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Entropy of Bogoliubov Automorphisms of the Canonical Anticommutation Relations

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Abstract. We compute the entropy $h_{\omega_A}(\alpha_U)$ in the sense of Connes, Narnhofer and Thirring of Bogoliubov automorphisms α_U of the CAR-algebra with respect to invariant quasifree states ω_A with $0 \leqq A \leqq 1$ having pure point spectrum.

1. Introduction

In their recent paper [3] Connes, Narnhofer, and Thirring extended the definition of entropy for automorphisms of finite von Neumann algebras studied in [4] to the case of automorphisms of C^*-algebras invariant with respect to a given state. In the present paper we shall compute this for Bogoliubov automorphisms of the CAR-algebra with respect to invariant quasifree states. Recall, for more details see Sect. 4, that if H is a complex Hilbert space and $f \to a(f)$ is a representation of H in the CAR-algebra $\mathscr{A}(H)$ satisfying the canonical anticommutation relations then each unitary operator U on H defines a Bogoliubov automorphism α_U of $\mathscr{A}(H)$ by $\alpha_U(a(f)) = a(Uf)$. If $A \in B(H)$ satisfies $0 \leqq A \leqq 1$ and $AU = UA$, then α_U is invariant with respect to the (gauge invariant) quasifree state ω_A defined by A. In the case $A = \frac{1}{2}1$, i.e. ω_A is the unique tracial state τ on $\mathscr{A}(H)$, then the entropy $h_\tau(\alpha_U)$ is the same as that of the extension of α_U to the GNS-representation of τ as defined in [4]. A. Connes suggested to us that the formula for the entropy should be

$$h_\tau(\alpha_U) = \frac{\log 2}{2\pi} \int_0^{2\pi} m(U)(\theta) d\theta, \tag{1}$$

where $m(U)$ is the multiplicity function of the absolutely continuous part U_a of U, a conjecture which initiated the present work. More generally, if U_a acts on the

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subspace H_a of H we can write H_a as a direct integral

$$H_a = \int_{\mathbf{T}}^{\oplus} H_\theta d\theta$$

with $d\theta$ the Lebesgue measure on the unit circle $\mathbf{T}$. Correspondingly, the operator A commuting with U decomposes on H_a by the formula

$$A_a = A | H_a = \int_{\mathbf{T}}^{\oplus} A(\theta) d\theta.$$

Let η denote the real function on $[0,1]$ defined by $\eta(0)=0$, $\eta(t) = -t\log t$, $t \in (0,1]$. Then if A has pure point spectrum we shall prove the formula

$$h_{\omega_A}(\alpha_U) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Tr}\,(\eta(A(\theta)) + \eta(1 - A(\theta)))d\theta, \qquad (2)$$

where Tr denotes the usual trace on $B(H_\theta)$.

For general A we leave it as an open problem whether (2) is true. It is implicit in (2) that the entropy is unaffected by the singular part of U. We shall in particular show that if the spectral measure of U is singular with respect to Lebesgue measure than $h_\varphi(\alpha_U) = 0$ for all α_U-invariant states φ.

In addition to giving a formula for the entropy of a large class of automorphisms and invariant states (2) also yields an example of entropy in a more technically complicated situation than in previous calculations [1, 2, 3, 4, 7]. Namely in those cases the computation is based on the existence of a natural maximal abelian subalgebra which is globally invariant under the automorphism. In the case of Bogoliubov automorphisms we cannot in general expect this.

The proof of (2) is divided into five sections. The first, Sect. 2, contains a characterization of the Lebesgue integral on the circle by its properties on a class of functions which in our applications will be multiplicity functions of unitary operators with Lebesgue spectral measures. In Sect. 3 we prove some basic general results on entropy that will be needed in the sequel following closely the theory developed in [3]. In Sect. 4 we study the canonical anticommutation relations in more detail and develop the basic techniques on entropy in the case of quasifree states and Bogoliubov automorphisms. In Sect. 5 we consider the case when the spectral measure of U has nonzero singular part. In particular we show (2) in the simple case when the multiplicity function of the Lebesgue part of U is constant on a finite number of arcs of rational length. Then the proof is completed in Sect. 6, first for the case when A is a scalar operator, in which case the characterization in Sect. 2 is used, and then in the general case.

2. Lebesgue Measure on the Circle

In this section we show a result on the Lebesgue measure on the circle, which will be used to give the formula for the entropy of Bogoliubov automorphisms with respect to quasifree states defined by scalar operators.

Let $\mathscr{C}$ be the additive semigroup of functions $f: \mathbf{T} \to \{0\} \cup \mathbb{N}$ which are measurable with respect to Lebesgue measure $d\theta$ on $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. For further use we denote by $\mathbb{1}$ the constant function equal to 1 to $\mathbf{T}$ and by $T_n : \mathscr{C} \to \mathscr{C}$ $(n \in \mathbb{N})$

the map

$$(T_n f)(\rho) = \sum_{\substack{z \in \mathbf{T} \\ z^n = \rho}} f(z).$$

Let $\mu:\mathscr{C} \to \mathbb{R}^+$ – the nonnegative reals – which satisfies the following conditions:

(i) $\mu(n\mathbb{1}) = n, n \in \{0\} \cup \mathbb{N}$.

(ii) $f \leqq g \Rightarrow \mu(f) \leqq \mu(g)$.

(iii) $f_j \nearrow f \Rightarrow \mu(f_j) \nearrow \mu(f), j \in \mathbb{N}$.

(iv) $\mu(T_n f) = n\mu(f)$.

(v) $\mu(f) = \mu(g)$ if f and g are equal a.e. (with respect to Lebesgue measure).

In our applications $\mathscr{C}$ will consist of multiplicity functions of unitaries and μ will be a scalar multiple of the entropy of the corresponding Bogoliubov automorphism.

Theorem 2.1. *Let $\mu:\mathscr{C} \to \mathbb{R}^+$ be a map satisfying conditions* (i)–(v) *above. Then we have*

$$\mu(f) = \frac{1}{2\pi} \int_0^{2\pi} f d\theta.$$

The proof of this fact will be divided onto a few lemmas. We use the notation $d\lambda = \frac{1}{2\pi} d\theta$.

Lemma 2.2. *Given $\varepsilon > 0$ there is $\delta > 0$ such that if $f \in \mathscr{C}$, $f \leqq \mathbb{1}$, and if*

$$\int_0^{2\pi} (\mathbb{1} - f_n) d\lambda \leqq \delta$$

then $\mu(f) > 1 - \varepsilon$.

Proof. Assume to the contrary that there are $f_n \in \mathscr{C}$, $f_n \leqq \mathbb{1}$, such that

$$\int (\mathbb{1} - f) d\lambda \leqq 10^{-n},$$

and $\mu(f_n) \leqq 1 - \varepsilon$, $n \in \mathbb{N}$. Let

$$g_n = \inf_{k \geqq n} f_k \in \mathscr{C}.$$

We have

$$\int (\mathbb{1} - g_n) d\lambda \leqq \sum_{k \geqq n} \int (\mathbb{1} - f_k) d\lambda \leqq 2 \cdot 10^{-n},$$

and $g_n \leqq f_n$, $g_n \nearrow g$ with g equal to $\mathbb{1}$ almost everywhere. Hence $\mu(f_n) \geqq \mu(g_n)$, $\mu(g_n) \nearrow \mu(g)$. Thus $\mu(g) = \mu(\mathbb{1}) = 1$, contradicting $\mu(f_n) \leqq 1 - \varepsilon$. □

Lemma 2.3. *Given $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. If $f \in \mathscr{C}$ satisfies $T_q f = p\mathbb{1}$ almost everywhere $(p, q \in \mathbb{N})$, and $g \in \mathscr{C}$ satisfies $g \leqq f$ and*

$$\int g d\lambda \geqq (1 - \delta) \int f d\lambda,$$

then we have

$$\mu(g) \geqq (1 - \varepsilon) \int f d\lambda$$

(δ depends only on ε).

Proof. Since $d\lambda$ is the Haar measure $\int T_q f d\lambda = q \int f d\lambda$, and by (iv) $\mu(T_n f) = n\mu(f)$, we may replace f and g by $T_q f$ and $T_q g$ respectively. So what must be proved is that there is $\delta > 0$ depending only on ε such that whenever $g \leqq p\mathbb{1}$ for some $p \in \mathbb{N}$ and

$$\int g d\lambda \geqq (1-\delta)p,$$

then we have

$$\mu(g) \geqq (1-\varepsilon)p.$$

Using (iv) this follows from Lemma 2.2 and the fact that there is $g_1 \in \mathscr{C}$ such that $g_1 \leqq \mathbb{1}$ and $T_p g_1 = g$. □

Lemma 2.4. *Let $g \in \mathscr{C}$ be upper semicontinuous. Then we have*

$$\mu(g) = \int g d\lambda.$$

Proof. Let $g_n = g \wedge (n\mathbb{1})$. Since $g_n \nearrow g$ it will be sufficient to prove the lemma for the g_n's, i.e. we may assume g is bounded, say $g \leqq n\mathbb{1}$. Let $\varepsilon > 0$. Let $X_k = g^{-1}([k, \infty))$. Then X_k is compact. It is easily seen that there are $q_k \in \mathbb{N}$ and an open set $Y_k \supset X_k$ such that the boundary of Y_k is contained in the set

$$\{e^{2\pi i s/q_k} : 1 \leqq s \leqq q_k,\ s \in \mathbb{N}\},$$

and $\lambda(Y_k)(1-\delta) \leqq \lambda(X_k)$, δ being as in Lemma 2.3. Let χ_k be the characteristic function χ_{Y_k} of Y_k, and let

$$f = \sum_{1 \leqq k \leqq n} \chi_k, \quad q = q_1, q_2, \ldots, q_n.$$

Then $f \geqq g$, $T_q f = p\mathbb{1}$ for some $p \in \mathbb{N}$, and

$$(1-\delta) \int f d\lambda \leqq \int g d\lambda,$$

so that by Lemma 2.3.

$$\mu(g) \geqq (1-\varepsilon) \int f d\lambda \geqq (1-\varepsilon) \int g d\lambda.$$

On the other hand

$$\mu(g) \leqq \mu(f) = \frac{1}{q}\mu(T_q f) = \frac{p}{q} = \int f d\lambda \leqq \frac{1}{1-\delta} \int g d\lambda.$$

Since $\varepsilon > 0$ is arbitrary and we may choose $\delta < \varepsilon$ it follows that

$$\mu(g) = \int g d\lambda. \quad \square$$

Proof of Theorem. By general measure theory there is a sequence (f_n) in $\mathscr{C}$, $f_1 \leqq f_2 \leqq \cdots$ such that the f_n's are upper semicontinuous, and if $g = \lim_n f_n$ then $g = f$ almost everywhere. Thus we have

$$\int f d\lambda = \int g d\lambda = \lim_n \int f_n d\lambda = \lim_n \mu(f_n) = \mu(g) = \mu(f). \quad \square$$

3. Some General Entropy Results

We collect in this section entropy results which do not involve quasifree states, and which are more or less direct consequences of the theory developed in [3]. To fix our notation we recall the definitions in [3].

