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Generalized Apollonian Packings

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Abstract. In this paper we generalize the classical two-dimensional Apollonian packing of circles to the case where the circles are no more tangent. We introduce two elements of $SL(2,\mathbb{C})$ as generators: R and T that are hyperbolic rotations of $\frac{2\pi}{3}$ and $\frac{2\pi}{N}$ $(N=2,3,4,\ldots)$, around two distinct points. The limit set of the discrete group generated by R and T provides, for $N=7,8,9,\ldots$ a generalization of the Apollonian packing (which is itself recovered for $N=\infty$). The values $N=2,3,4,5$ produce a very different result, giving rise to the rotation groups of the cube for $N=2$ and 4, and the icosahedron for $N=3$ and 5. For $N=6$ the group is no longer discrete. To further analyze this structure for $N \geqq 7$, we move to the Minkowski space in which the group acts on a one sheeted hyperboloid. The circles are now represented by points on this variety and generate a crystal on it.

I. Introduction

In a classical construction of an Appolonian packing, one starts with a curvilinear triangle and constructs the inscribed circle, thus creating three curvilinear triangles out of the original one. This process is then repeated with each of the resulting curvilinear triangles and their descendants. The method for producing the inscribed circles can be realized with inversions [1] or Möbius maps [2].

In this latter case, one sees that the Apollonian packing is the limit set of a discrete subgroup of $SL(2,\mathbb{C})$. We present here a technique for generating non-tangential disk packings as limit sets of discrete groups which include as a special case the Apollonian packing. Extension to higher dimensional sphere packings will be presented elsewhere.

Disk and sphere packings are natural models for porous media [3,4] and a

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parametrized family of such packings would be a valuable tool for making models of various media having different porosities. The group invariance would be useful in the study of flows in such models. These considerations, prompted by the experimental results of [5] were the motivation for our initial investigations.

More recently, the fact that the computer pictures of our packings bear some resemblance to X-ray scatter plots of crystals and quasi-crystals [6] has lead us to wonder if there might be some connection between our work and these areas. The family of groups related to our packings contains the finite groups of symmetries of the cube and icosahedron; so there is an (unexpected) relation to crystals. In addition, the typical fractals generated by our infinite group only realize their total symmetry in the limiting process. This is somewhat akin to the long-range symmetry of quasi-crystals [7]. Furthermore the fact that there is a one-to-one correspondence between these fractals and a regular lattice drawn on a one sheeted space-like hyperboloid may give some support to these considerations.

The paper is organized as follows:

In Sect. II, we give the basic geometric construction and derive the related family of groups which depends on a parameter α. We show that the group can be generated by two rotations, one of $2\pi/3$ and one of 2α, and that the group always contains the alternating group on four elements.

In Sect. III we study the disk packing aspects. We prove that $\alpha = \frac{\pi}{N}$ for $N \in \mathbb{N}$, and $N \geqq 7$ is a necessary condition to have a packing.

In Sects. IV and V, we consider a representation of our groups as groups of isometries on a four-dimensional Minkowski space and represent the action of the group in terms of Chebyshev polynomials.

In Sects. VI and VII, we study the case $\alpha = \pi/N$ for $2 \leqq N \leqq 5$ and $N = 6$ respectively. In the former case, we obtain the group of the cube and icosehadron; in the latter case the group is not discrete.

In Sect. VIII, we indicate higher dimensional extensions.

II. The Basic Construction

Consider three circles with equal radii r and centers at cubic roots of unity: $1, \omega, \omega^2$. Label the circles X_1, X_2, X_3 in counterclockwise order starting with the one whose center is at 1. We seek a fourth circle X_0 and three Möbius maps T_1, T_2, T_3, such that

$$T_i X_j = X_j, \quad j \neq i, \quad i,j = 1,2,3, \tag{II.1}$$

$$T_i X_i = X_0, \tag{II.2}$$

$$\begin{aligned} T_2 &= R T_1 R^{-1}, \\ T_3 &= R^{-1} T_1 R, \end{aligned} \tag{II.3}$$

where R is a rotation of $+\frac{2\pi}{3}$ around the origin. It follows that X_0 will have center at 0 because it is invariant under R:

$$R X_0 = R T_1 X_1 = R T_1 R^{-1} R X_1 = T_2 R X_1 = T_2 X_2 = X_0. \tag{II.4}$$

One can construct T_1 as a composition:

$$T_1 = SJ, \tag{II.5}$$

where S is the reflection through the axis of symmetry of X_2 and X_3 and J is the inversion with respect to a circle orthogonal to X_2 and X_3. Some elementary computations (see Appendix A) reveal that

$$T_1 z = \frac{(I-1)}{2}\frac{2z - I}{z + I}, \tag{II.6}$$

where I is the center of the circle of inversion:

$$I = \frac{1 + \sqrt{9 - 8r^2}}{2}. \tag{II.7}$$

The *fixed points* of T_1 are the points of intersection of the inversion circle J with the axis of symmetry of X_2 and X_3. These are f and f^* (see Fig. 1),

$$f = -\tfrac{1}{2} + i\beta, \tag{II.8}$$

where

$$\beta = \sqrt{\frac{3}{4} - r^2} = \frac{1}{2}\frac{\sin\alpha}{\sqrt{\cos^2\alpha - 2/3}}. \tag{II.9}$$

For the moment we shall consider that r vary from zero where all X_i are reduced to points, to $r_{\max} = \frac{\sqrt{3}}{2}$, where all X_i are tangent. In this range of values of r, no X_i intersect (except for $r = r_{\max}$).

The angle α in Fig. 1 which is the argument of $(f + I)$ is defined by:

$$\cos\alpha = \frac{I - \frac{1}{2}}{k} \tag{II.10}$$

or in terms of r:

$$\cos\alpha = \frac{1}{2}\sqrt{\frac{9 - 8r^2}{3(1 - r^2)}} \quad \text{or} \quad r^2 = \frac{3}{2}\frac{1 - 2\cos 2\alpha}{1 - 3\cos 2\alpha}. \tag{II.11}$$

The radius ρ_0 of the circle X_0 (of center 0) is easily computed (see Appendix A). We also introduce the circle X_∞ defined by:

$$X_\infty = T_1^{-1} X_1. \tag{II.12}$$

One checks that the center of X_∞ is 0. Therefore X_∞ is invariant under R and we see that:

$$X_\infty = T_i^{-1} X_i, \quad i = 1, 2, 3. \tag{II.13}$$

The radius ρ_∞ of (X_∞) is given in Appendix A. In particular we have:

$$\rho_0 \rho_\infty = 1 - r^2 = \frac{1}{2(3\cos 2\alpha - 1)}. \tag{II.14}$$

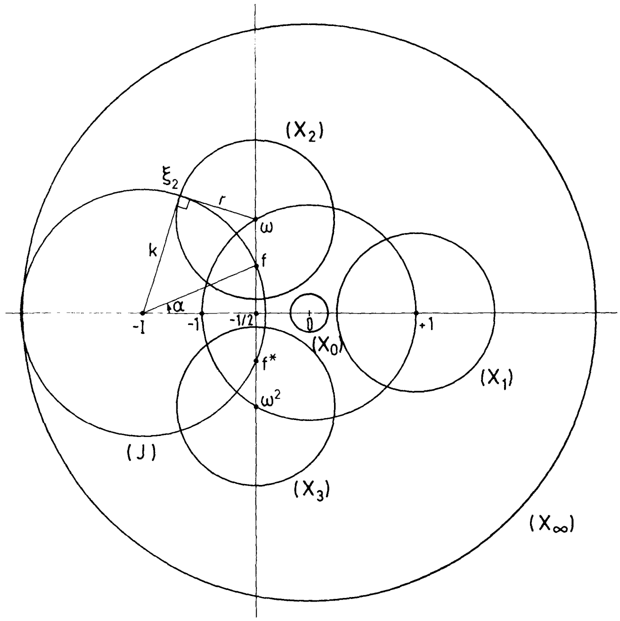

Fig. 1. The basic construction. $\alpha = \pi/8$, $r = 0.744$, $I = 1.568$, $k = 1.157$, $\rho_0 = 0.164$, $\rho_\infty = 2.706$

