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Gibbs' Functionals on Subshifts

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Abstract. We consider functionals on one dimensional subshifts which have prescribed Randon–Nikodym derivative under transportation by conjugating homeomorphisms, and investigate their relation to Ruelle's transfer operator. In particular we show that two-sided functionals essentially are products of a functional which are supported on stable and unstable leaves. We also prove the meromorphicity of the Fourier transform of correlation functions for Axiom A follows in a more general setting

In this paper we study Gibbs' functionals on one dimensional lattice systems and relate them to the eigenspaces to Ruelle's Perron Frobenius operator. The knowledge we have about its spectrum enables us to give a classification of Gibbs' functionals in terms of eigenfunctionals up to a remainder which corresponds to the essential spectrum and therefore remains inaccessible to the technique employed here. However as the essential spectrum is an artefact of the Banach space of functions we are working with, it seems conceivable that a more sensible choice of a function space might remove this difficulty. Using interactions, previous work on this subject was done by Ruelle in [12] and [13]. Some of his results will be translated into a setting using an exponentially decreasing potential instead of interactions whereby one has to rely on Sinai's representation of two-sided functions by cohomologous one-sided ones. Similar to the case of a measure, we define Gibbs' functionals by their behaviour when transported by conjugating homeomorphisms. This characterisation of Gibbs' measures was first considered by Capocaccia [7] who also showed that given a point and its image the germs of conjugating homeomorphisms are locally unique. The need of extending this notion to functionals arose in [12] where the correlation function of Axiom A diffeomorphisms was examined and the region of meromorphicity of its Fourier transform determined. It thereby turned out that using Gibbs' functionals, the residues of simple poles acquire an intriguingly simple form. In the case of poles of higher order the expressions of the negative terms in the Laurent expansion become more involving; however the coefficients can again be expressed through Gibbs' functionals.

In the first section we define Gibbs' functionals using exponentially fast decaying functions or potentials. In Sect. 2 we show then that one-sided functionals are given by the eigenfunctionals of the transfer operator, and in Sect. 3 we establish the connection between the two- and one-sided functionals, proving that up to a remainder term they are given by products of one-sided functionals. In the final part we extend the result of [13] about meromorphicity of the correlation function for suspended flows to the case where the eigenspaces of Ruelle's operator no longer have to be one-dimensional. For simple (or semi-simple) eigenvalues of the transfer operator, Ruelle [13] introduced Gibbs' functionals for Axiom A flows. However it seems that there is little hope of defining Gibbs' functionals for multiple eigenvalues, at least not in a way which would result in a good theorem. In our context this without serious consequence.

1. Introduction

Let (Ω, T) be a Smale space (a compact metric space with a local product structure, see [11]) and denote by $d_\Omega(\cdot,\cdot)$ its metric. A map ψ from some open $U \subset \Omega$ into Ω is called *conjugating*, if $d_\Omega(T^k\psi(x), T^k(x)) \to 0$ for $|k| \to \infty$ uniformly in $x \in U$. With a properly chosen metric the distance actually decrease in a uniformly exponential way. Let F be a (real) valued Hölder continuous function on Ω and define

$$r(x) = \exp \sum_{k \in \mathbb{Z}} (FT^k\psi(x) - FT^k(x)).$$

Since the distance $d_\Omega(T^k\psi(x), T^k(x))$ decreases exponentially fast, the sum converges uniformly in $x \in U$. We say a probability measure μ on Ω is a *Gibbs' state* if for all conjugating homeomorphism ψ defined on some U_ψ, the measure $\psi\mu$ is absolutely continuous with respect to μ and the Radon Nikodym derivative (RN-derivative) satisfies $(d\psi\mu/d\mu)(x) = r(x)$ for $x \in U_\psi$. Se also [11] Chapter 7. For a continuous function $F: \Omega \to \mathbb{R}$ the *pressure* $P(F, T)$ is defined by the variational principle

$$P(F, T) = \sup_\rho (h_T(\rho) + \rho(F)),$$

where ρ are T-invariant probability measures on Ω. By $h_T(\rho)$ we denote the measure theoretic entropy of ρ with respect to T. Measures which attain the supremum are called equilibrium states for F. For details see Walters [16], where also equivalent definitions for pressure are given and discussed. If F is Hölder continuous and T topologically mixing there exists a unique equilibrium state which is also a Gibbs' state (see [11] Chap. 7). Conversely, a Gibbs' state is an equilibrium state and hence T-invariant, provided T is a topologically mixing transformation.

We set $C_\beta(\Omega)$ for complex Hölder continuous functions on Ω with Hölder exponent $\beta \in (0, 1]$. From now on μ no longer needs to be a measure but can be a functional, and from now on μ will be an element in the dual of $C_\beta(\Omega)$ unless stated otherwise. This generalisation was for Axiom A diffeomorphisms first considered by Ruelle in [12] and then in [13] extended to suspended flows. For $\mu, \nu \in C_\beta(\Omega)^*$ we say formally $r \in C_\beta(\Omega)$ is the RN-derivative $d\mu/d\nu$ if the identity $\mu = r\nu$ is satisfied. Besides this we shall take the liberty of writing $\int \chi d\mu = \int \chi r d\nu$ for (test) functions $\chi \in C_\beta(\Omega)$, as one is used to do for measures. The following definition extends the notion of Gibbs' states to functionals which under conjugating homeomorphisms have a prescribed RN-derivative.

Definition. 1. A functional $\mu \in C_\beta(\Omega)^*$ is a *Gibbs' functional* for $F \in C_\beta(\Omega)$ if

$$\mu((\chi \circ \psi) r) = \mu(\chi),$$

for all $\chi \in C_\beta(\Omega)^*$ with support in $\psi(U_\psi)$ and for all conjugating homeomorphisms ψ defined in some open $U_\psi \subset \Omega$.

We shall restrict mainly to subshifts of finite type. Let **A** be some finite set with the discrete topology, let A be an $|\mathbf{A}| \times |\mathbf{A}|$-matrix of zeros and ones and define

$$\Sigma = \left\{ z \in \prod_{i \in \mathbb{Z}} \mathbf{A} : A[z_i, z_{i+1}] = 1 \; \forall i \in \mathbb{Z} \right\},$$

which carries the product topology. We call Σ a subshift of finite type and define on it a (two-sided) shift transformation by $\sigma(z)_i = z_{i+1}, i \in \mathbb{Z}$. We say **A** is the alphabet of Σ. For positive $\rho < 1$ one defines a metric on Σ by $d(x, y) = \rho^k$, where $k = k(x, y) = \max\{j : x_i = y_i \text{ for all } |i| \leqq j\}$. The topology on Σ is then generated by the open-closed sets

$$U(x_{-n} \cdots x_n) = \{z \in \Sigma : z_i = x_i, |i| \leqq n\},$$

where $x_{-n} \cdots x_n$ is a word in Σ of length $2n+1$, $n = 1, 2, \ldots$. The $U(x_{-n} \cdots x_n)$ are usually called cylinders. Note that Σ has dimension zero. Throughout this paper we shall assume that (Σ, σ) is topologically mixing, that is A^n is positive for large n. The variation of a complex function f on Σ is a function defined by

$$\mathrm{var}_n f(x) = \sup\{|f(x) - f(y)| : y \in \Sigma \text{ satisfying } k(x, y) \geqq n\},$$

$n = 1, 2, \ldots$. If the variation decays fast enough, such that for some continuous and strictly positive function u on Σ

$$\|f\|_u = \sup_{x \in \Sigma} \sup_{n \in \mathbb{Z}} \mathrm{var}_n(x) \exp 2 \cdot \min(u^{-n}(x), u^n(x))$$

is finite, where $u^n = u + u\sigma + \cdots + u\sigma^{n-1}$ and $u^{-n} = u\sigma^{-1} + \cdots + u\sigma^{-n}$, we define a norm $|||\cdot|||_u = \|\cdot\|_\infty + \|\cdot\|_u$ and denote by $C_u(\Sigma)$ the space of complex functions on Σ which are finite with respect to this norm. In fact, $C_u(\Sigma)$ is a Banach algebra. We have a filtration $C_u(\Sigma) \subset C_{u'}(\Sigma)$, $0 < u' \leqq u$, and furthermore, $C_u(\Sigma)$ is dense in $C_{u'}(\Sigma)$, with $\bigcap_{u > 0} C_u(\Sigma)$ dense in $C_{u'}(\Sigma)$ (in the $|||\cdot|||_{u'}$-norm). Such a continuous and strictly positive u is called modulus of continuity. We shall assume that the variation of u itself decays exponentially fast.

A conjugating homeomorphism ψ defined on $U(z_k \cdots z_l) \subset \Sigma$ for some Σ-word $z_k \cdots z_l$, $k < l$, replaces the z-string by another word $z'_k \cdots z'_l$, $z'_k = z_k$, $z'_l = z_l$. As one can see this defines a conjugating map from U to $\psi(U)$. In fact all conjugating homeomorphisms on Σ are of this form. The RN-derivative of a Gibbs' functional for $f \in C_u(\Sigma)$ transported by ψ is given by $\exp \sum_{k \in \mathbb{Z}} (f\sigma^k\psi - f\sigma^k)$.

To discuss Gibbs' functionals on Σ it is necessary (or convenient) to consider the one-sided, left infinite and right infinite subshifts defined by

$$\Sigma_0 = \left\{ x \in \sum_{i \leqq 0} \mathbf{A} : A[x_{i-1}, x_i] = 1 \,\forall i \leqq 0 \right\},$$

$$\Sigma_1 = \left\{ y \in \prod_{i \geqq 1} \mathbf{A} : A[y_i, y_{i+1}] = 1 \,\forall i \geqq 1 \right\},$$

and the one-sided shift transformations shift $\sigma^{-1}: \Sigma_0 \to \Sigma_0$ which is induced by σ^{-1} on the two-sided subshift is onto and locally a homeomorphism (and finite, at most $|\mathbf{A}|$, to one) and $\sigma: \Sigma_1 \to \Sigma_1$ induced by σ. As above in the two-sided case we define variation and the Banach algebras $C_{u_0}(\Sigma_0)$ and $C_{u_1}(\Sigma_1)$ of functions which are finite with respect to the norms $|||f_i|||_{u_i} = \|f_i\|_\infty + \|f_i\|_{u_i}$, where the Hölder constants $\|f_i\|_{u_i}$ are here given by $\sup_{x \in \Sigma_i} \sup_{n \geqq 1} \mathrm{var}_n f_i(x) \exp u_i^{\pm n}(x)$, $i = 1$.

For one-sided Hölder continuous functions $f_i \in C_\theta(\Sigma_i)$, $i = 0, 1$, one defines according to Ruelle [11] Perron–Frobenius type operators (transfer matrices) $L_i: C_{u_i}(\Sigma_i) \to C_{u_i}(\Sigma_i)$ by: $(L_0\chi)(x) = \sum_{x' \in \sigma x} \chi(x') \exp f_0(x')$, $x \in \Sigma_0$, where the summation is over all $x' \in \Sigma_0$ satisfying $\sigma^{-1}x' = x$; and similarly $(L_1\chi)(y) = \sum_{y' \in \sigma^{-1}y} \chi(y') \exp f_1(y')$ for $y \in \Sigma_1$, $y' \in \sigma^{-1}y = \{u' \in \Sigma_1 : \sigma y' = y\}$.