Let $\mathfrak{A}$ be a unital C^*-algebra, $C_1,\dots,C_k$ finite dimensional C^*-algebras, and $\gamma_j:C_j\to\mathfrak{A}$ a unital completely positive map, $j=1,\dots,k$. Let φ be a state on $\mathfrak{A}$ and $P:\mathfrak{A}\to B$ a unital positive map of $\mathfrak{A}$ into a finite dimensional abelian C^*-algebra B such that there is a state μ on B for which $\mu\circ P=\varphi$. If $p_1,\dots,p_r$ are the minimal projections in B then there are states φ_i, $i=1,\dots,r$, of $\mathfrak{A}$ such that

$$P(x)=\sum_{i=1}^{r}\varphi_i(x)p_i,\quad x\in\mathfrak{A}. \tag{1}$$

Since $\mu\circ P=\varphi$,

$$\varphi=\sum_{i=1}^{r}\mu(p_i)\varphi_i \tag{2}$$

is φ written as a convex combination of the φ_i. In the notation of [3]

$$\varepsilon_\mu(P)=\sum\mu(p_i)S(\varphi\,|\,\varphi_i),$$

where $S(\varphi\,|\,\varphi_i)$ is the relative entropy of the two states φ and φ_i, see [3, 6, 9]. The entropy defect $s_\mu(P)$ is given by

$$s_\mu(P)=S(\mu)-\varepsilon_\mu(P),$$

where $S(\mu)=-\sum_{i=1}^{r}\mu(p_i)\log\mu(p_i)$ is the entropy of μ.

Let B_j, $j=1,\dots,k$, be a C^*-subalgebra of B and $E_j:B\to B_j$ a μ-invariant conditional expectation. Then the quadruple (B,E_j,P,μ) is called an *abelian model* for $(\mathfrak{A},\varphi,\gamma_1,\dots,\gamma_k)$, and its entropy is defined to be

$$S\left(\mu\,\middle|\,\bigvee_{j=1}^{k}B_j\right)-\sum_{j=1}^{k}s_\mu(\rho_j),$$

where $\rho_j=E_j\circ P\circ\gamma_j:C_j\to B_j$. The sup of the entropies of all such abelian models is denoted by

$$H_\varphi(\gamma_1,\dots,\gamma_k).$$

If α is a φ-invariant automorphism of $\mathfrak{A}$ let $\gamma:C\to\mathfrak{A}$ be a unital completely positive map of a finite dimensional C^*-algebra C, and denote by

$$h_{\varphi,\alpha}(\gamma)=\lim_{k\to\infty}\frac{1}{k}H_\varphi(\gamma,\alpha\circ\gamma,\dots,\alpha^{k-1}\circ\gamma).$$

The entropy of α with respect to φ is

$$h_\varphi(\alpha)=\sup_\gamma h_{\varphi,\alpha}(\gamma).$$

In the above discussion we have implicitly assumed that the state μ is faithful. We shall use this assumption explicitly in the proof of our next lemma. But the reader should have no great difficulties in extending the proof to the possible situation of a nonfaithful μ.

Lemma 3.1. *Let φ be a pure state on the unital C^*-algebra $\mathfrak{A}$, and suppose α is a φ-invariant automorphism of $\mathfrak{A}$. Then $h_\varphi(\alpha)=0$.*

Proof. Let notation be as above with (B,E_j,P,μ) as the given abelian model. Since μ is faithful and P is given by (1) it follows from (2) that $\varphi_i=\varphi$ for all i since φ is pure. Thus $P(x)=\varphi(x)1$ and therefore $\rho_j(y)=\varphi\circ\gamma_j(y)1, j=1,\dots,k$. Consequently $\varepsilon_\mu(\rho_j)=0$, and so $s_\mu(\rho_j)=S(\mu|B_j)$. Thus the entropy for the abelian model is

$$S\left(\mu\,\middle|\,\bigvee_{j=1}^{r} B_j\right)-\sum_{j=1}^{r} S(\mu|B_j)\leqq 0,$$

whence

$$H_\varphi(\gamma_1,\dots,\gamma_k)=0.$$

Since this holds for all choices of γ's, $h_\varphi(\alpha)=0$. □

Lemma 3.2. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state, and α a φ-invariant automorphism. Suppose $\mathscr{B}$ is a C^*-subalgebra of $\mathfrak{A}$ such that there is an expectation $E:\mathfrak{A}\to\mathscr{B}$ satisfying $\varphi\circ E=\varphi$, and $\alpha E=E\alpha$. Then $\alpha|\mathscr{B}$ is an automorphism of $\mathscr{B}$ and*

$$h_\varphi(\alpha|\mathscr{B})\leqq h_\varphi(\alpha).$$

Proof. If C is a finite dimensional C^*-algebra and $\gamma:C\to\mathfrak{A}$ is completely positive then $E\alpha^n\gamma=\alpha^n E\gamma$, so by [3, Proposition III.6(b)] and the assumption $\varphi E=\varphi$,

$$H_\varphi(E\gamma,\alpha E\gamma,\dots,\alpha^{k-1}E\gamma)=H_\varphi(E\gamma,E\alpha\gamma,\dots,E\alpha^{k-1}\gamma)\leqq H_\varphi(\gamma,\alpha\gamma,\dots,\alpha^{k-1}\gamma).$$

Thus we have

$$h_{\varphi;\alpha|\mathscr{B}}(E\gamma)\leqq h_{\varphi,\alpha}(\gamma),$$

proving the lemma, as it is obvious that $\alpha(\mathscr{B})=\mathscr{B}$. □

Lemma 3.3. *Let $\mathfrak{A}$ be a C^*-algebra, φ a state, and α a φ-invariant automorphism of $\mathfrak{A}$. For each $j\in\mathbb{N}$ let $\mathfrak{A}_j$ be a C^*-subalgebra of $\mathfrak{A}$ and $E_j:\mathfrak{A}\to\mathfrak{A}_j$ an expectation such that $\alpha E_j=E_j\alpha$. Suppose $\mathfrak{A}_1\subset\mathfrak{A}_2\subset\cdots$ is increasing such that*

(i) $\mathfrak{A}=\left(\bigcup_{i=1}^{\infty}\mathfrak{A}_j\right)^{-}$, *norm closure.*
(ii) $E_{j+1}E_j=E_jE_{j+1}=E_j, j\in N$.
(iii) $E_j\to\mathrm{id}$ *pointwise-norm.*

Then $\alpha|\mathfrak{A}_j$ is an automorphism of $\mathfrak{A}_j$ for $j\in N$ and

$$h_\varphi(\alpha)\leqq\underline{\lim}\,h_\varphi(\alpha|\mathfrak{A}_j).$$

If moreover $\varphi\circ E_j=\varphi$ for all j then

$$h_\varphi(\alpha)=\lim h_\varphi(\alpha|\mathfrak{A}_j).$$

Proof. Since $\alpha E_j=E_j\alpha$, $\alpha|\mathfrak{A}_j\in\mathrm{Aut}(\mathfrak{A}_j)$ for all j. Let C be a finite dimensional C^*-algebra, $d=\dim C$, and suppose $\gamma:C\to\mathfrak{A}$ is a completely positive map. Since $E_j\to\mathrm{id}$, the identity map on $\mathfrak{A}$, positive in norm and C is finite dimensional, $E_j\gamma\to\gamma$ in norm. Furthermore we have

$$\varphi E_j=(\varphi\alpha)E_j=(\varphi E_j)\alpha,$$

so that α is φE_j-invariant for all j. Let

$$\varepsilon_j=\|E_j\gamma-\gamma\|.$$

Then $\varepsilon_j\to 0$ as $j\to\infty$. By [3, Proposition IV.3] we have

$$|H_\varphi(E_j\gamma,\alpha E_j\gamma,\dots,\alpha^{k-1}E_j\gamma)-H_\varphi(\gamma,\alpha\gamma,\dots,\alpha^{k-1}\gamma)|\leqq 6k\varepsilon_j(\tfrac{1}{2}+\log(1+d\varepsilon_j^{-1})),$$

whence, letting $k \to \infty$,

$$|h_{\varphi,\alpha|\mathfrak{A}_j}(E_j\gamma) - h_{\varphi,\alpha}(\gamma)| \leqq 6\varepsilon_j(\tfrac{1}{2} + \log(1 + d\varepsilon_j^{-1})). \tag{3}$$