For the reader's convenience, we recall that a Möbius transformation:

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad ad - bc = 1 \tag{II.15}$$

transforms the circle of center γ and radius r into a circle of center γ' and radius r' with

$$\gamma' = \frac{a}{c} - \frac{1}{c} \frac{1}{(c\gamma + d)} \frac{1}{1 - \dfrac{r^2}{\left|\gamma + \dfrac{d}{c}\right|^2}},$$

$$r' = \frac{r}{\left| |c\gamma + d|^2 - r^2 |c|^2 \right|}. \tag{II.16}$$

Equation (II.16) allows to check that the center of X_∞ is 0. In particular we have

$$T_1 = \begin{bmatrix} \dfrac{2(I-1)}{\sqrt{6I(I-1)}} & \dfrac{-I(I-1)}{\sqrt{6I(I-1)}} \\ \dfrac{2}{\sqrt{6I(I-1)}} & \dfrac{2I}{\sqrt{6I(I-1)}} \end{bmatrix} = \begin{bmatrix} \cos\alpha - \sqrt{\cos^2\alpha - 2/3} & -\dfrac{1}{6} \dfrac{1}{\sqrt{\cos^2\alpha - 2/3}} \\ 2\sqrt{\cos^2\alpha - 2/3} & \cos\alpha + \sqrt{\cos^2\alpha - 2/3} \end{bmatrix}. \tag{II.17}$$

We notice that T_1^{-1} is given by the same matrix, choosing the other determination in the square root of (II.7). Therefore, the other determination generates the same group, with X_0 and X_∞ interchanged.

Mobiüs transformations in the form of (II.15) are commonly classified according to their trace: $\mathrm{Tr}\, T = a + c$, [8]. We are interested in the values of r that make T_1 be either parabolic $\{(\mathrm{Tr}\, T_1)^2 = 4\}$ or elliptic $\{(\mathrm{Tr}\, T_1)^2 \in [0,4)\}$ in which case T_1 is a rotation by 2α. We have the following:

Proposition 1. *T_1 is parabolic if and only if $r^2 = 3/4$. T_1 is elliptic if and only if either $r^2 \geqq 9/8$ or $r^2 < 3/4$.*

In the range $0 \leqq r^2 \leqq 3/4$, α decreases from $\frac{\pi}{6}$ to 0. The case $r^2 = 3/4$ corresponds to mutually tangent circles X_1, X_2, X_3 with X_0 the inscribed circle and X_∞ the circumscribed circle. The images of X_0 under the members of the group produce the classical Apollonian packing that has been investigated and extensively studied, [1, 9, 10, 11]. In the cases $0 \leqq r^2 < 3/4$ it is necessary that the T_i's generate a discrete group if we are to obtain a packing from the image of X_0 under the group elements. In these cases 2α, the angle of rotation, should be a rational multiple of 2π. That is, a necessary condition to have a packing is

$$\alpha = \frac{k\pi}{N}, \quad k, N \in \mathbb{N}.$$

We shall prove later on, that for $0 \leqq r^2 \leqq \frac{3}{4}$, it is necessary to choose $k = 1$, otherwise the orbit of X_1 under the cyclic group generated by T_1 will consist of overlapping circles. Also in this case N is necessarily at least 7. In any case we shall restrict our discussion to

$$\alpha = \frac{\pi}{N} \tag{II.18}$$

and distinguish two cases: (i) $N = 2, 3, 4, 5, 6$. (ii) $N \geqq 7$. In case (ii) to be discussed in the next section, the limit set of the group generated by T_1, T_2, T_3 appears to consist of infinitely many non-overlapping circles which form a packing of X_∞ the common image of $T_i^{-1} X_i$. In case (i) the geometry of the circles $\{X_i\}$ does not play a central role. We note here a common feature of all cases.

Proposition 2. *$T_i T_j^{-1}$ generate an isomorphic copy of A_4, the alternating group on 4 elements.*

Proof. From the fact that

$$T_i^{-1} X_i = T_j^{-1} X_j = X_\infty \quad \text{for} \quad 0 \leqq r \leqq \frac{\sqrt{3}}{2} \tag{II.19}$$

one checks immediately that $T_i T_j^{-1}$ acts for $i \neq j$ as a cyclic permutation on X_i, X_j, X_0 and leaves the remaining circle invariant. The group relation implied by this being analytic in r, will remain true for any complex value of r. From:

$$T_i T_j^{-1} X_i = X_0; \quad T_i T_j^{-1} X_j = X_i; \quad T_i T_j^{-1} X_0 = X_j; \quad T_i T_j^{-1} X_k = X_k; \\ i \neq j, \quad \{i, j = 1, 2, 3\}. \tag{II.20}$$

we see that the group generated by $T_i T_j^{-1}$ is isomorphic to A_4 and:

$$\begin{aligned} a &= T_1 T_2^{-1} = (1,0,2)(3), \\ b &= T_3 T_2^{-1} = (2,3,0)(1), \\ c &= T_1 T_3^{-1} = (1,0,3)(2). \end{aligned} \tag{II.21}$$

The twelve elements of the group isomorphic to A_4 are:

$$\{I, a, a^2, b, b^2, c, c^2, ab, ba, ca, bc, ac^2\} \approx A_4. \tag{II.22}$$

Here explicitly, we have

$$\begin{gathered} a = (1,0,2)(3); \quad a^2 = a^{-1} = (2,0,1)(3); \quad b = (2,3,0)(1); \quad b^2 = b^{-1} = (0,3,2)(1), \\ c = (1,0,3)(2); \quad c^2 = c^{-1} = (3,0,1)(2); \quad ab = (1,0)(3,2); \quad ba = (3,0)(2,1), \\ ca = (3,1)(2,0); \quad ac = ba; \quad bc = (2,3,1)(0); \quad cb = a; \quad a^2 c = (3,1,2)(0). \\ abc = b. \end{gathered} \tag{II.23}$$

It follows that R corresponds to bc. Therefore the elements of order 2 of the group are ab, ba, ca and the $T_i T_j^{-1}$ are order 3 elements. One can construct easily the multiplication table of this group.

Proposition 3. *Similarly $T_i^{-1} T_j$ generate an isomorphic copy of A_4.*

Proof. Replace X_0 by X_∞ everywhere in (II.20).

We shall denote by $\langle T_1, T_2, T_3 \rangle$ the group generated by T_1, T_2, T_3. We notice that $T_i T_j^{-1}$ or $(T_i^{-1} T_j)$ is a finite subgroup of $\langle\ \rangle$, isomorphic to A_4 which is the group of symmetries of the tetrahedron. This remark will be useful later on.

III. Packing Aspects

In this section, we shall consider the case where $0 \leqq r < \frac{\sqrt{3}}{2}$, T_1 is elliptic. If we want the group to be discrete, α should be of the form

$$\alpha = \frac{k\pi}{N}.$$

If we want the images of X_1 under the successive powers of T_1 not to overlap then necessarily $N \geqq 7$ and $k = 1$,

$$\alpha = \frac{\pi}{N}, \quad N = 7, 8, 9, \ldots. \tag{III.1}$$

To prove statement (III.1) we need several steps which are outlined in the following.

To study more conveniently the action of T_1 on X_1, we have to move to a system of coordinates in which T_1 is diagonal. The diagonalized form of T_1 reads:

$$D = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}. \tag{III.2}$$

Let Λ be the transformation which diagonalizes T_1:

$$T_1 = \Lambda D \Lambda^{-1}. \tag{III.3}$$

With f and f^* (II.8) being the fixed points of T_1, we have

$$\Lambda(z)=\frac{fz+f^*}{z+1},$$

$$\Lambda^{-1}(z)=-\frac{z-f^*}{z-f}. \tag{III.4}$$

The study of the maps of X_1 under the various powers of T_1 is therefore equivalent to the study of the maps of $\Lambda^{-1}X_1$ under the powers of D.