Gibbs' functionals $\mathfrak{G}_0, \mathfrak{G}_1$ on the one-sided shifts Σ_0, Σ_1 are defined similar to the two-sided case above. The advantage however is that instead of an infinite sum over the integers from $-\infty$ to $+\infty$ we have to deal with finite sums (if we forget for a moment that we have to use Proposition 3 below), where the number of non-zero terms depends on the particular conjugating homeomorphism we are considering. We examine the right infinite case Σ_1 in more detail, results obtained there have also a formulation for the left sided case. Put T_n, $n \geqq 1$, for the set of Σ_1-words $\eta = \eta_1 \cdots \eta_n$ and $U(\eta)$ for the cylinder $\{y \in \Sigma_1 : \eta_i = y_i \,\forall i \leqq n\}$. A conjugating homeomorphism ψ_1 is of the form $\psi_1(\eta y) = \tilde{\eta} y$, where $\eta, \tilde{\eta} \in T_n$ and $y \in \Sigma_1$ is such that $\eta y, \tilde{\eta} y$ are allowed sequences. The RN-derivative of a Gibbs' functional v on Σ_1 is then given by

$$(d\psi_1 v/dv)(\eta y) = \exp(f_1{}^n(\tilde{\eta} y) - f_1{}^n(\eta y)).$$

In the case of Σ_0 one proceeds in the same way. Before passing on to the two-sided functionals we shall in the next section linger more on one-sided ones, in particular on $\mathfrak{G}_1$.

2. The One-sided Case

The main result of this section is Proposition 2 which classifies one-sided Gibbs' functionals as eigenfunctionals of Ruelle's operator. We restrict our attention to the right-sided case and drop the index 1 whenever possible. Let u, strictly positive and continuous, be a modulus of continuity for functions on Σ and let $f \in C_u(\Sigma)$. For real f Ruelle's Perron–Frobenius theorem [11] tells us that the largest eigenvalue λ_0 of the associated operator $L = L_f$ is real, positive and simple if (Σ, σ) is topologically mixing, while the rest of the spectrum is contained in a disc of radius strictly smaller than λ_0. The pressure of f as defined by the variational principle equals $\log \lambda_0$. In the case of complex valued f, L has isolated eigenvalues of finite multiplicity in the annulus $\{z \in \mathbb{C}: e^{P(\mathbb{R}f-u)} < |z| \leqq e^{P(\mathbb{R}f)}\}$ and an essential spectrum which is contained in the closed disc with radius $e^{P(\mathbb{R}f-u)}$, where $P(\mathbb{R}f)$ is the pressure of the real part of f (see [8] Lemma 2). Let $\mathbb{E}_\lambda$ be the eigenspace in $C_u(\Sigma)$ to the eigenvalue λ, $|\lambda| > e^{P(\mathbb{R}f-u)}$, and $\mathbb{E}_\lambda{}^*$ the corresponding eigenspace in the dual $C_u(\Sigma)^*$. As complex vectorspaces $\dim \mathbb{E}_\lambda = \dim \mathbb{E}_\lambda{}^* = l$. In $\mathbb{E}_\lambda{}^*, \mathbb{E}_\lambda$ we choose orthogonal bases $v_{\lambda,r}, N_{\lambda,r}, r = 1, \ldots, l$, normalised so that $v_{\lambda,r}(N_{\lambda',s}) = \delta_{\lambda,\lambda'}\delta_{r,s}$ ($\delta_{r,s}$ is the Kronecker symbol: $\delta_{r,s} = 1$ if $r = s$ and 0 otherwise). Hence we obtain the decompositions ($^\perp$ stands for transposition)

$$\begin{aligned} L^* v &= \sum_\lambda \lambda v(N_\lambda{}^\perp) L_\lambda v_\lambda + R^*(v), \\ L\chi &= \sum_\lambda \lambda N_\lambda{}^\perp L_\lambda v_\lambda(\chi) + R(\chi), \end{aligned} \tag{2-1}$$

where the spectral radii of the remainder $R: C_u(\Sigma) \to C_u(\Sigma)$, $R^*: C_u(\Sigma)^* \to C_u(\Sigma)^*$ are less or equal to $e^{P(\mathbb{R}f-u)}$, N_λ is the vector of eigenfunctions $(N_{\lambda,1}, N_{\lambda,2}, \ldots)^\perp$ and v_λ stands for the eigenfunctionals $(v_{\lambda,1}, v_{\lambda,2}, \ldots)^\perp$. The matrices L_λ are assumed to be in Jordan normal form with ones in the diagonal. We denote by σ^* the adjoint to the shift given by $(\sigma^* v)(\chi) = v(\chi \circ \sigma)$, $v \in C_u(\Sigma)^*, \chi \in C_u(\Sigma)$.

Proposition 2. *Let u be a modulus of continuity, $f \in C_u(\Sigma)$ and λ be in the discrete spectrum of L, then the functionals $v_{\lambda,s}$, $s = 1, \ldots, 1$ are Gibbs' functionals. Moreover, any Gibbs' functional $v \in C_u(\Sigma)^*$ has the representation*

$$v = \sum_\lambda v(N_\lambda)^\perp v_\lambda + \mathbb{P}^* v,$$

where $\mathbb{P}^*: C_u(\Sigma)^* \to C_u(\Sigma)^*$ is a projection and $L\mathbb{P}^*$ has spectral radius $\leqq e^{P(\mathbb{R}f-u)}$.

Proof. The first assertion was shown in [12], but nevertheless we shall give a proof. For $a, b, c \in \mathbf{A}$ satisfying $A[a,c] = A[b,c] = 1$ we define a conjugating homeomorphism $\psi_c: U(ac) \to U(bc)$ in the obvious way by putting $\psi_c(acy) = bcy$, for $acy \in U(ac)$. Since v is Gibbs, we have for (test) functions $\chi \in C_u(\Sigma)$,

$$\int_{U(ac)} \chi(ay)\, dv(ay) = \int_{U(bc)} \chi(ay) e^{f(ay) - f(by)}\, dv(by).$$

One obtains by summming over all $a \in \mathbf{A}$ satisfying $A[a,c] = 1$,

$$\sum_a \int_{U(ac)} \chi(y)\, dv(y) = \int_{U(bc)} (L\chi)(\sigma y) e^{-f(y)}\, dv(y),$$

which holds independent of $b \in \mathbf{A}$, $A[b,c] = 1$. Let $N(c)$ be the number of predecessors of c, that is the number of b which satisfy $A[b,c] = 1$ ($N(c) \geqq 1$) and sum over b:

$$\int_{\sigma^{-1}U(c)} \chi(y)\, dv(y) = N(c)^{-1} \int_{\sigma^{-1}U(c)} (L\chi)(\sigma y) e^{-f(y)}\, dv(y).$$

Put $g(acy) = f(acy) + \log N(c) \in C_u(\Sigma)$, and sum over $c \in \mathbf{A}$ for which we get

$$v(\chi) = \int_\Sigma (L\chi)(\sigma y) e^{-g(y)}\, dv(y) = (\sigma^* e^{-g} v)(L\chi).$$

We see that the condition $v = L^* \sigma^* e^{-g} v$ is necessary for v to be Gibbs. That it is sufficient follows from the fact that the equations also allow to be read from right to left. With (2-1) for L^* we decompose ν as follows

$$\nu = \sum_\lambda \lambda(\sigma^* e^{-g}\nu)(N_\lambda{}^\perp)L_\lambda \nu_\lambda + R^*(\sigma^* e^{-g}\nu), \tag{2-2}$$

where the spectral radius of $R^*: C_u(\Sigma)^* \to C_u(\Sigma)^*$ is less or equal to $e^{P(\mathbb{R}f-u)}$. This representation of ν shows that up to some remainder term $\mathbb{P}^*\nu$, which lies in a linear subspace of $C_u(\Sigma)$ to which restricted L has spectral radius $\leqq e^{P(\mathbb{R}f-u)}$, one-sided Gibbs' functionals are linear combinations of eigenfunctionals ν_λ. To conclude the proof we have to show that the functionals $\nu_{\lambda,r}$ are Gibbs'. To this end let κ, λ be discrete eigenvalues of L, recall that $\lambda L_\lambda{}^\perp N_\lambda = LN_\lambda$ (we write L for the unit matrix times L) and evaluate

$$\begin{aligned}(\sigma^* e^{-g}\nu_{\kappa,r})(\lambda L_\lambda{}^\perp N_\lambda)_s &= (L^*\kappa^{-1}L_\kappa{}^{-1}\nu_\kappa)_r((\lambda L_\lambda{}^\perp N_\lambda \circ \sigma)_s e^{-g})\\ &= \kappa^{-1}\lambda(L_\kappa{}^{-1}\nu_\kappa)_r(L((L_\lambda{}^\perp N_\lambda\circ\sigma)_s e^{-g}))\\ &= \kappa^{-1}\lambda \sum_{c\in A}\int_{U(c)} \sum_{a,A[a,c]=1} (L_\lambda{}^\perp N_\lambda(cx))_s e^{f(acx)-g(acx)} d(L_\kappa{}^{-1}\nu_\kappa)_r(cx)\\ &= \kappa^{-1}\lambda(L_\kappa{}^{-1}\nu_\kappa)_r(L_\lambda{}^\perp N_\lambda)_s = \begin{cases}0 & \text{if } \kappa \neq \lambda\\ \delta_{r,s} & \text{if } \kappa = \lambda.\end{cases}\end{aligned}$$

In addition we have $R^*\sigma^* e^{-g}\nu_{\kappa,r} = 0$ as can be seen from the following identities:

$$\begin{aligned}\nu_{\kappa,r}((R\chi)\circ\sigma e^{-g}) &= (L^*\kappa^{-1}L_\kappa{}^{-1}\nu_\kappa)_r((R\chi)\circ\sigma e^{-g})\\ &= (\kappa^{-1}L_\kappa{}^{-1}\nu_\kappa)_r(L((R\chi)\circ\sigma e^{-g}))\\ &= (\kappa^{-1}L_\kappa{}^{-1}\nu_\kappa)_r(R\chi) = 0\end{aligned}$$

for all $\chi \in C_u(\Sigma)$. Thus $L^*\sigma^* e^{-g}$ leaves eigenfunctionals of L^* invariant, which proves the first half of the proposition. Inserting this into (2-2) proves the second half. □

3. The Two-Sided Functionals

In this section we connect two-sided to one-sided Gibbs' functionals and prove in Proposition 4 the two-sided equivalent of Proposition 2 for the two-sided case. Two functions $f, g \in C_{u'}(\Sigma)$ are said to be cohomologous if there exists a Hölder continuous $w \in C_{u'}(\Sigma)$ such that $f - g = w - w\sigma$ for some positive u'. A function that is cohomologous to zero is a cocycle. Put $\mathbb{G}_f$ for the Gibbs' functionals for $f \in C_u(\Sigma)$. One easily sees that $\mathbb{G}_f = \mathbb{G}_g$ if g is cohomologous to f and that Gibbs' functionals are therefore determined by the equivalence class of cohomologous functions a particular f is in. This leaves some freedom to choose f, however only in the two-sided case. One-sided functional generally do change when adding to f a cocycle. The following classical result by Sinai [15] is essential for the description of two-sided Gibbs' functionals. It asserts that Hölder continuous functions are cohomologous to functions which are constraint in the local unstable direction.

