_α with $\alpha \in M$, where

$$\varepsilon f_\alpha = e_\alpha \quad (\alpha \in M). \tag{7}$$

Thus there holds the multiplication rule

$$[f_\alpha, f_\beta] = (\beta - \alpha)f_{\alpha+\beta} + \delta_{\alpha,-\beta}c(\alpha)\mathscr{C} \tag{8a}$$

with $c(\alpha)$ in $\mathbb{F}$. Anticommutativity provides the conditions

$$c(0) = 0, \quad c(-\alpha) = -c(\alpha), \tag{8b}$$

the Jacobi identity provides the conditions

$$c(\alpha+\beta)=(\beta-\alpha)^{-1}\{-(\alpha+2\beta)c(\alpha)+(2\alpha+\beta)c(\beta))\} \quad (\alpha,\beta\in M,\alpha\neq\beta). \tag{8c}$$

Together they characterize the 2-cocycles of $L(M/\mathbb{F})$ with trivial action. Well known examples of 2-cocycles are obtained by setting

$$c_1(\alpha)=\alpha \tag{8d}$$

or

$$c_2(\alpha)=\alpha^3. \tag{8e}$$

Supposing now that the module M is cyclic $\neq 0$:

$$M\simeq \mathbb{Z}b_1 \quad (0\neq b_1\in\mathbb{F}),$$

then the conditions (8b), (8c) enable us to compute $c(jb_1)$ uniquely as soon as the values of $c(b_1)$ and $c(2b_1)$ are known. This is because

$$\begin{aligned} c(0)&=0,\\ c(-b_1)&=-c(b_1),\\ c(-2b_1)&=-c(2b_1),\\ (n+1)b_1&=nb_1+b_1 \quad \text{if} \quad n\in\mathbb{Z}^{>0},\quad n1_{\mathbb{F}}\neq 1_{\mathbb{F}}\\ (-n)b_1&=-(nb_1) \quad \text{if} \quad n\in\mathbb{Z}^{>0}. \end{aligned}$$

But the $\mathbb{F}$-linear combinations of the 2-cocycles c_1, c_2 can be used to match any prescribed values of $c(b_1), c(b_2)$ so that every 2-cocycle is an $\mathbb{F}$-linear combination of c_1, c_2.

At characteristic 2 we have $2b_1=0$ so that $\mathbb{F}c_2=\mathbb{F}c_1$, at characteristic 3 we have $2b_1=-b_1$ so that $\mathbb{F}c_2=\mathbb{F}c_1$. At all other characteristics the two cocycles c_1, c_2 are linearly independent and form an $\mathbb{F}$-basis of the 2-cocycle module.

We remark that the 2-coboundaries are provided by the $\mathbb{F}$-multiples of c_1. Hence we have

Theorem 1. *The Virasoro algebra $V(\mathbb{Z}1_{\mathbb{F}}/\mathbb{F})$ corresponding to the Cartan-Witt Lie algebra $L(\mathbb{Z}1_{\mathbb{F}})$ has the basis elements*

$$e_\alpha(\alpha\in M), \quad \mathscr{C} \tag{9a}$$

for $M=\mathbb{Z}1_{\mathbb{F}}$ with multiplication rules

$$[e_\alpha,e_\beta]=(\beta-\alpha)e_{\alpha+\beta}+\beta^3\delta_{\alpha,-\beta}\mathscr{C} \quad (\alpha,\beta\in M), \tag{9b}$$

$$[e_\alpha,\mathscr{C}]=0 \quad (\alpha\in M) \tag{9c}$$

if the characteristic of $\mathbb{F}$ is not 2 or 3.

There holds the isomorphism

$$V(\mathbb{Z}1_{\mathbb{F}}/\mathbb{F})\cong L(\mathbb{Z}1_{\mathbb{F}}/\mathbb{F}) \quad (\mathrm{char}(\mathbb{F})=2,3). \quad \square$$

The basic multiplication rules (9a–9c) of the generalized Virasoro algebra express the fact that $V(M/\mathbb{F})$ is given in terms of fine gradings (see [8]) which happens to be a Cartan decomposition. Its grading group is an abelian group isomorphic to M.

Remark. The multiplication rule (9b) can be replaced by any rule

$$[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta} + (\gamma_1\beta + \gamma_2\beta^3)\delta_{\alpha,-\beta}\mathscr{C} \tag{9e}$$

with constants γ_1, γ_2 in $\mathbb{F}$ such that $\gamma_2 \neq 0$. It is customary to use the normalization

$$[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta} + \tfrac{1}{12}(\beta^3 - \beta)\delta_{\alpha,-\beta}\mathscr{C}, \tag{9f}$$

because in this case the elements e_0, e_{-1}, e_1 form the $\mathbb{F}$-basis of the 3-dimensional simple Lie algebra isomorphic to $sl(2, \mathbb{F})$ and for natural number arguments $\beta \neq 2$ we obtain the smallest possible integer expression.