Using formulae (II.16), and (II.8), (II.9), we find the center $\tilde{\gamma}$ and radius $\tilde{r}$ of $\Lambda^{-1}X_1$ to be

$$\tilde{\gamma}_1=-\frac{3}{3-2r^2}\left(\frac{1}{2}+i\sqrt{\frac{3}{4}-r^2}\right);\quad |\tilde{\gamma}_1|=\frac{3}{3-2r^2}\sqrt{1-r^2}=\frac{\sqrt{3\cos 2\alpha-1}}{\sqrt{2}\cos 2\alpha},$$

$$\tilde{r}_1=\frac{r\sqrt{3-4r^2}}{3-r^2}=\frac{\sin\alpha}{\cos 2\alpha}\sqrt{2\cos 2\alpha-1}. \tag{III.5}$$

D rotates $\Lambda^{-1}X_1=\tilde{X}_1$ by 2α. From $\tilde{0}_1$ the center of rotation of D, one sees the circle $\tilde{X}_1$ under an angle 2ϕ:

$$\sin\phi=\frac{\tilde{r}_1}{|\tilde{\gamma}_1|}=\frac{r}{3}\sqrt{\frac{3-4r^2}{1-r^2}} \tag{III.6}$$

or, using (II.14):

$$\sin\phi=\sin\alpha\sqrt{\frac{2(2\cos 2\alpha-1)}{3\cos 2\alpha-1}}, \tag{III.7}$$

α being of the form $\frac{k}{N}\pi$ (k and N relatively prime), the rotation D defines exactly N angular sectors of angle $\frac{2\pi}{N}$ (see Fig. 2). If we want the various images of $\tilde{X}_1$ under D not to overlap, it is necessary that $\tilde{X}_1$ be entirely contained in one of these sectors, that is:

$$0<2\phi<\frac{2\pi}{N}. \tag{III.8}$$

We remark that N has to be $\geqq 7$ in order for α to be $<\frac{\pi}{6}$. Then (III.8) implies, using (III.7):

$$\sin\frac{k\pi}{N}\sqrt{\frac{2\left(2\cos\frac{2k\pi}{N}-1\right)}{3\cos\frac{2k\pi}{N}-1}}<\sin\frac{\pi}{N}, \tag{III.9}$$

and

$$N\geqq 7\ (k,N)\ \text{relatively prime}, \tag{III.10}$$

$$0\leqq\frac{k\pi}{N}<\frac{\pi}{6}. \tag{III.11}$$

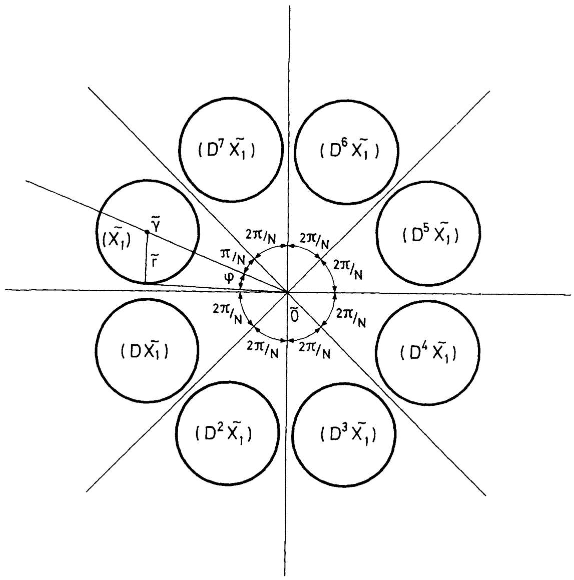

Fig. 2. The action of D on $\tilde{X}_1$ for $N=8$

From (III.11) we see that

$$\frac{\pi}{N} < \frac{\pi}{6k} \quad \text{which implies} \quad \frac{\pi}{N} \leqq \frac{\pi}{6k+1}. \tag{III.12}$$

If we square (III.9) we get

$$\frac{1-4\sin^2\dfrac{k\pi}{N}}{1-3\sin^2\dfrac{k\pi}{N}} \, \frac{\sin^2\dfrac{k\pi}{N}}{\sin^2\dfrac{\pi}{N}} < 1 \tag{III.13}$$

with

$$\frac{\pi}{N} \leqq \frac{\pi}{6k+1}.$$

Equation (III.13) implies that $k=1$. The proof is a consequence of the following lemma.

Lemma. *For $k \geqq 2$ and $0 \leqq \xi \leqq \dfrac{\pi}{6k+1}$, $k \in \mathbb{N}$, we have*

$$\frac{1-4\sin^2 k\xi}{1-3\sin^2 k\xi} \cdot \frac{\sin^2 k\xi}{\sin^2 \xi} > 1.$$

The proof of this lemma can be found in Appendix B.

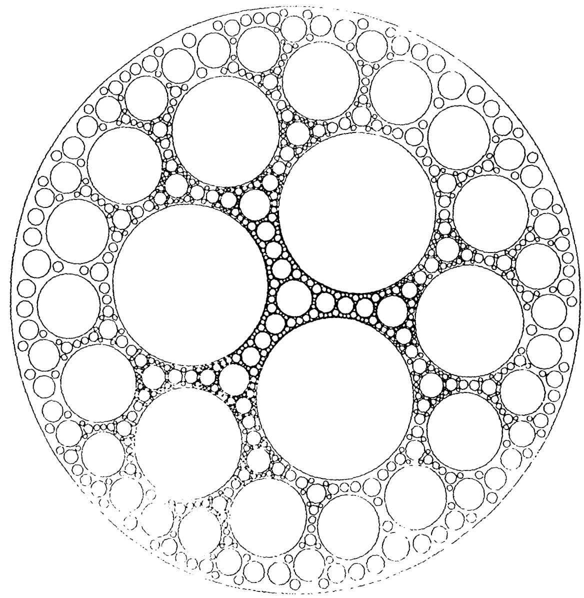

Fig. 3. The limit set for $N = 8$

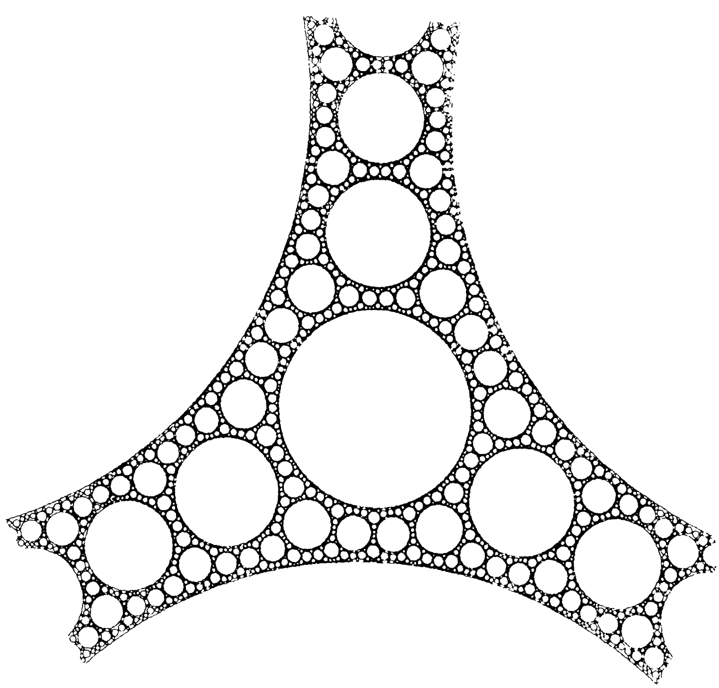

Fig. 4. A blow-up of Fig. 3

We have performed computer graphical experiments with α of the form $\frac{\pi}{N}$, $N \geqq 7$, and constructed the successive maps of X_1 under the full group $\langle T_1, T_2, T_3 \rangle$ see Figs. 3 and 4, we have never observed overlapping of circles. However we presently have no proof of the following conjectures, although some hints toward a proof will be discussed in the later section.

Conjecture 1. For $\alpha = \frac{\pi}{N}$, $N \geqq 7$, the circles obtained by applying the full group $\langle T_1, T_2, T_3 \rangle$ to X_1 do not cross each other, and each circle which is obtained an infinite number of times remains inside X_∞. We therefore obtain a packing of X_∞.

Conjecture 2. For $\alpha = \frac{k\pi}{N}$, $k \geqq 2$ (k, N relatively prime) the circles obtained in the above fashion form a nowhere dense subset of $\mathbb{C}$. That is $\langle T_1 T_2 T_3 \rangle$ is discontinuous.

