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Parabolic Problems for the Anderson Model

I. Intermittency and Related Topics

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Abstract. The objective of this paper is a mathematically rigorous investigation of intermittency and related questions intensively studied in different areas of physics, in particular in hydrodynamics. On a qualitative level, intermittent random fields are distinguished by the appearance of sparsely distributed sharp "peaks" which give the main contribution to the formation of the statistical moments. The paper deals with the Cauchy problem $(\partial/\partial t)u(t,x) = Hu(t,x)$, $u(0,x) = u_0(x) \geq 0$, $(t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d$, for the Anderson Hamiltonian $H = \kappa\Delta + \xi(\cdot)$, where $\xi(x)$, $x \in \mathbb{Z}^d$, is a (generally unbounded) spatially homogeneous random potential. This first part is devoted to some basic problems. Using percolation arguments, a complete answer to the question of existence and uniqueness for the Cauchy problem in the class of all nonnegative solutions is given in the case of i.i.d. random variables. Necessary and sufficient conditions for intermittency of the fields $u(t,\cdot)$ as $t \to \infty$ are found in spectral terms of H. Rough asymptotic formulas for the statistical moments and the almost sure behavior of $u(t,x)$ as $t \to \infty$ are also derived.

Contents

Introduction

During the last decade the conception of intermittency became popular in different areas of physics, in particular in hydrodynamics. But until now it obstinately resists a mathematically rigorous foundation. From a qualitative point of view, intermittent random fields are distinguished by the formation of strongly pronounced spatial structures: sharp peaks, foliations, and others. In the papers [13–15] Ya. B. Zel'dovich et al. proposed a "rigorous" and constructive definition of asymptotic (as $t \to \infty$) intermittency which may be successfully applied to a large class of evolution problems. This definition works well in several physically interesting situations for both stationary (i.e. time independent) and nonstationary random media including magnetic and temperature fields in turbulent flows, linearized schemes of chemical kinetics, and others (for details see [15]).

However, the papers [13–15] and related articles in physics literature provide only rough outlines concerning mathematical analysis of the subject. Moreover, such outlines have been made under special assumptions (e.g. Gaussian-like random inputs) and, what is even more significant, they are far from being complete. A thorough investigation of more general models will reveal a number of new facts and enlighten deep connections with other branches of the theory of disordered systems (perlocation, localization).

The paper will be divided into two separate parts. In both we will restrict ourselves to consideration of stationary random media on the lattice (the latter for the sake of technical simplification). More precisely, we will study the Cauchy problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \kappa \Delta u(t,x) + \xi(x) u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{Z}^d, \end{aligned} \tag{0.1}$$

where κ denotes a positive constant, Δ is the finite difference Laplacian acting on functions $f : \mathbb{Z}^d \to \mathbb{R}$ according to the formula

$$\Delta f(x) = \sum_{|y-x|=1} [f(y) - f(x)],$$

and $\Xi = \{\xi(x); x \in \mathbb{Z}^d\}$ denotes a field of independent indentically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathscr{F}, \mu)$. We are mainly interested in the behavior of the solution to (0.1) for nonnegative random initial functions u_0 which are either localized [e.g. $u_0(x) = \delta_0(x)$] or spatially homogeneous random fields (e.g. $u_0 \equiv 1$). We will refer to κ and Ξ as to the *diffusion coefficient* and the *random medium (random potential)*, respectively.

The operator

$$H = \kappa \Delta + \xi(\cdot)$$

on the right of Eq. (0.1) coincides with Anderson's tight binding Hamiltonian with diagonal disorder ([2, 9, 11]). However, in contrast to the quantum mechanical problem described by the Schrödinger equation

$$\begin{aligned} i\frac{\partial \psi}{\partial t} &= H\psi, \\ \psi(0,x) &= \psi_0(x) \in l^2(\mathbb{Z}^d), \end{aligned} \tag{0.2}$$

which is always welll-posed, the *parabolic problem* (0.1) *for the Anderson model* is not always solvable. Moreover, the qualitative behavior of the solutions to problem (0.1) significantly differs from that of the ψ-functions of problem (0.2).

Let us briefly outline the program of our paper. In the present first part we will discuss some general questions, study existence and uniqueness for problem (0.1) and derive rough asymptotic formulas for the almost sure (a.s.) behavior (as $t \to \infty$) of the solution as well as for its statistical moments expressing the effect of intermittency. The second part will be devoted to a more detailed analysis of the solution to (0.1), where we will use more refined techniques (in the first place cluster expansions). We also intend to include an investigation of the spectral aspects of the problem and its connections with the theory of localization.

Frequently Used Notation. We denote by $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z}^d$, $\mathbb{R}^d$, and $\mathbb{R}_+$ the set of natural numbers, the d-dimensional integer lattice, the d-dimensional Euclidean space, and the nonnegative half-axis, respectively. As norm on $\mathbb{Z}^d$ we take

$$|z| = \sum_{i=1}^{d} |z^i|, \qquad z = (z^1, \dots, z^d) \in \mathbb{Z}^d;$$

i.e. $|z|$ is the Euclidean length of the shortest path connecting 0 with z along the edges of the lattice $\mathbb{Z}^d$. Let δ_x denote the Kronecker symbol, i.e. $\delta_x(y) = 1$ if $y = x$ and $\delta_x(y) = 0$ otherwise ($x, y \in \mathbb{Z}^d$). If B is a finite set, then $|B|$ will stand for the number of its elements.

Throughout this paper $\Xi = \{\xi(x); x \in \mathbb{Z}^d\}$ stands for a spatially homogeneous random field on a probability space $(\Omega, \mathscr{F}, \mu)$. Expectation with respect to μ will be indicated by $\langle \cdot \rangle$. The indicator function of an event A will be denoted by $\mathbb{1}(A)$. Throughout Sect. 2 we will assume that the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are mutually independent. By F we will denote the (right-continuous) distribution function of $\xi = \xi(0)$; $\operatorname{ess\,sup} \xi = \sup\{r : F(r) < 1\}$ we will stand for the essential supremum of ξ.

Given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}$, we will write

$$f(t) \ll g(t)$$

if $g(t) - f(t) \to \infty$ as $t \to \infty$. We further denote by x^+ and x^- the positive and negative parts of $x \in \mathbb{R}$, respectively. We set $\log_+ x = \log x$ if $x > e$ and $\log_+ x = 1$ otherwise.

1. General Questions

1.1. Intermittency

We first introduce the notion of intermittency in a form which is most convenient for our purposes. Let $\{\eta(t, x); x \in \mathbb{Z}^d\}$, $t \geq 0$, be a family of nonnegative spatially homogeneous random fields on a joint probability space $(\Omega, \mathscr{F}, \mu)$. Suppose that the functions

$$\Lambda_p(t) = \log\langle \eta(t, 0)^p \rangle, \qquad t \geq 0, \, p \in \mathbb{N},$$

are finite.

Definition. The random fields $\{\eta(t,x); x \in \mathbb{Z}^d\}$ will be called *intermittent* (asymptotically as $t \to \infty$) if they are ergodic and

$$\Lambda_1(t) \ll \frac{\Lambda_2(t)}{2} \ll \frac{\Lambda_3(t)}{3} \ll \dots . \tag{1.1}$$

If the (finite) Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{\Lambda_p(t)}{t}, \qquad p \in \mathbb{N},$$

exist and

$$\lambda_1 < \frac{\lambda_2}{2} < \frac{\lambda_3}{3} < \dots ,$$

then (1.1) will clearly be fulfilled. The definition of asymptotic intermittency proposed by Zel'dovich et al. [15] is based on this stronger requirement.

Intermittency means that there is an anomalous, as compared with Gaussian, ratio between successive statistical moments. Indeed, condition (1.1) tells us that the growth rate (as $t \to \infty$) of the moment $\langle \eta(t,0)^p \rangle$ increases "progressively" with its number p: for large t the second moment is much larger than the square of the first moment, the fourth is much larger than the product of the first and the third and the square of the second, and so on.

The interpretation of this effect consists in the following. As a rule, the physically relevant characteristics of the random field are of global nature and, because of ergodicity, may be expressed in terms of moments (e.g. mean mass concentration or mean energy). It follows from (1.1) that asymptotically as $t \to \infty$ the main contribution to each moment function is carried by higher and higher and more and more widely spaced "overshoots" ("peaks") of the random field. Thus, for large t the overwhelming part of the mass (or energy) of the field $\eta(t,\cdot)$ is concentrated in these "peaks." To make this more transparent, let us choose level functions l_p, $p \in \mathbb{N}$, so that

$$\Lambda_1(t) \ll l_1(t) \ll \frac{\Lambda_2(t)}{2} \ll l_2(t) \ll \frac{\Lambda_3(t)}{3} \ll \dots$$

and consider the events

$$E_p(t) = \{\eta(t,0) > \exp(l_p(t))\}.$$

On the one hand, Chebychev's inequality tells us that

$$\mu(E_p(t)) \le \exp\{\Lambda_p(t) - p l_p(t)\} \to 0 \qquad \text{as } t \to \infty;$$

i.e. the density of the random level set $L_p(t) = \{x \in \mathbb{Z}^d : \eta(t,x) > \exp(l_p(t))\}$ of overshoots with amplitude exceeding $\exp(l_p(t))$ is asymptotically small. On the other hand,

$$\begin{aligned}\langle \eta(t,0)^{p+1} \mathbb{1}(\Omega \backslash E_p(t)) \rangle &\le \exp\{(p+1) l_p(t)\} \\ &= \exp\{(p+1) l_p(t) - \Lambda_{p+1}(t)\} \langle \eta(t,0)^{p+1} \rangle \\ &= \bar{o}(\langle \eta(t,0)^{p+1} \rangle).\end{aligned}$$

Consequently,

$$\langle \eta(t,0)^{p+1} \rangle \sim \langle \eta(t,0)^{p+1} \mathbb{1}(E_p(t)) \rangle \qquad \text{as } t \to \infty.$$

But this means that for large t the overwhelming contribution to the $(p+1)$-st moment is carried by the high overshoots of the field $\eta(t,\cdot)$ on the random set $L_p(t)$. As a characteristic feature of intermittency, the sets $L_1(t) \supset L_2(t) \supset \ldots$ describe a hierarchy of higher and higher and more and more widely spaced peaks which are primarily responsible for the formation of the successive statistical moments.

Let us finally mention that for each $p \in \mathbb{N}$ the condition

$$\frac{\Lambda_p(t)}{p} \ll \frac{\Lambda_{p+1}(t)}{p+1} \tag{1.2}$$

already implies that

$$\frac{\Lambda_q(t)}{q} \ll \frac{\Lambda_{q+1}(t)}{q+1} \quad \text{for all } q \geq p$$

provided that all functions Λ_q, $q \geq p$, are finite. Indeed, for each $t \geq 0$ the real function $p \mapsto \Lambda_p(t)$ is convex and satisfies $\Lambda_0(t) = 0$. [This is the cumulant generating function of $\log \eta(t,0)$.] Therefore

$$\begin{aligned} q\Lambda_{q+1}(t) - (q+1)\Lambda_q(t) &= \sum_{k=0}^{q-1} \{(\Lambda_{q+1}(t) - \Lambda_q(t)) - (\Lambda_{k+1}(t) - \Lambda_k(t))\} \\ &\geq \sum_{k=0}^{p-1} \{(\Lambda_{p+1}(t) - \Lambda_p(t)) - (\Lambda_{k+1}(t) - \Lambda_k(t))\} \\ &= p\Lambda_{p+1}(t) - (p+1)\Lambda_p(t) \end{aligned}$$

for all $q \geq p$. Because of (1.2), the expression on the right tends to infinity as $t \to \infty$, and we arrive at the desired assertion.

1.2. Linearized Models of Chemical Kinetics

We next discuss the relationship between the parabolic problem (0.1) for the Anderson model and (linearized) problems of chemical kinetics. On a particle level linearized chemical processes may be described by Markov branching models on the lattice $\mathbb{Z}^d$. Let us consider the following model. Suppose that at time zero there is a single particle in the system which starts to move according to the laws of a time-continuous random walk and after a random time splits into two or dies. Each of its descendants evolves according to the same law but independent of all other particles. This evolution depends on the "diffusion coefficient" κ of the underlying random walk and the realizations of two spatially homogeneous random fields $\{\xi_+(x); x \in \mathbb{Z}^d\}$ and $\{\xi_-(x); x \in \mathbb{Z}^d\}$ determining the branching rates. During the time interval dt any particle at site $x \in \mathbb{Z}^d$ jumps to one of its $2d$ neighboring sites with equal probability κdt, splits into two with probability $\xi_+(x)dt$, and dies with probability $\xi_-(x)dt$. Let $\eta(t,y)$ denote the number of particles occupying site y at time t. Let further $\mathbb{P}_x$ and $\mathbb{E}_x$ denote, respectively, the probability law and expectation for the "diffusion" and branching mechanism, where the index x indicates that we start at time $t=0$ with a single particle at

$x \in \mathbb{Z}^d$. Note that the measures $\mathbb{P}_x$, $x \in \mathbb{Z}^d$, depend on the realizations of the random medium $(\xi_-(\cdot), \xi_+(\cdot))$. Finally, let μ be the probability measure associated with the random medium $(\xi_-(\cdot), \xi_+(\cdot))$.

The moment generating function

$$v_z(t,x;y) = \mathbb{E}_x z^{\eta(t,y)}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d,\ y \in \mathbb{Z}^d,\ 0 < |z| \le 1,$$

satisfies the Skorokhod equation

$$\begin{aligned} \frac{\partial v_z}{\partial t} &= \kappa \Delta v_z + \xi_+(x) v_z^2 - (\xi_+(x) + \xi_-(x)) v_z + \xi_-(x), \\ v_z(0,x;y) &= z \quad \text{if} \quad x = y \quad \text{and} \\ v_z(0,x;y) &= 1 \quad \text{otherwise}, \end{aligned} \tag{1.3}$$

where the Laplace operator acts on the x-variable (see e.g. [3, 12]). Differentiating (1.3) with respect to z at $z = 1$, one obtains the moment equations for the particle field $\eta(t,y)$. In particular, $m(t,x) = \mathbb{E}_x \eta(t,0)$ satisfies

$$\begin{aligned} \frac{\partial m}{\partial t} &= \kappa \Delta m + \xi(x) m, \\ m(0,x) &= \delta_0(x), \end{aligned} \tag{1.4}$$

where $\xi(x) = \xi_+(x) - \xi_-(x)$. In this way we arrived at the main equation of our paper. Because of symmetry of the Anderson operator, it is not difficult to see that $m(t,x)$ coincides with $\mathbb{E}_0 \eta(t,x)$. That is, $m(t,x)$ is the mean number of particles at site x at time t provided that we start at time 0 with a single particle at the origin. This interpretation allows to pass to spatially homogeneous intitial configurations which lead to constant initial data in (1.4).

Let us mention the following important feature of the above equation. If the random variables $\xi_-(x)$ and $\xi_+(x)$, $x \in \mathbb{Z}^d$, are mutually independent and

$$\mu(\xi_+(0) > \xi_-(0)) > 0 \tag{1.5}$$

(i.e. if with arbitrarily small but positive probability the birth rate exceeds the death rate), then

$$\liminf_{t \to \infty} \frac{\log m(t,x)}{t} > 0 \qquad \mu\text{-a.s. for each } x \in \mathbb{Z}^d$$

(see Sect. 4). In other words, (1.5) implies that the branching process is supercritical. Moreover, one can show that under the same assumptions

$$\mathbb{P}_0\left(\liminf_{t \to \infty} \frac{\log \eta(t,x)}{t} > 0 \right) > 0 \qquad \mu\text{-a.s. for each } x \in \mathbb{Z}^d.$$

1.3. Anderson Localization

Let us finally point out the connection between problem (0.1) and the theory of Anderson localization. On a qualitative level, this connection indicates the presence of intermittency for $m(t, \cdot)$ as well as for the underlying particle fields

$\eta(t,\cdot)$, although it does not give the chance to study this effect quantitatively (in the sense of our definition given in Sect. 1.1).