Let $\delta > 0$ and choose C and γ such that if $h_\varphi(\alpha) < \infty$

$$|h_\varphi(\alpha) - h_{\varphi,\alpha}(\gamma)| < \delta,$$

and if $h_\varphi(\alpha) = \infty$, $h_{\varphi,\alpha}(\gamma) \geqq n$. By (3) we have for this choice of γ

$$\begin{aligned}|h_{\varphi,\alpha|\mathfrak{A}_j}(E_j\gamma) - h_\varphi(\alpha)| &\leqq |h_{\varphi,\alpha|\mathfrak{A}_j}(E_j\gamma) - h_{\varphi,\alpha}(\gamma)| + |h_{\varphi,\alpha}(\gamma) - h_\varphi(\alpha)| \\ &< 6\varepsilon_j(\tfrac{1}{2} + \log(1 + d\varepsilon_j^{-1})) + \delta,\end{aligned}$$

when $h_\varphi(\alpha) < \alpha$, whence

$$h_\varphi(\alpha|\mathfrak{A}_j) \geqq h_{\varphi,\alpha|\mathfrak{A}_j}(E_j\gamma) > h_\varphi(\alpha) - 6\varepsilon_j(\tfrac{1}{2} + \log(1 + d\varepsilon_j^{-1})) - \delta.$$

If $h_\varphi(\alpha) = \infty$ we similarly obtain $h_\varphi(\alpha|\mathfrak{A}_j) \geqq n - 6\varepsilon_j(\frac{1}{2} + \log(1 + d\varepsilon_j^{-1}))$. Since $6\varepsilon_j(\frac{1}{2} + \log(1 + d\varepsilon_j^{-1})) \to 0$ as $j \to \infty$, we have

$$\underline{\lim}\, h_\varphi(\alpha|\mathfrak{A}_j) \geqq h_\varphi(\alpha) - \delta.$$

Since δ is arbitrary the first conclusion of the lemma follows.

If $\varphi E_j = \varphi$ for all j then the converse inequality $h_\varphi(\alpha) \geqq \overline{\lim}\, h_\varphi(\alpha|\mathfrak{A}_j)$ is a consequence of Lemma 3.2. □

One of the challenging open problems concerning noncommutative entropy is whether it is additive on tensor product, i.e. is

$$h_{\varphi\otimes\psi}(\alpha \otimes \beta) = h_\varphi(\alpha) + h_\psi(\beta)?$$

We next show the easy half of this problem.

Lemma 3.4. *Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be two C^*-algebras with states φ' and φ'' respectively. Let α' and α'' be φ' and φ''-invariant automorphisms of $\mathfrak{A}'$ and $\mathfrak{A}''$. Then*

$$h_{\varphi'\otimes\varphi''}(\alpha' \otimes \alpha'') \geqq h_{\varphi'}(\alpha') + h_{\varphi''}(\alpha'').$$

Proof. Let $(\mathfrak{A}', \varphi', \gamma')$ be given with an abelian model (B', E_j', P', μ') and subalgebras B_j' with E_j' the μ'-invariant expectation of B' onto B_j'. Assume we have a similar setup for $(\mathfrak{A}'', \varphi'', \gamma'')$. Since relative entropy and entropy of states are additive on tensor products we have additivity of $\varepsilon_\mu, S(\mu), s_\mu$. We may assume $B' = \bigvee_{i=1}^k B_i'$, $B'' = \bigvee_{i=1}^k B_i''$, and so $B' \otimes B'' = \bigvee_{i=1}^k B_i' \otimes B_i''$. Thus

$$S\left(\mu' \otimes \mu'' \,\middle|\, \bigvee_{i=1}^k B_i' \otimes B_i''\right) = S(\mu') + S(\mu'').$$

It follows easily that the entropy of the tensor product of the abelian model for $(\mathfrak{A}' \otimes \mathfrak{A}'', \varphi' \otimes \varphi'', (\alpha')^j \circ \gamma' \otimes (\alpha'')^j \circ \gamma'', j = 0, \ldots, k-1)$ is the sum of the entropies of the two abelian models for $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively. Taking sup over all tensor product abelian models as above we get

$$H_{\varphi'}(\gamma', \alpha'\gamma', \ldots, (\alpha')^{k-1}\gamma') + H_{\varphi''}(\gamma'', \alpha''\gamma'', \ldots, (\alpha'')^{k-1}\gamma''). \tag{4}$$

However, to get

$$H_{\varphi'\otimes\varphi''}(\gamma' \otimes \gamma', (\alpha' \otimes \alpha'')(\gamma' \otimes \gamma''), \ldots, (\alpha' \otimes \alpha'')^{k-1}(\gamma' \otimes \gamma'')) \tag{5}$$

we take the sup over a larger family of abelian models. Thus the expression in (5) is greater than that in (4). Similarly, taking the sup of all possible γ's the conclusion of the lemma follows. □

4. Bogoliubov Automorphisms and Quasifree States

Let H be a complex Hilbert space. The CAR-algebra $\mathscr{A}(H)$ over H is a C^*-algebra with the property that there is a linear map $f \to a(f)$ of H into $\mathscr{A}(H)$ whose range generates $\mathscr{A}(H)$ as a C^*-algebra and satisfying the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f, g)1, \quad f, g \in H,$$
$$a(f)a(g) + a(g)a(f) = 0,$$

where $(\cdot, \cdot)$ is the inner product on H, and 1 is the unit of $\mathscr{A}(H)$. If $0 \leqq A \leqq 1$ is an operator on H then the *quasifree* state ω_A on $\mathfrak{A}(H)$ is defined by its values on products of the form $a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)$ given by

$$\omega_A(a(f_n)^* \cdots a(f_1)^* a(g_1) \cdots a(g_m)) = \delta_{nm} \det((Ag_i, f_j)). \qquad (1)$$

If U is a unitary operator on H then U defines an automorphism α_U of $\mathscr{A}(H)$, called a *Bogoliubov automorphism* (or quasifree or one-particle automorphism) determined by

$$\alpha_U(a(f)) = a(Uf).$$

If U and A commute it is an easy consequence of the above definition of ω_A that α_U is ω_A-invariant. More generally if T is a contraction on H commuting with A then there is a unique unital completely positive map $\alpha_T : \mathscr{A}(H) \to \mathscr{A}(H)$ such that $\alpha_T(a(f)) = a(Tf)$, and $\omega_A \alpha_T = \omega_A$, see [5]. If P is a projection commuting with A then α_P is an expectation of $\mathscr{A}(H)$ onto $\mathscr{A}(PH)$.

We shall mostly be concerned with the case when A has pure point spectrum, say $(f_n)_{n \in \mathbb{N}}$ is an orthonormal basis for H such that $Af_n = \lambda_n f_n, n \in \mathbb{N}$. Define recursively $V_0 = 1$, $V_n = \prod_{i=1}^{n} (1 - 2a(f_i)^* a(f_i))$, $e_{11}^{(n)} = a(f_n)a(f_n)^*$, $e_{12}^{(n)} = a(f_n)V_{n-1}$, $e_{21}^{(n)} = V_{n-1}a(f_n)^*$, $e_{22}^{(n)} = a(f_n)^* a(f_n)$. Then the $(e_{ij}^{(n)} : i = 1, 2)$ form a complete set of 2×2 matrix units generating a factor $M_2(\mathbb{C})_n$ of type I_2, and for distinct n, m $e_{ij}^{(m)}$ and $e_{kl}^{(n)}$ commute. It follows in particular that $\mathscr{A}(H) \cong \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})_n$ and that ω_A is a product state, $\omega_A = \bigotimes_{n=1}^{\infty} \omega_{\lambda_n}$ with respect to this tensor product factorization, where ω_λ is the state on $M_2(\mathbb{C})$ given by

$$\omega_\lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (1 - \lambda)a + \lambda d.$$

The Bogoliubov automorphism α_{-1} is ω_A-invariant for all quasifree states ω_A. Its fixed point algebra is denoted by $\mathscr{A}(H)_e$ and is the even CAR-algebra. It is generated by even products of $a(f)$'s and $a(g)^*$'s. $\mathscr{A}(H)$ is the direct sum of $\mathscr{A}(H)_e$ and the spectral subspace of -1 for α_{-1}. If $H = H_1 \oplus H_2$ then

$$\mathscr{A}(H_2)_e \subset \mathscr{A}(H_1)' \cap \mathscr{A}(H),$$

because operators $a(f)^{\#}, f \in H_1$, and $a(g)^{\#}, g \in H_2$, anticommute, where $a^{\#}$ denotes a^* or a. Thus even products of the $a(g)^{\#}$'s will commute with $a(f)^{\#}$. Thus for each finite even dimensional subspace K_n of H_1 the C^*-algebra generated by $\mathscr{A}(K_n)$ and $\mathscr{A}(H_2)_e$ is isomorphic to $\mathscr{A}(K_n) \otimes \mathscr{A}(H_2)_e$. Since $\mathscr{A}(H_1)$ is the norm closure of the union of such $\mathscr{A}(K_n)$'s it follows from the uniqueness of the tensor product norm that the C^*-algebra generated by $\mathscr{A}(H_1)$ and $\mathscr{A}(H_2)_e$ is isomorphic to $\mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e$.

Let $A = A_1 \oplus A_2, A_i \in B(H_i), 0 \leqq A_i \leqq 1$. Then

$$\omega_A | \mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e = \omega_{A_1} | \mathscr{A}(H_1) \otimes \omega_{A_2} | \mathscr{A}(H_2)_e. \tag{2}$$

Indeed, in (1) let $m = n$ and use the anticommutation relations to rearrange the factors in the defining equation (1) so that $f_1, \ldots, f_k \in H_1$, $f_{k+1}, \ldots, f_n \in H_2$, $g_1, \ldots, g_l \in H_1$, $g_{l+1}, \ldots, g_n \in H_2$. Since $(Ag_i, f_j) = 0$ if one of g_i and f_j is in H_1 and the other in H_2, the matrix $((Ag_i, f_j))$ is a block matrix

$$\begin{pmatrix} ((A_1 g_i, f_j)) & 0 \\ 0 & ((A_2 g_i, f_j)) \end{pmatrix},$$

where $((A_1 g_i, f_j))$ is a $k \times l$ matrix and $((A_2 g_i, f_j))$ and $(n-k) \times (n-l)$ matrix. The determinant of this matrix is zero unless $k = l$, a fact easily verified by induction. If $k = l$ the determinant equals the product

$$\omega_{A_1}(a(f_k)^* \cdots a(f_1)^* a(g_1) \cdots a(g_k)) \omega_{A_2}(a(f_n)^* \cdots a(f_{k+1})^* a(g_{k+1}) \cdots a(g_n)),$$

from which (2) follows.