Now let us turn to the case that there is a basis $b_1, b_2, \ldots, b_d$ of the additive abelian group M with $d > 1$. In that case the values $c(b_1), c(b_2), \ldots, c(b_d)$ suffice to compute uniquely all other values in agreement with (8b), (8c). This is because

$$\begin{aligned} c(0) &= 0, && \\ c(-b_i) &= -c(b_i) && (1 \leqq i \leqq d), \\ 2b_1 &= (b_1 + b_i) + (b_1 - b_i) && (1 < i \leqq d), \\ (j+1)b_i &= jb_i + b_i && (j \in \mathbb{Z}^0, j1_{\mathbb{F}} \neq 1_{\mathbb{F}}), \\ (-j)b_i &= -(jb_i). && \end{aligned}$$

We observe that c_1, c_2 are linearly independent over $\mathbb{F}$ so that there is a unique $\mathbb{F}$-linear combination c' of c_1, c_2 satisfying

$$(c - c')(b_j) = 0 \quad \text{for} \quad j = 1, 2. \tag{9g}$$

As above the 2-coboundaries are provided by the $\mathbb{F}$-multiples of c_1.

We remark that the Jacobi identity implies the 2-parametric formula

$$\begin{aligned} (c - c')(\xi_1 b_1 + \xi_2 b_2 + b) &= (c - c')(\xi_1 b_1 + (\xi_2 b_2 + b)) \\ &= (\xi_2 b_2 + b - \xi_1 b_1)^{-1}(b - \xi_2 b_2)^{-1}(2\xi_1 b_1 + \xi_2 b_2 + b)(2\xi_2 b_2 + b)(c - c')(b) \\ &= (\xi_1 b_1 + b - \xi_2 b_2)^{-1}(b - \xi_1 b_1)^{-1}(2\xi_2 b_2 + \xi_1 b_1 + b)(2\xi_1 b_1 + b)(c - c')(b) \end{aligned}$$

$$\left(0 \neq b \in \sum_{j=1}^{d} \mathbb{F}b_j; \xi_1, \xi_2 \in \mathbb{Z}1_{\mathbb{F}}\right) \tag{9h}$$

for $d > 2$. It implies that $c = c'$ in case $\operatorname{char}(\mathbb{F}) \neq 2$ so that

Theorem 2. *If the field $\mathbb{F}$ is not of characteristic 2 then the generalized Virasoro algebra $V(M/\mathbb{F})$ corresponding to the simple Lie algebra $L(M/\mathbb{F})$ over a submodule M of $\mathbb{F}$ has the basis elements (9a) with multiplication rules (9b), (9c). But for* $\operatorname{char}(\mathbb{F}) = 2$ *any value assignment of $c(b_1), \ldots, c(b_d)$ in $\mathbb{F}$ leads to a 2-cocycle c so that the center of $V(M/\mathbb{F})$ has $\mathbb{F}$-dimension $d - 1$.* □

2. The Automorphism Group of the Generalized Virasoro Algebra for Characteristic Zero

For the study of the structure and of the representations of the generalized Virasoro algebras over a field $\mathbb{F}$ of characteristic 0 we use the **additive orderings** of a module M as a tool. Any total ordering of M with the property that the sum of two positive elements of M is positive is said to be an additive ordering. We observe that $\alpha > 0$ implies that $-\alpha < 0$, because $\alpha + (-\alpha) = 0$. We also observe that additive orderings only exist if the additive group M contains no elements of finite order > 1, i.e. M is torsion free.

The positive elements of an additively ordered submodule $M \neq 0$ form a halfmodule not containing 0. Conversely, any semimodule S contained in M that does not contain 0, is contained in a maximal subsemimodule S' of M of the same kind. Of course both satisfy the cancellation property so that they are halfmodules. And if the element y of M is not contained in S then by assumption, the subsemimodule $\mathbb{Z}^{>0}y \cup (\mathbb{Z}^{>0}y + S') \cup S'$ containing both y and S' also contains 0. Hence either $y = 0$ or there holds an equation $\lambda y + z = 0$ with $\lambda \in \mathbb{Z}^{>0}, z \in S'$. If also $-y$ is not contained in S' then there also holds an equation $\lambda'(-y) + z' = 0$ with $\lambda' \in \mathbb{Z}^{>0}, z' \in S'$ so that $0 = \lambda z' + \lambda' z \in S'$, a contradiction. It follows that $-y \notin S'$. Hence S' is the positivity set for the additive ordering

$$a > \Longleftrightarrow a - b \in S' \quad (a, b \in M)$$

of M. Thus it follows that any semimodule S contained in M that does not contain 0 is contained in the positivity subset of some additive ordering of M. In particular, every non-zero element of M can be made positive for some additive ordering of M.

We can define an additive ordering of a rank r module M by using an appropriate $\mathbb{Z}$-basis $b_1, b_2, \ldots, b_r$ of $\mathbb{Q}M$ to which we apply the lexicographic ordering of M

$$\sum \xi_i b_i > \sum \eta_i b_i \quad (\xi_i, \eta_i \in \mathbb{Q}; 1 \leqq i \leqq r)$$

if and only if for some index j

$$\xi_j > \eta_j \quad \text{and} \quad \xi_i = \eta_i, \quad (1 \leqq i < j).$$

Extending the range of the coefficients $\xi_1, \xi_2, \ldots, \xi_r$ beyond $\mathbb{Q}$ to the reals the positive linear combinations $\sum_{r=1}^{r} \xi_i \times b_i$ define a **halfflag** of the $\mathbb{R}$-linear space $\mathbb{R} \bigotimes_{\mathbb{Q}} M$ relative to the basis $b_1, b_2, \ldots, b_r$.