Proposition 3. *For $f \in C_u(\Sigma)$ there exist w_1 and f_1 in $C_{(1/2)u}(\Sigma)$ such that $f_1 = f + w_1 - w_1\sigma$; we have $f_1(x) = f_1(y)$ whenever $x_i = y_i$, $i \leqq 0$. Moreover if f is real valued w_1, f_1 both can be chosen to be real.*

For a proof see [8] Proposition 1. The same statement for some $w_0 \in C_{(1/2)u}(\Sigma)$

applies to $f_0 = f + w_0 - w_0\sigma$ which has the property that $f_0(x) = f_0(y)$ whenever $x_i = y_i$ for $i \geqq 1$. In particular (f_0, f_1) can be identified with an element in $C_{u_0}(\Sigma_0) \times C_{u_1}(\Sigma_1)$, where the one-sided moduli of continuity u_0, u_1 are cohomologous to u and such that functions in $C_{(1/2)u}(\Sigma)$ which depend only on negative or positive coordinates can be identified with elements in $C_{u_0}(\Sigma_0)$ and $C_{u_1}(\Sigma_1)$. We shall repeatedly use one-sided functions in a two-sided context with the obvious meaning. Without loss of generality we can assume that u_0, u_1 are positive (see [8]). Put $\mathbb{G}_0$ for the one-sided Gibbs' functionals on Σ_0 associated to f_0 and $\mathbb{G}_1 \subset C_{u_1}(\Sigma_1)$ for those associated to $f_1\sigma$ and let L_0, L_1 be the transfer operators on $C_{u_0}(\Sigma_0), C_{u_1}(\Sigma_1)$ with weight functions f_0 and $f_1\sigma$. For a discrete eigenvalue λ of L_1 let as in the previous section $v_{\lambda,s}, N_{\lambda,s}$, $s = 1, \dots, l$, be normalised orthogonal bases in $\mathbb{E}_{1,\lambda}{}^*, \mathbb{E}_{1,\lambda}$ and $\mu_{\kappa,r}, M_{\kappa,r}$, $s = 1, \dots, k$, normalised orthogonal bases spanning the eigenspaces of $L_0{}^*, L_0$ to the discrete eigenvalue κ. Put $\tau(z) = (w_0 - w_1)\sigma(z) \in C_{(1/2)u}(\Sigma)$ and define a map $\tilde{\tau}$ from $C_{u_0}(\Sigma_0) \times C_{u_1}(\Sigma_1)$ into $C_{(1/2)u}(\Sigma)$ by $\tilde{\tau}(\mu, v) = e^{\tau}\mu v$.

As (Σ, σ) is a hyperbolic space we can identify the local stable and unstable "leaves" with Σ_0 and Σ_1 respectively. The transfer operators L_0 and L_1 associated to the functions f_0 and $f_1\sigma$ act in the stable and unstable directions and can be interpreted as operators in $C_{(1/2)u}(\Sigma)$ as (the comma separates the zero'th coordinate from the first)

$$(L_1\chi)(x, y) = \sum_{\eta \in A} \chi(x, \eta y) e^{f_1\sigma(\eta y)},$$

$\chi \in C_{(1/2)u}(\Sigma)$, and similarly for L_0. Denote by $\mathbb{F}_i \subset C_{(1/2)u}(\Sigma)^*$ the subspace of functionals on which the induced operator $L_i{}^*$ has spectral radius $\leqq e^P, P = P(\mathbb{R}f - u)$, $i = 0, 1$, and put $\mathbb{F} = \mathbb{F}_0 \cup \mathbb{F}_1$.

Proposition 4. *The image of $\mathbb{G}_0 \times \mathbb{G}_1$ under $\tilde{\tau}$ are two-sided Gibbs' functionals for f. Moreover any two-sided Gibbs' functional ω for f which can be extended to $C_{(1/2)u}(\Sigma)^*$ has the representation*

$$\omega = \sum_{\kappa,\lambda} \sum_{r,s} \omega(e^{-\tau} M_{\kappa,r} N_{\lambda,s}) \tilde{\tau}(\mu_{\kappa,r}, v_{\lambda,s}) + \tilde{\omega},$$

where the summation κ, λ is over the discrete spectrum of $L_0{}^*$ and $L_1{}^*$ and $\tilde{\omega} \in \mathbb{F}$.

Proof. We first show the second part of the proposition. For ω Gibbs' and a conjugating homeomorphism ψ defined on $U_\psi \subset \Sigma$ the RN-derivative $(d\psi\omega/d\omega)(z) = r(z)$ is given by

$$r(z) = \exp \sum_{k \in \mathbb{Z}} (f\sigma^k\psi(z) - f\sigma^k(z)),$$

$z \in U_\psi$. Without loss of generality we may assume that U_ψ is a cylinder set $\{z \in \Sigma \colon z_i = z'_i \text{ for } i = k, \dots, l\}$ for some Σ-word $z'_k \cdots z'_l$, with $k \leqq 0 < l$ say. Let $z''_k \cdots z''_l$ be a Σ-word with $z''_k = z'_k, z''_l = z'_l$, then a conjugating homeomorphism ψ is defined by $\psi(z)_i = z''_i$, $i = k, \dots, 1$ and $\psi(z)_i = z_i$ otherwise, maps U_ψ onto $U(z''_k \cdots z''_l)$ and induces conjugating homeomorphisms ψ_i that map some open $U_{i,\psi}$ into Σ_1 by $\psi_1(y)_i = y_i$, $i > l$, and $\psi_1(y)_i = z''_i$, $i = 1, \dots, l$, for $y \in U(z'_0 \cdots z'_l)$; and similarly for $\psi_0 \colon \psi_0(x)_i = x_i$, $i < k$, and $\psi_0(x)_i = z''_i$, $k \leqq i \leqq 0$, for $x \in \{x \in \Sigma_0 \colon x_i = z'_i, k \leqq i \leqq 0\}$. Put

$$r_0 = \exp \sum_{k \leqq 0} (f_0\sigma^k\psi_0 - f_0\sigma^k) \in C_{u_0}(\Sigma_0), \quad r_1 = \exp \sum_{k \leqq 1} (f_1\sigma^k\psi_1 - f_1\sigma^k) \in C_{u_1}(\Sigma_1),$$

which are defined on $U_{0,\psi}$, $U_{1,\psi}$ respectively. With $\tau = w_0\sigma - w_1\sigma$ the RN-derivative $(d\psi\omega/d\omega)(z)$ assumes that form

$$r(z) = r_0(x)r_1(y)\exp(\tau\psi(z) - \tau(z)), \tag{3-1}$$

where $xy = z \in U_\psi$. By assumption ω can be extended to a functional on $C_{(1/2)u}(\Sigma)$ which we again denote by ω. Since $C_u(\Sigma)$ is dense in $C_{(1/2)u}(\Sigma)$ this extension is unique, which makes $e^{-\tau}\omega$ a well defined element in $C_{(1/2)u}(\Sigma)^*$ that has RN-derivative $(d\psi e^{-\tau}\omega/de^{-\tau}\omega)(xy) = r_0(x)r_1(y)$ when transported by ψ. Let $\chi \in C_{(1/2)u}(\Sigma)$ be independent of positive coordinates and define on Σ_1 a functional $\hat{\omega}(\chi,\cdot)$ by putting $\hat{\omega}(\chi,\chi') = \omega(e^{-\tau}\chi\cdot\chi')$ where $\chi' \in C_{(1/2)u}(\Sigma)$ depends only on coordinates >0. Since $\hat{\omega}(\chi,\cdot)$ is Gibbs' by Proposition 2 it can be written as

$$\hat{\omega}(\chi,\cdot) = \sum_\lambda \sum_s \hat{\omega}(\chi, N_{\lambda,s})\nu_{\lambda,s}(\cdot) + \mathbb{P}_1{}^*\hat{\omega}(\chi,\cdot), \tag{3-2}$$

where the summation is over the discrete spectrum of $L_1{}^*$ and $\mathbb{P}_1{}^*$ is a projection operator such that $L_1{}^*\mathbb{P}_1{}^*$ has spectral radius $\leq e^P$. Now as χ varies $\hat{\omega}(\cdot, N_{\lambda,s})$ is a functional on Σ_0 and, as we have seen, Gibbs'. Thus by Proposition 2

$$\hat{\omega}(\cdot, N_{\lambda,s}) = \sum_\kappa \sum_r \hat{\omega}(M_{\kappa,r}, N_{\lambda,s})\mu_{\kappa,r}(\cdot) + \mathbb{P}_0{}^*\hat{\omega}(\cdot, N_{\lambda,s}), \tag{3-3}$$

where the spectral radius of $L_0{}^*\mathbb{P}_0{}^*$ is bounded by e^P. The second half of the proposition follows now from (3-2) and (3-3) since $\hat{\omega} = e^{-\tau}\omega$ for functions which are products of right-sided and left-sided functions. Note that $\mathbb{F}_i = \mathbb{P}_i{}^*C_{(1/2)u}(\Sigma)^*$, $i = 0, 1$.

It remains to show the first part of the statement. For $(\mu,\nu) \in \mathbb{G}_0 \times \mathbb{G}_1$ we obviously have $e^\tau\mu\nu \in C_{(1/2)u}(\Sigma)^* \subset C_u(\Sigma)^*$. As described above a conjugating homeomorphism $\psi: U_\psi \to \Sigma$ induces conjugating homeomorphisms ψ_0, ψ_1 in Σ_0 and Σ_1. Now as μ and ν are Gibbs' it follows from (3-1) that $e^\tau\mu\nu$ is Gibbs' as well. □

If we call $e^\tau\mu_{\kappa,r}\nu_{\lambda,s}$ pure Gibbs' functionals, then with an additional normalising condition on $\mathbb{G}_0$ and $\mathbb{G}_1$ $\tilde{\tau}$ is a bijection onto the pure two-sided functionals whose inverse is given by ω. A trivial but nonetheless interesting consequence of the last proposition is the following corollary which asserts a sort of local product structure for equilibrium states. We take a weighted product of transversal measures on stable and unstable leaves whereby the weight function turns out to be Hölder continuous of class $C_{(1/2)u}(\Sigma)$ and not of class $C_u(\Sigma)$, since its derivation involved Proposition 3 where we lost some regularity.