During the last years substantial progress has been achieved in analyzing the spectral properties of the multidimensional Anderson Hamiltonian

$$H = \kappa\Delta + \xi(\cdot) \quad \text{in } l^2(\mathbb{Z}^d)$$

with i.i.d. random potential $\xi(x)$, $x \in \mathbb{Z}^d$, see Pastur [11] for a most complete survey. Above all, almost sure completeness of the point spectrum and exponential decay of eigenfunctions of H in the band tails, or throughout the whole spectrum $\mathrm{Sp}(H)$ of H provided the coupling constant $\sigma = \kappa^{-1}$ is large, have been established under mild conditions on the distribution of $\xi(0)$ (Hölder continuity of the distribution function and existence of a finite moment), see Fröhlich et al. [5] and Martinelli and Scoppola [9].

At least formally, the solution of (1.4) admits the spectral representation

$$m(t,\cdot) = \int_{-\infty}^{\infty} e^{t\lambda} dE(\lambda)\delta_0, \tag{1.6}$$

where $\{E(\lambda); \lambda \in \mathbb{R}\}$ denotes the spectral family associated with the Hamiltonian H. According to the above we can choose $\lambda_0 < \sup \mathrm{Sp}(H)$ so that the part of the spectrum of H in (λ_0, ∞) is pure point. Let λ_i and ψ_i $(i = 1, 2, \dots)$ denote the corresponding set of (random) eigenvalues and normalized (random) eigenfuntions. Then

$$m(t,\cdot) = \sum_i e^{t\lambda_i}\psi_i(0)\psi_i(\cdot) + \int_{-\infty}^{\lambda_0} e^{t\lambda} dE(\lambda)\delta_0. \tag{1.7}$$

Since

$$\left\| \int_{-\infty}^{\lambda_0} e^{t\lambda} dE(\lambda)\delta_0 \right\|_{l^2(\mathbb{Z}^d)} \le \exp\{t\lambda_0\},$$

this shows that asymptotically as $t \to \infty$ is will be sufficient to take into account only that part of the spectrum in the spectral decomposition (1.6) which belongs to the immediate proximity of its upper bound. Hence we conclude from (1.7) and the exponential decay of the ψ_i's that for large t the random field $m(t,\cdot)$ is "essentially" localized on the "supports" of the eigenfunctions ψ_i in the upper band tail which are "sparsely distributed in space." This is the picture which we have in mind by saying that Anderson localization indicates the presence of intermittency on a qualitative level.

Unfortunately, it is by no means simple to give the above considerations a strong sense, since the theory in its present stage is essentially nonconstructive (cf. [5, 11]). Our aim is to do in some sense the converse: analyzing the asymptotics of $m(t,x)$ as $t \to \infty$ by means of direct probabilistic methods and using (1.7), we want to obtain information about the structure of the spectrum of H in the band tails.

2. Existence and Uniqueness

2.1. Main Result

In this section we study existence and uniqueness for the random Cauchy problem (0.1). We will assume throughout that the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are independent and identically distributed. We will mostly consider random initial data u_0 which almost surely belong to the *class* $\mathcal{U}_0$ of functions $\varphi : \mathbb{Z}^d \to \mathbb{R}_+$ which do not vanish identically and satisfy

$$(\mathcal{U}_0) \qquad \limsup_{|x|\to\infty} \frac{\log_+ \varphi(x)}{|x| \log |x|} < 1 .$$

This assumption on the growth at infinity is trivially satisfied for bounded initial data u_0. Is is also fulfilled for (not a.s. vanishing) nonnegative homogeneous random fields $\{u_0(x); x \in \mathbb{Z}^d\}$ with

$$\left\langle \left(\frac{\log_+ u_0(0)}{\log_+ \log_+ u_0(0)} \right)^d \right\rangle < \infty .$$

If the random variables $u_0(x)$, $x \in \mathbb{Z}^d$, are i.i.d., then this moment condition is also necessary for $(\mathcal{U}_0)$ (cf. Lemma 2.5 below).

Problem (0.1) is closely related to the Feynman-Kac functional

$$u(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d . \tag{2.1}$$

Here and in the following $(x(t), \mathbb{P}_x)$ denotes symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$; $\mathbb{P}_x$ is the conditional probability law of the process $\{x(t); t \geq 0\}$ given $x(0) = x$, and $\mathbb{E}_x$ stands for expectation with respect to $\mathbb{P}_x$. Set $\xi = \xi(0)$.

We are now in a position to formulate our main result.

Theorem 2.1. *Assume that the initial datum u_0 belongs to class $\mathcal{U}_0$* a.s.
a) *If*

$$\left\langle \left(\frac{\xi^+}{\log_+ \xi} \right)^d \right\rangle < \infty , \tag{2.2}$$

then a.s. *problem* (0.1) *has a unique nonnegative solution u. This solution admits the Feynman-Kac representation* (2.1).
b) *If*

$$\left\langle \left(\frac{\xi^+}{\log_+ \xi} \right)^d \right\rangle = \infty \tag{2.3}$$

and either $d \geq 2$ or

$$d = 1 \quad \text{and} \quad \langle \log(1 + \xi^-) \rangle < \infty , \tag{2.4}$$

then a.s. *there is no nonnegative solution to problem* (0.1).

We remark that the assertion of Theorem 2.1 will remain true if one considers solutions u on $[0,T] \times \mathbb{Z}^d$, $0 < T < \infty$ (instead of $\mathbb{R}_+ \times \mathbb{Z}^d$). The assumption about the nonnegativity of the solution cannot be dropped. It is not hard to construct functions $u : \mathbb{R}_+ \times \mathbb{Z}^d \to \mathbb{R}$, $u \not\equiv 0$, which satisfy (0.1) for $\xi(x) \equiv 0$ and $u_0(x) \equiv 0$. If the solution u is allowed to attain both positive and negative values, uniqueness can only be established in the so-called Täcklind classes (see [4, Chap. 3, Sect. 2] for the spatially continuous case).

In dimension $d \geq 2$, Theorem 2.1 gives a complete answer to the question of existence and uniqueness for problem (0.1) in the class of *all nonnegative* solutions. In particular, uniqueness is established without *any* restriction on the growth of the solutions at infinity. An analysis of the proof (Sect. 2.3) shows that assumption (2.4) can be weakened slightly.

The assertion of Theorem 2.1 substantially relies on the *percolation behavior* of the random medium. To establish uniqueness via a Harnack type inequality we will use the fact that a.s. all connected components of the level set $\{x \in \mathbb{Z}^d : \xi(x) \leq \alpha\}$ are finite provided that α is sufficiently negative (cf. Lemma 2.3). In dimension $d \geq 2$, the level set $\{x \in \mathbb{Z}^d : \xi(x) > \alpha\}$ contains a.s. a (unique) infinite connected component W^+ having positive asymptotic density provided that α is sufficiently negative. Roughly speaking, this allows to restrict in the Feynman-Kac formula (2.1) the motion of the random walk $x(\cdot)$ to the set W^+ in order to derive "explosion" of the Feynman-Kac functional, i.e. nonexistence of the solution to (0.1), under assumption (2.3). This is the reason why we do not need to impose any restriction on the lower tail behavior of the distribution of ξ in dimension $d \geq 2$. In contrast to this, assertion b) about nonexistence will not be true in the one-dimensional case without an additional assumption like (2.4). For any i.i.d. random field $\{\xi(x); x \in \mathbb{Z}\}$ with given tail behavior at $+\infty$ satisfying (2.3) one can change the tail behavior at $-\infty$ in such a way that a.s. a nonnegative solution u of (0.1) exists. This is a kind of *"shielding effect."* If the potential $\xi(x)$ has very large negative peaks, then these peaks will "shield" the large positive peaks avoiding in this way "explosion" of the Feynman-Kac functional (2.1) (cf. the remark at the end of Sect. 2.3). The fact that existence (nonexistence) may be expressed in terms of the finiteness (infiniteness) of the moment (2.2) is also caused by the almost sure behavior of the random medium. As we will see in Sect. 2.3, the clue is that a.s. the upper limit of $\xi(x)/(|x| \log |x|)$ as $|x| \to \infty$ does not attain finite positive values. It is either 0 or ∞ depending on whether (2.2) or (2.3) is satisfied (Lemma 2.5).

The proof of Theorem 2.1 will be broken down into several steps. In Sect. 2.2 we will consider existence and uniqueness of the non-random Cauchy problem in terms of finiteness of the associated Feynman-Kac functional. After that we will show that for the random potential Ξ this functional is a.s. finite (infinite) under assumption (2.2) [(2.3), (2.4)]. In Sect. 2.3 this will be done under the additional restriction that the random medium Ξ is bounded from below. This restriction will then be removed in Sect. 2.4.

2.2. *The Deterministic Cauchy Problem*

Let $q : \mathbb{Z}^d \to \mathbb{R}$ and $v_0 : \mathbb{Z}^d \to \mathbb{R}_+$ be arbitrary functions. The following (rather standard) lemma enlightens the connection between the Cauchy problem

$$\begin{aligned} \frac{\partial v}{\partial t} &= \kappa \Delta v + qv \quad \text{on } \mathbb{R}_+ \times \mathbb{Z}^d, \\ v\,|_{t=0} &= v_0 \quad \text{on } \mathbb{Z}^d, \end{aligned} \tag{2.5}$$

and the Feynman-Kac functional

$$\underline{v}(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t q(x(s))\, ds \right\} v_0(x(t)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d. \tag{2.6}$$

Lemma 2.2 (Existence). *Problem* (2.5) *admits at least one nonnegative solution* v *iff*

$$\underline{v}(t,x) < \infty \quad \textit{for all} \quad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d. \tag{2.7}$$

If (2.7) *is fulfilled, then* $\underline{v}$ *is the minimal nonnegative solution of* (2.5).

Roughly speaking, nonexistence for (2.5) is caused by "explosion" of the Feynman-Kac functional (2.6). This may occur since q and v_0 are allowed to grow to infinity arbitrarily fast. Such trouble cannot occur for the associated intitial boundary value problems in finite regions of $\mathbb{Z}^d$ with time-continuous boundary conditions for which existence, uniqueness and their Feynman-Kac representation are obvious.

Given a natural number N, we introduce the cube

$$Q_N = \{z = (z^1, z^2, \dots, z^d) \in \mathbb{Z}^d : |z^i| < N \quad \text{for} \quad 1 \le i \le d\}$$

and its boundary

$$\partial Q_N = Q_{N+1} \backslash Q_N .$$

By τ_N we denote the first hitting time of ∂Q_N:

$$\tau_N = \inf\{t \ge 0 : x(t) \in \partial Q_N\}.$$

Proof of Lemma 2.2. a) Assuming (2.7), we show that $\underline{v}$ solves (2.5). An application of the strong Markov property to (2.6) yields for each N,

$$\begin{aligned} \underline{v}(t,x) = {} & \mathbb{E}_x \exp\left\{ \int_0^t q(x(s))\, ds \right\} v_0(x(t))\, \mathbb{1}(\tau_N > t) \\ & + \mathbb{E}_x \exp\left\{ \int_0^{\tau_N} q(x(s))\, ds \right\} \underline{v}(t-\tau_N, x(\tau_N))\, \mathbb{1}(\tau_N \le t), \end{aligned} \tag{2.8}$$

$(t,x) \in \mathbb{R}_+ \times (Q_N \cup \partial Q_N)$. Assume for the moment that $\underline{v}$ is time-continuous. Then (2.8) is the unique solution of the initial boundary value problem associated with (2.5) in the cylinder $\mathbb{R}_+ \times (Q_N \cup \partial Q_N)$ with prescribed continuous boundary function $\underline{v}$ on $\mathbb{R}_+ \times \partial Q_N$. Since N is arbitrary, this proves that $\underline{v}$ solves (2.5).

It only remains to check that $\underline{v}(t,x)$ is continuous in t for each $x \in \mathbb{Z}^d$. It suffices to consider the point $x = 0$. An application of (2.8) for $x = 0$ and $N = 1$ yields

$$\begin{aligned} \underline{v}(t,0) = {} & \exp\{tq(0)\} v_0(0) \mathbb{P}_0(\tau_1 > t) \\ & + (2d)^{-1} \sum_{|x|=1} \mathbb{E}_0 \exp\{\tau_1 q(0)\} \underline{v}(t-\tau_1, x) \mathbb{1}(\tau_1 \le t). \end{aligned}$$

Since τ_1 is exponentially distributed with respect to $\mathbb{P}_0$, this expression is indeed continuous in t.

b) Let v be a nonnegative solution of (2.5). We show that $v \geq \underline{v}$. Considering v as the solution of the associated initial boundary value problem in the cylinder $\mathbb{R}_+ \times (Q_N \cup \partial Q_N)$, we conclude that the Feynman-Kac representation (2.8) holds also for v instead of $\underline{v}$. Hence,

$$v(t,x) \geq \mathbb{E}_x \exp\left\{ \int_0^t q(x(s))\, ds \right\} v_0(x(t))\, \mathbb{1}(\tau_N > t), \qquad (t,x) \in \mathbb{R}_+ \times Q_N\,.$$

Letting $N \to \infty$, we get $v \geq \underline{v}$ on $\mathbb{R}_+ \times \mathbb{Z}^d$. □

We will call the potential q *percolating from below*, if for each $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{Z}^d : q(x) \leq \alpha\}$ contains an infinite connected component. Otherwise q will be called *non-percolating from below*. (Two points x and y in $\mathbb{Z}^d$ are considered to be neighbors if $|x - y| = 1$.)

Lemma 2.3 (Uniqueness). *If q is non-percolating from below, then the Cauchy problem* (2.5) *admits at most one nonnegative solution.*

Proof. 1. Since q is assumed to be non-percolating from below, we can and will fix $\alpha < 0$ so that all connected components of the level set $\{q \leq \alpha\}$ are finite. For each $N \in \mathbb{N}$, let us denote by Γ_N the set of paths

$$\gamma : 0 = x_0 \to x_1 \to \ldots \to x_n \tag{2.9}$$

(of successively neighboring points $x_0, x_1, \ldots, x_n$) in $\mathbb{Z}^d$ such that $x_0, \ldots, x_{n-1} \in Q_N$ and $x_n \in \partial Q_N$. Given a path of the form (2.9), we will denote the sequence $\{x_0, \ldots, x_{n-1}\}$ by the same symbol γ. We define the subsequences

$$\gamma_- = \{x \in \gamma : q(x) \leq \alpha\}, \qquad \gamma_+ = \{x \in \gamma : q(x) > \alpha\}\,.$$

One readily checks that the finiteness of all connected components of the level set $\{q \leq \alpha\}$ is equivalent to

$$\lim_{N\to\infty} \min_{\gamma\in\Gamma_N} |\gamma_+| = \infty\,. \tag{2.10}$$

($|\gamma_+|$ denotes the length of the sequence γ_+.)
2. Acccording to Lemma 2.2, the class of nonnegative solutions of problem (2.5) contains the minimal solution (provided that this class is not empty). To prove uniqueness in this class, it therefore suffices to verify that each nonnegative solution v of (2.5) with initial datum $v_0 = 0$ vanishes identically. It will clearly be enough to check that $v(t,0) = 0$ for all $t \in \mathbb{R}_+$. To this end we will use the Feynman-Kac representation

$$v(t,0) = \mathbb{E}_0 \exp\left\{ \int_0^{\tau_N} q(x(s))\, ds \right\} v(t - \tau_N, x(\tau_N))\, \mathbb{1}(\tau_N \leq t) \tag{2.11}$$

[cf. (2.8)]. We intend to derive from (2.11) the Harnack type inequality

$$v(T,0) \geq \left(\frac{T}{t}\right)^{\min_{\gamma\in\Gamma_N} |\gamma_+|-1} \exp\{(\alpha - 2\kappa d)\,(T - t)\}\, v(t,0) \tag{2.12}$$

which is valid for $0 < t < T$ and all natural numbers N for which the minimum in the exponent on the right is not zero. Because of (2.10), this then implies that $v(t,0) = 0$ for all $t \in (0,T)$.