Definition 1. *The halfflag of the $\mathbb{R}$-linear space M relative to a well ordered basis $b_1, b_2, \ldots$ is the union of disjoint halfmodules $H_1, H_2, H_3, \ldots$*

$$H_1\colon \quad \lambda_1 b_1 + \sum_{i>1} \lambda_i b_i, \quad \lambda_1 > 0$$

$$H_2\colon \quad \lambda_2 b_2 + \sum_{i>2} \lambda_i b_i, \quad \lambda_2 > 0$$

$$H_3\colon \quad \lambda_3 b_3 + \sum_{i>3} \lambda_i b_i, \quad \lambda_3 > 0$$

$$\vdots$$

□

We realize that the $\mathbb{R}$-linear subspaces $\mathbb{R}H_1, \mathbb{R}H_2, \mathbb{R}H_3, \ldots$ generated by the $H_1, H_2, H_3, \ldots$ form the flag

$$\langle H_1 \rangle \supset \langle H_2 \rangle \supset \langle H_3 \rangle \supset \cdots$$

corresponding to the $\mathbb{R}$ – basis $\mathrm{b}_1, b_2, b_3, \ldots$ of M, and that for this interpretation H_i is an open halfplane of $\langle H_i \rangle$.

The halfflag remains invariant under positive triangular basis transformations

$$b_i \to b_i' = \alpha_{ii} b_i + \sum_{k>1} \alpha_{ik} b_k \quad (\alpha_{ii} > 0, \alpha_{ik} \in \mathbb{R} \text{ if } i < k).$$

Any other linear basis transformation of M carries the given halfflag into another halfflag distinct from the given one.

For example the numbers $\alpha + \beta i$ for which either α is a positive real rumber or $\alpha = 0$ and β is a non-negative real number, form a halfflag consisting of a halfplane and a halfline. Arbitrary halflags are obtained as the images of non-singular affine transformation fixing the origin.

We see that the complex numbers a with positive real part are "infinitely larger" than the purely imaginary numbers b in as much as $a > \lambda b$ for all rational integers.

Two positive elements a, b of an additively ordered module m are said to be **comparable** (in magnitude) if there are natural numbers κ, λ such that $\kappa a > b, a > \lambda b$. This relation is an equivalence relation. The number of comparability classes is said to be the **rank** of the additive ordering. It is at most equal to the rank of the module M, i.e. to the $\mathbb{Q}$-dimension of $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} M$.

The additive orderings of a rank r module M which were defined above by means of a $\mathbb{Q}$-basis of the $\mathbb{Q}$-module $\mathbb{Q} \underset{\mathbb{Z}}{\otimes} M$ may be said to be the **regular**, additive orderings. They are characterized by the equality of the order rank and r.

An example of an irregular additive ordering of a torsion free module of rank 2 is provided by the set of numbers $a + b\sqrt{2}$ $(a, b \in \mathbb{Z})$ whith natural ordering on the real line where the module rank is 2, but the rank of the ordering is 1. Essentially the same situation is provided by the Gaussian integers $a + bi$ $(a, b \in \mathbb{Z})$ with "unnatural ordering" $a + bi > c + di \Longleftrightarrow a - c + \sqrt{2}(b - d) > 0$.

Irregular additive orderings exist in case $r > 1$. They are obtained upon using an epimorphism μ of $\mathbb{R} \underset{\mathbb{Z}}{\otimes} M$ on $\mathbb{R}^{1 \times \rho}$ with $\rho \in \mathbb{Z}$, $1 \leqq \rho < r$, such that μ restricts to a monomorphism on M.

We obtain the corresponding irregular additive ordering of M by retrenchment of the lexicographic additive ordering of $\mathbb{R}^{1 \times \rho}$:

$$a > b \quad (a, b \in M) \Longleftrightarrow \mu(a - b) = 0, \ldots, 0, \xi_{i+1}, \ldots, \xi_\rho)$$
$$(i \in \mathbb{Z},\ 0 \leqq i < \rho,\ \xi_j \in \mathbb{R}\ (i < j \leqq \rho),\ \xi_{i+1} > 0).$$

We apply additive orderings of M to the study of finite dimensional subalgebras of the Virasoro algebra $V(M/\mathbb{F})$ over fields of zero characteristic and to the study of the automorphism group of $V(M/\mathbb{F})$ over $\mathbb{F}$.

Every additive ordering of M gives rise to a corresponding **triangular decomposition**

$$V(M/\mathbb{F}) = V(M/\mathbb{F})_+ \oplus V(M/\mathbb{F})_0 \oplus V(M/\mathbb{F})_-$$

of the generalized Virasoro algebra into the direct sum of the $\mathbb{F}$-subalgebras

$$V(M/\mathbb{F})_+ = \bigoplus_{\substack{\alpha > 0 \\ \alpha \in M}} \mathbb{F} e_\alpha,$$

$$V(M/\mathbb{F})_0 = \mathbb{F} e_0 + \mathbb{F}\mathscr{C},$$

$$V(M/\mathbb{F})_- = \bigoplus_{\substack{\alpha<0\\ \alpha\in M}} \mathbb{F}e_\alpha$$

such that

$$[V(M/\mathbb{F})_0, V(M/\mathbb{F})_\pm] = V(M/\mathbb{F})_\pm,$$
$$[V(M/\mathbb{F})_0, V(M/\mathbb{F})_0 = 0.$$

The triangular decomposition is a coarsening of the basic grading, but it is not itself a grading. It is useful in dealing with representations of $V(M/\mathbb{F})$. In particular it implies the existence of highest weight representations [10] for the algebra $V(M/\mathbb{F})$.