Corollary 5. *The equilibrium state μ on Σ is up to a normalising factor of the form $e^\tau\mu\nu$, where μ,ν are the unique Gibbs' measures on Σ_0, Σ_1 and τ is as above.*

Proposition 6. ([12] Proposition 2.2) *L_0 and L_1 have the same discrete eigenvalues with the same multiplicity and the eigenspaces are isomorphic.*

Proof. For reason of completeness we bring a proof which is modelled after [12] Proposition 2.2, adapting it to our purpose. Let τ be as before and define an operator $D: C_{u_1}(\Sigma_1)^* \to C_{u_0}(\Sigma_0)$ by (comma parts the zero'th coordinate from the first)

$$(D\nu)(x) = \int e^{\tau(x,y)}d\nu(y),$$

$\nu \in C_{u_1}(\Sigma_1)^*$, for which we can do the following transformations: (Set $\tau(x, y) = -\infty$ whenever $xy \notin \Sigma$.)

$$\begin{aligned} DL_1^*(\nu)(x) &= \int (L_1 e^\tau)(x, y) d\nu(y) \\ &= \int \sum_{\eta \in A} \exp(\tau(x, \eta y) + f_1 \sigma(\eta y)) d\nu(y) \\ &= \int \sum_{\eta \in A} \exp(w_0(x\eta, y) - w_1(x\eta, y) + f(x\eta, y) + w_1(x\eta, y) - w_1 \sigma(x\eta, y)) d\nu(y) \\ &= \int (L_0 e^\tau)(x, y) d\nu(y) = (L_0 D\nu)(x). \end{aligned}$$

This shows that $L_0 D = DL_1^*$, and similarly one shows that $L_1 D^* = D^* L_0^*$, where D^* adjoint to D. One sees that D maps $\mathbb{E}_{1,\lambda}^*$ into $\mathbb{E}_{0,\lambda}$ for discrete eigenvalues λ. A similar statement applies to D^*. We still have to show that D is injective on $\mathbb{E}_{1,\lambda}^*$. Let $\nu \in \mathbb{E}_{1,\lambda}^*$ be non-zero and $\chi \in C_{u_1}(\Sigma_1)$ such that $\nu(e^\tau \chi)$ does not vanish. Without losing any generality we can assume that $\chi(y)$ depends only on the first n coordinates for some n, that is χ is constant on cylinders $U(\eta), \eta \in T_n$. Moreover since $\nu(e^\tau \chi) \neq 0$ we have that also $(L_1^*\nu)(e^\tau \chi)$ is non-zero and

$$\begin{aligned} (L_1^{*n}\nu)(e^\tau \chi) &= \int_{\Sigma_1} \sum_{|\eta| = n} \chi(\eta) \exp(\tau(x, \eta y) + f_1{}^n \sigma(\eta y)) d\nu(y) \\ &= \sum_{|\eta| = n} \chi(\eta) \exp f_0{}^{(-n)}(x\eta) \int_{\Sigma_1} e^{\tau(x\eta, y)} d\nu(y) = L_0{}^n(\chi D\nu)(x) \end{aligned}$$

is non-zero which implies that $D\nu$ does not vanish. In the last equation χ is a locally constant function in $C_{u_0}(\Sigma_0)$. We also used $f_1 = f_0 - \tau + \tau\sigma^{-1}$ and $f_1{}^n \sigma(\eta y) = f_0{}^{(-n)}(x\eta) - (\tau - \tau\sigma^n)(x, \eta y)$, where $f_0{}^{(-n)} = f_0 + \cdots + f_0 \sigma^{1-n}$. By the same argument one shows that $D^* \mathbb{E}_{0,\lambda}^*$ is injective. Hence $D\mathbb{E}_{1,\lambda}^* \subset \mathbb{E}_{0,\lambda}$ and $D^* \mathbb{E}_{0,\lambda}^* \subset \mathbb{E}_{1,\lambda}$ which implies that $\mathbb{E}_{0,\lambda}$ and $\mathbb{E}_{1,\lambda}$ are isomorphic. □

This proposition is particularly interesting for evaluating one-sided functions. Let $\chi \in C_{(1/2)u}(\Sigma)$ be independent of positive coordinates so that we can identify it with a function in $C_{u_0}(\Sigma_0)$ and let $\nu \in C_{u_1}(\Sigma_1)^*$ be an eigenfunctional to the eigenvalues λ and $\mu \in C_{u_0}(\Sigma_0)^*$ Gibbs. Then

$$\tilde{\tau}(\mu, \nu)(\chi) = \mu\nu(e^\tau \chi) = \mu(\nu(e^\tau)\chi) = \mu(M\chi),$$

where $M = D\nu \in C_{u_0}(\Sigma_0)$ is an eigenfunction of L_0 to the eigenvalue λ, and a similar result holds if χ depends only on positive coordinates. For f real, the unique Gibbs' measure on Σ_0 is up to a normalising factor given by $M\mu$, where $M = D\nu, \mu$ and ν span the one dimensional eigenspaces in $C_{u_0}(\Sigma_0), C_{u_0}(\Sigma_0)^*$ and $C_{u_1}(\Sigma_1)^*$ to the eigenvalue $e^{P(f)}$ which they share. Proofs and details to this classical result can be found in [1].

Let μ_κ, ν_λ span $\mathbb{E}_{0,\kappa}^*, \mathbb{E}_{1,\lambda}^*$ and be such that $L_0^* \mu_\kappa = \kappa L_{0,\kappa} \mu_\kappa$, $L_1^* \nu_\lambda = \lambda L_{1,\lambda} \nu_\lambda$, where $L_{0,\kappa}, L_{1,\lambda}$ are in Jordan normal form with 1's in the diagonal. If we choose normalized and orthogonal bases $M_\lambda, N_\lambda, \mu_\lambda, \nu_\lambda$ in $\mathbb{E}_{0,\lambda} \mathbb{E}_{1,\lambda}, \mathbb{E}_{0,\lambda}^*, \mathbb{E}_{1,\lambda}^*$ such that $M_\lambda = D\nu_\lambda$ and $N_\lambda = D^* \mu_\lambda$, the linear maps are related by $L_{0,\lambda}{}^\perp = L_{1,\lambda}$ since

$$\lambda L_{0,\lambda}{}^\perp M_\lambda = L_0 M_\lambda = L_0 D\nu_\lambda = DL_1^* \nu_\lambda = \lambda L_{1,\lambda} D\nu_\lambda = \lambda L_{1,\lambda} M_\lambda.$$

Define a collection of Gibbs' functionals by

$$\mu_{\kappa\lambda} = e^\tau \mu_\kappa \nu_\lambda{}^\perp,$$

where $\mu_{\kappa\lambda}$ is a $k \times l$-matrix of functionals on Σ with entries $(\mu_{\kappa\lambda})_{ij} = e^{\tau}(\mu_{\kappa})_i(\nu_\lambda)_j$ and k, l are the dimensions of $\mathbb{E}_{0,\kappa}$ and $\mathbb{E}_{1,\lambda}$. Notice that $(\mu_{\kappa\lambda})_{ij} \in C_{1/2u}(\Sigma)^*$. If λ is a simple eigenvalue then $\mu_{\lambda\lambda}$ is in fact the derivative of λ. For this see [12] where we also take Proposition 2.3 in the following form:

Lemma 7. *Let* $\mathbb{E}_{0,\kappa}{}^*$, $\mathbb{E}_{1,\lambda}{}^*$ *be eigenspaces of* $L_0{}^*, L_1{}^*$, *then* σ^* *restricted to* $\tilde{\tau}(\mathbb{E}_{0,\kappa}{}^* \times \mathbb{E}_{1,\lambda}{}^*)$ *has the eigenvalue* $\kappa\lambda^{-1}$ *and satisfies*

$$\sigma^* \mu_{\kappa\lambda} = \kappa\lambda^{-1} e^{\tau}(L_{0,\kappa}\mu_\kappa)(L_{1,\lambda}{}^{-1}\nu_\lambda)^{\perp} = \kappa\lambda^{-1} L_{0,\kappa}\mu_{\kappa\lambda} L_{1,\lambda}{}^{-1\perp}.$$

4. The Zeta Function

Given $f \in C_u(\Sigma)$ (Σ can be a one or two-sided subshift) the zeta function is then defined by

$$\zeta(f) = \exp \sum_{m \in \mathbb{N}} \zeta_m/m,$$

where $\zeta_m(f) = \sum_{x \in F(m)} \exp f^m(x)$ and $\mathbb{F}(m) = \{x \in \Sigma : \sigma^m x = x\}$ are the periodic points of period m. The pressure of the real part of f is a given by the variational principle and equals (cf. [1])

$$\lim_{m \in \mathbb{N}} m^{-1} \log \sum_{x \in F(m)} \exp \mathbb{R} f^m(x).$$

Thus we see that whenever $P(\mathbb{R}f) < 0$ the summation over m converges to an analytic and non-zero function and according to [8] Theorem 4 can meromorphically be continued as follows:

Theorem 8. *$\zeta(f)$ is a non-zero and analytic function in* $\{f \in C_u(\Sigma) : P(\mathbb{R}f) < 0\}$ *and has a meromorphic extension to the halfplane* $\{f \in C_u(\Sigma) : P(\mathbb{R}f - u) < 0\}$.

Note that zeta function and pressure do not change by adding a cocycle to f. In the one-sided case replacing f by $f + \log z$, $z \in \mathbb{C} \setminus \{0\}$, has the effect that the eigenvalues of L_f are scaled by z and the pressure of the real part becomes $P(\mathbb{R}f + \log|z|) = P(\mathbb{R}f) + \log|z|$. Ruelle introduced a generalized zeta function which we denote here by

$$d(z, f) = \zeta(f + \log z) = \exp \sum_{m \in \mathbb{N}} z^m \zeta_m/m,$$

(see also [11] Chapter 5.29) and for which the following result holds:

Corollary 9. *$d(z, f)$ is a non-zero analytic function for* $|z| < e^{-P(\mathbb{R}f)}$ *and can meromorphically be extended to* $|z| < e^{-P(\mathbb{R}f - u)}$ *with poles at* $1/\lambda(f)$, *where* $\lambda(f)$ *are eigenvalues of* L_f *(counting multiplicities).*

Consider the one-sided right infinite case and put $\mathbb{P}_\lambda(\cdot) = N_\lambda{}^{\perp}\nu_\lambda(\cdot)$ for the projection onto the eigenspace $\mathbb{E}_{1,\lambda}$. We shall need the following corollary which also has a formulation in the left sided case.

Corollary 10. *Given* $f'' \in C_{u_1}(\Sigma_1)$ *then* $(1 - zL_{f''})^{-1} = \sum_\lambda N_\lambda{}^{\perp}(1 - z\lambda L_{1,\lambda})^{-1}\nu_\lambda + K_z$, *where* K_z *is holomorphic for* $|z| < e^{-P(\mathbb{R}f'' - u_1)}$.