Let us fix an arbitrary path $\gamma \in \Gamma_N$ of the form (2.9) with $\gamma_+ \neq \emptyset$. The contribution of the random walk $x(\cdot)$ along the path γ to the expectation (2.11) equals

$$\chi_\gamma(t) = (2d)^{-n}\,\mathbb{E}_0 \exp\left\{\sum_{i=0}^{n-1} q_i\sigma_i\right\} v\left(t - \sum_{i=0}^{n-1}\sigma_i, x_n\right) \mathbb{1}\left(\sum_{i=0}^{n-1}\sigma_i \leq t\right),$$

where $q_i = q(x_i)$ and $\sigma_0, \sigma_1, \ldots$ denote the waiting times of the random walk $x(\cdot)$ between consecutive jumps. The random variables $\sigma_0, \sigma_1, \ldots$ are independent and exponentially distributed with parameter $2\kappa d$. To prove (2.12), it will be enough to show that

$$\chi_\gamma(T) \geq \left(\frac{T}{t}\right)^{|\gamma_+|-1} \exp\{(\alpha - 2\kappa d)(T-t)\}\,\chi_\gamma(t)\,. \tag{2.13}$$

3. It remains to prove (2.13). Since $\gamma_+ \neq \emptyset$, there exists m, $0 \leq m \leq n-1$, such that

$$x_m \in \gamma_+ \quad \text{and} \quad q(x_m) = \min_{x\in\gamma_+} q(x)\,.$$

We have

$$\begin{aligned}
\chi_\gamma(T) &= \kappa^n \int\limits_{\sum_{i=0}^{n-1} s_i \leq T} \!\!\cdots\!\! \int \sum_{i=0}^{n-1} ds_i \exp\left\{(q_m - 2\kappa d)\sum_{i=0}^{n-1} s_i\right\} \\
&\quad \times v\left(T - \sum_{i=0}^{n-1} s_i, x_n\right) \exp\left\{\sum_{\substack{i=0\\ i\neq m}}^{n-1} (q_i - q_m)s_i\right\} \\
&= \kappa^n \int_0^T dr \exp\{(q_m - 2\kappa d)(T-r)\} v(r, x_n) \\
&\quad \times \int\limits_{\sum_{i\neq m} s_i \leq T-r} \!\!\cdots\!\! \int \prod_{i\neq m} ds_i \exp\left\{\sum_{i\neq m}(q_i - q_m)s_i\right\}.
\end{aligned} \tag{2.14}$$

Here and in the following we do not indicate explicitly that the domains of integration are supposed to be restricted to positive values of the integration variables. Since $0 < t < T$ and $q_m > \alpha$, it follows that

$$\chi_\gamma(T) \geq \exp\{(\alpha - 2\kappa d)(T - t)\}\, \kappa^n \int_0^t dr\, \exp\{(q_m - 2\kappa d)(t - r)\}\, v(r, x_n)$$

$$\times \int_{\sum_{j\in\gamma_-} s_j \leq t-r} \cdots \int \prod_{j\in\gamma_-} ds_j\, \exp\Big\{\sum_{j\in\gamma_-} (q_j - q_m)s_j\Big\}$$

$$\times \int_{\sum_{i\in\gamma_+\setminus\{m\}} s_i \leq T-r-\sum_{j\in\gamma_-} s_j} \cdots \int \prod_{i\in\gamma_+\setminus\{m\}} ds_i\, \exp\Big\{\sum_{i\in\gamma_+\setminus\{m\}} (q_i - q_m)\, s_i\Big\}. \tag{2.15}$$

Here we have identified the sequence γ_- (respectively γ_+) with the set of indices i, $0 \leq i \leq n-1$, satisfying $x_i \in \gamma_-$ (respectively $x_i \in \gamma_+$). Making a change of integration variables of the form

$$s_i = \frac{T - r - \sum_{j\in\gamma_-} s_j}{t - r - \sum_{j\in\gamma_-} s_j}\, \tilde{s}_i, \qquad i \in \gamma_+\setminus\{m\},$$

and remembering that $q_i - q_m \geq 0$ for $i \in \gamma_+\setminus\{m\}$, we find that the last integral on the right of (2.15) is larger than $(T/t)^{|\gamma_+|-1}$ times the same integral but with T replaced by t. Thus,

$$\chi_\gamma(T) \geq \left(\frac{T}{t}\right)^{|\gamma_+|-1} \exp\{(\alpha - 2\kappa d)(T - t)\}\, \kappa^n \int_0^t dr\, \exp\{(q_m - 2\kappa d)(t - r)\}$$

$$\times\, v(r, x_n) \int_{\sum_{i\neq m} s_i \leq t-r} \cdots \int \prod_{i\neq m} ds_i\, \exp\Big\{\sum_{i\neq m} (q_i - q_m)s_i\Big\}.$$

Comparing the expression on the right with (2.14) for T replaced by t, we arrive at the desired estimate (2.13). $\square$

Remark. Without any assumption on the lower tail behavior of the potential q like that given in Lemma 2.3 uniqueness does not hold. This is closely related to explosion of jump Markov processes on $\mathbb{Z}^d$ with "strong" drift. For example, the birth and death process on $\mathbb{Z}$ with generator

$$Gf(x) = e^{2x+1}[f(x+1) - f(x)] + e^{-2x+1}[f(x-1) - f(x)]$$

explodes after a finite positive random time ζ. Therefore the function $w(t,x) = \mathbb{P}_x(\zeta \leq t)$ is a nontrivial solution of

$$\frac{\partial w}{\partial t} = Gw, \qquad w(0,x) \equiv 0.$$

From this we conclude that

$$v(t,x) = e^{x^2} w(t,x)$$

is a nontrivial (nonnegative) solution of (2.5) for $\kappa = 1$, $q(x) = 2 - e^{2x+1} - e^{-2x+1}$ and $v_0(x) \equiv 0$.

2.3. The Cauchy Problem with Random Potential

To prove Theorem 2.1, we now apply the Lemmas 2.2 and 2.3 to the random potential $\Xi = \{\xi(x);\, x \in \mathbb{Z}^d\}$. As a basic result of percolation theory, this potential is a.s. non-percolating from below. (Concerning standard facts of percolation theory, the reader is referred to the monograph Kesten [6] and the survey papers Kesten [7] and Men'shikov et al. [10].) Hence, a.s. problem (0.1) admits at most one nonnegative solution. Existence of a nonnegative solution is equivalent to

$$\mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) < \infty \qquad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d \tag{2.16}$$

(Lemma 2.2). To decide whether or not (2.16) is fulfilled, we need to compare the speed of decay of the probability that the random walk $(x(t), \mathbb{P}_x)$ hits a point y in the time interval $[0,t]$ (and similar quantities) with the speed of growth of $\xi(y)$ as $|y| \to \infty$. The asymptotic behavior of these quantities will be considered in the next two lemmas.

Let $N(t)$ denote the number of jumps of the random walk $x(\cdot)$ in the time interval $[0,t]$. For each $x \in \mathbb{Z}^d$, $\{N(t);\, t \geq 0\}$ is a Poisson process with respect to $\mathbb{P}_x$ having intensity $2\kappa d$.

Lemma 2.4. *For each $x \in \mathbb{Z}^d$ and each $t > 0$ we have*

$$\mathbb{P}_x\Big(\max_{s\in[0,t]} |x(s)| \geq n \Big) \leq \exp\{-n \log n + \underline{O}(n)\} \qquad \text{as } n \to \infty, \tag{2.17}$$

and

$$\mathbb{P}_x(N(t) = n) \geq \exp\{-n \log n + \underline{O}(n)\} \qquad \text{as } n \to \infty. \tag{2.18}$$

Note that in contrast to Brownian motion the leading term of these bounds does not depend on t or κ.

Proof. Fix $t > 0$ arbitrarily. We first prove (2.17). It will be enough to consider the case $x = 0$. Let $x^1(t), \ldots, x^d(t)$ denote the components of $x(t)$. Applying symmetry, the reflection principle, and Chebychev's exponential inequality, we obtain the estimate

$$\begin{aligned}
\mathbb{P}_0\Big(\max_{s\in[0,t]} |x(s)| \geq n \Big) &= \mathbb{P}_0\Big(\max_{s\in[0,t]} (|x^1(s)| + \ldots + |x^d(s)|) \geq n \Big) \\
&\leq 2^d\, \mathbb{P}_0\Big(\max_{s\in[0,t]} (x^1(s) + \ldots + x^d(s)) \geq n \Big) \\
&\leq 2^{d+1}\, \mathbb{P}_0(x^1(t) + \ldots + x^d(t) \geq n) \\
&\leq 2^{d+1} e^{-\beta n} \mathbb{E}_0 \exp\{\beta(x^1(t) + \ldots + x^d(t))\}
\end{aligned}$$

for arbitrary $\beta > 0$. Since $(x^1(t) + \ldots + x^d(t), \mathbb{P}_x)$ is a symmetric random walk on $\mathbb{Z}$ with generator $\kappa d \Delta$, the expectation on the right equals $\exp\{2\kappa d t[\cosh\beta - 1]\}$. Hence

$$\mathbb{P}_0\Big(\max_{s\in[0,t]} |x(s)| \geq n \Big) \leq 2^{d+1} \exp\{-\beta n + 2\kappa d t[\cosh\beta - 1]\}\,.$$

Putting on the right $\beta = \log n$, we get the desired estimate (2.17). Since

$$\mathbb{P}_x(N(t) = n) = \frac{(2\kappa dt)^n}{n!} \exp\{-2\kappa dt\},$$

(2.18) follows from Stirling's formula for $n!$. □

Lemma 2.5. *If* (2.2) *is satisfied, then*

$$\limsup_{|x|\to\infty} \frac{\xi(x)}{|x| \log |x|} \leq 0 \quad \text{a.s.} \tag{2.19}$$

Otherwise

$$\limsup_{|x|\to\infty} \frac{\xi(x)}{|x| \log |x|} = \infty \quad \text{a.s.} \tag{2.20}$$

The remarkable fact of this lemma is that a.s. the upper bound on the left of (2.19) does not belong to $(0, \infty)$. We already mentioned in Sect. 2.1 that this is essential to get a complete answer to the question of existence of a nonnegative solution to (0.1) in terms of the finiteness of the moment (2.2).

Proof. According to the Borel-Cantelli lemma, we must only check for each $c > 0$ that

$$\sum_{|x|>2} [1 - F(c|x| \log |x|)] < \infty \tag{2.21}$$

iff (2.2) is satisfied. Since the number of sites $x \in \mathbb{Z}^d$ with $|x| = r$ grows like const r^{d-1} as $r \to \infty$, (2.21) is equivalent to the finiteness of

$$\sum_{r=3}^{\infty} r^{d-1} \sum_{n\geq r} [F(c(n+1) \log(n+1)) - F(cn \log n)]$$
$$= \sum_{n=3}^{\infty} \left(\sum_{r=3}^{n} r^{d-1} \right) [F(c(n+1) \log(n+1)) - F(cn \log n)].$$

Since $\sum_{r=3}^{n} r^{d-1}$ behaves like const n^d as $n \to \infty$, this is true iff

$$\sum_{n=3}^{\infty} \int_{cn \log n}^{c(n+1) \log(n+1)} \left(\frac{t}{\log t} \right)^d F(dt) < \infty,$$

i.e. iff (2.2) is satisfied. □

Proof of Theorem 2.1. a) Suppose that (2.2) is satisfied and the initial datum u_0 belongs to class $\mathcal{U}_0$ a.s. To complete the proof of part a) it remains to check that (2.16) is valid a.s. For arbitrary $(t, x) \in \mathbb{R}_+ \times \mathbb{Z}^d$ we have

$$\mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t))$$
$$\leq \sum_{n=0}^{\infty} \mathbb{P}_x \left(\max_{s\in[0,t]} |x(s)| = n \right) \exp\left\{ t \max_{|y|\leq n} \xi(y) + \max_{|y|\leq n} \log u_0(y) \right\}.$$

Applying the bounds (2.17) and (2.19) and remembering the definition of class $\mathscr{U}_0$, we find that the sum on the right is finite a.s.

b) We will show that, under the assumptions of part b), a.s.

$$\mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) = \infty \quad \text{for all } t > 0 \text{ and } x \in \mathbb{Z}^d . \tag{2.22}$$

To this end it will be enough to verify that a.s.

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \delta_y(x(t)) = \infty \quad \text{for all } t > 0 . \tag{2.23}$$

To see this, let us introduce the quasi-transition function

$$q(t, x, y) = \mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \delta_y(x(t))$$

and choose a random site $z \in \mathbb{Z}^d$ so that $u_0(z) > 0$ a.s. Then an application of the Markov property and the symmetry of the random walk yield

$$\begin{aligned} &\mathbb{E}_x \exp\left\{ \int_0^{4t} \xi(x(s))\, ds \right\} u_0(x(t)) \\ &\qquad \geq q(t, x, 0)\, q(2t, 0, 0)\, q(t, 0, z)\, u_0(z) \end{aligned}$$

and

$$q(2t, 0, 0) = \sum_y q^2(t, 0, y) \geq \left[\sup_y q(t, 0, y) \right]^2 .$$

Thus (2.23) indeed implies (2.22).

We are now going to prove (2.23) under the additional assumption that the random medium is bounded from below. After that we will consider the one-dimensional case under assumption (2.4). The proof in the general case of an unbounded from below random medium will be postponed to the next section. Suppose therefore that the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are bounded from below by some constant $\alpha < 0$. Then (2.23) is an easy consequence of (2.18) and (2.20). Indeed, for almost every realization of the random medium Ξ we find a sequence (y_n) in $\mathbb{Z}^d$ with

$$\frac{\xi(y_n)}{|y_n| \log |y_n|} \to \infty . \tag{2.24}$$

The contribution of the random walk $x(\cdot)$ to the expectation

$$\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \delta_{y_n}(x(t)) \tag{2.25}$$

along a path of length $|y_n|$ joining 0 with y_n which hits y_n before $t/2$ and stays there until time t can be estimated from below by

$$\begin{aligned} & e^{\alpha t/2}(2d)^{-|y_n|}\mathbb{P}_0(N(t/2)=|y_n|)\, e^{\xi(y_n)t/2}\mathbb{P}_{y_n}(x(s)=y_n \text{ for } s\in[0,t/2]) \\ &\quad = (2d)^{-|y_n|}\exp\{(t/2)\xi(y_n)-|y_n|\log|y_n|+\underline{O}(|y_n|)\}. \end{aligned} \tag{2.26}$$

Here we have used (2.18). Because of (2.24), the expression on the right tends to infinity as $n\to\infty$, and we arrive at (2.23).