Suppose next U_i is a unitary operator on $H_i, i = 1, 2$. If $f \in H_1, U_1 \oplus U_2 f = U_1 f$ with obvious identification of f and $f \oplus 0$. Thus $\alpha_{U_1 \oplus U_2}(a(f)) = \alpha_{U_1}(a(f))$ and similarly for $g \in H_2$. If $f_1, \ldots, f_n \in H_1, g_1, \ldots, g_m \in H_2$ we have, with # as before,

$$\begin{aligned} \alpha_{U_1 \oplus U_2}\left(\prod_{i=1}^{n} a(f_i)^{\#} \prod_{j=1}^{m} a(g_j)^{\#}\right) &= \prod_i a(U_1 f_i)^{\#} \prod_j a(U_2 g_j)^{\#} \\ &= \alpha_{U_1}\left(\prod_i a(f_i)^{\#}\right) \alpha_{U_2}\left(\prod_j a(g_j)^{\#}\right). \end{aligned}$$

It follows that

$$\alpha_{U_1 \oplus U_2} | \mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e = \alpha_{U_1} | \mathscr{A}(H_1) \otimes \alpha_{U_2} | \mathscr{A}(H_2)_e.$$

Lemma 4.1. *Let $H = H_1 \oplus H_2, 0 \leqq A_i \leqq 1$ be an operator on H_i, and U_i be a unitary operator on $H_i, i = 1, 2$. Suppose $A_i U_i = U_i A_i, i = 1, 2$. Then we have*

$$h_{\omega_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2}) \geqq h_{\omega_{A_1}}(\alpha_{U_1}).$$

Proof. Let $E: \mathscr{A}(H_1 \oplus H_2) \to \mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e$ be the expectation $E = \frac{1}{2}(\mathrm{id} + \alpha_{1 \oplus -1})$, 1 denoting the identity on both H_1 and H_2. Since $1 \oplus -1$ commutes with $A_1 \oplus A_2, \alpha_{1 \oplus -1}$ is $\omega_{A_1 \oplus A_2}$-invariant, as is E. Thus by Lemmas 3.2 and 3.4 we have

$$\begin{aligned} h_{\omega_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2}) &\geqq h_{\omega_{A_1 \oplus A_2}}(\alpha_{U_1 \oplus U_2} | \mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e) \\ &= h_{\omega_{A_1} \otimes \omega_{A_2}}(\alpha_{U_1} \otimes \alpha_{U_2} | \mathscr{A}(H_1) \otimes \mathscr{A}(H_2)_e) \\ &\geqq h_{\omega_{A_1}}(\alpha_{U_1} | \mathscr{A}(H_1)) + h_{\omega_{A_2}}(\alpha_{U_2} | \mathscr{A}(H_2)_e) \\ &\geqq h_{\omega_{A_1}}(\alpha_{U_1} | \mathscr{A}(H_1)). \quad \square \end{aligned}$$

Remark 4.2. We remark for later use that it follows from the discussion preceding the lemma that if $H=\bigoplus_{i\in J} H_i$, J finite, then $\mathscr{A}(H)_e \supset \bigotimes_{i\in J}\mathscr{A}(H_i)_e$, and if $U=\bigoplus_{i\in J} U_i$ is unitary and $A=\bigoplus_{i\in J} A_i$ satisfies $0\leqq A\leqq 1$, then

$$\omega_A\Big|\bigotimes_{i\in J}\mathscr{A}(H_i)_e=\bigotimes_{i\in J}(\omega_{A_i}|\mathscr{A}(H_i)_e)$$

and

$$\alpha_U\Big|\bigotimes_{i\in J}\mathscr{A}(H_i)_e=\bigotimes_{i\in J}(\alpha_{U_i}|\mathscr{A}(H_i)_e).$$

Furthermore, if G is the group of unitaries $U=\bigoplus_{i\in J} U_i$ with $U_i=\pm 1$, then $E=2^{-\operatorname{card} J}\sum_{U\in G}\operatorname{Ad} U$ is an ω_A-invariant expectation of $\mathscr{A}(H)$ onto $\bigotimes_{i\in J}\mathscr{A}(H_i)_e$.

Each unitary operator U is a direct sum $U=U_a\oplus U_s$ with U_a acting on a Hilbert space H_a and U_s on H_s; U_a has spectral measure absolutely continuous with respect to Lebesgue measure $d\theta$ on the circle $\mathbf{T}$ while U_s has spectral measure singular with respect to $d\theta$. U_a is called the *absolutely continuous part* of U and U_s the *singular part*. We shall in the sequel use the notation $m(U)$ to denote the multiplicity function of U_a, i.e. $m(U)=m(U_a)$ in our notation.

Lemma 4.3. *Let U and V be unitary operators and $\lambda\in[0,1]$. Then we have, identifying λ and $\lambda 1$,*
(i) *If there is a unitary operator W such that $V=WUW^{-1}$ then $h_{\omega_\lambda}(\alpha_U)=h_{\omega_\lambda}(\alpha_V)$.*
(ii) *If U and V have the same singular parts and $m(U)\geqq m(V)$, then $h_{\omega_\lambda}(\alpha_U)\geqq h_{\omega_\lambda}(\alpha_V)$.*

Proof. (i) is obvious, cf. [3, VII.5].

(ii) The assumption on U and V means that up to unitary equivalence we may assume V is the restriction of U to a reducing subspace, so that (ii) follows from Lemma 4.1. □

Lemma 4.4. *Let (U_n) be a sequence of unitary operators and U a unitary operator, all with Lebesgue spectrum. Suppose $(m(U_n))$ is an increasing sequence with pointwise limit $m(U)$. Then $(h_{\omega_\lambda}(\alpha_{U_n}))$ is an increasing sequence and*

$$h_{\omega_\lambda}(\alpha_U)=\lim_{n\to\infty} h_{\omega_\lambda}(\alpha_{U_n}).$$

Proof. Since the singular part of each unitary n question is zero the assumption on the multiplicity functions implies that we may assume U lives on a Hilbert space H and $U_n=U|H_n$, where $H_n\subset H_{n+1}\subset H$ are reducing subspaces for U for all $n\in\mathbb{N}$, and $H=\overline{\bigcup_{n=1}^{\infty} H_n}$. Thus the lemma follows from Lemmas 3.2 and 3.4 and the fact that the projections onto the H_n define expectations on $\mathscr{A}(H)$ satisfying the conditions in the lemmas. □

We conclude this section with the computation of $h_{\omega_A}(\alpha_U)$ in some simple cases.

Lemma 4.5. *For $i\in\{1,\dots,r\}$ let H_i be an infinite dimensional separable Hilbert space with identity 1_i and let U_i be a unitary operator on H_i such that for each i there are $p_i\in\mathbb{N}$ and a common $q\in\mathbb{N}$ for all i, such that U_i^q is unitarily equivalent to V^{p_i}, where V is a bilateral shift operator of multiplicity 1. Let $U=\bigoplus_{i=1}^{r} U_i$ and let $A=\bigoplus_{i=1}^{r} c_i 1_i$*

with $c_i \in [0,1]$. *Then we have the formula*

$$h_{\omega_A}(\alpha_U) = q^{-1}\sum_{i=1}^{r} p_i S(\omega_{c_i}) = q^{-1}\sum_{i=1}^{r} p_i(\eta(c_i) + \eta(1-c_i)).$$

Furthermore, the same formula holds for the restrictions of ω_A *and* α_U *to* $\mathscr{A}(H)_e$.

Proof. Let $(f_{ik})_{k\in\mathbb{Z}}$ be an orthonormal basis for H_i such that $Vf_{ik} = f_{i(k+1)}$, and let

$$N = \mathscr{A}([f_{11}, \ldots, f_{1p_1}, \ldots, f_{r1}, \ldots, f_{rp_r}]),$$

where $[f_1, \ldots, f_m]$ denotes the subspace spanned by the vectors $f_1, \ldots, f_m$. Since $Af_{ik} = c_i f_{ik}$ we can write N as a tensor product

$$N = \bigotimes_{i=1}^{p} M_2(\mathbb{C})_i,$$

where $p = \sum_{i=1}^{r} p_i$, and from our previous discussion

$$\omega_A | N = \left(\bigotimes_{i=1}^{p_1} \omega_{c_1}\right) \otimes \cdots \otimes \left(\bigotimes_{i=1}^{p_r} \omega_{c_r}\right).$$

The subspaces

$$(U^q)^k([f_{11}, \ldots, f_{rp_r}]) = \bigoplus_{i=1}^{r} V^{kp_i}([f_{i1}, \ldots, f_{ip_i}])$$

for $k\in\mathbb{Z}$ are mutually orthogonal since V is a bilateral shift and the spaces H_i are left invariant. Furthermore the subspaces span $H = \bigoplus_{i=1}^{r} H_i$. Thus the algebras $(\alpha_{U^q})^k(N)$ generate $\mathscr{A}(H)$, and their even parts are pairwise commuting. The diagonals in the algebras $M_2(\mathbb{C}_i)$ appearing in the definition of N lie in the centralizer of the corresponding ω_{c_j}, hence in the centralizer of ω_A. Furthermore, they lie in $\mathscr{A}(H)_e$ as follows from the construction of the $M_2(\mathbb{C})_i$. If N_k is the algebra generated by $N, \alpha_{U^q}(N), \ldots, (\alpha_{U^q})^{k-1}(N)$ it thus follows from [3, Corollary VIII.8] that

$$\begin{aligned}
\frac{1}{k} H_{\omega_A}(N, \alpha_{U^q}(N), \ldots, (\alpha_{U^q})^{k-1}(N)) &= \frac{1}{k} S(\omega_A | N_k) \\
&= \frac{1}{k}\sum_{i=0}^{k-1} S(\omega_A | (\alpha_{U^q})^i(N)) \\
&= S(\omega_A | N) \\
&= \sum_{i=1}^{r} p_i S(\omega_{c_i}).
\end{aligned}$$