5. Suspended Flows

Let (Σ, σ) be a two-sided topologically mixing subshift of finite type and $r \in C_u(\Sigma)$ a real and strictly positive function. From now on the modulus of continuity is $u = 2\gamma r$ for some positive constant γ. Of course any positive and continuous function would do. The reason for this particular choice lies in the fact that suspensions which via Markov partitions are derived from Axiom A flows on manifolds have ceiling functions with this regularity where γ is half the contraction parameter of the flow (see [8] Sect. 5). See [2] for a detailed account of the construction of Markov partitions and suspensions for Axiom A flows. We put

$$\Sigma_r = \{(x, t) \in \Sigma \times \mathbb{R} : 0 \leqq t \leqq r(x)\}$$

and $\varphi_t : \Sigma_r \to \Sigma_r$, t real, for the suspended flow which is defined by $\varphi_t(x; s) = (x, s + t)$ whenever $0 \leqq s, s + t \leqq r(x)$ and extended to $t \in \mathbb{R}$ by identifying $(x, r(x))$ with $(\sigma x, 0)$, where σ is the shift on the lattice Σ. The function r is frequently called the ceiling or return function of the flow φ_t. We use the product topology on Σ_r (for a metric see [6]) and assume that there is more than one closed orbit (weak mixing). The entropy of φ_t is the entropy of the "time-one" map φ_1.

Denote by Λ the Lebesgue measure on $\mathbb{R}$ and let μ be a σ-invariant probability measure on the discrete Σ, then $\bar{\mu} = \mu(r)^{-1}\mu \times \Lambda$ is a φ_t-invariant probability measure on Σ_r. The measure theoretic entropy $h^*(\bar{\mu})$ which is given by the time-one map equals by Abramov's formula $h(\mu)/\mu(r)$, where $h(\mu)$ is the measure theoretic entropy of Σ with respect to μ. We also have the identity $P(-h^*(\bar{\mu})r) = 0$. The pressure of a continuous $F : \Sigma_r \to \mathbb{R}$ is defined by the

$$P^*(F) = \sup_{\rho} (h^*(\rho) + \rho(F)).$$

where ρ are φ_t-invariant probability measures on Σ_r. A measure that attains the supremum is called an equilibrium or Gibbs' state. If μ is an equilibrium state on Σ to the function $f - P^*(F)r$, $f(x) = \int_0^{r(x)} F(x, t)dt$, then $\bar{\mu} = \mu \times \Lambda / \mu(r)$ is an equilibrium state on Σ_r to the function F and we have $P(f - P^*(F)r) = 0$ which also determines $P^*(F)$ uniquely [5].

Denote by $\lambda(\mathcal{O})$ the length of a closed orbit $\mathcal{O}$ under the flow φ_t. The zeta function for a continuous $F : \Sigma_r \to \mathbb{C}$ is then given by

$$\zeta^*(z) = \prod_{\mathcal{O}} \left(1 - \exp \int_0^{\lambda(\mathcal{O})} F(\varphi_t x_{\mathcal{O}}) - z)dt\right)^{-1},$$

where $x_{\mathcal{O}} \in \mathcal{O}$ is a point on the closed orbit $\mathcal{O}$, z a complex variable, and where the Euler product is over all closed orbits. Provided $f(x)$ is an element in $C_u(\Sigma)$ we can express the zeta function $\zeta^*(z)$ by the one introduced in Sect. 4. Hence $\zeta(f - zr) = \zeta^*(z)$ and it follows from Theorem 8 that $\zeta^*(z)$ has a meromorphic extension to z for which $P(\mathbb{R}(f - zr) - \gamma r) < 0$. Thus

Proposition 11. [8] *$\zeta^*(z)$ is non-zero and analytic for $\mathbb{R}z > P^*(\mathbb{R}F)$ and has a meromorphic extension to the halfplane $\{z \in \mathbb{C} : \mathbb{R}z > P^*(\mathbb{R}F)\text{-}\gamma\}$ with a pole whenever L_{f+zr} has 1 as eigenvalue.*

Let $v_0, v_1, r_0, r_1 \in C_{(1/2)u}(\Sigma)$ be functions chosen according to Proposition 3 so that $r_0 = r + v_0 - v_0\sigma$, $r_1 = r + v_1 - v_1\sigma$, depend only on coordinates $\leqq 0$, $\geqq 1$ respectively and can be identified with functions in $C_{u_0}(\Sigma_0)$, $C_{u_1}(\Sigma_1)$. Note that r_0 and r_1 are real but not necessarily strictly positive although r is. Let as in Sect. 3 f_0, f_1 be one-sided functions cohomologous to f, then according to [13] we can do the following construction. For complex numbers α, β we form transfer operators $L_{f'}, L_{f''}$ using the functions $f' = f_0 + \alpha r_0 \sigma^{-1}$ and $f'' = f_1\sigma + \beta r_1$. (In the next section one of the parameters either α or β will always be equal to zero.) For κ, λ discrete eigenvalues of $L_{f'}, L_{f''}$, let as before μ_κ, ν_λ be bases for the eigenspaces $\mathbb{E}_{0,\kappa}^* \subset C_{u_0}(\Sigma_0)^*$, $\mathbb{E}_{1,\lambda}^* \subset C_{u_1}(\Sigma_1)^*$ such that $L_{f'}{}^*\mu_\kappa = \kappa L_{0,\kappa}\mu_\kappa$, $L_{f''}{}^*\nu_\lambda = \lambda L_{1,\lambda}\nu_\lambda$, where $L_{0,\kappa}$, $L_{1,\lambda}$ are invertible metrices in Jordan normal form with 1's in the diagonal. Without loss of generality we may assume that the eigenspaces $\mathbb{E}_{0,\kappa}^*$, $\mathbb{E}_{1,\lambda}^*$ are irreducible and have dimensions k and l. By Proposition 1 μ_κ, ν_λ are one-sided Gibbs' functionals for the functions f' and f''. On the discrete system Σ we define a collection of functionals (which are not necessarily Gibbs') by

$$\mu_{\kappa\lambda} = e^{\tau'}\mu_\kappa \nu_\lambda{}^\perp,$$

($^\perp$ means transposition) where $\tau' = w_0\sigma + \alpha v_0 - w_1\sigma - \beta v_1 = \tau + \alpha v_0 - \beta v_1$ and $\mu_{\kappa\lambda}$ is a $k \times l$-matrix with entries $(\mu_{\kappa\lambda})_{i,j} = e^{\tau'}(\mu_\kappa)_i(\nu_\lambda)_j$. One easily verifies that

$$\sigma^*\mu_{\kappa\lambda} = \kappa\lambda^{-1}e^{\tau' + \tau''\sigma^{-1}}(L_{0,\kappa}\mu_\kappa)(L_{1,\lambda}{}^{-1}\nu_\lambda)^\perp = \kappa\lambda^{-1}e^{\tau''\sigma^{-1}}L_{0,\kappa}\mu_{\kappa\lambda}L_{1,\lambda}{}^{\perp^{-1}},$$

where $\tau'' = (\beta - \alpha)r$ is a function in $C_u(\Sigma)$ (this situation is unlike Lemma 7 where we could explicitly determine the spectrum of σ^* for Gibbs' functionals.).

Let $\alpha = \beta$ and ν_λ span the eigenspace of $L_{f'}{}^*$ ($f'' = f_1\sigma + \beta r_1$) to the eigenvalue λ, then by Proposition 6 $M_\lambda = D\nu_\lambda = \nu_\lambda(e^{\tau + \beta(v_0 - v_1)})$ is a vector of eigenfunctions of $L_{f'}$ ($f' = f_0 + \beta r_0 \sigma^{-1}$) to the eigenvalue λ and forms a basis in $\mathbb{E}_{0,\lambda}$, ($L_{f'}, L_{f''}$ have the same spectrum). Since by definition $\mu_{\kappa\lambda} = e^{\tau + \alpha v_0 - \beta v_1}\mu_\kappa \nu_\lambda{}^\perp$ we obtain according to the remark made following Proposition 6 that $M_\lambda{}^\perp \mu_\kappa = e^{(\beta - \alpha)v_0}\mu_{\kappa\lambda}$ as a functional on left-sided functions. Similarly $N_\lambda = \mu_\lambda(e^{\tau + \beta(v_0 - v_1)})$. We put $\varepsilon_m = 1$ if $m \geqq 0$ and -1 if $m < 0$ and summarise as follows (with the convention $\tau''^{-m} = \tau''\sigma^{-1} + \dots + \tau''\sigma^{-m}$ if $m \geqq 1$):

Proposition 12. *The functionals $(\mu_{\kappa\lambda})_{i,j} \in C_u(\Sigma)^*$, $\kappa = \kappa(\alpha)$, $\lambda = \lambda(\beta)$ have the properties:*
(i) $\sigma^{*m}\mu_{\kappa\lambda} = (\kappa/\lambda)^m \exp(\varepsilon_m \tau''^{-m}) L_{0,\kappa}{}^m \mu_{\kappa\lambda} L_{1,\lambda}{}^{\perp - m}$ *for m integer,*
(ii) $\mu_{\kappa\lambda} = e^{(\alpha - \beta)v_0} M_\lambda{}^\perp \mu_\kappa$ *acting on one-sided functions in $C_{u_0}(\Sigma_0)$,*
(iii) $\mu_{\kappa\lambda} = e^{(\alpha - \beta)v_1} N_\kappa \nu_\lambda{}^\perp$ *acting on one-sided functions in $C_{u_1}(\Sigma_1)$.*

6. Correlation Function

In this section we give a more complete result on the correlation function for Axiom A flows as was previously known . We essentially use the same method as was employed in [13] and [9] to link the correlation function of Hölder continuous functions to the (generalised) zeta function the poles of which determine the poles of the Fourier transform of the correlation function also called reasonances. The connection between the poles of the correlation function and the poles of the zeta function was originally suggested by [14]. Also in [13] the residues of the poles of the Fourier transformed correlation function is given an expressed through Gibbs' functionals, however only for one dimensional eigenspaces of the transfer operator L. Here we give a proof without this restriction. Moreover, as we make use of the variational principle we also yield a better estimate on the width of the strip in which the Fourier transform is meromorphic.

As in the previous section in modulus of continuity is a positive multiple of the ceiling function. Denote by $C_u(\Sigma_r)$ the set of all continuous complex functions F on Σ_r such that $\int_0^{r(x)} e^{\rho t}F(x,t)dt \in C_u(\Sigma)$ for all $\rho\in\mathbb{C}$, and define functionals $d\tilde{\mu}_{\kappa\lambda}=e^{(\beta-\alpha)t}d\mu_{\kappa\lambda}\times dt$, where as in the previous section α,β are complex parameters and κ,λ are discrete eigenvalues of $L_{f'}, L_{f''}$. If κ and λ both assume the same value, say 1, and are simple or semi-simple, these functionals deserve being called Gibbs, as they were introduced in [13], where it was also shown that they get multiplied by a factor $e^{(\beta-\alpha)s}$ if mapped under $\varphi_s{}^*$. If κ,λ are multiple eigenvalues the associated functionals no longer have this nice property. For the following we agree that the 1 in $\mu_{1\lambda},\mu_{\lambda 1}$ are eigenvalues 1 of the transfer operators L_0, L_1 with $f_0, f_1\sigma$ as weight functions (parameter are 0). For $F:\Sigma_r\to\mathbb{R}$ we put $f(\xi)=\int_0^{r(\xi)}(F(\xi,t)-P^*(F))dt$, where $P^*(F)$ is the pressure of F.