We now turn to the proof of (2.23) in the one-dimensional case under the assumptions (2.3) and (2.4). Again, by Lemma 2.5, we find a random sequence (y_n) such that (2.24) holds a.s. We will assume without loss of generality that $y_n>\kappa t/2$, $\xi(y_n)>2\kappa$, and $y_n\to\infty$. Let τ_n denote the first hitting time of y_n. We estimate the expectation (2.25) from below by the contribution of the random walk $x(\cdot)$ moving along the path $0\to 1\to\ldots\to y_n$, hitting y_n before $t/2$, and staying there until time t. Then we obtain

$$\begin{aligned} \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(s))\,ds\right\}\delta_{y_n}(x(t)) &\ge \exp\{t/2(\xi(y_n)-2\kappa)\}\,2^{-y_n} \\ &\quad\times \mathbb{E}_0\exp\left\{-\sum_{k=0}^{y_n-1}\xi^-(k)\sigma_k\right\}1\!\!1\left(\sum_{k=0}^{y_n-1}\sigma_k\le t/2\right), \end{aligned}$$

where $\sigma_0,\sigma_1,\ldots$ are the waiting times of the random walk $x(\cdot)$ (between consecutive jumps) which are independent and exponentially distributed with parameter 2κ. The expectation on the right may be estimated from below by

$$\begin{aligned} & e^{-t/2}\mathbb{P}_0\left(\sigma_k\le \frac{t}{2y_n}\frac{1}{1+\xi^-(k)} \text{ for } 0\le k<y_n\right) \\ &\ge e^{-t/2}\prod_{k=0}^{y_n-1}\frac{\kappa t}{2y_n}\frac{1}{1+\xi^-(k)} \\ &= \exp\left\{-t/2+y_n\log(\kappa t/2)-y_n\log y_n-\sum_{k=0}^{y_n-1}\log(1+\xi^-(k))\right\}. \end{aligned}$$

Because of assumption (2.4), the strong law of large numbers yields

$$\sum_{k=0}^{y_n-1}\log(1+\xi^-(k))=\underline{O}(y_n) \qquad \text{a.s.} \tag{2.27}$$

Combining the above estimates and taking thereby into account (2.24) and (2.27), we find that the expectation (2.25) tends to infinity as $n\to\infty$ a.s., and we are done. □

Remark. At the end of Sect. 2.1 we mentioned that in the one-dimensional case an additional assumption like (2.4) is needed to ensure nonexistence. If the negative peaks of the random potential are "much larger" than the positive peaks, then a "shielding effect" will force existence. To illustrate this phenomenon, fix $\varepsilon \in (0, 1)$ arbitrarily and assume that $\xi = \xi(0)$ satisfies

$$\left\langle \left(\frac{\xi^+}{\log_+ \xi} \right)^\varepsilon \right\rangle < \infty, \quad \left\langle \frac{\xi^+}{\log_+ \xi} \right\rangle = \infty \tag{2.28}$$

and

$$\mu(\xi \leq -t) \sim \frac{(\log\log t)^{1+\varepsilon}}{(\log t)^\varepsilon} \quad \text{as } t \to \infty. \tag{2.29}$$

The last condition implies

$$\langle (\log(1 + \xi^-))^\delta \rangle < \infty$$

for all $\delta \in (0, \varepsilon)$ and

$$\langle \log(1 + \xi^-) \rangle = \infty.$$

Let us introduce the abbreviations

$$\zeta_n^+ = \max_{|x| \leq n} \xi^+(x)$$

and

$$\zeta_n^- = \min \left\{ \max_{-n<x<-n/2} \xi^-(x), \max_{n/2<x<n} \xi^-(x) \right\}.$$

Then, applying the Borel-Cantelli lemma, we obtain

$$\lim_{n\to\infty} \frac{\zeta_n^+}{n^{1/\varepsilon} \log n} = 0 \quad \text{a.s.} \tag{2.30}$$

and

$$\liminf_{n\to\infty} \frac{\log \zeta_n^-}{n^{1/\varepsilon} \log n} > 0 \quad \text{a.s.} \tag{2.31}$$

[(2.30) follows from the first half of (2.28) similarly to the proof of (2.19), and (2.31) is a consequence of (2.29).] Hence, asymptotically (as $|x| \to \infty$) the random field $\xi(x)$ has a.s. negative peaks which are much larger than the positive peaks. This turns out to be sufficient to create the "shielding effect." Indeed, according to the Feynman-Kac formula,

$$\begin{aligned} u(t,x) &= \mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) \\ &= \sum_{n=0}^\infty \mathbb{E}_x \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t))\, \mathbb{1}\Big(\max_{s\in[0,t]} |x(s)| = n \Big). \end{aligned} \tag{2.32}$$

For each $n \geq 3$ we find lattice points z_n^- and z_n^+ in $(-n, -n/2)$ and $(n/2, n)$, respectively, such that

$$\xi^-(z_n^-) = \max_{-n<x<-n/2} \xi^-(x)$$

and

$$\xi^-(z_n^+) = \max_{n/2<x<n} \xi^-(x).$$

To estimate the expectation on the right of (2.32), we replace the potential $\xi(x)$ in the interval $[-n, n]$ by ζ_n^+ except for the points z_n^- and z_n^+ where we replace it by $\zeta_n^+ - \zeta_n^-$. In this way we obtain for $n \geq 3$:

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{\int_0^t \xi(x(s))\, ds\right\} u_0(x(t))\, \mathbb{1}\Big(\max_{s\in[0,t]} |x(s)| = n\Big) \\
&\quad \leq \exp\Big\{t\zeta_n^+ + \max_{|y|\leq n} \log u_0(y)\Big\} \\
&\quad\quad \times \mathbb{E}_x \exp\left\{-\int_0^t \zeta_n^-(\delta_{z_n^-}(x(s)) + \delta_{z_n^+}(x(s)))\, ds\right\} \mathbb{1}\Big(\max_{s\in[0,t]} |x(s)| = n\Big). \quad (2.33)
\end{aligned}$$

If $n > 2|x|$, then each path which contributes to the expectation on the right of (2.33) hits z_n^- or z_n^+ and stays there (at least) an exponentially distributed time σ (with parameter 2κ) before time t. Therefore

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{-\int_0^t \zeta_n^-(\delta_{z_n^-}(x(s)) + \delta_{z_n^+}(x(s)))\, ds\right\} \mathbb{1}\Big(\max_{s\in[0,t]} |x(s)| = n\Big) \\
&\quad \leq \mathbb{E}_0 \exp\{-\zeta_n^- \sigma\} = \frac{2\kappa}{2\kappa + \zeta_n^-} \quad (2.34)
\end{aligned}$$

for all $n > 2|x|$. Combining the estimates (2.32)–(2.34), remembering that u_0 belongs to class $\mathscr{U}_0$ a.s. $\Big($i.e. $\max_{|y|\leq n} \log u_0(y) \leq \underline{O}(n \log n)\Big)$, and using (2.30) and (2.31), we find that a.s.

$$u(t,x) < \infty \qquad \text{for all } t \text{ and } x.$$

In other words, with probability one the Cauchy problem (0.1) has exactly one nonnegative solution.

2.4. Unbounded from Below Random Potential

This section is devoted to the proof of (2.23) in the multidimensional case ($d \geq 2$) under assumption (2.3) without any restriction on the lower tail of the distribution of $\xi(x)$. Roughly speaking, we will show that one can choose $\alpha \in \mathbb{R}$ so negative that the following holds true. One finds a sequence (y_n) in $\mathbb{Z}^d$ for which (2.24) is satisfied and, moreover, for each n, 0 can be joined with y_n by a path of length not exceeding a fixed multiple of $|y_n|$ which belongs to $\{x : \xi(x) > \alpha\}$ except for a bounded number of points. To this end we will use percolation arguments. The contribution of the random walk $x(\cdot)$ to the expectation (2.25) along a path with the mentioned properties can then be estimated from below by an expression similar to (2.26) to obtain (2.23).

Throughout this section we assume that $d \geq 2$. Given $r \geq 1$, let $\mathscr{G}_r$ denote the graph whose vertex set is $\mathbb{Z}^d$ and whose edge set consists of all bonds connecting vertices $x, y \in \mathbb{Z}^d$ with $x \neq y$ and $|x-y| \leq r$. For a finite set $B \subset \mathbb{Z}^d$, we denote by $\partial_r B$ the *r-boundary* of B, i.e. the set of all vertices $x \in \mathbb{Z}^d \backslash B$ for which there exists $y \in B$ with $|y-x| \leq r$. We further define the *external r-boundary* $\partial_r^e B$ as the set of sites $y \in \partial_r B$ for which there exists an infinite non-selfintersecting $\mathscr{G}_1$-path $y = y_0 \to y_1 \to y_2 \to \dots$ with vertices entirely contained in $\mathbb{Z}^d \backslash B$. Note that $\partial_1^e B$ is not $\mathscr{G}_1$-connected for any non-empty finite $\mathscr{G}_1$-connected set B. But we have the following lemma.

Lemma 2.6. *Let B be a $\mathscr{G}_{2d}$-connected finite subset of $\mathbb{Z}^d$. Then $\partial_{2d-1}^e B$ is $\mathscr{G}_1$-connected.*

Proof. It will be enough to show that any two vertices $y, z \in \partial_d^e B$ can be joined by a $\mathscr{G}_1$-path in $\partial_{2d-1} B$. Let $Q(q) = [q^1 - 1/2, q^1 + 1/2] \times \dots \times [q^d - 1/2, q^d + 1/2]$ denote the unit cube in $\mathbb{R}^d$ with center $q = (q^1, \dots, q^d)$. Since

$$\bar{B} = B \cup \partial_d B$$

is $\mathscr{G}_1$-connected, the polyeder

$$\hat{B} = \bigcup_{q \in \bar{B}} Q(q)$$

is also connected. Moreover, the external boundary $\partial^e \hat{B}$ of $\hat{B}$ is connected [8, Chap. 8, Sect. 57, 2, Theorem 6]. There exists a $\mathscr{G}_1$-path $y = y_0 \to y_1 \to \dots \to y_k$ in $\partial_d^e B$ such that $Q(y_k)$ is a "surface cube" of the polyeder $\hat{B}$, i.e. one of the $2d$ surfaces of this cube belongs to $\partial^e \hat{B}$. Let us denote this surface by S_y. Analogously we find a $\mathscr{G}_1$-path $z = z_0 \to z_1 \to \dots \to z_l$ in $\partial_d^e B$ such that one of the surfaces of $Q(z_l)$, S_z, belongs to $\partial^e \hat{B}$. We can connect S_y with S_z by a finite sequence of neighboring surfaces $S_1, \dots, S_r$ in $\partial^e \hat{B}$ belonging to "surface cubes" with centers $q_1, \dots, q_r$, respectively. Clearly each of the vertices $y_k = q_0, q_1, \dots, q_r, q_{r+1} = z_l$ has distance d to B, and the distance between consecutive vertices, q_i and q_{i+1}, $0 \leq i \leq r$, does not exceed d. Hence, q_i and q_{i+1} $(0 \leq i \leq r)$ can be joined by a $\mathscr{G}_1$-path of length $\leq d$ with vertices in $\partial_{2d-1} B$. In this way we have constructed a $\mathscr{G}_1$-path in $\partial_{2d-1} B$ joining y and z. □

Given $\alpha \in \mathbb{R}$, let us define the level sets

$$A_\alpha^- = \{x \in \mathbb{Z}^d : \xi(x) \leq \alpha\}$$

and

$$A_\alpha^+ = \{x \in \mathbb{Z}^d : \xi(x) > \alpha\}.$$

For each $x \in \mathbb{Z}^d$, we denote by $W_{2d}^-(x)$ the $\mathscr{G}_{2d}$-connected component of A_α^- containing x. If $x \notin A_\alpha^-$ we set $W_{2d}^-(x) = \emptyset$. We know from percolation theory that the level α can be chosen so negative that the following assertions are satisfied:

(A1) Almost surely the set A_α^+ contains a unique infinite $\mathscr{G}_1$-connected component W^+, and $\mu(0 \in W^+) > 0$.

(A2) Almost surely all $\mathscr{G}_{2d}$-connected components $W_{2d}^-(x)$ of A_α^- are finite. Moreover, there exists a positive constant h such that

$$\langle \exp\{h|W_{2d}^-(x)|\}\rangle < \infty \quad \text{for all } x \in \mathbb{Z}^d.$$

Assertion (A1) can be found in Aizenman et al. [1, Proposition 5.3]. Assertion (A2) is contained in Kesten [6, Theorem 5.1]. Until the end of the present section we will assume (A1) and (A2). We need the following refinement of the second half of Lemma 2.5.

Lemma 2.7. *If* (2.3) *is satisfied, then*

$$\limsup_{|x|\to\infty,\, x\in W^+} \frac{\xi(x)}{|x|\,\log|x|} = \infty \qquad \text{a.s.}$$

Proof. To simplify notation, let us assume without loss of generality that $\alpha = 0$. Recall that $F(t) = \mu(\xi(0) \le t)$, $t \in \mathbb{R}$. We assume further that the random variables $\xi(x), x \in \mathbb{Z}^d$, are represented in the form

$$\xi(x) = (1-\zeta(x))\xi_-(x) + \zeta(x)\xi_+(x)\,.$$

Thereby $\{\xi_-(x), \xi_+(x), \zeta(x);\, x \in \mathbb{Z}^d\}$ is a family of completely independent random variables. The variables $\xi_\pm(x)$ are assumed to have distribution functions $F_\pm$ defined by

$$F_-(t) = \frac{F(t)}{F(0)} \wedge 1, \qquad 1 - F_+(t) = \frac{1-F(t)}{1-F(0)} \wedge 1,$$

and the random variables $\zeta(x)$ attain the values 0 and 1 with probability $F(0)$ and $1-F(0)$, respectively. Note that $\xi_-(x) \le 0$ a.s., $\xi_+(x) > 0$ a.s., and

$$A_0^+ = \{x \in \mathbb{Z}^d : \zeta(x) = 1\} \qquad \text{a.s.}$$

Thus W^+ depends on $\{\zeta(x);\, x \in \mathbb{Z}^d\}$ only. Therefore an application of the Borel-Cantelli lemma yields for each $c > 0$:

$$\begin{aligned} &\mu(\xi(x) > c|x|\,\log|x| \text{ for infinitely many } x \in W^+) \\ &\quad = \langle \mu(\xi_+(x) > c|x|\,\log|x| \text{ for infinitely many } x \in B)\,|_{B=W^+}\rangle \\ &\quad = 1 \end{aligned} \tag{2.35}$$

provided that

$$\sum_{x\in W^+\setminus\{0\}} [1 - F(c|x|\,\log|x|)] = \infty \qquad \text{a.s.}$$

But

$$\begin{aligned} &\sum_{x\in W^+\setminus\{0\}} [1 - F(c|x|\,\log|x|)] \\ &\quad = \sum_{n=1}^{\infty} \sum_{|x|=n} 1\!\!1(x \in W^+) \sum_{r\ge n} [F(c(r+1)\,\log(r+1)) - F(cr\,\log r)] \\ &\quad = \sum_{r=1}^{\infty} \sum_{1\le|x|\le r} 1\!\!1(x \in W^+)\, [F(c(r+1)\,\log(r+1)) - F(cr\,\log r)]\,. \end{aligned} \tag{2.36}$$

By the ergodic theorem and (A1),

$$r^{-d} \sum_{1\le|x|\le r} 1\!\!1(x \in W^+)$$

converges to some finite positive constant a.s. as $r \to \infty$. Therefore the expression on the right of (2.36) is infinite a.s. iff

$$\sum_{r=1}^{\infty} r^d[F(c(r+1)\log(r+1)) - F(cr\log r)] = \infty. \tag{2.37}$$

Since

$$r^d[F(c(r+1)\log(r+1)) - F(cr\log r)] \sim \int_{cr\log r}^{c(r+1)\log(r+1)} \left(\frac{t}{c\log t}\right)^d F(dt)$$

as $r \to \infty$, (2.37) is equivalent to (2.3). Thus (2.35) holds for each $c > 0$, and we are done. □

Given $x, y \in W^+$, we denote by $d_{W^+}(x,y)$ the distance between x and y in W^+, i.e. the minimal length of $\mathscr{G}_1$-paths joining x and y in W^+.