Lemma 1. *For any two elements x, y of the generalized Virasoro algebra $V(M/\mathbb{F})$ that satisfy an equation*

$$[x, y] = \lambda y$$

with λ in $\mathbb{F}$, we have either

$$x\in\mathbb{F}\mathscr{C}, \quad \lambda y = 0$$

or

$$x\notin\mathbb{F}\mathscr{C}, \quad \lambda y = 0, \quad y\in\mathbb{F}x + \mathbb{F}\mathscr{C}$$

or

$$\lambda y \neq 0; \quad x, y\in\mathbb{F}e_{-\alpha} + \mathbb{F}e_0 + \mathbb{F}e_\alpha + \mathbb{F}\mathscr{C}$$

for some $\alpha \neq 0$ of M. □

Proof. Let $x\notin\mathbb{F}\mathscr{C}, y\notin\mathbb{F}\mathscr{C}$. Using an additive ordering of M we present x, y in the form

$$x = \lambda_0\mathscr{C} + \sum_{i=1}^{\rho} \lambda_i e_{\alpha_i}, \quad y = \mu_0\mathscr{C} + \sum_{j=1}^{\sigma} \mu_j e_{\beta_j},$$

$$(\rho, \sigma\in\mathbb{Z}^{>0};\ \lambda_i, \mu_j\in\mathbb{F};\ 0\neq\lambda_i, 0\neq\mu_j, \alpha_i\in M, \beta_j\in M\ (0\leqq i\leqq\rho, 0\leqq j\leqq\sigma);$$
$$\alpha_1 < \alpha_2 < \cdots < \alpha_\rho, \beta_1 < \beta_2 < \cdots < \beta_\sigma).$$

If $\alpha_1 \neq \beta_1$ then the lowest term of $[x, y]$ relative to the additive ordering of M is $\lambda_1\mu_1(\beta_1 - \alpha_1)e_{\alpha_1+\beta_1}$ which is not zero. Hence $\lambda\neq 0$, $\alpha_1 = 0$ in case $\alpha_1\neq\beta_1$. Similarly $\lambda\neq 0$, $\alpha_\rho = 0$ in case $\alpha_\rho\neq\beta_\sigma$. Hence $\alpha_1 = \beta_1$, $\alpha_\rho = \beta_\sigma$ in case $\lambda = 0$. Therefore $[x, y - \lambda_1\mu_1^{-1}x] = 0$, where $y - \lambda_1\mu_1^{-1}x$ would have lowest term with M-index $> \alpha_1$ in case it would not be contained in $\mathbb{F}\mathscr{C}$. Hence $y\in\mathbb{F}x + \mathbb{F}\mathscr{C}$.

Next we deal with the case that $\lambda\neq 0, y\neq 0$. If $\rho = 1$ then we have

$$[x, y] = \lambda_1 \sum_{j=1}^{\sigma} \mu_j(\beta_j - \alpha_j)e_{\alpha_j+\beta_j} = \lambda\sum_{j=1}^{\sigma}\mu_j e_{\beta_j},$$

hence

$$0 = \alpha_1 < \beta_1, \quad \lambda_1\mu_j\beta_j = \lambda\mu_j, \quad \lambda_1\beta_j = \lambda, \quad \beta_j = \lambda_1^{-1}\lambda,$$
$$\sigma = 1, \quad x, y\in\mathbb{F}\mathscr{C} + \mathbb{F}e_0 + \mathbb{F}e_{\beta_1}.$$

It remains to deal with the case that $\rho > 1$. Now it is not possible that both

$\alpha_1 = 0$, $\alpha_\rho = 0$. Since there is the automorphism

$$\kappa\colon\ V(M/\mathbb{F}) \to V(M/\mathbb{F}),$$
$$\kappa(e_\alpha) = -e_{-\alpha} \quad (\alpha \in M),$$
$$\kappa(\mathscr{C}) = -\mathscr{C}$$

of $V(M/\mathbb{F})$ of order 2 over $\mathbb{F}$ which reverses the given additive ordering of M we can assume without loss of generality that $\alpha_1 = \beta_1$.

Since $[x, y] = \lambda y$ it follows that $\alpha_1 + \beta_2 = \alpha_1, \beta_2 = 0$. if $\alpha_\rho = \beta_\sigma$ then it follows by a similar argument that $\beta_{\sigma-1} = 0$. Hence $\sigma = 3, \alpha_1 = \beta_1 < \beta_2 = 0 < \beta_3 = \alpha_0$. If $\alpha_1 > -\beta_3$ then $\alpha_1 + \beta_3$ occurs among the β's. This is impossible because $0 < \alpha_1 + \beta_3 < \beta_3$. Hence $\alpha_1 \leqq -\beta_3$. Similarly it follows that $\alpha_\rho \geqq -\beta_1$. Hence $\beta_3 = -\beta_1$.