IV. Polyspherical Representations

In order to try to prove that our circles do not overlap and to gain further insight into our problem, we introduce polyspherical coordinates for our circles. A more precise name would be *tetracircular coordinates*. An introduction to this subject can be found in [12 or 13].

IV.1. Notations and Definitions. $S(a, r)$ denotes the inner part of the circle of center a and radius r if $r > 0$; if $r < 0$ it will represent the outer part. Let

$$X = S(a, r), \quad Y = S(b, s). \tag{IV.1}$$

Then the "distance" between X and Y is

$$\Delta(X, Y) = \frac{|a-b|^2 - r^2 - s^2}{2rs}. \tag{IV.2}$$

In particular

$$\Delta(X, X) = -1, \tag{IV.3}$$

$$|\Delta(X, Y)| > 1 \tag{IV.4}$$

if X and Y do not intersect. $|\Delta(X, Y)| < 1$ means that the two circles X and Y intersect and the angle of intersection is:

$$\cos\theta = -\Delta(X, Y). \tag{IV.5}$$

The circles are oriented clockwise when $r > 0$ and anticlockwise when $r < 0$. The tangent are oriented accordingly, and θ is the supplement of the angle between the two oriented tangents. Given four circles Z_1, Z_2, Z_3, Z_0 in the plane, one defines the matrix G according to:

$$G^{ij} = \Delta(Z_i, Z_j), \quad i, j = 0, 1, 2, 3. \tag{IV.6}$$

This is a real 4×4 symmetric matrix. If

$$\det G \neq 0 \tag{IV.7}$$

the matrix is invertible, and the four circles X_i are said to be *independent*.

Let us define the contravariant coordinates of a circle Y by:

$$y^i = \Delta(Y, Z_i) \quad i = 0, 1, 2, 3. \tag{IV.8}$$

The "polyspherical coordinate of Y" will be the covariant components of Y, using G^{ij} as metric tensor,

$$y_i = \sum_{j=0}^{j=3} G_{ij} y^j = G_{ij} y^j \quad \text{with} \quad G_{ij} G^{jk} = \delta_i^k. \tag{IV.9}$$

(We use the convention of summation over repeated indices, when in upper and lower position.)

Notation. For simplicity we shall denote by Y_c the set of covariant components of (Y):

$$Y_c = (y_0, y_1, y_2, y_3) \tag{IV.10}$$

and Y^c the set of contravariant components of (Y):

$$Y^c = (y^0, y^1, y^2, y^3). \tag{IV.11}$$

Here are some basic facts: the proofs follow the lines of [13] and can be found in Appendix C.

(i) For any given circle U, V

$$\Delta(U, V) = G_{ij} u^i v^j \tag{IV.12}$$

$$= G^{ij} u_i v_j \tag{IV.13}$$

$$= u^j v_j = u_j v^j. \tag{IV.14}$$

(ii) If $\varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^0$ are the curvature of Z_1, Z_2, Z_3, Z_0, then the curvature Y is given by

$$\varepsilon_Y = y_i \varepsilon^i. \tag{IV.15}$$

(iii) The polyspherical coordinate of $Z_{(i)}$ are the coordinates of the standard vector e_i:

$$Z_{(i)j} = \delta_{ij}. \tag{IV.16}$$

We now specialize to the case where $Z_i = X_i$, where X_1, X_2, X_3, X_0 are the circles of the basic construction. In this case one verifies that there is a positive constant $\gamma > 1$ so that

$$\Delta(X_i, X_j) = G^{ij} = \gamma \quad \text{for} \quad i \neq j. \tag{IV.17}$$

In fact with $\alpha = \dfrac{\pi}{N} (N > 6)$

$$\gamma_N = \frac{1 - 2\sin^2 \dfrac{\pi}{N}}{1 - 4\sin^2 \dfrac{\pi}{N}} > 1. \tag{IV.18}$$

γ reaches the value 1 for $N = \infty$ (the Apollonian case).

The matrix G can be rewritten:

$$G = -(1+\gamma)I + \gamma J, \tag{IV.19}$$

where I is the unit matrix and J is the matrix consisting of 1's. Since the spectrum of γJ is $\{4\gamma, 0, 0, 0\}$, we see that the spectrum of G is

$$\sigma(G) = \{3\gamma - 1, -(1+\gamma), -(1+\gamma), -(1+\gamma)\}. \tag{IV.20}$$

Therefore the signature of the quadratic form associated to Δ will always be $+---$. Our space will be the four dimensional Minkowski space. Relations (IV.14) tells us that:

$$\Delta(U, U) = -1 = u_i u^i. \tag{IV.21}$$

Our circles are therefore space-like vectors of norm 1 in the Minkowski space. Therefore we move or transfer our Möbius maps T_1, T_2, T_3 to this four dimensional setting. (The Minkowski four dimensional space.)

Remark IV.1. $\Delta(U, V)$ is clearly invariant under any Möbius transformations.

The basis: $(X_0), (X_1), (X_2), (X_3)$ is transformed by T_1^{-1} into a new basis: $(X_1), (X_\infty), (X_2), (X_3)$. Clearly the G matrix associated to this new basis is the same as the previous G matrix. It is useful to compute the contravariant component of (X_∞) in the initial basis, an easy calculation gives:

$$\Delta(X_0, X_\infty) = X^0_{(\infty)} = 1 + 3\gamma\tau, \tag{IV.22}$$

$$\begin{aligned} \Delta(X_1, X_\infty) &= \gamma = X^1_{(\infty)}, \\ \Delta(X_2, X_\infty) &= \gamma = X^{(2)}_\infty, \\ \Delta(X_3, X_\infty) &= \gamma = X^{(3)}_\infty, \end{aligned} \tag{IV.23}$$

where

$$\tau = \frac{2\gamma}{2\gamma - 1} = 2\cos\frac{2\pi}{N}, \tag{IV.24}$$

$$\gamma = \frac{\tau}{2(\tau - 1)} = \frac{2\cos\dfrac{2\pi}{N}}{2\cos\dfrac{2\pi}{N} - 1}. \tag{IV.25}$$

Proposition 1. *There are matrices $M_i, i = 1, 2, 3$ so that for any circle Y:*

$$(T_i Y)_c = M_i Y_c. \tag{IV.26}$$

Proof.

$$(T_i Y)_c = G^{-1}(T_i Y)^c = G^{-1}\begin{bmatrix} \Delta(X_0, T_i Y) \\ \Delta(X_1, T_i Y) \\ \Delta(X_2, T_i Y) \\ \Delta(X_3, T_i Y) \end{bmatrix}, \tag{IV.27}$$

and by invariance of the scalar product

$$\Delta(U, TV) = \Delta(T^{-1}U, V), \tag{IV.28}$$

$$(T_i Y)_c = G^{-1} \begin{bmatrix} \Delta(T_i^{-1} X_0, Y) \\ \Delta(T_i^{-1} X_1, Y) \\ \Delta(T_i^{-1} X_2, Y) \\ \Delta(T_i^{-1} X_3, Y) \end{bmatrix}. \tag{IV.29}$$

For clarity of the proof we specialize to $i=1$, choosing $T_i = T_1$ then

$$(T_1 Y)_c = G^{-1} \begin{pmatrix} \Delta(X_1, Y) \\ \Delta(X_\infty, Y) \\ \Delta(X_2, Y) \\ \Delta(X_3, Y) \end{pmatrix}. \tag{IV.30}$$

Each component of the column vector in (IV.30) is a linear combination of the Y_k, therefore $(T_1, Y)_c$ is of the form (IV.26).