Lemma 2.8. *There exists* $\varrho > 1$ *such that* a.s.

$$\limsup_{|y|\to\infty, y\in W^+} \frac{d_{W^+}(x,y)}{|x-y|} \leq \varrho$$

for all $x \in W^+$.

Proof. Because of the spatial homogeneity of the random medium and the Borel-Cantelli lemma, it will be enough to check that

$$\sum_{y\in\mathbb{Z}^d} \mu(0 \in W^+, y \in W^+, d_{W^+}(0,y) > \varrho|y|) < \infty \tag{2.38}$$

for sufficiently large ϱ.

Given $y \in \mathbb{Z}^d$, we fix an arbitrary realization of the random medium such that $0 \in W^+$ and $y \in W^+$. We choose a $\mathscr{G}_1$-path γ_y of length $|y|$ joining 0 with y. Let z_1 be the first vertex of γ_y which does not belong to W^+. Then $z_1 \in A_\alpha^-$. We denote by z_1^- and z_1^+ the last vertex of γ_y before z_1 and the first vertex of γ_y after z_1, respectively, which belong to $\partial_{2d-1}^e W_{2d}^-(z_1)$. Clearly $\partial_{2d-1}^e W_{2d}^-(z_1) \subset A_\alpha^+$. Since $z_1^- \in W^+$, Lemma 2.6 yields $\partial_{2d-1}^e W_{2d}^-(z_1) \subset W^+$. Moreover, we can join z_1^- and z_1^+ by a $\mathscr{G}_1$-path which is contained in $\partial_{2d-1}^e W_{2d}^-(z_1)$. We replace the part of the path γ_y between z_1^- and z_1^+ by this new path. We next choose z_2 to be the first vertex of γ_y after z_1^+ which does not belong to W^+ and repeat the above construction for z_2 (instead of z_1), and so on. In this way we find a non-selfintersecting $\mathscr{G}_1$-path $\tilde{\gamma}_y$ in W^+ joining 0 with y which is entirely contained in

$$\gamma_y \cup \bigcup_{z\in\gamma_y} \partial_{2d-1}^e W_{2d}^-(z).$$

(Here we identified γ_y with the set of its vertices.) From this we conclude that

$$d_{W^+}(0,y) \leq |y| + (2d)^{2d-1} \sum_{z\in\gamma_y} |W_{2d}^-(z)|$$

$(0 \in W^+, y \in W^+)$. Hence, in order to prove (2.38), it will be enough to check that

$$\sum_{y \in \mathbb{Z}^d} \mu\Big(\sum_{z \in \gamma_y} |W_{2d}^-(z)| > \varrho |y| \Big) < \infty \tag{2.39}$$

for sufficiently large ϱ.

Given $c > 0$ and $y \in \mathbb{Z}^d \backslash \{0\}$, let us denote by l_y the integer part of $c \log |y|$. Then we obtain

$$\begin{aligned} &\sum_{y \in \mathbb{Z}^d \backslash \{0\}} \mu\Big(\sum_{z \in \gamma_y} |W_{2d}^-(z)| > \varrho |y| \Big) \\ &\quad \le \sum_{y \in \mathbb{Z}^d \backslash \{0\}} \mu\Big(\max_{|z| \le |y|} |W_{2d}^-(z)| > l_y \Big) \\ &\quad + \sum_{y \in \mathbb{Z}^d \backslash \{0\}} \mu\Big(\sum_{z \in \gamma_y} |W_{2d}^-(z)| \, \mathbb{1}(|W_{2d}^-(z)| \le l_y) > \varrho |y| \Big) . \end{aligned} \tag{2.40}$$

The first sum on the right is finite provided that the constant c is sufficiently large. This is an easy consequence of Chebychev's exponential inequality and (A2). Set $r_y = 2d(l_y + 1)$ and $B_{r_y}(z) = \{\tilde{z} : |\tilde{z} - z| \le r_y\}$. For arbitrary n and $z_1, \dots, z_n \in \mathbb{Z}^d$ with $\min\limits_{1 \le i < j \le n} |z_i - z_j| > 2r_y$, the random variables

$$|W_{2d}^-(z_i)| \, \mathbb{1}(|W_{2d}^-(z_i)| \le l_y), \qquad 1 \le i \le n,$$

are mutually independent, since each of them depends on the values of the medium in the associated ball $B_{r_y}(z_i)$ only. To estimate the last sum on the right of (2.40), we therefore divide γ_y (considered as a set of vertices) into $m \le 2r_y$ pairwise disjoint subsets $\gamma_y^{(1)}, \dots, \gamma_y^{(m)}$ such that any two points in $\gamma_y^{(i)}$ have a distance $> 2r_y$ $(1 \le i \le m)$. Hence, for each i,

$$S_y^{(i)} = \sum_{z \in \gamma_y^{(i)}} |W_{2d}^-(z)| \, \mathbb{1}(|W_{2d}^-(z)| \le l_y)$$

is a sum of independent identically distributed random variables. Taking this into account and applying Chebychev's exponential inequality, we obtain

$$\begin{aligned} &\mu\Big(\sum_{z \in \gamma_y} |W_{2d}^-(z)| \, \mathbb{1}(|W_{2d}^-(z)| \le l_y) > \varrho |y| \Big) \\ &\quad \le \sum_{i=1}^{m} \mu\Big(S_y^{(i)} > \varrho \frac{|y|}{2r_y} \Big) \\ &\quad \le 2r_y \exp\Big\{ -h\varrho \frac{|y|}{2r_y} \Big\} \langle \exp\{h |W_{2d}^-(0)|\} \rangle^{\frac{|y|}{2r_y} + 1} . \end{aligned}$$

Together with (A2) this implies that the last sum of the right of (2.40) converges for large ϱ, and we arrive at (2.39). □

We are now finally ready to prove assertion (2.23) in the multidimensional case under assumption (2.3). Combining the Lemmas 2.7 and 2.8, we a.s. find $x \in W^+$, a sequence (y_n) in W^+ with $|y_n| \to \infty$, and $\mathscr{G}_1$-paths γ_n joining 0 with

y_n such that the following is satisfied. For each n, the path γ_n passes through x, its part between x and y_n entirely belongs to $W^+ \subset A_\alpha^+$, and its length l_n equals $|x| + d_{W^+}(x, y_n)$. Moreover,

$$\frac{\xi(y_n)}{l_n \log l_n} \to \infty . \tag{2.41}$$

We can now argue in the same way as in the case when the random potential is bounded from below by estimating the contribution of the random walk $x(\cdot)$ to the expectation (2.25) along the paths γ_n. We thereby use (2.41) instead of (2.24). This yields a.s. (2.23), and we are done.

3. Asymptotics of the Statistical Moments and Intermittency

In this section we study the rough time asymptotics of the moments $\langle u(t,x)^p \rangle$, $p = 1, 2, \dots$, and intermittency for the solution u to the random Cauchy problem (0.1).

In the following $\Xi = \{\xi(x);\, x \in \mathbb{Z}^d\}$ will denote an arbitrary spatially homogeneous potential (of not necessarily i.i.d. random variables). We will assume that the initial datum $\{u_0(x);\, x \in \mathbb{Z}^d\}$ is a nonnegative spatially homogeneous random field which is independent of Ξ and has finite positive moments of all orders:

$$0 < \langle u_0^p \rangle < \infty \qquad \text{for all } p \in \mathbb{N},$$

where $u_0 = u_0(0)$. Instead of (0.1) we will consider the associated Feynman-Kac solution u given by (2.1) which is allowed to attain the value $+\infty$.

Under the above hypotheses the random fields $u(t,\cdot)$, $t \geq 0$, are spatially homogeneous and, in particular, the moments $\langle u(t,x)^p \rangle$ do not depend on x. If the pair $(\xi(\cdot), u_0(\cdot))$ is ergodic, then the fields $u(t,\cdot)$, $t \geq 0$, are also ergodic.

Let G denote the cumulant generating function of $\xi = \xi(0)$:

$$G(t) = \log\langle \exp(t\xi) \rangle, \qquad t \in \mathbb{R}.$$

The function G takes values in $(-\infty, +\infty]$, is convex and vanishes at zero. Moreover,

$$G(t)/t \uparrow \operatorname{ess\,sup} \xi \qquad \text{as } t \uparrow \infty . \tag{3.1}$$

In particular, ξ is unbounded from above iff $G(t)/t \to \infty$ as $t \to \infty$.

Theorem 3.1. *For each $p \in \mathbb{N}$ and each $t \geq 0$ we have*

$$\exp\{G(pt) - 2\kappa dpt\} \langle u_0^p \rangle \leq \langle u(t,0)^p \rangle \leq \exp\{G(pt) \langle u_0^p \rangle . \tag{3.2}$$

In particular, $\langle u(t,0)^p \rangle < \infty$ iff $G(pt) < \infty$. If $G(t) < \infty$ for all $t > 0$ and either the random potential Ξ is unbounded from above (i.e. ess sup $\xi = \infty$*) or the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are independent and* ess sup $\xi \neq 0$, *then*

$$\lim_{t\to\infty} \frac{\log\langle u(t,0)^p \rangle}{G(pt)} = 1, \qquad p \in \mathbb{N}. \tag{3.3}$$

Proof. Taking in the Feynman-Kac formula (2.1) only into account the contribution of the path $x(\cdot)$ which stays at 0 during the whole time interval $[0,t]$, we get

$$u(t,0) \geq \exp\{t\xi(0) - 2\kappa dt\}\, u_0(0).$$

This yields the lower bound in (3.2). Applying Hölder's and Jensen's inequalities and Fubini's theorem, we obtain the upper bound:

$$\begin{aligned}
\langle u(t,0)^p\rangle &= \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) \right)^p \right\rangle \\
&\leq \left\langle \mathbb{E}_0 \exp\left\{ p \int_0^t \xi(x(s))\, ds \right\} u_0^p(x(t)) \right\rangle \\
&= \mathbb{E}_0 \left\langle \exp\left\{ p \int_0^t \xi(x(s))\, ds \right\} \right\rangle \langle u_0^p \rangle \\
&\leq \frac{1}{t} \int_0^t ds\, \mathbb{E}_0 \langle \exp\{pt\xi(x(s))\}\rangle \langle u_0^p\rangle \\
&= \exp\{G(pt)\} \langle u_0^p\rangle .
\end{aligned}$$

If the potential Ξ is unbounded from above and $G(t) < \infty$ for all $t > 0$, then $G(t)/t \uparrow \infty$ as $t \uparrow \infty$ and (3.3) is immediate from (3.2). It remains to consider the case when the potential Ξ consists of i.i.d. random variables which are bounded from above ($\operatorname{ess\,sup} \xi < \infty$) and satisfy $\operatorname{ess\,sup} \xi \neq 0$. Under these assumptions (3.3) follows from (3.1), the upper bound in (3.2), and the almost sure asymptotics of $u(t,0)$ as $t \to \infty$ [see (4.1) below]. □

Note that the leading term $G(pt)$ in the asymptotic expansion of $\log\langle u(t,0)^p\rangle$ does not depend on the diffusion coefficient κ. The diffusion constant only enters higher order terms. In the case when the "high peaks" of $\xi(\cdot)$ and hence that of $u(t,\cdot)$ are concentrated at single lattice sites we expect that

$$\log\langle u(t,x)^p\rangle = G(pt) - 2\kappa dpt + \bar{o}(t), \tag{3.4}$$

i.e. we are close to the lower bound in (3.2). In this case it is "most profitable" for the random walk in the Feynman-Kac representation (2.1) to stay the overwhelming part of time at a site with a single high potential peak, cf. also the proof of the lower bound in (3.2). Thereby the random walk will choose the most advantageous among all high peaks as a result of a competition between the amplitude of each such peak and its distance from the origin. On the other hand, if the typical size of the "islands of peaks" of $u(t,\cdot)$ increases in time (due to the appearance of "large islands" of high peaks of the potential $\xi(\cdot)$), then there is no necessity for the random walk to stay during a long time period in a fixed bounded region. Thus, in this case we expect that

$$\log\langle u(t,0)^p\rangle = G(pt) - \bar{o}(t) \tag{3.5}$$

which means that we are close to the upper bound in (3.2). Under the assumption that the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are mutually independent, we conjecture the asymptotics (3.4) (respectively (3.5)) to be valid if the tail $\mu(\xi > t)$, $t \to \infty$, decreases slower (respectively faster) than in the case of a double exponential distribution.

The bounds (3.2), (3.3) indicate the presence of intermittency for the family of random fields $\{u(t,x);\, x \in \mathbb{Z}^d\}$ as $t \to \infty$ in the situation when the random potential Ξ is unbounded from above. Indeed, suppose that the cumulant generating function G is finite and $G(t)/t \uparrow \infty$ as $t \uparrow \infty$. Then

$$G(t) \ll G(2t)/2 \ll G(3t)/3 \ll \dots \,. \tag{3.6}$$

This follows from the estimate

$$\begin{aligned} &pG((p+1)t) - (p+1)G(pt) \\ &\quad = \sum_{k=0}^{p-1} \{[G((p+1)t) - G(pt)] - [G((k+1)t) - G(kt)]\} \\ &\quad \geq G((p+1)t) - G(pt) - G(t) \\ &\quad = [G((p+1)t) - G(pt+1)] - [G(t) - G(1)] + [G(pt+1) - G(pt)] - G(1) \\ &\quad \geq G(pt+1) - G(pt) - G(1) \\ &\quad \geq \frac{G(pt)}{pt} - G(1) \to \infty \qquad \text{as } t \to \infty\,. \end{aligned} \tag{3.7}$$

Here we have used the convexity of G and $G(0) = 0$. Unfortunately, in general the bounds (3.2) are too rough to derive from (3.6) intermittency of $u(t,\cdot)$ as $t \to \infty$.

To formulate our result about intermittency, we introduce the functions

$$\Lambda_p(t) = \log\langle u(t,0)^p \rangle, \qquad t \geq 0,\ p \in \mathbb{N}\,.$$

We denote by λ_0 the upper bound of the spectrum of the Anderson Hamiltonian $H = \kappa\Delta + \xi(\cdot)$:

$$\lambda_0 = \sup \mathrm{Sp}(H)\,.$$

Note that λ_0 is non-random if the potential Ξ is ergodic. A random variable will be called degenerate if it is almost surely constant.