Theorem 13. *Let (Σ_r,φ_t) be a the flow obtained by suspending the strictly positive $r\in C_u(\Sigma)$, $u=\gamma r$, $\gamma>0$, and let $\bar{\mu}$ be the unique equilibrium state for some real $F\in C_u(\Sigma_r)$. Then, for $G,H\in C_{(1/2)u}(\Sigma_r)$ the Fourier transform $\mathbb{Q}(\omega)$ of the correlation function $Q(t)=\bar{\mu}((G\varphi_t)H)$*

(i) *is meromorphic in the strip $\{\omega\in\mathbb{C}:|\mathbb{I}\omega|<\gamma\}$, and*

(ii) *the poles are located at the values of ω_0 at which either $L_{f'}$ or $L_{f''}$ has 1 as eigenvalue, where $f'=f_0-i\omega r_0\sigma^{-1}$, $f''=f_1\sigma+i\omega r_1$, and $\mathbb{Q}(\omega)$ has locally the expansion*

$$\begin{array}{ll}\tilde{\mu}_{1\kappa}(G)(1-\kappa L_{0,\kappa})^{-1}\tilde{\mu}_{\kappa 1}(H) & \text{if} \quad \kappa=\kappa(i\omega_0)\in Sp(L_{f'}),\\ \tilde{\mu}_{\lambda 1}{}^{\perp}(G)(1-\lambda L_{1,\lambda})^{-1}\tilde{\mu}_{1\lambda}{}^{\perp}(H), & \text{if} \quad \lambda=\lambda(i\omega_0)\in Sp(L_{f''}),\end{array}$$

plus a contribution which is regular in a neighbourhood of $i\omega_0$. The matrices $L_{0,\kappa}$, $L_{1,\lambda}$ are in Jordan normal form with 1's *in their diagonals. We assume that the eigenspaces are irreducible and have basis as described above and count eigenvalues according to the number of irreducible subspaces into which the eigenspaces split.*

The next lemma will be needed to prove the theorem. It will be necessary to decompose complex functions $\mathbb{G}\in C_{(1/2)u}(\Sigma)$ into locally constant functions: $\mathbb{G}=\sum_{m\geq 0}\mathbb{G}_m$, where $\mathbb{G}_m\in C_{(1/2)u}(\Sigma)$ are stepfunctions constant on two-sided cylinders $U(\eta_{-m}\cdots\eta_m)$, so that $|\mathbb{G}_{m+1}(x)\leq\|\mathbb{G}\|_{(1/2)u}\exp-\min(u^m(x),u^{-m}(x))$ are such that $\mathrm{var}_k\,\mathbb{G}_m$ is small for $k<m$. For each symbol $\eta\in\mathbf{A}$ choose one-sided infinite sequences $\hat{x}\in\Sigma_0$, $\hat{y}\in\Sigma_1$ such that the composition $\hat{x}\eta\hat{y}$ is admissible in Σ. Define $\mathbb{G}_0(\eta)=\mathbb{G}(\hat{x}\eta\hat{y})$, $\eta\in\mathbf{A}$, and inductively for $m\geq 0$,

$$\mathbb{G}_m(\eta)=\mathbb{G}(\hat{x}\eta\hat{y})-\sum_{0\leq k<m}\mathbb{G}_k(\eta_{-k}\cdots\eta_k),$$

where $\eta=\eta_{-m}\cdots\eta_m$ are words in Σ of length $2m+1$ and $\hat{x},\hat{y}$ depend merely on η_{-m} and η_m. Clearly $\mathbb{G}_m\in C_{(1/2)u}(\Sigma)$ and satisfies by construction the following estimate

$$\mathrm{var}_k\,\mathbb{G}_m(x)\leq\begin{cases}2.\mathrm{var}_m\,\mathbb{G}(x) & \text{for} \quad k\leq m,\\ 0 & \text{for} \quad k>m.\end{cases}$$

Lemma 14. *Let $g \in C_{u_1}(\Sigma_1)$ and let $L_g: C_{u_1}(\Sigma_1) \to C_{u_1}(\Sigma_1)$ be the transfer operator associated to g here as an operator from $C_{(1/2)u}(\Sigma)$ to itself, where $u_1 > 0$ is cohomologous to u. Then for all $\chi \in C_{(1/2)u}(\Sigma)$, locally constant functions $\mathbb{G}_m$ as introduced in the last paragraph and $m \geqq 0$, the identity holds:*

$$\sum_{j \geqq m} L_g{}^j((\mathbb{G}_m \sigma^j)\chi) = L_g{}^m((\mathbb{G}_m \sigma^m)(1 - L_g)^{-1}\chi).$$

Proof. By the manipulations (as usual a comma parts the zero'th from the first coordinate)

$$\begin{aligned}\sum_{j \geqq m} (L_g{}^j \mathbb{G}_m \sigma^j \chi)(x, y) &= \sum_{j \geqq m} \sum_{|\eta| = j} \mathbb{G}_m(\eta_{j-m+1} \cdots \eta_j, y_1 \cdots y_m)\chi(x, \eta y) \exp g^j(\eta y) \\ &= \sum_{|\rho| = m} \mathbb{G}_m(\rho, y_1 \cdots y_m) \exp g^m(\rho y) \sum_{k \geqq 0} \sum_{|\delta| = k} \chi(x, \delta\rho y) \exp g^k(\delta\rho y) \\ &= L_g{}^m(\mathbb{G}_m \sigma^m (1 - L_g)^{-1}\chi)(x, y)\end{aligned}$$

for $(x, y) \in \Sigma_0 \times \Sigma_1$ satisfying $xy \in \Sigma$ and $\rho\delta = \eta \in T_j$, $(\rho, \delta) \in T_m \times T_k$, $j = m + k$, so that $x\eta y$ is admissible. □

A similar statement holds for some $g \in C_{u_0}(\Sigma_0)$, $L_g: C_{u_0}(\Sigma_0) \to C_{u_0}(\Sigma_0)$ and $m \geqq 0$.

Proof of Theorem 13. We proceed along the path led out in [13] and relate the Fourier transform $\mathbb{Q}(\omega) = \mu\nu(e^\tau r) \int_{\mathbb{R}} e^{i\omega t} Q(t) dt$ of the correlation function $Q(t)$ to the spectrum of the operators $L_{f'}$ and $L_{f''}$, where $f' = f_0 - i\omega r_0 \sigma^{-1}$, $f'' = f_1 \sigma + i\omega r_1$ are one-sided functions and $f(\xi) = \int_0^{r(\xi)} (F(\xi, t) - P^*(F)) dt$. The moduli of continuity u_0, u_1 of f', f'' are cohomologous to u. Since f is real, the largest eigenvalues of L_0, L_1 are single and real, namely 1, as the pressure of f is zero. Let μ, ν be probability measures on Σ_0, Σ_1 which span the eigenspaces to the eigenvalues 1. For $\mathbb{Q}(\omega) = \mu\nu(e^\tau q_\omega)$ we have in the sense of distributions (or by Fubini's theorem once we know that the integrals exist)

$$q_\omega(\xi) = \int_{\mathbb{R}} e^{i\omega t} \int_0^{r(\xi)} ((G\varphi_t)H)(\xi, s) ds dt = \mathbb{G}^*{}_\omega(\xi) \mathbb{H}_{-\omega}(\xi),$$

with $\mathbb{H}_\omega(\xi) = \int_0^{r(\xi)} e^{i\omega t} H(\xi, t) dt$ (similarly $\mathbb{G}_\omega$) and

$$\begin{aligned}\mathbb{G}^*{}_\omega(\xi) &= \int_{\mathbb{R}} e^{i\omega t} G\varphi_t(\xi, 0) dt \\ &= \sum_{j \in \mathbb{Z}} \int_{r^j(\xi)}^{r^{j+1}(\xi)} e^{i\omega t} G(\xi, t) dt \\ &= \sum_{j \geqq 0} \exp(i\omega(r_1{}^j - v_1)(\xi))(e^{i\omega v_1} \mathbb{G}_\omega)\sigma^j(\xi) \\ &\quad + \sum_{j < 0} \exp(-i\omega(r_0{}^j + v_0)(\xi))(e^{i\omega v_0} \mathbb{G}_\omega)\sigma^j(\xi),\end{aligned}$$

$(r_1{}^0 = 0)$ where $r_0{}^{-k} = r_0\sigma^{-1} + \cdots + r_0\sigma^{-k}$ for $k \geqq 1$. We decompose $e^{i\omega v_1}\mathbb{G}_\omega$ and $e^{i\omega v_0}\mathbb{G}_\omega$ as described above and split $\mathbb{Q}(\omega)$ (and thus the proof of the theorem) into three parts:

$$\mathbb{Q}(\omega) = \mathbb{Q}_1(\omega) + \mathbb{Q}_0(\omega) + \mathbb{Q}_*(\omega),$$

where the summation over j is: (i) $j \geqq m$, (ii) $j \leqq -m$, (iii) $|j| < m$. This procedure will become clear in a second. The first and second summand will have poles which can be related to the poles of the zeta function, while the third term turns out to be analytical in the region of interest.