We have to find the explicit form of the M_i. Let us for example, compute M_1. The columns of M_1 are $M_1 e_0, M_1 e_1, M_1 e_2, M_1 e_3$, from left to right. Now:

$$\begin{aligned} T_1 X_1 &= X_0 \quad \to M_1 e_1 = e_0 \\ T_1 X_2 &= X_2 \quad \to M_1 e_2 = e_2 \\ T_1 X_3 &= X_3 \quad \to M_1 e_3 = e_3 \\ T_1 X_0 &= X_{1,0} \to M_1 e_0 = e_{1,0}. \end{aligned} \tag{IV.31}$$

We only have to compute

$$M_1 e_0 = (T_1 X_0)_c. \tag{IV.32}$$

[Remember that (IV.14) tells us that the polyspherical coordinate of the basic circles are the basic vectors coordinates!]

$$(T_1 X_0)_c = G^{-1} (T_1 X_0)^c \tag{IV.33}$$

and

$$(T_1 X_0)^c = \begin{bmatrix} \Delta(X_0, T_1, X_0) \\ \Delta(X_1, T_1, X_0) \\ \Delta(X_2, T_1, X_0) \\ \Delta(X_3, T_1, X_0) \end{bmatrix} = \begin{bmatrix} \Delta(X_1, X_0) \\ \Delta(X_\infty, X_0) \\ \Delta(X_2, X_0) \\ \Delta(X_3, X_0) \end{bmatrix}, \tag{IV.34}$$

and finally using the fact that

$$G^{-1} = -\frac{1}{1+\gamma}\left[I - \frac{\gamma}{3\gamma - 1} J \right]. \tag{IV.35}$$

Therefore we get

$$(T_1 X_0)_c = \begin{bmatrix} \tau \\ -1 \\ \tau \\ \tau \end{bmatrix} \tag{IV.36}$$

or

$$M_1 e_0 = \tau e_0 - e_1 + \tau e_2 + \tau e_3 \tag{IV.37}$$

and

$$M_1 = \begin{bmatrix} \tau & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \tau & 0 & 1 & 0 \\ \tau & 0 & 0 & 1 \end{bmatrix}. \tag{IV.38}$$

In the same way we get:

$$\begin{aligned} M_2 e_0 &= \tau e_0 + \tau e_1 - e_2 + \tau e_3, \\ M_2 e_1 &= e_1, \\ M_2 e_2 &= e_0, \\ M_2 e_3 &= e_3, \end{aligned} \tag{IV.39}$$

and

$$\begin{aligned} M_3 e_0 &= \tau e_0 + \tau e_1 + \tau e_2 - e_3, \\ M_3 e_1 &= e_1, \\ M_3 e_2 &= e_2, \\ M_3 e_3 &= e_0. \end{aligned} \tag{IV.40}$$

Therefore

$$M_2 = \begin{bmatrix} \tau & 0 & 1 & 0 \\ \tau & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \tau & 0 & 0 & 1 \end{bmatrix}, \tag{IV.41}$$

and

$$M_3 = \begin{bmatrix} \tau & 0 & 0 & 1 \\ \tau & 1 & 0 & 0 \\ \tau & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{IV.42}$$

The eigenvalues of M_i are: $1, 1, e^{\pm 2i\pi/N}$. The eigenvectors corresponding to the eigenvalues 1 are trivial to find. The one corresponding to $e^{\pm 2i\pi/N}$ are for M_1:

$$-e^{\pm i\pi/N} e_0 + e^{\mp i\pi/N} e_1 \pm \frac{i\cos\dfrac{2\pi}{N}}{\sin\dfrac{\pi}{N}}(e_2 + e_3). \tag{IV.43}$$

Remark. It is fundamental to notice that all these formulae and the isomorphism between the M_i and T_i make sense *only* for $N \geqq 7$. If one would extend the previous results to $N = 2$ for instance, we should have $M_i^2 = I$, however:

$$M_1 = \begin{bmatrix} -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \tag{IV.44}$$

and $M_1^2 \neq I$ while $T_1^2 = I$. The eigenvalues of M are $1, 1, -1, -1$ and only 3 eigenvectors exist in this case e_2, e_3 and $e_0 + e_1 - e_2 - e_3$. Therefore the group generated by M_1 is infinite, being isomorphic to a translation group, while for T_1 it is finite of order 2.

To further study why the analytic continuation of the two groups $\langle T_1, T_2, T_3 \rangle$ and $\langle M_1, M_2, M_3 \rangle$ which are isomorphic for $N \geqq 7$ do not continue to be isomorphic for $N \leqq 6$, one must go back to the fundamental quadratic form generated by G.

$$X_c^T G X_c = -1. \tag{IV.45}$$

Using (IV.18), we see that the eigenvalues of G are such that the form is:

for	$N \geqq 7$	a one sheeted hyperboloid (signature $+---$),
for	$N = 6$	degenerate into a cone,
for	$N = 5$	a prolate ellipsoid,
for	$N = 4$	sphere,
for	$N = 3$	an oblate ellipsoid,
for	$N = 2$	a cylinder.

V. Representation of the Matrix Elements of the Group in Terms of Tchebycheff Polynomials

Clearly the group $\langle M_1, M_2, M_3 \rangle$ is generated by two generators. For instance M_1 and P, where P corresponds to the rotation R for the group $\langle T_1, T_2, T_3 \rangle$. P is simply the permutation:

$$P = (1, 2, 3)(0). \tag{V.1}$$

The most general element of the group can therefore be written:

$$M_1^{n_1} P M_1^{n_2} P M_1^{n_3} P \cdots. \tag{V.2}$$

It is therefore a product of elements of the form

$$E_n = M_1^n P. \tag{V.3}$$

[We only need to consider non-negative powers n, because $M_1^n = I$.] Let us find the action of E_n. We have

$$\begin{aligned} E_n e_0 &= M_1^n e_0, \\ E_n e_1 &= e_2, \\ E_n e_2 &= e_3, \\ E_n e_3 &= M_1^n e_1 = M_1^{n-1} M_1 e_1 = M_1^{n-1} e_0. \end{aligned} \tag{V.4}$$

Therefore it suffices to study $M_1^n e_0$ for $n \geqq 0$. We have the following result (see

Appendix C for the proof). For $n \geqq 0$

$$M_1^n e_0 = e_0 \frac{\sin \frac{2\pi}{N}(n+1)}{\sin \frac{2\pi}{N}} - e_1 \frac{\sin \frac{2\pi}{N} n}{\sin \frac{2\pi}{N}} + (e_2 + e_3) \frac{\cos \frac{2\pi}{N}}{\cos \frac{\pi}{N}} \frac{\sin (n+1) \frac{\pi}{N}}{\sin \frac{\pi}{N}} \frac{\sin \frac{n\pi}{N}}{\sin \frac{\pi}{N}} \tag{V.5}$$

$$= q_n e_0 - q_{n-1} e_1 + r_n(e_2 + e_3), \tag{V.6}$$

where q_n and r_n are polynomials in τ with integer coefficients. Actually q_n is the Tchebycheff polynomial of second kind $U_k\left(\frac{\tau}{2}\right)$.

Let us define a "word" w as being a certain product

$$w = E_{n_1} E_{n_2} \cdots E_{n_k}. \tag{V.7}$$

The "distance" between two circles is:

$$\Delta(X, Y) = \Delta(w_1 X_0, w_2 X_0), \tag{V.8}$$

where w_1, w_2 are words made from the group, those words map X_0 onto X and X_0 onto Y. Because of the invariance of Δ under any Möbius transformation

$$\Delta(X, Y) = \Delta(X_0, w_1^{-1} w_2 X_0) = \Delta(X_0, w X_0). \tag{V.9}$$

If we want that no circles intersect we must have

$$|\Delta(X_0, w X_0)| \geqq 1. \tag{V.10}$$

If furthermore we want to have all circles inside X_∞, that is no circle inside X_0, we must have, either:

$$\Delta(X_0, w X_0) = -1 \tag{V.11}$$

when $w X_0 \equiv X_0$ or

$$\Delta(X_0, w X_0) \geqq 1 \tag{V.12}$$

when $X_0 \neq w X_0$. The equal sign occurs only for $N = \infty$. Combining (VII.11) and (VII.14) we have:

$$\Delta(X_0, w X_0) = e_0^T G w e_0 \tag{V.13}$$

or writing

$$w e_0 = w_0 e_0 + w_1 e_1 + w_2 e_2 + w_3 e_3, \tag{V.14}$$

we find

$$\Delta(X_0, w X_0) = -w_0 + \gamma(w_1 + w_2 + w_3). \tag{V.15}$$

Clearly the w_i being polynomials in τ and γ being a rational in τ, all elements here belong to *cyclotomic* fields.