Theorem 3.2. *Let the assumptions introduced at the beginning of this section be satisfied. Suppose in addition that the pair $(\xi(\cdot), u_0(\cdot))$ is ergodic and $G(t) < \infty$ for all $t > 0$.*

a) *Under these assumptions the random fields $\{u(t,x);\, x \in \mathbb{Z}^d\}$ are intermittent as $t \to \infty$ if and only if*

$$\lim_{t\to\infty} \left[\frac{\Lambda_2(t)}{2} - \Lambda_1(t)\right] = \infty\,. \tag{3.8}$$

Otherwise

$$\limsup_{t\to\infty} \left[\frac{\Lambda_2(t)}{2} - \Lambda_1(t)\right] < \infty\,. \tag{3.9}$$

b) *Assertion* (3.8) *holds if and only if one of the following two conditions is satisfied:*

(i) $\operatorname{ess\,sup} \xi = \infty$;

(ii) $\operatorname{ess\,sup} \xi < \infty$ *and*

$$\lim_{t\to\infty} \left\langle \mathbb{E}_0 \exp\left\{ \int_0^t [\xi(x(s)) - \lambda_0]\, ds \right\} \right\rangle = 0. \tag{3.10}$$

In particular, this is fulfilled if ξ *is non-degenerate and*

(iii) $\lambda_0 = \operatorname{ess\,sup} \xi$.

Proof. 1. The ergodicity of $(\xi(\cdot), u_0(\cdot))$ implies the ergodicity of the fields $u(t,\cdot)$, $t \geq 0$. According to Theorem 3.1, the finiteness of $G(t)$ for all $t > 0$ is equivalent to the finiteness of the functions Λ_p, $p \in \mathbb{N}$. Together with the remark at the end of Sect. 1.1 this shows that (3.8) implies intermittency of $u(t,\cdot)$ as $t \to \infty$.

2. We show that

$$\begin{aligned} &\left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \right)^p \right\rangle \langle u_0 \rangle^p \\ &\leq \langle u(t,0)^p \rangle \\ &\leq \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \right)^p \right\rangle \langle u_0^p \rangle, \qquad t \geq 0,\ p \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Recall that $u_0(\cdot)$ and $\xi(\cdot)$ are assumed to be independent. To compute expectation $\langle\cdot\rangle$, we can therefore first apply expectation $\langle\cdot\rangle_{u_0}$ with respect to $u_0(\cdot)$ for each fixed realization of $\xi(\cdot)$ and then average over $\xi(\cdot)$. Taking this into account and using Hölder's inequality and Fubini's theorem, we find that

$$\begin{aligned} \langle u(t,0)^p \rangle &= \left\langle \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) \right)^p \right\rangle_{u_0} \right\rangle \\ &\geq \left\langle \left\langle \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t)) \right\rangle_{u_0}^p \right\rangle \\ &= \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \right)^p \right\rangle \langle u_0 \rangle^p . \end{aligned}$$

This is the lower bound in (3.11). To derive the upper bound, we introduce the quasi-transition function

$$q(t,x,y) = \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \delta_y(x(t)), \qquad (t,x,y) \in \mathbb{R}_+ \times \mathbb{Z}^d \times \mathbb{Z}^d . \tag{3.12}$$

Then we obtain

$$
\begin{aligned}
\langle u(t,0)^p\rangle &= \left\langle \left(\sum_z q(t,0,z)u_0(z) \right)^p \right\rangle \\
&= \sum_{z_1,\dots,z_p} \langle q(t,0,z_1)\dots q(t,0,z_p)\rangle \langle u_0(z_1)\dots u_0(z_p)\rangle \\
&\le \sum_{z_1,\dots,z_p} \langle q(t,0,z_1)\dots q(t,0,z_p)\rangle \langle u_0^p\rangle \\
&= \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\,ds \right\} \right)^p \right\rangle \langle u_0^p\rangle .
\end{aligned}
$$

3. Because of (3.11), to decide whether or not (3.8) or (3.9) is fulfilled, we can and will assume without loss of generality that $u_0 \equiv 1$. In this case the following fundamental identity is valid:

$$
\langle u(s+t,0)\rangle = \langle u(s,0)u(t,0)\rangle, \qquad s,t \ge 0 . \tag{3.13}
$$

To prove it, we note that the random quasi-transition function (3.12) is symmetric in x and y, spatially homogeneous, and satisfies the Chapman-Kolmogorov equation. Moreover,

$$
u(t,x) = \sum_y q(t,x,y) .
$$

Consequently,

$$
\begin{aligned}
\langle u(s+t,0)\rangle &= \sum_{y,z} \langle q(s,0,y)q(t,y,z)\rangle \\
&= \sum_{y,z} \langle q(s,y,0)q(t,y,z)\rangle \\
&= \sum_{y,z} \langle q(s,0,-y)q(t,0,z-y)\rangle \\
&= \langle u(s,0)u(t,0)\rangle .
\end{aligned}
$$

4. As a consequence of the fundamental identity (3.13), the functions Λ_p, $p = 1,2$, have the following properties:

$$
\Lambda_2(t) = \Lambda_1(2t), \qquad t \ge 0; \tag{3.14}
$$

$$
\Lambda_1 \text{ is convex and } \Lambda_1(0) = 0 . \tag{3.15}
$$

Assertion (3.14) is obvious from (3.13) for $s = t$. To prove (3.15), it will be enough to check that Λ_1 is continuous and

$$
2\Lambda_1(s+t) \le \Lambda_1(2s) + \Lambda_1(2t) \qquad \text{for all } s,t \ge 0 . \tag{3.16}
$$

An application of Hölder's inequality to the expectation on the right of (3.13) and a repeated application of this identity yield

$$
\langle u(s+t,0)\rangle^2 \le \langle u(s,0)^2\rangle \langle u(t,0)^2\rangle = \langle u(2s,0)\rangle \langle u(2t,0)\rangle .
$$

But this is equivalent to (3.16). It remains to check that the function

$$\langle u(t,0)\rangle = \mathbb{E}_0\left\langle \exp\left\{ \int_0^t \xi(x(s))\,ds \right\}\right\rangle, \qquad t \in [0,T],$$

is continuous for arbitrary $T > 0$. The function under the expectation signs on the right is continuous and bounded from above by

$$\sum_{x\in S_T} \exp\{T\xi^+(x)\},$$

where $S_T = \{x(t) : t \in [0,T]\}$. Since

$$\begin{aligned}\mathbb{E}_0\left\langle \sum_{x\in S_T} \exp\{T\xi^+(x)\}\right\rangle &= \mathbb{E}_0\,|S_T|\,\langle \exp\{T\xi^+(0)\}\rangle \\ &\le \mathbb{E}_0\,|S_T|\,(e^{G(T)}+1) < \infty,\end{aligned}$$

the desired continuity follows from Lebesgue's dominated convergence theorem.
5. We next show that condition (i) implies (3.8). We know from (3.1) and (3.2) that condition (i) is equivalent to

$$\lim_{t\to\infty} \Lambda_1(t)/t = \infty. \tag{3.17}$$

Proceeding as in the proof of (3.7) for $p = 1$, we conclude from (3.14), (3.15) and (3.17) that

$$\Lambda_2(t) - 2\Lambda_1(t) = \Lambda_1(2t) - 2\Lambda_1(t) \ge \frac{\Lambda_1(t)}{t} - \Lambda_1(1) \to \infty \qquad \text{as } t \to \infty,$$

i.e. we arrive at (3.8).
6. It remains to consider the case ess sup $\xi < \infty$. We claim that

$$\lim_{t\to\infty} \Lambda_1(t)/t = \lambda_0 < \infty. \tag{3.18}$$

Note that $\Lambda_1(t)/t$ is non-decreasing because of (3.15). Hence the limit on the left exists. It is obvious from the Feynman-Kac formula for $u(t,0)$ and the definition of Λ_1 that this limit does not exceed ess sup ξ. In particular, it is finite.

Let $\{E(\lambda);\, \lambda \in \mathbb{R}\}$ and $\{P_t;\, t \ge 0\}$ denote, respectively, the spectral family and the semi-group associated with the Hamiltonian H, and let $(\cdot,\cdot)$ be the inner product in $l^2(\mathbb{Z}^d)$. Then we obtain

$$\mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\,ds \right\} \ge (P_t\delta_0,\delta_0) = \int_{-\infty}^{\lambda_0} e^{\lambda t}\, d(E(\lambda)\delta_0,\delta_0).$$

Since $((E(\lambda_0) - E(\lambda))\,\delta_0,\delta_0) > 0$ with positive probability for each $\lambda < \lambda_0$, we conclude from this that

$$\lim_{t\to\infty} \Lambda_1(t)/t \ge \lambda_0.$$

To derive the opposite inequality, we observe that

$$\left\langle \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \right\rangle \sim \left\langle \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \mathbb{1}_{B_t}(x(t)) \right\rangle, \tag{3.19}$$

where B_t denotes the ball in $\mathbb{Z}^d$ with center 0 and radius t and $\mathbb{1}_{B_t}$ is the indicator function of B_t. This easily follows from (2.17) and the boundedness from above of ξ. But

$$\begin{aligned} \mathbb{E}_0 \exp & \left\{ \int_0^t \xi(x(s))\, ds \right\} \mathbb{1}_{B_t}(x(t)) \\ & = (P_t \mathbb{1}_{B_t}, \delta_0) \le (P_t \mathbb{1}_{B_t}, \mathbb{1}_{B_t}) \\ & \le e^{\lambda_0 t} |B_t| \le \text{const}\, t^d\, e^{\lambda_0 t}. \end{aligned} \tag{3.20}$$

Here we have used the spectral representation of $(P_t \mathbb{1}_{B_t}, \mathbb{1}_{B_t})$. Combining (3.19) with (3.20), we arrive at

$$\lim_{t\to\infty} \Lambda_1(t)/t \le \lambda_0 .$$

7. Besides of ess sup $\xi < \infty$ we assume without loss of generality that $\lambda_0 = 0$. (Otherwise one considers the functions $\Lambda_p(t) - \lambda_0 pt$ instead of $\Lambda_p(t)$ ($p = 1, 2$) corresponding to the potential $\xi(\cdot) - \lambda_0$.) Then, because of (3.18) and convexity, Λ_1 is non-decreasing and $\Lambda_1(t) \le 0$ for all t. Hence the limit

$$\lim_{t\to\infty} \Lambda_1(t) = \varrho$$

exists. Moreover, condition (3.10) is equivalent to $\varrho = -\infty$.

Let us first consider the case $\varrho = -\infty$. Because of (3.14) and (3.15), the function $\Lambda_2(t) - 2\Lambda_1(t)$ equals $\Lambda_1(2t) - 2\Lambda_1(t)$ and the latter is non-decreasing. Hence the limit

$$\lim_{t\to\infty} [\Lambda_2(t) - 2\Lambda_1(t)] = c$$

exists. To derive (3.8), we must show that $c = \infty$. Suppose the contrary. Then we obtain

$$\Lambda_1(2^n t) \le 2^n \Lambda_1(t) + (2^n - 1)c$$

successively for $n = 1, 2, \ldots$ Dividing both sides by $2^n t$ and letting $n \to \infty$, we conclude from this that

$$\lim_{s\to\infty} \frac{\Lambda_1(s)}{s} \le \frac{\Lambda_1(t) + c}{t}$$

for every $t > 0$. Since $\varrho = -\infty$, the expression on the right is negative for large t. But this contradicts (3.18) for $\lambda_0 = 0$. Hence $c = \infty$.

In the case $\varrho > -\infty$ we obtain

$$\lim_{t\to\infty} [\Lambda_2(t) - 2\Lambda_1(t)] = \lim_{t\to\infty} [\Lambda_1(2t) - 2\Lambda_1(t)] = -\varrho < \infty,$$

which yields (3.9) (also in the general case $u_0 \not\equiv 1$).

8. It remains to derive condition (ii) from condition (iii) in the case $\lambda_0 < \infty$. We know from (3.14) that

$$\left\langle \mathbb{E}_0 \exp\left\{ \int_0^{2t} [\xi(x(s)) - \lambda_0]\, ds \right\} \right\rangle = \left\langle \left(\mathbb{E}_0 \exp\left\{ \int_0^{t} [\xi(x(s)) - \lambda_0]\, ds \right\} \right)^2 \right\rangle .$$

Passing to the limit as $t \to \infty$ and taking into account condition (iii), we find that

$$\langle \eta \rangle = \langle \eta^2 \rangle ,$$

where

$$\eta = \mathbb{E}_0 \exp\left\{ \int_0^{\infty} [\xi(x(s)) - \lambda_0]\, ds \right\} .$$

Together with $0 \le \eta < 1$ this implies that $\eta = 0$ a.s., and we arrive at (3.10). □

Suppose that the field $\{\xi(x);\, x \in \mathbb{Z}^d\}$ is ergodic and bounded from above. Then it is easy so see that condition (iii) of Theorem 3.2 is fulfilled iff

$$\mu(\xi(x) > \lambda \text{ for } |x| < R) > 0 \qquad \text{for all } \lambda < \operatorname{ess\,sup} \xi \text{ and } R > 0 . \tag{3.21}$$

Let us consider several erxamples.

1. Let $\{\xi(x);\, x \in \mathbb{Z}^d\}$ be a field of non-degenerate i.i.d. random variables having finite exponential moments. Suppose that the initial datum u_0 satisfies the conditions formulated at the beginning of this section and $(\xi(\cdot), u_0(\cdot))$ is ergodic (e.g. $u_0 \equiv 1$). In this case (3.21) is satisfied and, consequently, the fields $\{u(t,x);\, x \in \mathbb{Z}^d\}$ are intermittent.

2. Let $\{\eta(x);\, x \in \mathbb{Z}^d\}$ be a homogeneous Gaussian random field. Suppose that this field is ergodic and its finite dimensional distributions are non-degenerate (i.e. equivalent to the Lebesgue measure). This is satisfied if the associated spectral measure is continuous and has an absolutely continuous component. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function which is bounded from above Lebesgue-almost everywhere:

$$h_0 = \operatorname{ess\,sup} h < \infty .$$

Define

$$\xi(x) = h(\eta(x)), \qquad x \in \mathbb{Z}^d . \tag{3.22}$$

This potential is ergodic, $\operatorname{ess\,sup} \xi = h_0$, and (3.21) is fulfilled.

3. Consider the shock noise

$$\eta(x) = \sum_{y \in \mathbb{Z}^d} k(x-y)\zeta(y), \qquad x \in \mathbb{Z}^d ,$$

where $\zeta(y)$, $y \in \mathbb{Z}^d$, are independent Poissonian random variables with parameter α, $\Sigma |k(x)| < \infty$, $\Sigma k(x) > 0$. Let $h : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function which is bounded from above, and define the potential $\xi(\cdot)$ by (3.22). Then it is again not difficult to show that (3.21) is fulfilled.

4. Let $\{\eta(x);\ x \in \mathbb{Z}^d\}$ be an ergodic random field such that a.s. $a < \eta(0) < b$ for some positive constants a and b. Define

$$\xi(x) = -\kappa \frac{\Delta\eta(x)}{\eta(x)}.$$

Since η is a generalized eigenfunctions of H to the eigenvalue 0, we have

$$\eta(0) = \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(s))\,ds\right\}\eta(x(t)), \qquad t \geq 0.$$

From this we conclude that $\lambda_0 = 0$ and condition (ii) of Theorem 3.2 is violated. Hence, the fields $\{u(t,x);\ x \in \mathbb{Z}^d\}$ are not intermittent. Moreover,

$$\limsup_{t\to\infty}\left[\frac{\Lambda_{p+1}(t)}{p+1} - \frac{\Lambda_p(t)}{p}\right] < \infty$$

for *all* $p \in \mathbb{N}$.