Hence we have

$$h_{\omega_A, \alpha_{U^q}}(N) = \sum_{i=1}^{r} p_i S(\omega_{c_i}).$$

To complete the argument let for each $n\in\mathbb{N}$,

$$M_n = \bigvee_{-n}^{n} (\alpha_{U^q})^j(N).$$

Then (M_n) is an increasing sequence of finite dimensional subalgebras of $\mathscr{A}(H)$ with dense union. Then by [3, Theorem VII.4]

$$h_{\omega_A}(\alpha_{U^q}) = \lim_{n\to\infty} h_{\omega_A,\alpha_{U^q}}(M_n). \tag{3}$$

Now the C^*-algebra generated by $M_n, \alpha_{U^q}(M_n), \ldots, (\alpha_{U^q})^{k-1}(M_n)$ equals the one generated by $(\alpha_{U^q})^j(N)$, $-n \leqq j \leqq n+k-1$. Our previous argument with N and the diagonals in the $M_2(\mathbb{C})_i$ shows that [3, Corollary VIII.8] implies

$$H_{\omega_A}(M_n, \alpha_{U^q}(M_n), \ldots, (\alpha_{U^q})^{k-1}(M_n)) = (2n+k)S(\omega_A \mid N),$$

and this holds also for $H_{\omega_A|\mathscr{A}(H)_e}$ and $M_n \cap \mathscr{A}(H)_e$. We conclude that

$$h_{\omega_A,\alpha_{U^q}}(M_n) = S(\omega_A \mid N),$$

and hence from (3) $h_{\omega_A}(\alpha_{U^q}) = S(\omega_A \mid N)$. Thus it follows from [3, VIII.5] that

$$h_{\omega_A}(\alpha_U) = \frac{1}{q} h_{\omega_A}(\alpha_{U^q}) = \frac{1}{q}\sum_{i=1}^{r} p_i S(\omega_{c_i}). \tag{4}$$

Finally, since the diagonals in the $M_2(\mathbb{C})_i$ lie in $\mathscr{A}(H)_e$ we get the inequality

$$h_{\omega_A|\mathscr{A}(H)_e}(\alpha_U \mid \mathscr{A}(H)_e) \geqq \frac{1}{q}\sum_{i=1}^{r} p_i S(\omega_{c_i}).$$

The opposite inequality follows from (4) and Lemma 3.2. □

Lemma 4.6. *Let U be a unitary operator on H with Lebesgue spectrum consisting of disjoint arcs $\exp(2\pi i[a_j, b_j])$ such that $b_j - a_j = \frac{p_j}{q}$, $p_j, q \in \mathbb{N}$, with $j \in J \subset \mathbb{N}$. Let $H_j = L^2(\exp(2\pi i[a_j, b_j]))$ considered as a subspace of $L^2(\mathbf{T}, d\theta)$, and write $U = \bigoplus_{j\in J} U_j$ with $U_j = U \mid H_j$. Suppose U_j has constant finite multiplicity n_j, and let $0 \leqq A_j \leqq 1$ act on H_j and commute with U_j. Writing $U_j = V_j \oplus \cdots \oplus V_j$ (n_j times) we assume $A_j = \bigoplus_{k=1}^{n_j} c_{jk} 1_j$, where 1_j is the identity on the space on which V_j acts. Let B_j denote the diagonal $n_j \times n_j$ matrix*

$$B_j = \begin{pmatrix} c_{j1} & & 0 \\ & \ddots & \\ 0 & & c_{jn_j} \end{pmatrix}.$$

Then $A_j = B_j \otimes 1_j$, and we have the formula

$$h_{\omega_A}(\alpha_U) = \sum_{j\in J} (b_j - a_j)\,\mathrm{Tr}_{n_j}(\eta(B_j) + \eta(1 - B_j)),$$

where Tr_{n_j} is the usual trace on $M_{n_j}(\mathbb{C})$. Furthermore, the same formula holds for the restrictions of ω_A and α_U to $\mathscr{A}(H)_e$.

Proof. We first assume J is finite, say $J = \{1, \ldots, r\}$. We may write

$$U = (\underbrace{V_1 \oplus \cdots \oplus V_1}_{n_1}) \oplus \cdots \oplus (\underbrace{V_r \oplus \cdots \oplus V_r}_{n_r})$$

$$A = (c_{11} 1_1 \oplus \cdots \oplus c_{1n_1} 1_1) \oplus \cdots \oplus (c_{r1} 1_r \oplus \cdots \oplus c_{rn_r} 1_r).$$

Now V_j^q is a bilateral shift of multiplicity p_j. Thus by Lemma 4.5

$$\begin{aligned} h_{\omega_A}(\alpha_U) &= \sum_{j=1}^{r} \frac{p_j}{q} \sum_{l=1}^{n_j} S(\omega_{c_{jl}}) \\ &= \sum_{j=1}^{r} \frac{p_j}{q} \mathrm{Tr}_{n_j}(\eta(B_j) + \eta(1-B_j)) \\ &= \sum_{j=1}^{r} (b_j - a_j) \mathrm{Tr}_{n_j}(\eta(B_j) + \eta(1-B_j)). \end{aligned}$$

If J is infinite we may assume $J = \mathbb{N}$. Let $W_r = \bigoplus_{j=1}^{r} U_j$. Then $W_r A = A W_r$, so if Q_r is the orthogonal projection of H onto $\bigoplus_{j=1}^{r} H_j$ then the expectations E_r of $\mathscr{A}(H)$ onto $\mathscr{A}\left(\bigoplus_{j=1}^{r} H_j\right)$ defined by Q_r satisfying the conditions of Lemma 3.3, cf. [5]. Thus by Lemma 3.3 the proof is complete. □

5. The Case of Singular Spectrum

In this section we study the case when the unitary operator U has a nontrivial singular part U_s. The main result shows that if $U = U_a \oplus U_s$ with U_a as in Lemma 4.6 then $h_{\omega_A}(\alpha_U) = h_{\omega_A}(\alpha_{U_a})$ with A as in that lemma. We first prove an operator theoretic lemma.

Lemma 5.1. *Let U be a unitary operator on H with spectral measure singular with respect to Lebesgue measure. Let P be a finite rank orthogonal projection onto a subspace of H, and let $\varepsilon > 0$ be given. Then there is $k_0 \in \mathbb{N}$ such that for each integer $k \geqq k_0$ there is a finite rank projection Q_k with the properties*
(i) $\|(1-Q_k)U^s P\| < \varepsilon$, *for* $0 \leqq s \leqq k$.
(ii) $\dim Q_k \leqq \varepsilon k$.

Proof. Since the spectral measure of U is a singular and P has finite rank there is a set $\sigma \subset \mathbf{T}$ such that the following hold for given $\delta > 0$:

(a) $\sigma = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$, where $\sigma_j\,(1 \leqq j \leqq N)$ are arcs, $\sigma_j = \{\exp(i\theta) : \alpha_j \leqq \theta \leqq \beta_j\}$ such that $0 \leqq \alpha_1 < \beta_1 \leqq \alpha_2 < \beta_2 \leqq \dots \leqq \alpha_N < \beta_N \leqq 2\pi$.

(b) $N \max(\beta_j - \alpha_j) < \delta$.

(c) If $E(\sigma)$ is the spectral projection of U for the set σ then $\|(1-E(\sigma))P\| < \delta$.

Indeed, since the spectral measure of U is singular it is immediate that (a), (c), and the condition $\sum(\beta_j - \alpha_j) < \delta$ can be satisfied. To get (b) it suffices to substitute the σ_j by arcs of almost the same length.

For a number $M > 0$ consider the MN arcs $\sigma_{1,1}, \dots, \sigma_{NM,M}$ obtained by subdividing each σ_j into M arcs of equal length. Let $E(\sigma_{j,M})$ be the corresponding spectral projections of U, which are pairwise orthogonal. Let then $\mathscr{X}_M$ be the subspace

$$\mathscr{X}_M = \bigoplus_{j=1}^{MN} E(\sigma_{j,M}) P(H).$$

We have $\dim \mathscr{X}_M \leqq MN \dim P$. If $f \in P(H)$ and $\|f\| = 1$, then we have with $\theta_j \in \sigma_{j,M}$, that the distance

$$\begin{aligned} d(U^s f, \mathscr{X}_M)^2 &\leqq \left\| \sum_{j=1}^{MN} e^{i\theta_j s} E(\sigma_{j,M}) f - U^s f \right\|^2 \\ &\leqq 2 \sum_{j=1}^{MN} \| e^{i\theta_j s} E(\sigma_{j,M}) f - U^s E(\sigma_{j,M}) f \|^2 + 2 \| U^s E(\sigma) f - U^s f \|^2 \\ &\leqq 2 \sum_{j=1}^{MN} \| (e^{i\theta_j s} 1 - U^s) E(\sigma_{j,M}) \|^2 \| E(\sigma_{j,M}) f \|^2 + 2\delta^2 \\ &\leqq 2 \max_{1 \leqq j \leqq MN} \| (e^{i\theta_j s} 1 - U^s) E(\sigma_{j,M}) \|^2 + 2\delta^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \| (e^{i\theta_j s} 1 - U^s) E(\sigma_{j,M}) \| &\leqq \sup_{z \in \sigma_{j,M}} | e^{i\theta_j s} - z^s | \\ &\leqq (\text{length}\, \sigma_{j,M}) \cdot s \\ &\leqq \frac{s}{M} \max_{1 \leqq j \leqq MN} (\beta_j - \alpha_j) \\ &< \frac{s\delta}{MN}. \end{aligned}$$