If $\alpha_i \neq \beta_j$ then $\alpha_i + \beta_j$ occurs among the β's. It follows that either $\rho = 2$ or $\rho = 3$, $\alpha_2 = 0$. In any event we have

$$x, y \in \mathbb{F}e_{-\alpha} + \mathbb{F}e_0 + \mathbb{F}e_\alpha + \mathbb{F}\mathscr{C}. \quad \square$$

We use Lemma 1 to determine the structure of the automorphism group $\mathrm{Aut}(V(M/\mathbb{F}))$. Firstly there is the normal subgroup $\mathrm{Aut}_{\mathbb{F}}(V(M/\mathbb{F}))$ formed by the automorphisms of $(V(M/\mathbb{F})$ over $\mathbb{F}$. The factor group is represented by the group $\mathrm{Aut}(\mathbb{F})'$ formed by the automorphisms ω' of $V(M/\mathbb{F})$ for which

$$\omega'(\lambda\mathscr{C}) = \omega(\lambda)\mathscr{C}, \quad \omega'(\lambda e_\alpha) = \omega(\lambda)e_\alpha \quad (\lambda \in \mathbb{F},\ \alpha \in M,\ \omega \in \mathrm{Aut}(\mathbb{F})).$$

They form a subgroup of $\mathrm{Aut}(V(M/\mathbb{F}))$ that is isomorphic to $\mathrm{Aut}(\mathbb{F})$ such that

$$\mathrm{Aut}(V(M/\mathbb{F})) = \mathrm{Aut}(\mathbb{F}) < \ltimes\ \mathrm{Aut}_{\mathbb{F}}(V(M/\mathbb{F})). \tag{10}$$

Theorem 3.

(a) *The center of the automorphism group* $\mathrm{Aut}(V(M/\mathbb{F}))$ *of the generalized Virasoro algebra* $V(M/\mathbb{F})$ *over* $\mathbb{F}$ *consists of the automorphisms*

$$\kappa = \kappa_\lambda\colon V(M/\mathbb{F}) \to V(M/\mathbb{F})$$
$$\kappa(e_0) = e_0 + \lambda\mathscr{C}$$
$$\kappa(e_\alpha) = e_\alpha \quad (0 \neq \alpha \in M\ (\lambda \in \mathbb{F}))$$
$$\kappa(\mathscr{C}) = \mathscr{C}$$

forming the one parametric central subgroup $\mathscr{Z}(\mathrm{Aut}_{\mathbb{F}}(M/\mathbb{F}))$.

(b) *The isomorphisms* $f\colon M \to \mathbb{F}\backslash 0$ *(additive to multiplicative) of the module* M *onto the multiplicative group* $\bar{\mathscr{M}}(M/\mathbb{F})$ *(a subgroup of the multiplicative group* $\mathbb{F}$*) form an abelian group under* $\mathbb{F}$*-multiplication that is isomorphic to the normal subgroup* $\mathscr{M}(M/\mathbb{F})$ *of* $\mathrm{Aut}_{\mathbb{F}}(V/\mathbb{F})$ *formed by the automorphisms*

$$\kappa\colon V(M/\mathbb{F}) \to V(M/\mathbb{F})$$
$$\kappa(e_\alpha) = f(\alpha)e_\alpha \quad (\alpha \in M)$$
$$\kappa(\mathscr{C}) = \mathscr{C}.$$

(c) *The* $\mathbb{F}$*-multipliers of* M *consisting of the elements* $\zeta \in \mathbb{F}$ *satisfying* $\zeta M = M$ *form a subgroup* $\bar{\mathscr{S}}(M/\mathbb{F})$ *of the multiplicative group of* $\mathbb{F}$ *that is isomorphic to the* *subgroup* $\mathscr{S}(M/\mathbb{F})$ *of* $\mathrm{Aut}_{\mathbb{F}}(M/\mathbb{F})$ *consisting of the scale change automorphisms*

$$\kappa'' = \kappa''_{\zeta}: V(M/\mathbb{F}) \to V(M/\mathbb{F})$$
$$\kappa''(e_\alpha) = e_{\zeta\alpha}$$
$$\kappa''(\mathscr{C}) = \zeta^3 \mathscr{C}.$$

(d) *The automorphism group of the generalized Virasoro algebra* $V(M/\mathbb{F})$ *over the characteristic zero field* $\mathbb{F}$ *is the semidirect product of* $\mathscr{S}(M/\mathbb{F})$ *over the direct product of the normal subgroup* $\mathscr{M}(M/\mathbb{F})$ *with the central subgroup* $\mathscr{Z}(\mathrm{Aut}_{\mathbb{F}}(V(M/\mathbb{F})))$,

$$\mathrm{Aut}_{\mathbb{F}}(V/\mathbb{F}) = \mathscr{S}(M/\mathbb{F}) \rtimes (\mathscr{M}(M/\mathbb{F}) \times \mathscr{Z}(\mathrm{Aut}_{\mathbb{F}}(V(M/\mathbb{F})))).$$