(i) Let us first determine $\mathbb{Q}_1$, and let ν be the probability measure on Σ_1 spanning the eigenspace of $L_1{}^*$ to the eigenvalue 1. Put

$$\mathbb{Q}_{1,m}(\omega) = \mu\nu\left(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega} \sum_{j \geqq m} \mathbb{G}_{1,\omega,m}\sigma^j \exp(i\omega r_1{}^j) \right),$$

where $\sum_{m \geqq 0} \mathbb{G}_{1,\omega,m} = e^{i\omega v_1}\mathbb{G}_\omega$ is a decomposition of the kind described above. By Lemma 14 we obtain for m fixed summing over $j \geqq m$

$$\begin{aligned}\mathbb{Q}_{1,m}(\omega) &= \sum_{j \geqq m} \mu(L_1{}^{*j}\nu)(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}(\mathbb{G}_{1,\omega,m}\sigma^j)\exp(i\omega r_1{}^j))\\ &= \mu\nu\left(\sum_{j \geqq m} L_{f''}{}^j(\mathbb{G}_{1,\omega,m}\sigma^j)e^{\tau - i\omega v_1}\mathbb{H}_{-\omega} \right)\\ &= \mu\nu(L_{f''}{}^m(\mathbb{G}_{1,\omega,m}\sigma^m)(1 - L_{f''})^{-1}e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}),\end{aligned}$$

where $f'' = f_1\sigma + i\omega r_1$. Drawing out $L_1{}^m$ yields (because $L_1{}^*\nu = \nu$):

$$\mathbb{Q}_{1,m}(\omega) = \mu\nu((\mathbb{G}_{1,\omega,m}\sigma^m)\exp(i\omega r_1{}^m)(1 - L_{f''})^{-1}e^{\tau - i\omega v_1}\mathbb{H}_{-1}),$$

where $\mathbb{G}_{1,\omega,m}\sigma^m \exp(i\omega r_1{}^m) \in C_{(1/2)u}(\Sigma)$ depends only on positive coordinates and can be identified with a function in $C_{u_1}(\Sigma_1)$. In the next step we apply Corollary 10. Put $\mathbb{Q}_{1,m,\lambda}$ for the contribution made by the eigenvalue $\lambda = \lambda(i\omega)$ of $L_{f''}$ which is

$$\mathbb{Q}_{1,m,\lambda}(\omega) = \mu\nu((\mathbb{G}_{1,\omega,m}\sigma^m)\exp(i\omega r_1{}^m)N_\lambda{}^\perp(1 - \lambda L_{1,\lambda})^{-1}\nu_\lambda(e^{-i\omega v_1}\mathbb{H}_{-\omega})).$$

We have $\mathbb{Q}_{1,m} = \sum_\lambda \mathbb{Q}_{1,m,\lambda} + \mathbb{X}_{1,m}$, where the summation is over the discrete spectrum of $L_{f''}$ and converges if we consider only finitely many λ's as we do if we decrease u slightly. The remainder $\mathbb{X}_{1,m}$ is of the form ($K = K_{z(i\omega)}$ as in Corollary 10)

$$\begin{aligned}\mathbb{X}_{1,m}(\omega) &= \mu(L_1{}^{*m}\nu)(\mathbb{G}_{1,\omega,m}\sigma^m \exp(i\omega r_1{}^m)K(e^{\tau - i\omega_1}\mathbb{H}_{-\omega}))\\ &= \mu\nu\left(\sum_{|\eta| = m} (\mathbb{G}_{1,\omega,m}\sigma^m K(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}))(\eta\cdot)\exp f''^m(\eta\cdot) \right).\end{aligned}$$

Now $|\mathbb{G}_{1,\omega,m}(\xi)| \leqq e^{\|u\|_\infty}\| e^{i\omega v_1}\mathbb{G}_\omega \|_{(1/2)u}(\exp -u^m(\xi) + \exp -u^{-m}(\xi))$, and therefore

$$|\mathbb{X}_{1,m}| \leqq c_1\mu\nu\left(\sum_{|\eta| = m} (e^{-u^m} + e^{-u^{-m}})\sigma^m(\cdot, \eta\cdot)\exp \mathbb{R}f''^m(\eta\cdot) \right),$$

where $c_1 \leqq e^{\|u\|_\infty}\| e^{i\omega v_1}\mathbb{G}_\omega \|_{(1/2)u}\| K(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}) \|_\infty$. As u is cohomologous to u_1 which depends only on positive coordinates and since $u^{-m}\sigma^m = u^m$, we get the following inequation (with matrix maximum norm):

$$\begin{aligned}|\mathbb{X}_{1,m}| &\leqq c_2 \sum_{|\eta| = m} \left(\exp \sup_{x \in U(\eta)} (\mathbb{R}f''^m - u_1{}^m\sigma^m)(x) + \exp \sup_{x \in U(\eta)} (\mathbb{R}f''^m - u_1{}^m)(x) \right)\\ &\leqq 2c_2 \| A^N \| e^{2N\|\mathbb{R}f'' - u_1\|_\infty} \sum_{|\eta| = m} \exp(\mathbb{R}f''^m - u_1{}^m)(\eta^\infty)\\ &\leqq c_3 \sum_{|\eta| = m} \exp(\mathbb{R}f'' - u_1)^m(\eta^\infty),\end{aligned}$$

$c_2, c_3 > 0$. Here we used that (Σ, σ) is topologically mixing and $A^N > 0$ for some N. The last two summations are over those η for which the periodic points η^∞ obtained by concatenating η with itself are admissible. We also used that $u_1{}^m\sigma^m(\eta^\infty) = u_1{}^m(\eta^\infty)$. Thereby with the variational principle

$$|\mathbf{X}_{1,m}| \leqq c_4 e^{mP(f - \mathbb{I}\omega r - u)},$$

$c_4 > 0$, uniformly in m, and since by assumption $P(f - \mathbb{I}\omega r - u) < 0$ the sum $\sum_{m \geqq 0} \mathbf{X}_{1,m}$ converges for $\mathbb{I}\omega > -\gamma$ to an analytic function $\mathbf{X}_1$. To reformulate the expression $\mathbf{Q}_{1,m,\lambda}$ recall that $L_{0,\lambda} = L_{1,\lambda}{}^\perp$ and the identities $\mu_{\lambda 1}{}^\perp = e^{i\omega v_1} N_\lambda{}^\perp v$, $\sigma^{*m}\mu_{\lambda 1} = \exp(-i\omega r^{-m})(\lambda L_{0,\lambda})^m \mu_{\lambda 1}$ which corresponds to the choice $(\alpha, \beta) = (i\omega, 0)$ and $\mu_{1\lambda}{}^\perp = e^{\tau - i\omega v_1}\mu v_\lambda$ for which we put $(\alpha, \beta) = (0, i\omega)$ (λ is tied to the parameter value $i\omega$). We obtain by Proposition 12,

$$\begin{aligned}
\mathbf{Q}_{1,m,\lambda}(\omega) &= \mu_{\lambda 1}{}^\perp((e^{-i\omega v_1}\mathbf{G}_{1,\omega,m})\sigma^m \exp(i\omega r^m))(1 - \lambda L_{1,\lambda})^{-1}\mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega}) \\
&= \mu_{\lambda 1}{}^\perp(e^{-i\omega v_1}\mathbf{G}_{1,\omega,m})(\lambda L_{1,\lambda})^m(1 - \lambda L_{1,\lambda})^{-1}\mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega}) \\
&= \mu_{\lambda 1}{}^\perp(e^{-i\omega_1}\mathbf{G}_{1,\omega,m})(1 - \lambda L_{1,\lambda})^{-1}\mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega}) - \mathbf{V}_{1,m,\lambda}(\omega),
\end{aligned}$$

involving the remainder

$$\begin{aligned}
\mathbf{V}_{1,m,\lambda}(\omega) &= \mu_{\lambda 1}{}^\perp(e^{-i\omega v_1}\mathbf{G}_{1,\omega,m}) \sum_{0 \leqq p < m} (\lambda L_{1,\lambda})^p \mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega}) \\
&= v(\mathbf{G}_{1,\omega,m}\sigma^m e^{-i\omega r_1{}^m} N_\lambda{}^\perp) \sum_{0 \leqq p < m} (\lambda L_{1,\lambda})^{p-m} \mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega})
\end{aligned}$$

which still has to be estimated. There are constants c_5, c_6, c_7 depending on λ but not on m such that $\|N_{\lambda,r}\|_\infty \leqq c_5$, $|(\mu_{1\lambda}(\mathbf{H}_{-\omega}))_r| \leqq c_6$, $r = 1, \ldots, l$, l being the dimension of $\mathbf{E}_{1,\lambda}$, and in the matrix maximum norm $\|L_{1,\lambda}{}^{-p}\| \leqq c_7 p^l$, $p \geqq 1$. With the identity $L_1{}^{*m}v = v$ we get by the same argument as in estimating $\mathbf{X}_{1,m}$ that

$$\begin{aligned}
|\mathbf{V}_{1,m,\lambda}| &\leqq l c_5 c_6 c_7 m^l v\left(\sum_{|\eta| = m} e^{\mathbb{R}f''(\eta\cdot)}(e^{-u_1{}^m\sigma^m} + e^{-u_1{}^m})(\eta\cdot)\right) \sum_{1 \leqq p \leqq m} |\lambda|^{-p} \\
&\leqq c_8 l m^l e^{mP(\mathbb{R}f'' - u_1)} \sum_{1 \leqq p \leqq m} |\lambda|^{-p} \leqq c_9 l m^l e^{-m\rho},
\end{aligned}$$

uniformly in m, where $c_8, c_9 > 0$ and $\rho = \log|\lambda| - P(f - \mathbb{I}\omega r - u)$ is positive as $|\lambda|$ is strictly larger than the radius $e^{P(f - \mathbb{I}\omega r - u)}$ of the essential spectrum of $L_{f''}$. Thus the series $\sum_{m \geqq 0} \mathbf{V}_{1,m,\lambda}$ converges to a function $\mathbf{W}_{1,\lambda}$ which is analytic for $\mathbb{I}\omega > -\gamma$. Since $\sum_{m \geqq 0} e^{-i\omega v_1}\mathbf{G}_{1,\omega,m} = \mathbf{G}_\omega$ we finally get

$$\sum_{m \geqq 0} \mathbf{Q}_{1,m,\lambda} = \mu_{\lambda 1}{}^\perp(\mathbf{G}_\omega)(1 - \lambda L_{1,\lambda})^{-1}\mu_{1\lambda}{}^\perp(\mathbf{H}_{-\omega}) + \mathbf{W}_{1,\lambda}(\omega),$$

and

$$\mathbf{Q}_1(\omega) = \sum_\lambda \tilde{\mu}_{\lambda 1}{}^\perp(G)(1 - \lambda L_{1,\lambda})^{-1}\tilde{\mu}_{1\lambda}{}^\perp(H) + \mathbf{Y}_1(\omega).$$

The summation $\sum_\lambda \mathbf{W}_{1,\lambda}$ is finite and converges absolutely if we replace u by a slightly smaller $u' > 0$ since $L_{f''}$ then will have only finitely many discrete eigenvalues in the somewhat bigger space $C_{u'}(\Sigma_1)$, On the other hand γ, which depends continuously on u, increases to its original value as u' approaches u from below and we conclude that the remainder term $\mathbb{Y}_1 = \mathbb{X}_1 + \sum_\lambda \mathbb{W}_{1,\lambda}$ is analytic for $|\mathbb{I}\omega| < \gamma$.