This gives

$$\begin{aligned} d(U^s f, \mathscr{X}_M) &\leqq \left(2 \left(\frac{s\delta}{MN} \right)^2 + 2\delta^2 \right)^{1/2} \\ &\leqq \sqrt{2} \frac{s\delta}{MN} + \sqrt{2}\delta. \end{aligned}$$

Since this estimate is uniform in $f \in PH$, $\|f\| = 1$, we have actually proved that for Q the orthogonal projection onto $\mathscr{X}_M$ we have

$$\| (1 - Q) U^s P \| < \sqrt{2} \left(\frac{s}{MN} + 1 \right) \delta,$$

and

$$\dim Q \leqq MN \dim P.$$

Given k let us take Q_k to be the orthogonal projection onto $\mathscr{X}_M$, and with ε as in the statement of the lemma, let

$$M = \left[\frac{\varepsilon k}{N \dim P} \right].$$

Then $\dim Q_k \leqq \varepsilon k$. On the other hand

$$\max_{1 \leqq s \leqq k} \sqrt{2} \left(\frac{s}{\left[\frac{\varepsilon k}{N \dim P} \right] N} + 1 \right) \delta = \sqrt{2} \left(\frac{k}{\left[\frac{\varepsilon k}{N \dim P} \right] N} + 1 \right) \delta$$

$$\leqq \sqrt{2}\left(\frac{k}{\frac{\varepsilon k}{2N \dim P}N}+1\right)\delta$$

$$=\sqrt{2}\left(\frac{2\dim P}{\varepsilon}+1\right)\delta$$

$$<\varepsilon,$$

if $\frac{\varepsilon k}{N \dim P} \geqq 1$, so that $\left[\frac{\varepsilon k}{N \dim P}\right] \geqq 1$, and

$$\delta < \frac{\varepsilon}{\sqrt{2}\left(\frac{2\dim P}{\varepsilon}+1\right)}.$$

Thus choosing δ sufficiently small and then

$$k_0 = \left[1 + \frac{N \dim P}{\varepsilon}\right],$$

it follows that Q_k chosen as above satisfies the conditions of the lemma. □

Before we show that the singular part does not affect the entropy, we digress for a moment to show that if U has singular spectrum the entropy of α_U taken with respect to any state φ such that $\varphi \circ \alpha_U = \varphi$, is zero. In addition to being a result of more general type its proof will make the proof of the lemma following it more transparent.

Theorem 5.2. *Let U be a unitary operator on H with spectral measure singular with respect to the Lebesgue measure. Let φ be a state on $\mathscr{A}(H)$ such that α_U is φ-invariant. Then $h_\varphi(\alpha_U) = 0$.*

Proof. Let P be an orthogonal projection in $B(H)$ of finite rank. Let $j: P(H) \to H$ be the inclusion map. Then there are, see [5] unital completely positive maps

$$\alpha_j: \mathscr{A}(P(H)) \to \mathscr{A}(H) \quad \text{with} \quad \alpha_j(a(f)) = a(jf),$$
$$\alpha_p: \mathscr{A}(H) \to \mathscr{A}(P(H)) \quad \text{with} \quad \alpha_p(a(f)) = a(pf).$$

If $P_n \nearrow 1$ is a sequence of such projections then with $j_n: P_n(H) \to H$ the inclusion,

$$\alpha_{j_n} \circ \alpha_{p_n} \to \mathrm{id}_{\mathscr{A}(H)}$$

in the pointwise-norm topology. By [3, Theorem V.2]

$$h_\varphi(\alpha_U) = \lim_n h_{\varphi,\alpha_U}(\alpha_{j_n}),$$

where

$$h_{\varphi,\alpha_U}(\alpha_j) = \lim_{k\to\infty} \frac{1}{k} H_\varphi(\alpha_j, \alpha_U\alpha_j, \ldots, \alpha_U^{k-1}\alpha_j).$$

Whence it suffices to show

$$h_{\varphi,\alpha_U}(\alpha_j) = 0.$$

Let P be as above. Since $\dim P < \infty$, given $\delta > 0$ there is $\eta > 0$ such that if $W_1, W_2 : P(H) \to H$ are isometries with $\| W_1 - W_2 \| < \eta$, then, see [5]

$$\| \alpha_{W_1} - \alpha_{W_2} \| < \delta,$$

where $\alpha_W(a(f)) = a(Wf)$. Let Q_k be as in Lemma 5.1. Denote by $\operatorname{pol}(Q_k U^s | P(H))$ the partial isometry W_2 appearing in the polar decomposition

$$Q_k U^s | P(H) = W_2 |(Q_k U^s | P(H))|.$$

Let $W_1 = U^s | P(H)$. Since

$$\| U^s P - Q_k U^s P \| < \varepsilon \quad \text{if} \quad 0 \leqq s \leqq k,$$

we can easily infer

$$\| U^s | P(H) - \operatorname{pol}(Q_k U^s | P(H)) \| \leqq 3\varepsilon,$$

if $\varepsilon \leqq \frac{1}{2}$, which we shall assume. Thus, choosing $\varepsilon < \frac{\eta}{3}$ we have for $k \geqq k_0$,

$$\| \alpha_{U^s | P(H)} - \alpha_{\operatorname{pol}(Q_k U^s | P(H))} \| < \delta \quad \text{for} \quad 0 \leqq s \leqq k.$$

By [3, Proposition IV.3] there is for given $\chi > 0$ and $\varepsilon > 0$, $k_0 \in \mathbb{N}$ such that if $k \geqq k_0$ and Q_k are as in Lemma 5.1 then

$$H_\varphi(\alpha_j, \alpha_U \alpha_j, \ldots, \alpha_U^{k-1} \alpha_j) \leqq k\chi + H_\varphi(\alpha_{\operatorname{pol}(Q_k j)}, \ldots, \alpha_{\operatorname{pol}(Q_k U^{k-1} j)}). \tag{1}$$

If we let $v : Q_k(H) \to H$ be the inclusion map then

$$\alpha_{\operatorname{pol}(Q_k j)} = \alpha_v \circ \alpha_{\operatorname{pol}(Q_k j)},$$

whence by [3, Proposition III.6(a) and 6(c)]

$$H_\varphi(\alpha_{\operatorname{pol}(Q_k j)}, \ldots, \alpha_{\operatorname{pol}(Q_k U^{k-1} j)}) \leqq H_\varphi(\alpha_v, \ldots, \alpha_v) = H_\varphi(\alpha_v). \tag{2}$$

On the other hand by [3, III.4]

$$H_\varphi(\alpha_v) = S(\varphi \circ \alpha_v),$$

where $\varphi \circ \alpha_v$ is a state on $\mathscr{A}(Q_k(H))$, a C^*-algebra of dimension less than $2^{k\varepsilon}$. Thus

$$H_\varphi(\alpha_v) \leqq \log 2^{k\varepsilon} = k\varepsilon \log 2.$$

Hence by (1) and (2)

$$\frac{1}{k} H_\varphi(\alpha_j, \alpha_U \alpha_j, \ldots, \alpha_U^{k-1} \alpha_j) \leqq \chi + \varepsilon \log 2.$$

Since χ and ε are arbitrary, $h_{\varphi, \alpha_U}(\alpha_j) = 0$. □

Lemma 5.3. *Let U be a unitary operator on H with absolutely continuous part U_a acting on H_a and singular part U_s acting on H_s. Let $A = A_a \oplus A_s$ commute with $U = U_a \oplus U_s$, $0 \leqq A \leqq 1$. Assume A_a and U_a are as in Lemma 4.6. Then $h_{\omega_A}(\alpha_U) = h_{\omega_{A_a}}(\alpha_{U_a})$ is given by the formula in Lemma 4.6. Furthermore the same hold for the restrictions to $\mathscr{A}(H)_e$.*

Proof. As in the proof of Lemma 4.6 we may restrict attention to the case when the spectrum of U_a consists of a finite number of disjoint arcs. Furthermore if the multiplicity of U_a on one of the arcs is infinite then both sides of the formula are infinite, hence we may assume each multiplicity is finite.