Let us consider the next word after w:

$$w' = E_n w. \tag{V.16}$$

Let us remark that we can consider any values $n = 1, 2, \ldots, N-1$ for n, because for $n=0$, w' and w give the same Δ. We now compute

$$\begin{aligned} w'_0 &= w_0 q_n + w_3 q_{n-1}, \\ w'_1 &= -[w_0 q_{n-1} + w_3 q_{n-2}], \\ w'_2 &= w_1 + w_0 r_n + w_3 r_{n-1}, \\ w'_3 &= w_2 + w_0 r_n + w_3 r_{n-1}. \end{aligned} \tag{V.17}$$

The q_n are the Tchebycheff polynomials of the second kind and verify

$$\begin{aligned} q_n &= \tau q_{n-1} - q_{n-2}, \\ r_{n+1} &= r_n + \tau q_n. \end{aligned} \tag{V.18}$$

Let us apply the previous results to the case $N=\infty$. In this case we cannot take only n positive, but for any $n \in \mathbb{Z}$, we have:

$$\begin{aligned} q_n &= n+1 \\ r_n &= n(n+1) \\ \gamma &= +1 \end{aligned} \tag{V.19}$$

and (V.17) reads

$$\begin{aligned} w'_0 &= (n+1)w_0 + n w_3, \\ w'_1 &= -[n w_0 + (n-1)w_3], \\ w'_2 &= w_1 + n(n+1)w_0 + n(n-1)w_3, \\ w'_3 &= w_2 + n(n+1)w_0 + n(n-1)w_3. \end{aligned} \tag{V.20}$$

If we compute $\Delta' - \Delta$ we get

$$\Delta' - \Delta = 2n[n w_0 + (n-2)w_3]. \tag{V.21}$$

The w_i being integers the difference of any Δ_w and $\Delta_{w'}$ has to be an *even* integer! Because $\Delta(X_0, X_0) = -1$, it then results that

$$\Delta_w = \text{odd integer}, \tag{V.22}$$

and therefore

$$|\Delta_w| \geqq 1 \tag{V.23}$$

as it should. We have rederived here a well known result [13]. Inside the cyclotomic field, Δ is a finite sum of integers over the various components of the field. We shall explore in forthcoming papers this structure.

VI. Recovery of the Platonician Groups

This case corresponds to $r^2 \geqq \frac{9}{8}$, and therefore when $\alpha = \frac{\pi}{N}$, N can only take the values 2, 3, 4, 5. From (II.8) the fixed points of T_1 are:

$$f_\pm = -\frac{1}{2} \pm \sqrt{r^2 - \frac{3}{4}} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \sqrt{\frac{1-\cos 2\alpha}{1-3\cos 2\alpha}}. \tag{VI.1}$$

They are both real and T_1 is elliptic. The values of r^2 and α of interest are:

α	$\frac{\pi}{5}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r^2	$\frac{3(3+\sqrt{5})}{2}$	$\frac{3}{2}$	$\frac{6}{5}$	$\frac{9}{8}$

We used the fact that $\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$ and $\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$. Since

$$f_+f_-=1-r^2<0, \tag{VI.2}$$

we can find a dilation factor d so that:

$$(df_+)(df_-)=1. \tag{VI.3}$$

In fact

$$d=\sqrt{2(1-3\cos 2\alpha)}. \tag{VI.4}$$

Proposition 1. *Let d given by (VI.4) and define*:

$$\hat{T}_i=DT_iD^{-1} \quad \text{or} \quad \hat{T}_i(z)=dT_1(d^{-1}z). \tag{VI.5}$$

Then $\langle \hat{T}_1, \hat{T}_2, \hat{T}_3\rangle$ is a group of rotations of the Riemann sphere.

Proof. We first recall that the necessary and sufficient condition for an elliptic Möbius transformation to be a rotation of the Riemann sphere is that

$$z_{f_1}z_{f_2}^*=-1, \tag{VI.6}$$

where z_{f_1} and z_{f_2} are the fixed points of the transformation. It then follows that $\hat{T}_1$ is a rotation of the Riemann sphere. Now, the fixed points of $\hat{T}_2$ are $wdf_\pm$ and:

$$(\omega df_+)^*(\omega df_-)=df_+df_-=-1. \tag{VI.7}$$

So $\hat{T}_2$ is also a rotation (of the same angle 2α), as is $\hat{T}_3$. Therefore $\hat{T}_1, \hat{T}_2, \hat{T}_3$ are three rotations of the same angle 2α of the Riemann sphere; we now determine the angles between the axes of rotation.

We represent the Riemann sphere, as a sphere of radius 1, which intersects the complex plane in its center. If $P=\left(\frac{2x}{|x|^2+1};\frac{2y}{|z|^2+1};\frac{|z|^2-1}{|z|^2+1}\right)$ is the point on the Riemann sphere corresponding to $z=x+iy$ in the complex plane, and if P_ω and P_{ω^2} correspond respectively to ωz and $\omega^2 z$, then:

$$PP_\omega=P_\omega P_{\omega^2}=P_{\omega^2}P=\frac{(|z|^2-1)^2-2|z|^2}{(|z|^2+1)^2}=\cos\gamma. \tag{VI.8}$$

Computing, with $z=df_-$, we get:

$$|df_-|=-df_-=\frac{1}{\sqrt{2}}\{\sqrt{1-3\cos 2\alpha}+\sqrt{3}\sqrt{1-\cos 2\alpha}\} \tag{VI.9}$$

and

$$\cos\gamma = \frac{\cos 2\alpha}{\cos 2\alpha - 1}, \tag{VI.10}$$

where γ is the *common* angle between the three axes of rotation corresponding to the rotations $\hat{T}_1, \hat{T}_2, \hat{T}_3$.

For our cases:

α	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{5}$
$\cos\gamma$	$\frac{1}{2}$	$\frac{1}{3}$	0	$-\frac{1}{\sqrt{5}}$

Proposition 2. *For* $\alpha = \frac{\pi}{4}$ *and* $\alpha = \frac{\pi}{2}$, $\langle \hat{T}_1, \hat{T}_2, \hat{T}_3 \rangle$ *is conjugate to the rotation group of a cube.*

Proof. For $\alpha = \frac{\pi}{2}$, $(\hat{T}_i)^2 = 1$ and the axes of rotation mutually meet at 60°. This is the case when the axes of rotation pass through the centers of two adjacent edges. See Fig. 5.

For $\alpha = \frac{\pi}{4}$, the axes of rotation are mutually orthogonal and pass through the centers of the faces of the cube.

For $N = 4$, we also give in Fig. 6, the set made of 4 circles which is invariant by $\langle T_1, T_2, T_3 \rangle$.

Proposition 3. *For* $\alpha = \frac{\pi}{3}$ *and* $\alpha = \frac{\pi}{5}$, $\langle \hat{T}_1, \hat{T}_2, \hat{T}_3 \rangle$ *is conjugate to the rotation group of an icosahedron or dodecahedron.*

Proof. Again, the angles are right. For $\alpha = \frac{\pi}{5}$ we consider a dodecahedron; the axes of rotation pass through the centers of three pentagons that share a common

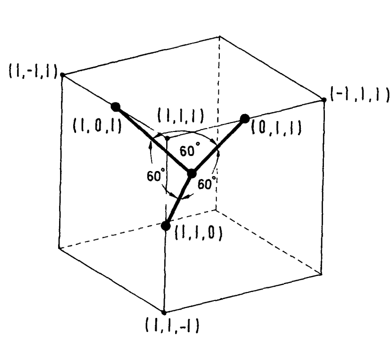

Fig. 5. The case $N = 2$

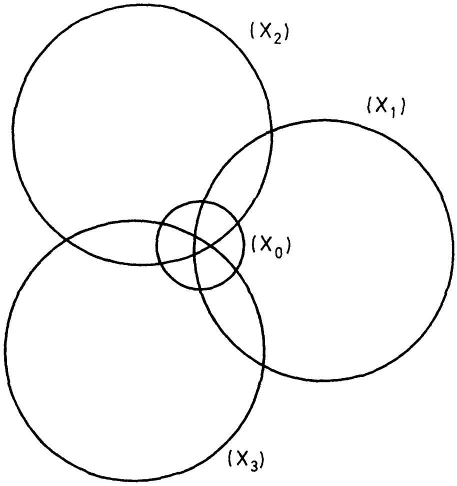

Fig. 6. The case $N = 4$

vertex. For $\alpha = \frac{\pi}{3}$, we consider an icosahedron; the axes of rotation pass through the centers of three triangles that are adjacent to a single common triangle.