5. We consider the previous example in the one-dimensional case. Instead of $a < \eta(0) < b$ we assume that $\eta(0) \geq 1$ a.s. and $\langle \eta(0)^p\rangle < \infty$, $\langle\eta(0)^{p+1}\rangle = \infty$ for some $p \geq 2$. Under these suppositions we obtain

$$\limsup_{t\to\infty}\left[\frac{\Lambda_{q+1}(t)}{q+1} - \frac{\Lambda_q(t)}{q}\right] \begin{cases} < \infty & \text{for } q < p, \\ = \infty & \text{for } q \geq p. \end{cases} \tag{3.23}$$

To prove this, we assume for simplicity that $\kappa = 1$ and $u_0 \equiv 1$. We show that a.s.

$$u(t,0) \leq \eta(0) \qquad \text{for all } t \tag{3.24}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t u(s,0)\,ds = c_0\eta(0), \tag{3.25}$$

where $c_0 = \langle\eta(0)\rangle/\langle\eta^2(0)\rangle$. The bound (3.24) follows from the observation that η is a time-independent solution of Eq. (0.1) and $u_0(x) \leq \eta(x)$ for all x. We introduce the functions

$$h_c(x) = \begin{cases} \displaystyle\sum_{0<y\leq x}\frac{1}{\eta(y-1)\eta(y)}\sum_{0\leq z<y}[\eta(z) - c\eta^2(z)] & \text{for } x \geq 0, \\ \displaystyle\sum_{x<y\leq 0}\frac{1}{\eta(y-1)\eta(y)}\sum_{y\leq z<0}[\eta(z) - c\eta^2(z)] & \text{for } x \leq 0, \end{cases}$$

$c \in \mathbb{R}$ The proof of (3.25) relies on the following two facts. Firstly, h_c satisfies

$$Gh_c = \frac{1}{\eta} - c,$$

where the operator G is defined by

$$Gf(x) = \frac{\eta(x+1)}{\eta(x)}[f(x+1) - f(x)] + \frac{\eta(x-1)}{\eta(x)}[f(x-1) - f(x)].$$

Secondly, applying Birkhoff's ergodic theorem, we find that a.s. h_c is bounded from below (above) for $c < c_0$ ($c > c_0$). After deriving the equation for $v(t,x) = \int_0^t u(s,x)\,ds/\eta(x)$, one easily checks that the function w_c defined by

$$\frac{1}{t}\int_0^t u(s,x)\,ds = \left[c - \frac{h_c(x)}{t} + \frac{w_c(t,x)}{t}\right]\eta(x) \tag{3.26}$$

satisfies

$$\frac{\partial w_c}{\partial t} = Gw_c, \qquad w_c\,|_{t=0} = h_c\,.$$

From this (and the bounds $h_c(x) - ct \le w_c(t,x) \le h_c(x) - (c-1)t$) we conclude that $w_c(t,0)$ remains bounded from below (above) as $t \to \infty$ for $c < c_0$ ($c > c_0$). Assertion (3.25) is a consequence of this and (3.26).

Since Λ_1 is convex (step 4 of the proof of Theorem 3.2), the limit of $\Lambda_1(t)$ as $t \to \infty$ exists. Taking this into account, we conclude from (3.24) and (3.25) that

$$\lim_{t\to\infty} \Lambda_1(t) = \lim_{t\to\infty} \log\left\langle \frac{1}{t}\int_0^t u(s,0)\,ds \right\rangle = \log(c_0\langle\eta(0)\rangle)\,.$$

Hence, since $\Lambda_q(t)/q$ is non-decreasing in q, all functions Λ_q are bounded from below as $t \to \infty$. From (3.24) and the moment assumptions on $\eta(0)$ we also know that

$$\limsup_{t\to\infty} \Lambda_q(t) < \infty \qquad \text{for } q \le p\,, \tag{3.27}$$

and we arrive at the first half of (3.23). Using (3.25), we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \langle u(s,0)^{p+1}\rangle\,ds \ge \left\langle \left(\liminf_{t\to\infty} \frac{1}{t}\int_0^t u(s,0)\,ds \right)^{p+1} \right\rangle = \infty\,.$$

Consequently,

$$\limsup_{t\to\infty} \Lambda_{p+1}(t) = \infty\,. \tag{3.28}$$

Combining (3.28) with (3.27) and taking into account the remark at the end of Sect. 1.1, we finally get the second half of assertion (3.23).

4. Almost Sure Asymptotics

The objective of this section is to derive the rough (logarithmic) asymptotics of the solution $u(t,x)$ to the random Cauchy problem (0.1). Our approach is based on the conception of "strong centers" suggested by Ya. B. Zel'dovich and described in [13, 14]. Let us explain its essence on a particle level in the supercritical case when $u(0,x) = \delta_0(x)$, i.e. when at time zero there is a single particle in the system located at the origin (cf. Sect. 1.2). The overwhelming portion of the multitude of particles occupying the origin at time $t \gg 1$ has been generated at one of that sites x at which $\xi(x) > a(t) \gg 1$ (strong centers). Here $a(t)$ denotes an unboundedly increasing function depending on the tail $\mu(\xi > r)$, $r \to \infty$. Some of the descendants of the original particle will reach such a strong center before time t, create at this site new particles with high reproduction rate, and after that, due to diffusion, a part of the latter will return to the origin until time t. This process is accompanied by a competition between two factors: the greater the distance away from 0 the stronger the occurring centers (high local maxima of the potential) but the smaller the probability for a single particle (and each of its descendants) to move from the origin to such a center and back.

Quantitatively, the asymptotics of $u(t,0)$ as $t \to \infty$ depends on the ratio between the growth of the function

$$M(R) = \max_{|x|\le R} \xi(x)$$

and the decay of large deviation probabilities of the random walk $x(\cdot)$ in the spirit of the Lemmas 2.5 and 2.4, respectively.

We distinguish between four qualitatively different zones which may be described in terms of the growth rate of $M(R)$ as $R \to \infty$ and, in the case of i.i.d. random variables $\xi(x)$, $x \in \mathbb{Z}^d$, also in terms of the speed of decay of $\mu(\xi > r)$ as $r \to \infty$:

a) $M(R) = \underline{O}(R^\alpha)$, $0 < \alpha < 1$. Roughly speaking, in this case the upper tail of the distribution of $\xi = \xi(0)$ has polynomial decay: An exact (non-random) asymptotics of $M(R)$ is not available. One must take into account strong centers at a distance $R(t) = \underline{O}(t^\beta)$, $\beta \approx (1-\alpha)^{-1} > 1$, from the origin, i.e. in the domain of superlarge deviations.
b) $M(R) = \underline{O}((\log R)^\alpha)$, $\alpha > 0$. This is the situation of fractional exponential tails, $\log(1 - F(r))^{-1} = \underline{O}(r^{1/\alpha})$. The main contribution to $u(t,0)$ is given by strong centers in the standard large deviation zone $R(t) = \underline{O}(t)$. This includes the Gaussian case popular in physics [14, 15].
c) $M(R) = \underline{O}((\log\log R)^\alpha)$, $\alpha > 0$. To derive the rough asymptotics of $u(t,0)$, it suffices to take into account strong centers in an arbitrary zone $R(t) = \underline{O}(t^\beta)$, $0 < \beta < 1$, of moderate deviations. For $\alpha \le 1$ the main contribution to $u(t,0)$ does no longer come from isolated strong centers but results from slightly lower local maxima of the potential which are surrounded by other high peaks of comparable amplitude. Because of its intermediate stage between "essentially unbounded" and bounded potentials, this zone is most interesting from the point of view of localization theory and, in particular, in connection with the estimation of the Lifshitz tails of the integrated density of states [11].
d) $M(R) = \underline{O}(1)$, i.e. the potential $\xi(\cdot)$ is a.s. bounded from above. In this case the strong centers form "large" (i.e. unboundedly increasing as $t \to \infty$) islands. Of course, the structure of that islands which give the main contribution to $u(t,0)$ depends on the detailed behavior of the distribution of ξ near the upper bound λ_0 of the spectrum of $H = \kappa\Delta + \xi(\cdot)$. But, at least in the i.i.d. case, these details will only appear in the higher order terms of the growth of $u(t,0)$ as $t \to \infty$. Concerning the rough asymptotics, the answer is simple:

$$\lim_{t\to\infty} \frac{\log u(t,x)}{t} = \lambda_0 \qquad \text{a.s. for each } x \in \mathbb{Z}^d. \tag{4.1}$$

If the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are independent, then $\lambda_0 = \operatorname{ess\,sup} \xi$. The proof of these facts is left to the reader.

We now turn to the formulation of the precise result. To this end we assume that the potential $\Xi = \{\xi(x);\, x \in \mathbb{Z}^d\}$ consists of i.i.d. random variables which have a *continuous* distribution function and are *unbounded from above* (i.e. $F(r) < 1$ for all $r \in \mathbb{R}$). We consider the nonnegative solution u to the random Cauchy problem (0.1) for a.s. not identically vanishing nonnegative initial data u_0 satisfying

$$\limsup_{|x|\to\infty} \frac{\log \log_+ u_0(x)}{\log |x|} < 1 \qquad \text{a.s.} \tag{4.2}$$

Condition (4.2) is slightly stronger than assumption ($\mathscr{U}_0$) in Sect. 2.1. In the one-dimensional case we further impose condition (2.4) on the lower tail behavior of the potential Ξ. To formulate our result, we introduce the non-decreasing continuous function

$$\varphi(r) = \log \frac{1}{1 - F(r)}, \qquad r \in \mathbb{R},$$

and its inverse

$$\psi(s) = \min\{r : \varphi(r) \geq s\}; \qquad s > 0.$$

The function ψ is strictly increasing, and $\varphi(\psi(s)) = s$ for each $s > 0$.

Theorem 4.1. *Under the above assumptions, with probability one for each $x \in \mathbb{Z}^d$ the nonnegative solution $u(t,x)$ to the random Cauchy problem (0.1) has the following asymptotic behavior as $t \to \infty$.*

a) *If*

$$\lim_{s\to\infty} \frac{\log \psi(s)}{s} = 1/\gamma \tag{4.3}$$

for some $\gamma > d$, then

$$\varphi\left(\frac{\log u(t,x)}{t}\right) \sim \frac{\gamma}{\gamma - d}\, d \log t\,.$$

b) *If*

$$\lim_{s\to\infty} \frac{\log \psi(s)}{s} = 0 \tag{4.4}$$

and

$$\lim_{s\to\infty} [\psi(\theta s) - \psi(s)] = \infty \quad \textit{for each } \theta > 1\,, \tag{4.5}$$

then

$$\varphi\left(\frac{\log u(t,x)}{t}\right) \sim d \log t\,.$$

c) *If*

$$\lim_{\theta\downarrow 1} \limsup_{s\to\infty} [\psi(\theta s) - \psi(s)] = 0\,, \tag{4.6}$$

then

$$\frac{\log u(t,x)}{t} = \psi(d \log t) + \underline{O}(1)\,.$$

Note that the subdivision of Theorem 4.1 into the cases a)–c) does conform with the classification introduced before. In all three cases $1-F(r) \leq \text{const}\, r^{-\gamma}$ for some $\gamma > d$ which implies (2.2) and therefore existence of a unique nonnegative solution u to (0.1). The assumption $\gamma > d$ in (4.3) cannot be removed, since $0 < \gamma < d$ implies (2.3) in which case the solution does not exist.

Before proving Theorem 4.1, we derive two lemmas on the growth of the maximum $M(R)$ and the decay of large deviation probabilities for the random walk $x(\cdot)$.

The functions φ and ψ have the following remarkable property. If ξ has distribution function F, then $\eta = \varphi(\xi)$ is exponentially distributed with mean one. If η is exponentially distributed with mean one, then $\xi = \psi(\eta)$ has distribution function F. (Here the continuity of F is needed.)

In dimension $d \geq 2$ we consider the level sets A_α^+ and A_α^- introduced in Sect. 2.4 and choose α so negative that the percolation assumptions (A1) and (A2) of that section are satisfied. For each $n \in \mathbb{N}$ we set $W_n^+ = \{x \in W^+ : |x| \leq n\}$, where W^+ denotes the unique infinite cluster in A_α^+.

Lemma 4.2. a) *We have* a.s.

$$\varphi\Big(\max_{|x|\leq n} \xi(x)\Big) \sim d \log n \qquad \text{as } n \to \infty .$$

b) *If* $d \geq 2$, *then* a.s.

$$\varphi\Big(\max_{x\in W_n^+} \xi(x)\Big) \sim d \log n \qquad \text{as } n \to \infty .$$

Proof. a) Since φ is non-decreasing, it will be enough to show that the field $\eta(x) = \varphi(\xi(x))$, $x \in \mathbb{Z}^d$, of independent exponentially distributed random variables with mean one satisfied a.s.

$$\max_{|x|\leq n} \eta(x) \sim d \log n .$$

Since the maximum on the left is non-decreasing in n and $\log 2^n \sim \log 2^{n+1}$, it suffices to check that for each $\varepsilon \in (0, 1)$ a.s.

$$\max_{|x|\leq 2^n} \eta(x) > (1+\varepsilon)\, d \log 2^n \qquad \text{only for finitely many } n$$

and

$$\max_{|x|\leq 2^n} \eta(x) < (1-\varepsilon)\, d \log 2^n \qquad \text{only for finitely many } n .$$

But this follows from the Borel-Cantelli lemma by standard estimates.
b) Because of assertion a), it only remains to check that

$$\max_{x\in W_{2^n}^+} \eta(x) < (1-\varepsilon)\, d \log 2^n \qquad \text{only for finitely many } n, \tag{4.7}$$

a.s. for each $\varepsilon \in (0, 1)$. Similarly to the proof of Lemma 2.7, we find mutually independent random variables $\eta_-(x)$, $\eta_+(x)$, $\zeta(x)$, $x \in \mathbb{Z}^d$, such that a.s.

$$\eta(x) = (1-\zeta(x))\eta_-(x) + \zeta(x)\eta_+(x), \qquad x \in \mathbb{Z}^d ,$$

$\eta_-(x) \le \varphi(\alpha)$, $\eta_+(x) > \varphi(\alpha)$, $\mu(\eta_+(x) > r) = \exp\{-(r - \varphi(\alpha))\}$ for $r \ge \varphi(\alpha)$, $\zeta(x)$ attains the values 0 and 1 with probability $1 - e^{-\varphi(\alpha)}$ and $e^{-\varphi(\alpha)}$, respectively, and $A_\alpha^+ = \{x : \zeta(x) = 1\}$. Since $\eta(x) = \eta_+(x)$ for $x \in W^+$ and $\{\eta_+(x); x \in \mathbb{Z}^d\}$ and W^+ are independent, we may apply the Borel-Cantelli lemma with respect to the conditional law of μ given $\{\zeta(x); x \in \mathbb{Z}^d\}$ to reduce the proof of (4.7) to the verification of

$$\sum_n (1 - e^{\varphi(\alpha)} 2^{-(1-\varepsilon)dn})^{|W_{2^n}^+|} < \infty \qquad \text{a.s.}$$

But this follows from the estimate

$$(1 - e^{\varphi(\alpha)} 2^{-(1-\varepsilon)dn})^{|W_{2^n}^+|} \le \exp\{-2^{-(1-\varepsilon)dn} |W_{2^n}^+|\}$$

and the observation that, according to Birkhoff's ergodic theorem,

$$|W_{2^n}^+| \sim C\, 2^{nd}$$

a.s. as $n \to \infty$ for some constant $C > 0$. □

As before, let $(x(t), \mathbb{P}_x)$ denote symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$, and let $N(t)$ be the number of jumps of $x(\cdot)$ during the time interval $[0, t]$. By τ_x and $\tau(R)$ we denote the first hitting times of the site $x \in \mathbb{Z}^d$ and the set $\{y \in \mathbb{Z}^d : |y| \ge R\}$, respectively. Recall that in dimension $d = 1$ assumption (2.4) is imposed.