If we can prove the lemma for U_a and A_a as in Lemma 4.5 the general case follows from that case just as Lemma 4.6 followed from Lemma 4.5. So we assume $U_a = \bigoplus_{i=1}^{r} U_i$ with $U_i^q = V^{p_i}$, where V is bilateral shift of multiplicity 1, and $A_a = \bigoplus_{i=1}^{r} c_i 1_i$, $c_i \in [0,1]$. Let X be as in the proof of Lemma 4.5. Thus X has an orthonormal basis

$$\{f_{11}, \dots, f_{1p_1}, \dots, f_{r1}, \dots, f_{rp_r}\},$$

where $f_{ik} \in H_i$, $Vf_{ik} = f_{i(k+1)}$. To simplify notation let $W = U^q$, so that

$$W = W_a \oplus W_s, \quad W_a = U_a^q = \bigoplus_{i=1}^{r} V^{p_i}, \quad W_s = U_s^q.$$

For $n \in \mathbb{N}$ let

$$X_n = \bigvee_{j=-n}^{n} W_a^j X.$$

Then $\bigcup_{n=0}^{\infty} X_n$ is dense in H_a. Choose an increasing sequence (Y_n) of finite dimensional subspaces of H_s with union dense in H_s. Then $\bigcup_n X_n \oplus Y_n$ is dense in H, so by [3, Theorem VII.4]

$$h_{\omega_A}(\alpha_W) = \lim_n h_{\omega_A, \alpha_W}(\mathscr{A}(X_n \oplus Y_n)).$$

We use notation similar to that used in the proof of Theorem 5.2. Let

$$j_{X_n}: X_n \to H_a, \quad j_{Y_n}: Y_n \to H_a, \quad j_n = j_{X_n} \oplus j_{Y_n}: X_n \oplus Y_n \to H$$

be the inclusion maps, and let

$$\alpha_{j_{X_n}}: \mathscr{A}(X_n) \to \mathscr{A}(H_a) \subset \mathscr{A}(H)$$

etc. be the inclusion maps of the corresponding algebras. Fix $n \in \mathbb{N}$ and let P be the orthogonal projection onto Y_n. Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, and Q_k for $k \geqq k_0$ be as in Lemma 5.1. Then

$$\| W_a^m j_{X_n} \oplus W_s^m j_{Y_n} - W_a^m j_{X_n} \oplus \operatorname{pol}(Q_k W_s^m j_{Y_n}) \|, \quad 1 \leqq m \leqq k,$$

is small, so we can by [3, Proposition IV.3] assume

$$H_{\omega_A}(\alpha_{j_n}, \alpha_W \alpha_{j_n}, \dots, \alpha_W^{k-1} \alpha_{j_n}) \leqq k\varepsilon + H_{\omega_A}(\alpha_{j_{X_n} \oplus \operatorname{pol}(Q_k j_{Y_n})}, \dots, \alpha_{W_a^{k-1} j_{X_n} \oplus \operatorname{pol}(Q_k W_s^{k-1} j_{Y_n})}).$$

Let $v: Q_k(H_s) \to H_s$ be the inclusion, and let $i_n: X_n \to X_n$ be the identity map. Then we have

$$\alpha_{W_a^m j_{X_n} \oplus \operatorname{pol}(Q_k W_s^m j_{Y_n})} = \alpha_{W_a^m j_{X_n} \oplus v} \circ \alpha_{i_n \oplus \operatorname{pol}(Q_k W_s^m j_{Y_n})}.$$

It follows from [3, Proposition III.6(a)] that

$$\begin{aligned} &H_{\omega_A}(\alpha_{j_{X_n} \oplus \operatorname{pol}(Q_k j_{Y_n})}, \dots, \alpha_{W_a^{k-1} j_{X_n} \oplus \operatorname{pol}(Q_k W_s^{k-1} j_{Y_n})}) \\ &\leqq H_{\omega_A}(\alpha_{j_{X_n} \oplus v}, \dots, \alpha_{W_a^{k-1} j_{X_n} \oplus v}) \\ &= H_{\omega_A}(\alpha_{j_{X_n} \oplus v}, \alpha_{W_a \oplus 1_s} \circ \alpha_{j_{X_n} \oplus v}, \dots, \alpha_{W_a \oplus 1_s}^{k-1} \circ \alpha_{j_{X_n} \oplus v}), \end{aligned}$$

where 1_s is the identity on H_s. We may as in [3] identity $\alpha_{J_{X_n}\oplus v}$ with $M_n = \mathscr{A}(X_n \oplus Q_k(H_s))$. Then the last expression in the above inequality becomes

$$H_{\omega_A}(M_n, \alpha_{W_a\oplus 1_s}(M_n), \ldots, \alpha_{W_a\oplus 1_s}^{k-1}(M_n)).$$

Let $Z = \bigoplus_{j=1}^{k-1} W_a^j X_n$. We may assume Z has even dimension, so $\mathscr{A}(Z)$ is a factor. Then Z has an orthonormal basis of eigenvectors for A, so ω_A factors between $\mathscr{A}(Z)$ and its relative commutant $\mathscr{A}(Z)^c$ in $\mathscr{A}(H)$. Let $M = \mathscr{A}(Z \oplus Q_k(H_s))$. Then $\mathscr{A}(Z)^c \cap M \cong \mathscr{A}(Q_k(H_s))$. Since also $\mathscr{A}(Z)$ is the tensor product with ω_A a product state of $2n+k$ copies of $\mathscr{A}(X)$ we have, since $\dim \mathscr{A}(Q_k(H_s)) \leqq 2^{k\varepsilon}$,

$$\begin{aligned} S(\omega_A | M) &= (2n+k) S(\omega_A | \mathscr{A}(X)) + S(\omega_A | \mathscr{A}(Z)^c \cap M) \\ &\leqq (2n+k) S(\omega_A | \mathscr{A}(X)) + k\varepsilon \log 2. \end{aligned}$$

Since M contains the algebra generated by $M_n, \alpha_{W_a\oplus 1_s}(M_n), \ldots, \alpha_{W_a\oplus 1_s}^{k-1}(M_n)$, it follows from [3, Lemma VIII.1] that

$$H_{\omega_A}(M_n, \alpha_{W_a\oplus 1_s}(M_n), \ldots, \alpha_{W_a\oplus 1_s}^{k-1}(M_n)) \leqq (2n+k) S(\omega_A | \mathscr{A}(X)) + k\varepsilon \log 2.$$

We thus have, going back in the proof

$$\frac{1}{k} H_{\omega_A}(\alpha_{j_n}, \alpha_W \alpha_{j_n}, \ldots, \alpha_W^{k-1} \alpha_{j_n}) \leqq \varepsilon + \frac{2n+k}{k} S(\omega_A | \mathscr{A}(X)) + \varepsilon \log 2.$$

We therefore conclude that

$$h_{\omega_A, \alpha_W}(\alpha_{j_n}) \leqq S(\omega_A | \mathscr{A}(X)),$$

whence

$$h_{\omega_A}(\alpha_W) \leqq S(\omega_A | \mathscr{A}(X)),$$

and therefore

$$h_{\omega_A}(\alpha_U) = \frac{1}{q} h_{\omega_A}(\alpha_W) \leqq \frac{1}{q} S(\omega_A | \mathscr{A}(X)).$$

Now from the proof of Lemma 4.5 we have

$$h_{\omega_{A_a}}(\alpha_{U_a}) = \frac{1}{q} S(\omega_{A_a} | \mathscr{A}(X)).$$

By Lemma 4.1 $h_{\omega_{A_a}}(\alpha_{U_a}) \leqq h_{\omega_A}(\alpha_U)$, hence they are equal, completing the proof of the lemma for $h_{\omega_A}(\alpha_U)$.

To see that the entropy is the same for the restriction to the even algebra $\mathscr{A}(H)_e$ we know by Lemmas 3.1 and 4.1 and the first part that

$$\begin{aligned} h_{\omega_{A_a}|\mathscr{A}(H)_e}(\alpha_{U_a} | \mathscr{A}(H)_e) &\leqq h_{\omega_A|\mathscr{A}(H)_e}(\alpha_U | \mathscr{A}(H)_e) \\ &\leqq h_{\omega_A}(\alpha_U) \\ &= h_{\omega_{A_a}}(\alpha_{U_a}). \end{aligned}$$

But by Lemma 4.6 $h_{\omega_{A_a}|\mathscr{A}(H)_e}(\alpha_{U_a} | \mathscr{A}(H)_e) = h_{\omega_{A_a}}(\alpha_{U_a})$, completing the proof for the restrictions to the even algebra. □

6. The Entropy Formula

We prove two formulas. When the quasifree state is of the form ω_λ we first express the entropy by the multiplicity function of U. Then we prove the general formula when A has pure point spectrum. Recall that we use the notation $m(U)$ to denote the multiplicity function of the absolutely continuous part U_a of U. To express the formulas we use direct integral theory as described in the introduction based on U_a.

Theorem 6.1. *With U a unitary operator on the Hilbert space H and $0 \leqq \lambda \leqq 1$, we have*

$$h_{\omega_\lambda}(\alpha_U) = \frac{1}{2\pi}(\eta(\lambda) + \eta(1-\lambda)) \int_0^{2\pi} m(U(\theta))d\theta .$$

Proof. If $\lambda = 0$ or 1 then ω_λ is a pure state, see e.g. [8], so $h_{\omega_\lambda}(\alpha_U) = 0$ by Lemma 3.1. Since $\eta(\lambda) + \eta(1-\lambda) = 0$ when $\lambda \in \{0, 1\}$ the formula holds in this case. Assume $0 < \lambda < 1$. Since by [3, VII.5] $h_{\omega_\lambda}(\alpha_U^n) = |n| h_{\omega_\lambda}(\alpha_U)$, it follows from Lemmas 4.3, 4.4 and 4.5 that all the conditions of Theorem 2.1 are satisfied. Thus the formula holds when U has Lebesgue spectrum.