Proof. Any automorphism ω of $V(M/\mathbb{F})$ over $\mathbb{F}$ preserves Eqs. (9b) so that

$$[\omega(e_0), \omega(e_\alpha)] = \alpha\omega(e_\alpha) \quad (\alpha \in M),$$
$$\omega(e_0) = f(0)e_\alpha + \lambda \quad (f(0), \lambda \in \mathbb{F} 0 \neq f(0)) \tag{11a}$$
$$\omega(e_\alpha) = f(\alpha)e_{g(\alpha)}\lambda \quad (0 \neq f(\alpha) \in \mathbb{F}, 0 \neq g(\alpha) \in M, \lambda \in \mathbb{F} 0 \neq \alpha \in M) \tag{11b}$$

according to Lemma 1. Because of the automorphism property of ω we have

$$g(\alpha + \beta) = g(\alpha) + g(\beta) \quad (\alpha, \beta \in M)$$

where we set $g(0) = 0$ and $g(M) = M$, also

$$f(\alpha + \beta) = f(\alpha) f(\beta), \tag{11c}$$

finally

$$g(\alpha) = \zeta\alpha \quad (\alpha \in M) \tag{11d}$$

for some ζ of $\mathbb{F}$, and

$$\omega(\mathscr{C}) = \zeta^3 \mathscr{C}. \tag{11e}$$

Hence it follows that ζ is an $\mathbb{F}$-multiplier of M and that f is an additive to multiplicative monomorphism of M into the multiplicative group of $\mathbb{F}$.

The analysis given here implies (1), (2), (3). Conversely, if f is an isomophism (additive to multiplicative) of M into $\mathbb{F}\backslash 0$ and ζ is a multiplier and λ is some element of $\mathbb{F}$ then the $\mathbb{F}$-linear mapping ω of $V(M/\mathbb{F})$ on $V(M/\mathbb{F})$ defined by (11a), (11b) and (11c) is an automorphism over $\mathbb{F}$ so that (4) is verified. □

Theorem 3 may be summarized by the remark that the Cartan decomposition of $V(M/\mathbb{F})$ implied by the rules (9a–9c) is unique in the characteristic zero case so that $\mathrm{Aut}(V(M/\mathbb{F}))$ is the automorphism group of the $\mathbb{F}$-grading of $V(M/\mathbb{F})$ implied by (9a–9c). The diagonal subgroup is the group $\mathscr{S}(M/\mathbb{F})$, the stabilizer is the direct product of $\mathscr{S}(M/\mathbb{F})$ and $\mathscr{Z}(\mathrm{Aut}_{\mathbb{F}}(V(M/\mathbb{F}))$.

3. The Finite Dimensional Subalgebras of the Generalized Virasoro Algebras of Zero Characteristic

Theorem 4. *The finite dimensional subalgebras of the generalized Virasoro algebra* $V(M/\mathbb{F})$ *over a zero characteristic field* $\mathbb{F}$ *are* dim 1: *Any non-zero element of* $V(M/\mathbb{F})$ *generates a one dimensional subalgebra*

dim 2: $\mathbb{F}X + \mathbb{F}\mathscr{C}$ $\quad (0 \neq X \notin \mathbb{F}\mathscr{C})$

$\mathbb{F}(e_0 + \lambda\mathscr{C}) + \mathbb{F}e_\alpha$ $\quad (\lambda \in \mathbb{F}, 0 \neq \alpha \in M)$

dim 3: $\mathbb{F}[e_\alpha, e_{-\alpha}] + \mathbb{F}e_\alpha + \mathbb{F}e_{-\alpha}$ $\quad (0 \neq \alpha \in M)$

$\mathbb{F}\mathscr{C} + \mathbb{F}e_0 + \mathbb{F}e_\alpha$

dim 4: $\mathbb{F}e_0 + \mathbb{F}e_\alpha + \mathbb{F}e_{-\alpha} + \mathbb{F}\mathscr{C}$ $\quad (0 \neq \alpha \in M)$.

Proof. The proof follows from Lemma 1.

4. Realization by Differential Operations

As is well known the Lie algebra $L(\mathbb{Z}1_{\mathbb{F}}/\mathbb{F})$ is faithfully represented over any field $\mathbb{F}$ by the linear differential operators

$$\gamma(e_\alpha) = t^{\alpha+1}\frac{d}{dt}$$

which are applied e.g. to the elements of the rational function field $\mathbb{F}(t)$ in the variable t over $\mathbb{F}$.

The attempt to represent $L(M/\mathbb{F})$ in a similar way depends on a suitable definition of the power t^α for exponents α in M. That can be done in symbolic terms upon creating symbolic Laurent rings over M, M being the exponent range.

Over the complex number field we define the analytic function t^α of the complex variable t for any exponent $\alpha \in \mathbb{C}$ setting

$$t^\alpha = \exp(\alpha \log t).$$

However, the logarithmic function is not unique. Therefore we introduce the variable substitution

$$t = \exp(2\pi i s)$$

in terms of which we obtain the faithful representation of the Lie algebra $L(M/\mathbb{C})$ by the linear differential operators

$$\gamma(e_\alpha) = \frac{\exp(2\pi i \alpha s)}{2\pi i}\frac{d}{ds} \quad (\alpha \in M).$$

This construction can be used to introduce the higher rank Virasoro loops and Virasoro–Kac–Moody algebras. From the point of view of representation theory those Virasoro algebras $V(M/\mathbb{C})$ are of particular interest for which the submodule M of $\mathbb{C}$ is a vector lattice of the complex plane. Otherwise the Verma representation spaces either will have infinite multiplicities for some weights or the weights derived from a given highest weight will not form a discrete set.

Guided by those considerations we have unitary Verma representation spaces of $V(M/\mathbb{C})$, (cf. [10, 12]).