Lemma 4.3. a) *For arbitrary $R > 0$ and $t > 0$ we have*

$$\mathbb{P}_0(\tau(R) \le t) \le 2^{d+1} \exp\left\{ - R \log \frac{R}{\kappa d t} + R \right\}.$$

b) *If $d \ge 2$, then there exists $\varrho > 1$ such that for each $t > 0$* a.s.

$$\mathbb{E}_0 \exp\left\{ \int_0^{\tau_x} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_x \le t) \ge \exp\{-\varrho |x| \log |x|\} \tag{4.8}$$

and

$$\mathbb{E}_x \exp\left\{ \int_0^{\tau_0} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_0 \le t) \ge \exp\{-\varrho |x| \log |x|\} \tag{4.9}$$

for all sufficiently large $x \in W^+$. In dimension $d = 1$, (4.8) and (4.9) hold for all sufficiently large $|x|$ provided that $\langle \log(1 + \xi^-)\rangle < \infty$.

Proof. a) From the proof of Lemma 2.4 we know that

$$\begin{aligned} \mathbb{P}_0(\tau(R) \le t) &= \mathbb{P}_0\Big(\max_{s \in [0,t]} |x(s)| \ge R \Big) \\ &\le 2^{d+1} \exp\{-R\beta + 2\kappa d t [\cosh \beta - 1]\} \end{aligned}$$

for all positive β, R, t. Choosing

$$\beta = \log \frac{R}{\kappa d t},$$

we get the desired upper bound for $R > \kappa dt$. For $R \le \kappa dt$ the assertion is trivial.

b) We only prove (4.8). The proof of (4.9) is similar. We first consider the case $d \ge 2$. Let z by a (random) site in the infinite cluster W^+. Then an application of the strong Markov property with respect to τ_z yields

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ \int_0^{\tau_x} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_x \le t) \\
&\quad \ge \mathbb{E}_0 \exp\left\{ \int_0^{\tau_z} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_z = \tau(|z|) \le t/2) \\
&\quad \times \mathbb{E}_z \exp\left\{ \int_0^{\tau_x} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_x \le t/2)
\end{aligned}$$

for all x with $|x| > |z|$. Since the first factor on the right is positive and does not depend on x, it only remains to estimate the second one. Let γ be a path in W^+ joining z with x and having length $|\gamma| = d_{W^+}(z,x)$. Note that $\xi(x) \ge -|\alpha|$ along γ. Let $\sigma_0, \sigma_1, \sigma_2, \ldots$ denote the waiting times of the random walk between consecutive jumps. To obtain the desired lower bound, we take only into account the contribution of the random walk along the path γ:

$$\begin{aligned}
&\mathbb{E}_z \exp\left\{ \int_0^{\tau_x} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_x \le t/2) \\
&\quad \ge (2d)^{-|\gamma|} \exp\{-|\alpha|\, t/2\}\, \mathbb{P}_z(\sigma_0 + \ldots + \sigma_{|\gamma|-1} \le t/2) \\
&\quad \ge (2d)^{-|\gamma|} \exp\{-|\alpha|\, t/2\}\, \mathbb{P}_z(N(t/2) = |\gamma|) \\
&\quad = \exp\{-|\alpha|\, t/2\} \frac{(\kappa t/2)^{|\gamma|}}{|\gamma|!}\, e^{-\kappa dt} \\
&\quad \ge \exp\{-|\gamma| \log|\gamma| + |\gamma| \log(\kappa t/2) - (|\alpha| + 2\kappa d)\, t/2\}\,.
\end{aligned}$$

Here we have used that $N(t/2)$ is Poissonian with parameter κdt. Assertion (4.8) now easily follows if we take into account that, according to Lemma 2.8, there exists $\varrho > 1$ such that a.s.

$$|\gamma| = d_{W^+}(z,x) < \frac{\varrho}{2}\, |x|$$

for all sufficiently large $x \in W^+$.

In dimension $d = 1$ it suffices to consider positive sites x. For $x > t$ we obtain

$$\begin{aligned}
&\mathbb{E}_0 \exp\left\{ \int_0^{\tau_x} \xi(x(s))\, ds \right\} \mathbb{1}(\tau_x \le t) \\
&\ge 2^{-x}\, \mathbb{E}_0 \exp\left\{ -\sum_{k=0}^{x-1} \sigma_k \xi^-(k) \right\} \mathbb{1}\left(\sum_{k=0}^{x-1} \sigma_k \le t \right) \\
&\ge 2^{-x} e^{-t}\, \mathbb{P}_0\left(\sigma_k \le \frac{t}{x} \frac{1}{1+\xi^-(k)} \quad \text{for } 0 \le k < x \right) \\
&\ge 2^{-x} e^{-t} \sum_{k=0}^{x-1} \left(\frac{\kappa t}{x} \frac{1}{1+\xi^-(k)} \right) \\
&= \exp\left\{ -x \log \frac{2x}{\kappa t} - t - \sum_{k=0}^{x-1} \log(1+\xi^-(k)) \right\},
\end{aligned}$$

where we have used that the random variables $\sigma_0, \sigma_1, \dots$ are independent and exponentially distributed with parameter 2κ. Because of assumption (2.4), the strong law of large numbers yields that a.s. the sum on the right is of order $\underline{O}(x)$ as $x \to \infty$, and we arrive at (4.8) for large x. $\square$

Proof of Theorem 4.1. It suffices to consider $x = 0$.
a) *Upper bound.* Given a sequence (R_n) with $0 < R_n \uparrow \infty$, we apply the Feynman-Kac formula (2.1) to obtain

$$\begin{aligned}
u(t,0) &= \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t))\, \mathbb{1}(\tau(R_0) > t) \\
&\quad + \sum_{n=1}^{\infty} \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} u_0(x(t))\, \mathbb{1}(\tau(R_{n-1}) \ge t < \tau(R_n)).
\end{aligned}$$

Estimating the integrals in the exponents and $u_0(x(t))$ by $t \max\limits_{|x|\le R_n} \xi(x)$ and $\max\limits_{|x|\le R_n} u_0(x)$ $(n = 0, 1, 2, \dots)$, respectively, and applying Lemma 4.3 a), we arrive at

$$\begin{aligned}
u(t,0) &\le \exp\left\{ t \max_{|x|\le R_0} \xi(x) + \max_{|x|\le R_0} \log u_0(x) \right\} \\
&\quad + 2^{d+1} \sum_{n=1}^{\infty} \exp\left\{ t \max_{|x|\le R_n} \xi(x) + \max_{|x|\le R_n} \log u_0(x) - R_{n-1} \log \frac{R_{n-1}}{\kappa d t} + R_{n-1} \right\}. \quad (4.10)
\end{aligned}$$

We shall choose $R_n = R_n(t)$ $(n = 0, 1, 2, \dots)$ so that the first term on the right gives the desired bound and the second (remainder) term tends to 0 as $t \to \infty$.

Let us first consider case a). We choose

$$R_n(t) = (n+1)\, t^{(1+\varepsilon)\frac{\gamma}{\gamma-d}},$$

where ε is an (arbitrarily) small positive number. Assumption (4.3) implies that

$$\varphi(r) \sim \gamma \log r . \tag{4.11}$$

Together with Lemma 4.2 a) this gives

$$\max_{|x| \le R} \xi(x) = R^{(1+\bar{o}(1)) \frac{d}{\gamma}} . \tag{4.12}$$

Using this and (4.2), one easily checks that a.s.

$$\max_{|x| \le R_0(t)} \log u_0(x) = \bar{o}\Big(t \max_{|x| \le R_0(t)} \xi(x)\Big) \tag{4.13}$$

provided that $\varepsilon > 0$ is sufficiently small (depending on the concrete realization of $u_0(\cdot)$). It is also not difficult to check that a.s.

$$\begin{aligned} \exp\Big\{ t \max_{|x| \le R_n(t)} \xi(x) + \max_{|x| \le R_n(t)} \log u_0(x) \\ - R_{n-1}(t) \log \frac{R_{n-1}(t)}{\kappa d t} + R_{n-1}(t) \Big\} \le e^{-nt} \end{aligned} \tag{4.14}$$

for sufficiently large t and all $n = 1, 2, \ldots$ Substituting (4.13) and (4.14) in (4.10), we find that a.s.

$$\frac{\log u(t,0)}{t} \le \max_{|x| \le R_0(t)} \xi(x) \, (1 + \bar{o}(1)) .$$

From this, (4.11) and Lemma 4.2 a) we conclude that

$$\varphi\Big(\frac{\log u(t,0)}{t}\Big) \le \varphi\Big(\max_{|x| \le R_0(t)} \xi(x)\Big) \, (1 + \bar{o}(1)) \sim d \log R_0(t) .$$

Since $\varepsilon > 0$ can be chosen arbitrarily small in the definition of $R_0(t)$, this yields the desired upper bound.

In case b) we choose

$$R_n(t) = (n+1) \, t^{1+\varepsilon}$$

with small (random) $\varepsilon > 0$. Then

$$\max_{|x| \le R_0(t)} \log u_0(x) = \bar{o}(t) . \tag{4.15}$$

Lemma 4.2 a) and assumption (4.4) together imply that a.s.

$$\max_{|x| \le R} \xi(x) = \bar{o}(R^\delta) \qquad \text{for each } \delta > 0 . \tag{4.16}$$

This and (4.2) together again imply (4.14). Substituting (4.14) and (4.15) in (4.10), we arrive at

$$\frac{\log u(t,0)}{t} \le \max_{|x| \le R_0(t)} \xi(x) + \bar{o}(1) \qquad \text{a.s.}$$

Together with assumption (4.5) and Lemma 4.2 a) we conclude from this that a.s.

$$\varphi\left(\frac{\log u(t,0)}{t}\right) \leq \varphi\Big(\max_{|x|\leq R_0(t)} \xi(x)\Big)\,(1+\bar{o}(1)) \sim d\,\log R_0(t)\,,$$

which yields the desired upper bound.

It remains to consider case c). We first note that assumption (4.6) implies

$$\limsup_{s\to\infty} \frac{\psi(s)}{\log s} < \infty\,. \tag{4.17}$$

Combining Lemma 4.2 a) with (4.6), we find that a.s.

$$\max_{|x|\leq R} \xi(x) = \psi(d\,\log R) + \underline{O}(1)\,. \tag{4.18}$$

In particular, a.s.

$$\max_{|x|\leq R} \xi(x) \leq C\,\log\log R \tag{4.19}$$

for large R, where C denotes a positive constant. We choose

$$R_n(t) = (n+1)\,t\,\log\log t\,.$$

Then we again obtain (4.15) and, because of this and (4.19), also inequality (4.14). Substituting (4.14), (4.15) and (4.18) in (4.10) and taking into account (4.6), we finally arrive at

$$\frac{\log u(t,0)}{t} \leq \psi(d\,\log R_0(t)) + \underline{O}(1) = \psi(d\,\log t) + \underline{O}(1) \qquad \text{a.s.}$$

b) *Lower bound.* If $d \geq 2$, then we choose for each (sufficiently large) $R > 0$ a site $x_R \in W_R^+$ so that

$$\xi(x_R) = \max_{x\in W_R^+} \xi(x)\,.$$

If $d = 1$, then we take x_R from $\{x \in \mathbb{Z} : |x| \leq R\}$ instead of W_R^+. In both cases Lemma 4.2 tells us that a.s.

$$\varphi(\xi(x_R)) \sim d\,\log R\,. \tag{4.20}$$

We choose a random site $z \in \mathbb{Z}^d$ so that $u_0(z) > 0$ a.s. Applying twice the strong Markov property to the Feynman-Kac formula (2.1), we obtain for $t > 3$:

$$
\begin{aligned}
u(t,0) \geq\ & \mathbb{E}_0 \exp\left\{ \int_0^{\tau_{x_R}} \xi(x(s))\, ds \right\} 1\!\!1(\tau_{x_R} \leq 1) \\
& \times \mathbb{E}_{x_R} \exp\left\{ \int_0^{t-3} \xi(x(s))\, ds \right\} 1\!\!1(x(s) = x_R \text{ for } s \in [0, t-3]) \\
& \times \mathbb{E}_{x_R} \exp\left\{ \int_0^{\tau_0} \xi(x(s))\, ds \right\} 1\!\!1(\tau_0 \leq 1) \\
& \times \inf_{1 \leq r \leq 3} \mathbb{E}_0 \exp\left\{ \int_0^{r} \xi(x(s))\, ds \right\} \delta_z(x(r)) u_0(z) \\
\geq\ & C \exp\{(t-3)\,(\xi(x_R) - 2\kappa d)\}\, \mathbb{E}_0 \exp\left\{ \int_0^{\tau_{x_R}} \xi(x(s))\, ds \right\} 1\!\!1(\tau_{x_R} \leq 1) \\
& \times \mathbb{E}_{x_R} \exp\left\{ \int_0^{\tau_0} \xi(x(s))\, ds \right\} 1\!\!1(\tau_0 \leq 1),
\end{aligned}
$$

where C denotes an a.s. positive random constant. Combining this with Lemma 4.3 b), we find that a.s.

$$
u(t,0) \geq C \exp\{(t-3)\,(\xi(x_R) - 2\kappa d) - 2\varrho R \log R\} \tag{4.21}
$$

for all sufficiently large R and all $t > 3$. To obtain the desired lower bounds we shall choose $R = R(t)$ in a suitable manner.

In case a) we take

$$
R(t) = t^{(1-\varepsilon)\frac{\gamma}{\gamma - d}}
$$

for arbitrarily small $\varepsilon \in (0,1)$. It then follows from (4.21) and (4.12) that a.s.

$$
\frac{\log u(t,0)}{t} \geq \xi(x_{R(t)})\,(1 + \bar{o}(1)).
$$

Together with (4.11) and (4.20) this implies that

$$
\varphi\left(\frac{\log u(t,0)}{t} \right) \geq d(\log R(t))\,(1 + \bar{o}(1)).
$$

Since ε can be taken arbitrarily small, this gives the desired lower bound.

In the cases b) and c) we choose

$$
R(t) = t^{1-\varepsilon},
$$

where again $\varepsilon \in (0,1)$ is small. Because of (4.16), we conclude in both cases from (4.21) that a.s.

$$
\frac{\log u(t,0)}{t} \geq \xi(x_{R(t)}) + \underline{O}(1).
$$

In case b), this together with assumption (4.5) and (4.20) implies that a.s.

$$\varphi\left(\frac{\log u(t,0)}{t}\right) \geq d(\log R(t))\,(1+\bar{o}(1)),$$

and we arrive at the desired lower bound. In case c) we obtain a.s.

$$\frac{\log u(t,0)}{t} \geq \psi(d\,\log R(t)) + \underline{O}(1) = \psi(d\,\log t) + \underline{O}(1).$$

Here we have also used assumption (4.6). □

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