Remark 1. For $N = 2, 3, 4, 5$ we can also consider $\alpha = \frac{k\pi}{N}$, we then discover that we obtain nothing new. The result is obvious for $N = 2, 3, 4$. For $N = 5$ one could think that $\alpha = \frac{\pi}{5}$ and $\alpha = \frac{2\pi}{5}$ would give different results, however computing $\cos\gamma$, one finds $\cos\gamma = -\frac{1}{\sqrt{5}}$ for $\alpha = \frac{\pi}{5}$ and $\cos\gamma = +\frac{1}{\sqrt{5}}$ for $\alpha = \frac{2\pi}{5}$. This shows that the axes of rotations are in fact the same in both cases.

Remark 2. One should be able to prove that the only cases where a finite group is generated is for $N = 2, 3, 4, 5$.

VII. The Case $\alpha = \frac{\pi}{6}$

We come now to the case $\alpha = \frac{\pi}{6}$. The fixed point of T_1 are, making use of (II.8) and (II.9): ω and ω^2. So the fixed points of T_2 are ω^2 and 1 and those of T_3 are 1 and ω.

Let us send:

$$\begin{cases} \omega \to \infty \\ \omega^2 \to 0 \\ 1 \to \omega^2 \end{cases}$$

with the transformation:

$$\begin{cases} f(z) = -\dfrac{z-\omega^2}{z-\omega} \\ f^{-1}(z) = \omega\dfrac{(z+\omega)}{z+1} \end{cases} \tag{VII.1}$$

define

$$\tilde{T}_i = f \circ T_i \circ f^{-1}. \tag{VII.2}$$

So

$$\tilde{T}_1 z = e^{2i\pi/6} z,$$

$$\tilde{T}_2 z = -\omega\frac{z}{z+1},$$

$$\tilde{T}_3 z = e^{-2i\pi/6} z - \omega.$$

The group $\langle \tilde{T}_1, \tilde{T}_3 \rangle$ is a *classical doubly periodic* group, whose general element has the form

$$Sz = ze^{2i\pi n/6} + m_+ e^{2i\pi/6} + m_- e^{-2i\pi/6}; \quad m_+, m_- \in \mathbb{Z}, \quad n = 1,2,3,4,5,6. \tag{VII.3}$$

The only limit point is ∞ cf. [8].

Claim. The group $(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$ has $\tilde{\mathbb{C}}$ as its limit and is thus not discontinuous.

Clearly the lattice is not invariant by $\tilde{T}_2$ because

$$\tilde{T}_2 z = -\frac{\omega z}{z+1} = -\omega + \frac{\omega}{z+1} \tag{VII.4}$$

while the various transformations $z \to z+1$, $z \to \omega z, z \to z - \omega$ leave the lattice invariant, the transformation $z \to \dfrac{1}{z}$ does not.

VIII. Higher Dimensional Extensions

One can try to extend the method introduced here to higher dimensions by starting with $n+1$ n-dimensional spheres $X_1, \ldots, X_n$ with centers at the vertices of a regular simplex in $\mathbb{R}^n$ and looking for a sphere X_0 and conformal maps $T_1, \ldots, T_n$ satisfying $T_i X_i = X_0$, $T_i X_j = X_j$ for $j \neq i$. In $\mathbb{R}^3$ the spheres $X_0, X_1, \ldots, X_4$ turn out to be mutually tangent and a packing investigated by Boyd [13] is the result. In this case $T_i^6 = I$. In $\mathbb{R}^4$ we believe that there is a non-tangential packing generated by maps that satisfy $T_i^5 = I$. We believe that these exhaust the finite dimensional packings that can be constructed with this technique. There is, however, an infinite dimensional packing generated by maps satisfying $T_i^4 = I$. Details of these constructions will appear elsewhere.

IX. Conclusion

We have studied a class of discrete subgroups of $SL(2, \mathbb{C})$ generated by two rotations. These subgroups fall into two classes: finite and infinite. The finite subgroups give the symmetry groups of the cube and icosahedron. The infinite subgroups have as their limit sets packings of circles. The action of the infinite groups can be analyzed in a four-dimensional Minkowski space and some preliminary analysis was made.

The next investigations will be to analyse the structure of the lattice on the one sheeted Minkowski hyperboloid, in particular its projection on the three dimensional Euclidean space. Also the generalization to higher dimensions, as well as the construction of discrete groups depending on more than one parameter, will be analyzed.

Appendix A

Denote the center and radius of the circle of inversion by $-I$ and k respectively.

The circle of inversion J and X_2 being by construction orthogonal at ξ_2 (see Fig. 1), we have:

$$r^2 + k^2 = \tfrac{3}{4} + (I - \tfrac{1}{2})^2. \tag{A.1}$$

For real x, we must have:

$$(Jx + I)(x + 1) = k^2. \tag{A.2}$$

For the center of (X_0) to be at 0, we force, (using $JX_1 = SX_0$):

$$\frac{J(1+r) + J(1-r)}{2} = -1. \tag{A.3}$$

From these we obtain:

$$I = \frac{1 \pm \sqrt{9 - 8r^2}}{2}. \tag{A.4}$$

We take the "+" sign. Later we will remark on the other choice of sign. We also have:

$$Jz = \begin{cases} \dfrac{k^2(x+I)}{(x+I)^2 + y^2} - I \\[2ex] \dfrac{k^2 y}{(x+I)^2 + y^2} \end{cases} \quad \text{and} \quad Sz = \begin{cases} -(1+x) \\[2ex] y \end{cases}, \tag{A.5}$$

where the upper line refers to the real part, while the lower one to the imaginary. With

$$I = \frac{1 + \sqrt{9 - 8r^2}}{2} \tag{A.6}$$

we derive from (II.5) and (A.5):

$$T_1 z = \frac{I-1}{2}\,\frac{2z - I}{z + I}. \tag{A.7}$$

From (A.1) and (A.6) we obtain the radius of the circle of inversion k:

$$k^2 = (I - \tfrac{1}{2})^2 + \tfrac{3}{4} - r^2 = 3(1 - r^2). \tag{A.8}$$

One also checks that:

$$\rho_0 = \frac{k^2 r}{(I+1)^2 - r^2} = \frac{2r^2 - 3 + \sqrt{9 - 8r^2}}{2r},$$

or in invariant form

$$\rho_0 = \frac{\sqrt{\cos 2\alpha + 1}\sqrt{3\cos 2\alpha - 1} - \sqrt{3}\cos 2\alpha}{\sqrt{2(2\cos 2\alpha - 1)(3\cos 2\alpha - 1)}}.$$

In the same way:

$$\rho_\infty = \frac{k^2 r}{r^2 - (2 - I)^2} = \frac{\sqrt{\cos 2\alpha + 1}\sqrt{3\cos 2\alpha - 1} + \sqrt{3}\cos 2\alpha}{\sqrt{2(2\cos 2\alpha - 1)(3\cos 2\alpha - 1)}}.$$

Appendix B

We are to prove that for $k \geqq 2$ and $0 \leqq \xi \leqq \dfrac{\pi}{6k+1}$, $k \in \mathbb{N}$, we have

$$\frac{1 - 4\sin^2 k\xi}{1 - 3\sin^2 k\xi} \cdot \frac{\sin^2 k\xi}{\sin^2 \xi} > 1. \tag{B.1}$$

With $\chi = \sin^2 k\xi$ we have $\sin \xi = \sin\left\{\dfrac{\sin^{-1}\sqrt{\chi}}{k}\right\}$ and the following inequality equivalent to B.1:

$$\frac{\sin^{-1}\sqrt{\chi}}{\sin^{-1}\left(\sqrt{\dfrac{\chi(1-4\chi)}{1-3\chi}}\right)} < k, \quad 0 \leqq \chi \leqq \sin^2\left(\frac{\pi}{6 + \dfrac{1}{k}}\right) \tag{B.2}$$

Now, for $u \geqq 0$, $\sin u \leqq u$ so $u \leqq \sin^{-1} u$. Also, for $0 \leqq u \leqq 1/2$, $\dfrac{\sin^{-1} u}{u} < \dfrac{2}{\sqrt{3}}$ by the Mean Value Theorem. This gives

$$\frac{\sin^{-1}\sqrt{\chi}}{\sin^{-1}\left(\sqrt{\dfrac{\chi(1-4\chi)}{1-3\chi}}\right)} \leqq \frac{\sin^{-1}\sqrt{\chi}}{\sqrt{\chi}}\sqrt{\frac{1-3\chi}{1-4\chi}} < \frac{2}{\sqrt{3}}\sqrt{\frac{1-3\chi}{1-4\chi}} \tag{B.3}$$

for $0 \leqq \chi \leqq \sin^2\left(\dfrac{\pi}{6 + \dfrac{1}{k}}\right) < \dfrac{1}{4}$.