Definition 2. *Let $V(M/\mathbb{C})$ be a generalized Virasoro algebra with triangular decomposition (11). The $\mathbb{C}$-representation space m of $V(M/\mathbb{C})$ is said to be a* **Verma module** *of $V(M/\mathbb{C})$ if there is a non-zero element u_0 of m such that*

$$m = U(V(M/\mathbb{C}))_- u_0 + \mathbb{F} u_0,$$
$$U(V(M/\mathbb{C}))_0 u_0 = \mathbb{F} u_0$$
$$U(V(M/\mathbb{C}))_+ u_0 = 0$$

and the equation $Xu_0 = 0$ for $X \in U(V(M/\mathbb{C}))_-$ implies $X = 0$. □

It follows that m has a spectral decomposition

$$m = \oplus m_\lambda$$

into the direct sum of eigenspaces m_λ of $H = \mathbb{C}e_0 + \mathbb{C}\mathscr{C}$ such that λ is a $\mathbb{C}$-linear mapping of the Cartan subalgebra H in $\mathbb{C}$ and

$$m_\lambda \neq 0, \quad hu = \lambda(h)u, \quad (h \in H, u \in m_\lambda). \tag{12a}$$

Among the values $\lambda(e_0)$ there is a highest one, say λ_0, such that $m_{\lambda_0} = \mathbb{F}u_0$.

It follows that the mappings λ of H into $\mathbb{C}$ are $\mathbb{C}$-linear, they are the **weights** of the representation space. It is shown in the customary way that the universal enveloping algebra $U(V)$ of $V(M/\mathbb{C})$ has the direct module decomposition

$$U(V) = U(V)_+ \oplus U(V)_0 \oplus U(V)_-, \tag{12b}$$

where

$U(V)_+$ has the $\mathbb{C}$-basis elements $\mathscr{C}^{v_0} e_0^{v_1} e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_s}$,
$U(V)_-$ has the $\mathbb{C}$-basis elements $e_{-\alpha_s} e_{-\alpha_{s-1}} \cdots e_{-\alpha_2} e_{-\alpha_1} e_0^{v_1} \mathscr{C}^{v_0}$,
$U(V)_0$ has the $\mathbb{C}$-basis elements $e_0^{v_1} \mathscr{C}^{v_0}$,
$v_0, v_1 \in \mathbb{Z}^{\geqq 0}, v_0 + v_1 \geqq 0; s \in \mathbb{Z}^{\geqq 0}; \alpha_1, \alpha_2, \ldots, \alpha_s \in M, 0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_s)$

for any given additive total ordering of M. Moreover, any Verma representation space with highest weight λ_0 and proper halfplane P obtained by translation of the negative halfplane by λ_0 is $u_0 = m(V, P, \mathscr{C})$ operator epimorphic image of $U(V)$ subject to the relations

$$U(V)_+ u_{\lambda_0} = 0 \tag{12c}$$

and (11a) for $u = u_{\lambda_0}$. Note that

$$\lambda(\mathscr{C}) = \lambda_0(\mathscr{C}) \tag{12d}$$

for all weights λ.

Definition 3. *The Verma representation space is said to be* **unitary** *if there exists a positive definite invariant symmetric sesquilinear form*

$$m \times m \to \mathbb{C}$$
$$u \times v \to (u, v) \quad (u, v \in M) \tag{13a}$$

so that the mapping (12a) satisfies the bilinearity condition:

$$(u_1 + u_2, v) = (u_1, v) + (u_2, v),$$
$$(u, v_1 + v_2) = (u, v_1) + (u, v_2), \tag{13b}$$
$$(\xi u, v) = \xi(u, v) \tag{13c}$$

as well as the symmetry condition

$$(u, v), \quad (v, u) \text{ are complex conjugate one to another} \tag{13d}$$

as well as the $V(M/\mathbb{C})$-invariance condition

$$(e_\alpha u, v) = (u, e_{-\alpha} v) \quad (\alpha \in M) \tag{13e}$$

as well as the positivity conditions

$$(u_{\lambda_0}, u_{\lambda_0}) = 1, \tag{13f}$$
$$(u, u) > 0 \quad \text{if} \quad u \neq 0, \quad (u, v, u_1, u_2, v_1, v_2 \in m, \xi \in \mathbb{C}). \quad \square \tag{13g}$$

The decomposition (12a) already implies the unique existence of an invariant symmetric sesquilinear form (12a) (subject to (13b–13f)) on the Verma representation space $m(V, P, \lambda_0)$. If there is a unitary Verma representation space with λ_0 as highest weight then the invariant symmetric sesquilinear form defined on $m(V, P, \lambda_0)$ is non-negative:

$$(u, u) \geqq 0 \quad (u \in m). \tag{13h}$$

Conversely, if (12g) holds for $m = m(V, P, \lambda_0)$ then the elements v of m satisfying

$$(v, v) \geqq 0 \tag{13i}$$

form the invariant subspace $m^\perp$ is a unitary Verma representation space with highest weight λ_0. The problem is to determine for which P, λ_0, the universal Verma representation space satisfies the non-negativity condition (13h).

5. Examples

In this section we give various examples illustrating a range of behaviour patterns.