(ii) For $\mathbb{Q}_0$ we use the decomposition $e^{i\omega v_0}\mathbb{G}_\omega = \sum_{m \geqq 0} \mathbb{G}_{0,\omega,m}$ where the locally constant functions $\mathbb{G}_{0,\omega,m}$ satisfy the same inequalities as $\mathbb{G}_{1,\omega,m}$. For fixed m we obtain (this time the summation is over $j \leqq -m$)

$$\mathbb{Q}_{0,m}(\omega) = \mu\nu(\mathbb{G}_{0,\omega,m}\sigma^{-m}\exp(-i\omega r_0{}^{-m})(1-L_{f'})^{-1}e^{\tau - i\omega v_0}\mathbb{H}_{-\omega}),$$

where $f' = f_0 - i\omega r_0 \sigma^{-1}$ and $\mathbb{G}_{0,\omega,m}\sigma^{-m}\exp(-i\omega r_0{}^{-m}) \in C_{(1/2)u}(\Sigma)$ depends only on coordinates $\leqq 0$. Using Corollary 10 we get with $-i\omega$ as parameter value for κ by the same argument as in (i) the following result:

$$\begin{aligned}\mathbb{Q}_{0,m}(\omega) &= \sum_\kappa (\mu_{1\kappa}(\mathbb{G}_\omega)(1-\kappa L_{0,\kappa})^{-1}\mu_{\kappa 1}(\mathbb{H}_{-\omega}) + \mathbb{W}_{0,\kappa}(\omega)) + \mathbb{X}_0(\omega) \\ &=: \sum_\kappa \tilde{\mu}_{1\kappa}(G)(1-\kappa L_{0,\kappa})^{-1}\tilde{\mu}_{\kappa 1}(H) + \mathbb{Y}_0(\omega),\end{aligned}$$

where $\mathbb{X}_0$, $\mathbb{W}_{0,\kappa}$ and $\mathbb{Y}_0$ are analytic for $\mathbb{I}\omega < \gamma$ and the summation is over the discrete spectrum of $L_{f'}$.

(iii) Finally $\mathbb{Q}_*$. The summation here is over $|j| < m$ where, for fixed m, we get by the same argument as in estimating $\mathbb{X}_{1,m}$ the following bounds (as f and r are real)

$$\begin{aligned}&\left|\mu\nu\left(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}\sum_{0 \leqq j < m}\mathbb{G}_{1,\omega,m}\sigma^j \exp(i\omega r_1{}^j)\right)\right| \\ &\leqq c_{10}\sum_{0 \leqq j < m}(L_1{}^{*j}\nu)(|\mathbb{G}_{1,\omega,m}\sigma^j|\exp(-(\mathbb{I}\omega)r_1{}^j)) \\ &= c_{11}\sum_{0 \leqq j < m}\sum_{|\eta| = j}\exp(\mathbb{R}f''^j - u_1{}^m)(\eta^\infty) \\ &\leqq c_{12}\sum_{0 \leqq j < m} e^{-(m-j)\inf u} e^{jP(f - \mathbb{I}\omega r - u)},\end{aligned}$$

with $c_{10} \leqq \| e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}\|_\infty$ and c_{11}, c_{12} independent of m. Since $P(f - \mathbb{I}\omega r - u) < 0$ and $\inf u$ is strictly positive the double sum over j and $m > j$ converges absolutely. Therefore

$$Z_1(\omega) = \mu\nu\left(e^{\tau - i\omega v_1}\mathbb{H}_{-\omega}\sum_{m \geqq 0}\sum_{0 \leqq j < m}\mathbb{G}_{1,\omega,m}\sigma^j\exp(i\omega r_1{}^j)\right)$$

is holomorphic in ω for $\mathbb{I}\omega > -\gamma$, and by the same argument one shows that

$$Z_0(\omega) = \mu\nu\left(e^{\tau - i\omega v_0}\mathbb{H}_{-\omega}\sum_{m \geqq 1}\sum_{1 \leqq j < m}\mathbb{G}_{0,\omega,m}\sigma^{-j}\exp(-i\omega r_0{}^{-j})\right)$$

is holomorphic for $\mathbb{I}\omega < \gamma$.

The three paragraphs (i), (ii), (iii) together yield

$$\mathbb{Q}(\omega) = \sum_\kappa \tilde{\mu}_{1\kappa}(G)(1-\kappa L_{0,\kappa})^{-1}\tilde{\mu}_{\kappa 1}(H) + \sum_\lambda \tilde{\mu}_{\lambda 1}{}^\perp(G)(1-\lambda L_{1,\lambda})^{-1}\tilde{\mu}_{1\lambda}{}^\perp(H) + \mathbb{Y}(\omega),$$

where the remainder $\mathbb{Y} = \mathbb{Y}_0 + \mathbb{Y}_1 + Z_0 + Z_1$ is holomorphic in the strip $|\mathbb{I}\omega| < \gamma$. The eigenvalues κ and λ depend on ω and therefore $\mathbb{Q}(\omega)$ has a pole whenever either $L_{f'}$ or $L_{f''}$ has eigenvalúe 1. Thus, counting multiplicities, the poles of $\mathbb{Q}(\omega)$ coincide with the poles of $\zeta(f - i\omega r) + \zeta(f + i\omega r)$. This proves the theorem. □

7. Further Remarks

(I) We give a short description of the Margulis measures for two-sided suspensions. For $f = 0$ and $\alpha = \beta = -h$, where h is the topology entropy of φ_t, one can define measures $\tilde{\mu}, \tilde{v}$ supported on the local weak stable and unstable leaves by $d\tilde{\mu} = e^{h(v_0 - t)} d\mu_1 dt$ and $d\tilde{v} = e^{h(t - v_1)} dv_1 dt$, where μ_1, v_1 are the measures on Σ_0, Σ_1 which span the one-dimensional eigenspaces of $L_{f'}$ and $L_{f''}$ to the eigenvalues 1. Similarly as in [4] Proposition 3.3 one shows that $\varphi^*_{-s}\tilde{\mu} = e^{-hs}\tilde{\mu}, \varphi^*_s \tilde{v} = e^{-hs}\tilde{v}$, $s \geqq 0$, where v_0, v_1 account for the fact that the ceiling function r in our case is two-sided. These identities are the scaling properties of Margulis transversal measures. To show that $\tilde{\mu}, \tilde{v}$ are indeed transversal measures we observe that the strong stable leave through a point $(xy, t) \in \Sigma_r$ is locally given by $\{(x'y, t + v_1(x'y) - v_1(xy)) : x' \in \Sigma_0$ close enough to $x\}$ where the identification $(z, r(z)) = (\sigma z, 0)$ applies (the size of the neighbourhood over which x' varies depends on $\| v_1 \|_\infty$). Let $\chi_x, \chi_{x'}$ be two functions defined on the weak unstable leaves characterised by the left-infinite sequences $x, x' \in \Sigma_0$ and whose support is sufficiently small. If $\chi_x, \chi_{x'}$ are equivalent under "sliding" along the strong stable foliation we have that $\chi_x(y, t) = \chi_{x'}(y, t + v_1(x'y) - v_1(xy))$, assuming $0 \leqq t + v_1(x'y) - v_1(xy) \leqq r(x'y)$. Hence one easily verifies that $\tilde{v}(\chi_x) = \tilde{v}(\chi_{x'})$ as the factor e^{-hv_1} drops out. The same conclusion applies to $\tilde{\mu}$, which therefore is a measure transversal to the strong unstable foliation.

Margulis originally proved the existence of these measures for Anosov flows. The generalization to Axiom A flows is due to Bowen and Marcus [4] whose approach we essentially followed here. If we glue $\tilde{\mu}$ and $\tilde{v}$ along the flow we get the measure $\tilde{\mu}_{11}$ which up to a normalising factor is the measure of maximal entropy and which in this sense is locally of product form.

(II) The transformation property (Ruelle [13] p. 107), $\varphi^*_s \tilde{\mu}_{\kappa\lambda} = e^{(\beta - \alpha)s} \tilde{\mu}_{\kappa\lambda}$ for semisimple κ, λ, was with $f = 0$ and $\beta = -h$ recently used by Pollicott [10] to show that for Axiom A attractors the non-weighted zeta function has an analytic extension to a halfplane $\mathbb{R}z > h - \varepsilon$, for some $\varepsilon > 0$, with the exception of a single pole at h (the topological entropy of the flow). This is a consequence of the fact that the eigenvalues of φ^*_s have moduli either 1 or bounded away from 1. It readily follows that the Fourier transform of the correlation function associated to the measure of maximal entropy is analytic in a uniform strip containing the real axis, apart from a pole at 0 whose residue essentially is the limit integral.

(III) It is natural to ask how Theorem 13 can be extended to Axiom A flows in general. By a well known result of Bowen [2] an Axiom A flow is semi-conjugated to a suspended flow whereby the ceiling function essentially measures the time it takes for points on small pieces of hypersurfaces which are transversal to the flow to flow up to the next one in order. As pointed out such a ceiling function has the regularity we assumed r to have. However the construction of the partitions is subject to some arbitrariness, as points on the forward and backward projected boundary set usually have several distinct symbolic descriptions. With regard to the zeta function the over-counting of periodic orbits which results from this ambiguity was eliminated by Manning and Bowen through considering auxiliary suspensions (see [3]). The difficulty here is that a functional $\tilde{\mu}_{\kappa\lambda}$ does not necessarily correspond to anything alike on the original system. One needs a Bowen–Manning formula for correlation functions (although for Anosov flows on three dimensional manifolds one can do without). For diffeomorphisms partial results in this direction were obtained in [12] Sect. 5.

(IV) There is no pole at 0 if we consider the expression $\bar{\mu}((G\varphi_t)H) - \bar{\mu}(G)\bar{\mu}(H)$ as the correlation function of F, G, the Fourier transform of which is analytic in a strip containing the real axis if the zeta function $\zeta(f + zr)$ has an analytic extension to the halfplane $\mathbb{R}z \geqq P(F) - \varepsilon$ for some positive ε with the exception of a single pole at $P(F)$. By the theorem of Payley Wiener the correlation function decays then exponentially fast, provided it satisfies some L^2 integrability condition. By suspending a locally constant function Ruelle [14] constructed an Axiom A flow which does not mix exponentially fast. The Fourier transform of its correlation function has poles arbitrarily close to the real axis which is expressed in the fact that the zeta function has poles arbitrarily close to the line $\mathbb{R}z = P(F)$. However the following corollary tells us that Axiom A diffeomorphisms always mix exponentially fast as the Fourier transform of $Q(T)$ is periodic. A proof without invoking the zeta function is in [3] 1.26.

Corollary 15. ([12]) *Suppose (Σ, σ) is topologically mixing and let μ be the unique Gibbs' state for some real valued $f \in C_\theta(\Sigma)$, for some constant $\theta > 0$, whose pressure is P. Then for given $G, H \in C_{(1/2)\theta}(\Sigma)$ the Fourier transform $\mathbb{Q}(\omega)$ of the "discrete time" correlation function*

$$Q(T) = \mu((G\sigma^T)H), \quad T \in \mathbb{Z},$$

is meromorphic in the strip $\{\omega \in \mathbb{C} : |\mathbb{I}\omega| < \theta\}$. Furthermore if κ is an eigenvalue of L_0^* (and L_1^*) with an eigenspace whose basis is linearly mapped by the (Jordan) matrix $L_{0,\kappa}$ (and $L_{1,\kappa}$), $\mathbb{Q}(\omega)$ is up to a function analytic for $|\mathbb{I}\omega| < \theta$ equal to

$$\sum_{\kappa} (\mu_{1\kappa}(G)(1 - \kappa e^{-i\omega - P} L_{0,\kappa})^{-1} \mu_{\kappa 1}(H) + \mu_{\kappa 1}{}^{\perp}(G)(1 - \kappa e^{i\omega - P} L_{1,\kappa})^{-1} \mu_{1\kappa}{}^{\perp}(H)).$$

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