Let now $U = U_a \oplus U_s$ be the decomposition of U into absolutely continuous and singular parts acting on H_a and H_s respectively. Let $\varepsilon > 0$ be given. By measure theory there is a unitary V with Lebesgue spectrum on H_a such that its multiplicity function satisfies

$$m(V) = \sum_{j=1}^{r} d_j \chi_{X_j} \geqq m(U), \tag{1}$$

where X_j is an arc of the form $\exp(2\pi i[a_i, b_i])$ with $b_i - a_i$ rational, and

$$\int_0^{2\pi} m(U(\theta))d\theta + \varepsilon > \int_0^{2\pi} m(V(\theta))d\theta. \tag{2}$$

We then have, by the first paragraph of the proof and Lemmas 4.1, 4.3, 5.3 in that order

$$\begin{aligned}\frac{1}{2\pi}(\eta(\lambda) + \eta(1-\lambda)) \int_0^{2\pi} m(U(\theta))d\theta &= h_{\omega_\lambda}(\alpha_{U_a}) \\ &\leqq h_{\omega_\lambda}(\alpha_{U_a \oplus U_s}) \\ &\leqq h_{\omega_\lambda}(\alpha_{V \oplus U_s}) \\ &= h_{\omega_\lambda}(\alpha_V) \\ &= \frac{1}{2\pi}(\eta(\lambda) + \eta(1-\lambda)) \int_0^{2\pi} m(V(\theta))d\theta \\ &< \frac{1}{2\pi}(\eta(\lambda) + \eta(1-\lambda)) \left[\int_0^{2\pi} m(U(\theta))d\theta + \varepsilon\right].\end{aligned}$$

Since ε is arbitrary the formula follows. □

We use direct integral as described in the introduction. If A commutes with $U = U_a \oplus U_s$ then $A = A_a \oplus A_s$. We have

$$A_a = \int_{\mathbf{T}}^{\oplus} A(\theta)d\theta$$

with $A(\theta)\in B(H_\theta)$, where $H_\theta=0$ if $m(U(\theta))=0$. Suppose A has pure point spectrum, so that

$$A=\sum_{j\in J}\lambda_j e_j,$$

with J finite or countably infinite, and (e_j) as an orthogonal family of projections with sum 1, and $0\leqq\lambda_j\leqq 1$. Denote by

$$U_j=U\,|\,e_j(H).$$

We denote by Tr the usual trace on $B(H_\theta)$. Writing $e_j=\int^{\oplus}e_j(\theta)d\theta$, and $c(\lambda)=\eta(\lambda)+\eta(1-\lambda)$ we have

$$\operatorname{Tr}(c(A(\theta)))=\sum c(\lambda_j)\operatorname{Tr}(e_j(\theta))=\sum c(\lambda_j)m(U_j(\theta)).$$

Thus the following lemma is immediate from Theorem 6.1.

Lemma 6.2. *With the above notation and assumptions we have*

$$\sum_{j\in J}h_{\omega_{\lambda_j}}(\alpha_{U_j})=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Tr}(\eta(A(\theta))+\eta(1-A(\theta)))d\theta.$$

Theorem 6.3. *Let $0\leqq A\leqq 1$ be an operator with pure point spectrum acting on H. Let U be a unitary operator on H commuting with A. Then we have*

$$h_{\omega_A}(\alpha_U)=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Tr}(\eta(A(\theta))+\eta(1-A(\theta)))d\theta.$$

Proof. If $A=\sum_{i=1}^{\infty}\lambda_i e_i$ the projections $p_n=\sum_{i=1}^{n}e_i$ define expectations on $\mathscr{A}(H)$ satisfying the conditions of Lemma 3.3. Hence to show the formula we may assume A has finite spectrum.

By Lemma 6.2 it suffices to show

$$h_{\omega_A}(\alpha_U)=\sum_{j\in J}h_{\omega_{\lambda_j}}(\alpha_{U_j}),$$

where $A=\sum_{j\in J}\lambda_j e_j$, J finite. Let $\mathscr{A}(H)_e$ be the even algebra. An inspection of the proof of Theorem 6.1 shows that the results used in the proof all hold for $\omega_\lambda|\mathscr{A}(H)_e$ and $\alpha_U|\mathscr{A}(H)_e$. We thus have

$$h_{\omega_\lambda}(\alpha_U|\mathscr{A}(H)_e)=h_{\omega_\lambda}(\alpha_U).$$

We therefore have, using Lemma 3.2 and Remark 4.2 together with Lemma 3.4,

$$\begin{aligned}h_{\omega_A}(\alpha_U)&\geqq h_{\omega_A}(\alpha_U|\mathscr{A}(H)_e)\\&\geqq\sum_{j\in J}h_{\omega_{\lambda_j}}(\alpha_U|\mathscr{A}(e_jH)_e)\\&=\sum_{j\in J}h_{\omega_{\lambda_j}}(\alpha_{U_j}).\end{aligned}$$

It therefore remains to show the converse inequality. For this we may assume the absolutely continuous part U_a of U acts as a multiplication operator on a subset X of the circle $\mathbf{T}$, and that $H_a=L^2(X,d\theta)$ considered as a subspace of

$L^2(\mathbf{T}, d\theta)$. We first consider the case when $m(U)$ is bounded. Thus

$$m(U) = \sum_{n=1}^{N} n\chi_{X_n},$$

where $X_n = m(U)^{-1}(n)$. Since A has finite spectrum we can subdivide each X_n into a disjoint union of Borel sets X_{nr} on which $A(\theta)$ is constant, i.e. has same finite spectrum counted with multiplicity. Thus we have

$$A_a = \bigoplus_{n,r} \int_{X_{nr}}^{\oplus} A(\theta) d\theta = \bigoplus_{n,r} A_{nr} \otimes 1_{nr},$$

where 1_{nr} is the identity on $L^2(X_{nr}, d\theta)$.

Let $\varepsilon > 0$ be given. Choose $Y_{nr} \subset X_{nr}$ a closed set, and choose open sets $O_{nr} \supset Y_{nr}$ which are disjoint and with each O_{nr} a finite union of arcs of the form $\exp(2\pi i[a_i, b_i])$ with $b_j - a_j$ rational and the size of O_{nr} to be determined below.

Let P_{nr} be the orthogonal projection of $L^2(\mathbf{T}, d\theta)$ onto $L^2(Y_{nr}, d\theta)$, and let

$$e = \bigoplus_{nr} P_{nr}.$$

Let V_a be the unitary multiplication operator on $H_1 = L^2\left(\bigcup_{n,r} O_{nr}, d\theta\right)$ with multiplicity function

$$m(V_a) = \sum_{n,r} n\chi_{O_{nr}}$$

and let

$$V = V_a \oplus U_s.$$

Define $\tilde{A}$ by $\tilde{A} = \tilde{A}_a \oplus A_s$, where

$$\tilde{A}_a = \bigoplus_{nr} A_{nr} \otimes 1_{O_{nr}},$$

where $1_{O_{nr}}$ is the identity on $L^2(O_{nr}, d\theta)$. Then we have $\tilde{A}e = Ae$ and $Ve = Ue$, and furthermore by Theorem 6.1 and Lemma 6.2

$$h_{\omega_{\tilde{A}}}(\alpha_V) = \sum h_{\omega_{\lambda_j}}(V_j),$$

where $V_j = V|e_j(H)$, since V and $\tilde{A}$ satisfy the assumptions of Lemma 4.6.

We now make our choice of the size of the O_{nr}'s, namely we choose them so close to the Y_{nr}'s that

$$h_{\omega_{\lambda_j}}(\alpha_{V_j}) < h_{\omega_{\lambda_j}}(\alpha_{(Ue)_j}) + \varepsilon 2^{-j},$$

where $(Ue)_j = Ue|e_j(H)$. This can be done since A has finite spectrum, $Ue = Ve$, and the theorem is true for V. We thus have

$$\begin{aligned} h_{\omega_A}(\alpha_{Ue \oplus U_s}) &= h_{\omega_A}(\alpha_{V_a e \oplus U_s}) \\ &\leqq h_{\omega_{\tilde{A}}}(\alpha_{V_a \oplus U_s}) && \text{(by Lemma 4.1)} \\ &= h_{\omega_{\tilde{A}_a}}(\alpha_{V_a}) && \text{(by Lemma 5.3)} \\ &= \sum_j h_{\omega_{\lambda_j}}(\alpha_{V_j}) && \text{(by Lemma 6.2)} \end{aligned}$$

$$< \sum_j (h_{\omega_{\lambda_j}}(\alpha_{Ue)_j}) + \varepsilon 2^{-j})$$
$$\leqq \sum_j h_{\omega_{\lambda_j}}(\alpha_{U_j}) + \varepsilon \qquad \text{(by Lemma 4.1).}$$

This holds for all $\varepsilon > 0$, so we conclude

$$h_{\omega_A}(\alpha_{Ue \oplus U_s}) \leqq \sum_j h_{\omega_{\lambda_j}}(\alpha_{U_j}).$$

The projections e can be constructed to form an increasing sequence when $\varepsilon \searrow 0$. Since each e commutes with A it defines an ω_A invariant expectation on $\mathscr{A}(H)$, so by Lemma 3.3 we have

$$h_{\omega_A}(\alpha_U) = \lim_{\varepsilon \to 0} h_{\omega_A}(\alpha_{Ue \oplus U_s}),$$

hence we have $h_{\omega_A}(\alpha_U) \leqq \sum_j h_{\omega_{\lambda_j}}(\alpha_{U_j})$, as we wanted to show, proving the theorem when U_a has bounded spectrum.

In the general case let P_N be the projection of H_a onto the spectral subspace where $m(U) \leqq N$. Then $P_N \nearrow 1$ as $N \to \infty$. Since the theorem holds for $\bigcup P_N$ by the first part of the proof it follows again from Lemma 3.3 that it holds for U. □

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References

1. Besson, O.: The entropy of quantum Markov states. Lecture Notes in Mathematics vol **1136**, pp. 81–89. Berlin, Heidelberg, New York: Springer 1985
2. Choda, M.: Entropy for *-endomorphisms and relative entropy for subalgebras (to appear)
3. Connes, A., Narnhofer, H. Thirring, W.: Dynamical entropy of C^*-algebras and von Neumann algebras. Commun. Math. Phys. **112**, 691–719 (1987)
4. Connes, A., Størmer, E.: Entropy of automorphisms of II_1 von Neumann algebras. Acta. Math. **134**, 289–306 (1975)
5. Evans, D.: Completely positive quasifree maps on the CAR-algebra. Commun. Math. Phys. **70**, 53–68 (1979)
6. Kosaki, M.: Interpolation theory and the Wigner–Yanase–Dyson–Lieb concavity. Commun. Math. Phys. **87**, 315–329 (1982)
7. Pimser, M., Popa, S.: Entropy and index for subfactors. Ann. Scient. Éc. Norm. Sup. **19**, 57–106 (1986)
8. Powers, R. T., Størmer, E.: Free states of the canonical anticommutation relations. Commun. Math. Phys. **16**, 1–33 (1970)
9. Pusz, W., Woronowicz, S.: Form convex functions and the WYDL and other inequalities. Lett. Math. Phys. **2**, 505–512 (1978)

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