5.1. $\mathbb{F} = \mathbb{C}, M = \{a + ib \mid a, b \in \mathbb{Z}\}$ (Gaussian integers) with multiplication normalized to

$$[e_{a+ib}, e_{c+id}] = (c - a + i(d - b))e_{a+c+i(b+d)} + \delta_{a+c,0}\delta_{b+d,0}((c + id) + (c + id)^3)\mathscr{C}$$

such that $\mathbb{C}e_0 + \mathbb{C}e_i + \mathbb{C}e_{-i} \cong sl(2, \mathbb{C})$.

Here the multiplier group $\mathscr{S}(M/\mathbb{C})$ is cyclic of order 4 generated by the multiplication of M by the imaginary unit. The weights of each Verma module form a discrete set. The multiplicity of the highest weight $\lambda(e_0)$ always is one. For any other weight, say $\lambda(e_0) + a + ib$ with $(a, b) \neq (0, 0)$, the multiplicity is finite only if the additive ordering of M is regular and $a + ib$ belongs to the smallest comparability class.

5.2. $\mathbb{F} = \mathbb{C}, M = \{a + ib \mid a, b \in \mathbb{Q}\}$ (Gaussian number field).

The multiplier group $\mathscr{S}(M/\mathbb{C})$ is formed by the multiplication of M by non-zero elements of M.

For regular additive orderings of M the weights of a Verma module form a countable set with accumulation points filling a halfplane of $\mathbb{C}$. Only the highest weight has finite multiplicity and that is one.

5.3. $\mathbb{F} = \mathbb{C}, M = \{a + \sqrt{2}b \mid a, b \in \mathbb{Z}\}$ (algebraic integers of $\mathbb{Q}\sqrt{2}$).

The multiplier group $\mathscr{S}(M/\mathbb{C})$ is the direct product of the group generated by the involution of M defined by multiplication by -1 and the infinite cyclic group multiplication generated by the $1 + \sqrt{2}$. If the natural additive ordering of $\mathbb{R}$ is adopted and the highest weight of a Verma module is real then the weights form a countable dense subset of a left halfine of $\mathbb{R}$ with infinite multiplicities for all but the highest weight.

5.4. $\mathbb{F} = \mathbb{C}, M = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

The multiplier group of M consists of the multiplications of M by non-zero elements of M.

Regarding multiplicities we make the same observations as in 5.3.

5.5. $d > 1$, $\operatorname{char}(\mathbb{F}) = 2$.

We obtain a Lie algebra $V(M/\mathbb{F})$ of finite dimension 2^{2d-1} with center of dimension $d-1$ over the field $\mathbb{F}$ of characteristic 2. Its faithful irreducible representations are of 2-power degree depending on a certain parameter set [13].

5.6. $d > 1$ $\operatorname{char}(\mathbb{F}) > 2$.

We obtain a Lie algebra $V(M/\mathbb{F})$ of finite dimension p^d over the field $\mathbb{F}$ of characteristic $p > 2$. Its faithful irreducible representations are of p-power degree depending on a certain parameter set [13].

Acknowledgements. The authors are grateful to Drs. V. G. Kac and P. Ramond for a number of helpful remarks.

References

1. Virasoro, M. A.: Spectrum generating and no ghost theorem for the dual resonance model. Phys. Rev. **D1**, 2933 (1970)
2. Cartan, E.: Les groupes de transformations continues infinis simples. Ann. Sci, École Norm. Sup. **26**, 93–161 (1909)
3. Chang, Ho-Jui: Über Wittsche Lie-Ringe. Hamb. Abh. **14**, 151–184 (1941)
4. Zassenhaus, H.: Über Lie'sche Ringe mit Primzahlcharakteristik. Hamb. Abh. **13**, 1–100 (1939)
5. Kostrikhin, A. I., Shafarevich, I. R.: Graded Lie algebras of finite characteristic. Izv. Akad. Nauk. SSSR, Ser. Math. **33**, 251–322 (1969)
6. Brower, R. C., Thorn, C. B.: Eliminating spurious states from the dual resonance model. Nucl. Phys. **B31**, 163–182 (1971)
7. Kac, V. G., Raina, A. K.: Highest weight representations of infinite dimensional Lie algebras, Advanced Series in Mathematical Physics Vol. 2. Singapore: World Scientific 1987
8. Patera, J., Zassenhaus, H.: On Lie gradings I. Linear Algebra Appl. **112**, 87–159 (1989)
9. Kac, V. G.: Infinite dimensional Lie algebras. Cambridge: Cambridge University Press 1987
10. Moody, R. V., Pianzola, A.: Lie algebras with triangular decompositions. Vol. 1, New York: J. Wiley 1990 (to appear)
11. Friedan, J., Qui, Z., Shenker, S.: Conformal invariance, unitarity and two dimensional critical exponents. In: Vertex Operators in Mathematics and Physics. Berlin, Heidelberg, New York:

Springer 1983; Math. Science Res. Inst. Publications. Lepowsky, J., Mandelstam, S., Singer, I. (eds.) **3**, 419–450 (1983)
12. Langlands, R.: On unitary representations of the Virasoro algebra. In: Infinite dimensional Lie algebras. pp. 141–159, Kass, S. N. (ed.) Singapore: World Scientific 1986
13. Zassenhaus, H.: The representations of Lie algebras of prime characteristic. Proc. Glasgow Math. Assoc. **2**, 1–36 (1954)

Communicated by A. Jaffe