Since $\dfrac{1-3\chi}{1-4\chi}$ is increasing, we see that it suffices to show that

$$\frac{1-3\sin^2\left(\dfrac{\pi}{6+\dfrac{1}{k}}\right)}{1-4\sin^2\left(\dfrac{\pi}{6+\dfrac{1}{k}}\right)} \leqq \frac{3}{4}k^2, \tag{B.4}$$

or

$$\sin^2\left(\frac{\pi}{6+\dfrac{1}{k}}\right) \leqq \frac{1}{12}\left\{3-\frac{1}{k^2-1}\right\}. \tag{B.5}$$

Using

$$\sin^2\frac{\pi}{6+\dfrac{1}{k}} = \frac{1}{2}\left[1-\cos\frac{2\pi}{6+\dfrac{1}{k}}\right] \tag{B.6}$$

$$\frac{1}{2} = \cos\frac{\pi}{3} \tag{B.7}$$

and

$$\cos p - \cos q = 2\sin\left(\frac{p+q}{2}\right)\sin\left(\frac{q-p}{2}\right). \tag{B.8}$$

We see that (B.5) is equivalent to

$$\cos\frac{2\pi}{6+\dfrac{1}{k}} - \frac{1}{2} \geqq \frac{1}{6}\cdot\frac{1}{k^2-1} \tag{B.9}$$

and that

$$\cos\frac{2\pi}{6+\dfrac{1}{k}} - \frac{1}{2} \geqq 2\sin\left\{\frac{25\pi}{78}\right\}\sin\left\{\frac{\pi}{36k+6}\right\}. \tag{B.10}$$

Now,

$$\sin\frac{\pi}{36k+6} > \frac{\pi}{36k+6}\cdot\cos\frac{\pi}{78} \tag{B.11}$$

since $\dfrac{\sin\chi}{\chi} = \cos\eta$ for some $0<\eta<\chi$ and $0<\dfrac{\pi}{36k+6}<\dfrac{\pi}{28}$ for $k \geqq 2$.

Therefore,

$$\cos\frac{2\pi}{6+\dfrac{1}{k}} - \frac{1}{2} \geqq 2\sin\frac{25\pi}{78}\cos\frac{\pi}{78}\cdot\frac{\pi}{36k+6} > \frac{5.2}{36k+6} = \frac{1}{6}\cdot\frac{5.2}{6k+1}. \tag{B.12}$$

Finally, the function $f(z)=\dfrac{6z+1}{z^2-1}$ is decreasing for $z \geqq 2$, so $\dfrac{5.2}{6k+1} \geqq \dfrac{1}{k^2-1}$ for $k \geqq 2$.

Appendix C

With $X=S(a,r)$ and $Y=S(b,s)$ as in (IV.1) we define

$$v(Y)=\frac{1}{s}\begin{bmatrix}|b|^2-s^2\\ \frac{1}{2}\\ \operatorname{Re} b\\ \operatorname{Im} b\end{bmatrix} \quad \text{and} \quad u(X)=\frac{1}{r}\begin{bmatrix}\frac{1}{2}\\ |a|^2-r^2\\ -\operatorname{Re} a\\ -\operatorname{Im} a\end{bmatrix}. \tag{C.1}$$

Then,

$$\Delta(X,Y)=v(Y)^T u(X). \tag{C.2}$$

From this we obtain the Darboux–Frobenius formula: if X_1, X_2, X_3, X_4, X_5 and Y_1, Y_2, Y_3, Y_4, Y_5 are circles in $\mathbb{R}^2$, then

$$\det(\Delta(X_i, Y_j))=0. \tag{C.3}$$

This is because $\Delta(X_i, Y_j)=v(Y_j)^T u(X_i)$ and $u(X_1),\dots,u(X_5)$ are linearly dependent in $\mathbb{R}^4$.

Now take $X_1=Y_1=Z_0$, $X_2=Y_2=Z_1$, $X_3=Y_3=Z_2$, $X_4=Y_4=Z_3$ and $X_5=Z$, $Y_5=Y$. (C.3) becomes (cf. IV.6)

$$\det\begin{bmatrix} G & (Y^c)^T\\ Z^c & \Delta(Y,Z)\end{bmatrix}=0. \tag{C.4}$$

By expansion

$$\Delta(Y,Z)\det G-(Y^c)^T(\operatorname{adj.} G)Z^c=0, \quad \operatorname{adj.} G=G^{-1}\det G. \tag{C.5}$$

If $\det G \neq 0$, we obtain

$$\Delta(Y,Z)=(Y^c)^T G^{-1} Z^c \tag{C.6}$$

which establishes (IV.12). Assertions (IV.13–IV.14) follow from (IV.9).

For example, we see immediately that

$$(Y^c)^T G^{-1}(Y^c)=Y_c G Y_c=-1. \tag{C.7}$$

Assertion (IV.16) is a trivial consequence of the definition (IV.9).

For assertion (IV.15), assume $Y=S(b,s)$ and that Z is a line whose distance from b is d and whose distance from the center of Z_i is d_i. Then, the extension of (IV.2) to the case of lines gives

$$\Delta(Y,Z)=d\eta, \tag{C.8}$$

where

$$\eta=\frac{1}{s} \tag{C.9}$$

and

$$\Delta(Z,Z_i)=d_i\varepsilon_i \quad \text{(without summation)}, \tag{C.10}$$

where

$$\varepsilon_i = \text{curvature of } Z_i = \frac{1}{\text{radius of } Z_i}. \tag{C.11}$$

Now,

$$d\eta = \Delta(Y, Z) = (Y^c)^T G^{-1} Z^c. \tag{C.12}$$

Letting $Z \to \infty$ we see that $\frac{d}{d_i} \to 1$. So, the d on the left of (C.12) can cancel with each d_i in Z^c yielding

$$\eta = (Y^c)^T G^{-1} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \tag{C.13}$$

which is what (IV.15) says.

Appendix D

The proof proceeds by induction on n. When $n = 0$, we obtain $M_1^0 e_0 = e_0$. For $n = 1$,

$$M_1 e_0 = \tau e_0 - e_1 + (e_2 + e_3)\tau. \tag{D.1}$$

If we assume

$$M_1^k e_0 = q_k(\tau) e_0 + p_k(\tau) e_1 + r_k(\tau)(e_2 + e_3), \tag{D.2}$$

we will discover that

$$p_{k+1}(\tau) = -q_k(\tau), \tag{D.3}$$

$$q_{k+1}(\tau) = \tau q_k(\tau) + q_{k-1}(\tau), \tag{D.4}$$

$$r_{k+1}(\tau) = r_k(\tau) + \tau q_k(\tau) = \sum_{j=0}^{k} \tau q_j(\tau). \tag{D.5}$$

Since $q_0 = 1$ and $q_1(\tau) = \tau$, we see that

$$q_k(\tau) = U_k\left(\frac{\tau}{2}\right), \tag{D.6}$$

where

$$U_k(\chi) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad \chi = \cos\theta, \tag{D.7}$$

are the Chebyshev polynomials of the second kind.

Standard manipulation now gives (V.5).

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