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Symmetry Groups and Non-Abelian Cohomology

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Dedicated to Res Jost and Arthur Wightman

Abstract. We consider the implementation of symmetry groups of automorphisms of an algebra of observables in a reducible representation whose multipliers in general are non-commuting operators in the commutant of the representation. The multipliers obey a non-abelian cocycle relation which generalizes the 2-cohomology of the group. Examples are given from the theory of spin algebras and continuous tensor products. For type *I* representations we show that the multiplier can be chosen to lie in the centre, giving an isomorphism with abelian theory.

1. Introduction

We start with Wigner's formulation of symmetry in quantum mechanics [1], which was used with serene power for the Poincaré group $\mathbb{P}_+^\uparrow$ [2]. The states $\{\boldsymbol{\psi}\}$ of a system are taken to form the unit sphere $\mathscr{H}_1$ in a projective Hilbert space $\mathscr{H}$: so if $\mathscr{H}$ is a Hilbert space and $\psi \in \mathscr{H}$ with $\|\psi\| = 1$, then the state $\boldsymbol{\psi}$ is the unit ray through ψ:

$$\boldsymbol{\psi} = \{\lambda\psi : |\lambda| = 1, \lambda \in \mathbb{C}\} \in \mathscr{H}_1 . \tag{1.1}$$

We furnish $\mathscr{H}_1$ with a transition probability:

$$\mathscr{P}(\boldsymbol{\psi}, \boldsymbol{\phi}) = |\langle \psi, \phi \rangle|^2 , \qquad \psi \in \boldsymbol{\psi},\ \phi \in \boldsymbol{\phi} . \tag{1.2}$$

A symmetry operation is an isometry $\mathbf{U}$, that is, a bijection $\mathbf{U} : \mathscr{H}_1 \to \mathscr{H}_1$ preserving $\mathscr{P}$:

$$\mathscr{P}(\mathbf{U}\boldsymbol{\psi}, \mathbf{U}\boldsymbol{\gamma}) = \mathscr{P}(\boldsymbol{\psi}, \boldsymbol{\phi}) \quad \text{for all} \quad \boldsymbol{\psi}, \boldsymbol{\phi} \in \mathscr{H}_1 . \tag{1.3}$$

The set of isometries is a group, denoted $\mathrm{Aut}\,\mathscr{H}_1$.

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If we have a group of symmetries, G, we obtain a projective representation of G, i.e. a homomorphism $\mathbf{U}: G \to \mathrm{Aut}\mathscr{H}_1$. Each isometry $\mathbf{U}(g)$ can be implemented by a unitary or anti-unitary operator U_g on $\mathscr{H}$ [3], which is ambiguous up to a phase, i.e. an element of $U(1)$. Let us take the unitary case. Then the map $g \to U_g$ is a multiplier representation:

$$U_g U_h = \omega(g,h) U_{gh}, \qquad g,h \in G, \tag{1.4}$$

where $\omega(g,h) \in U(1)$ is the multiplier. The associativity of U_g, U_h, U_k leads to the *cocycle relation*

$$\omega(g,h)\omega(gh,k) = \omega(h,k)\omega(g,hk), \qquad g,h,k \in G. \tag{1.5}$$

If U_g is replaced by

$$U'_g = \mu(g) U_g, \qquad \mu(g) \in U(1), \tag{1.6}$$

we do not change the projective isometry: $\mathbf{U}_g = \mathbf{U}'_g$. The multiplier ω' for U' is related to ω by

$$\omega'(g,h) = \mu(g)\mu(h)\mu(gh)^{-1}\omega(g,h) \tag{1.7}$$

and we say ω and ω' are equivalent if (1.6) holds for some map $\mu: G \to U(1)$.

A cocycle of the form

$$\omega_0(g,h) = \mu(g)\mu(h)\mu(gh)^{-1} \tag{1.8}$$

is called trivial, or a coboundary. Using point-wise multiplication in $U(1)$, the set of cocycles form a group Z^2 and the set of coboundaries form a subgroup B^2.

The second cohomology group, $H^2(G, U(1))$, defined as the quotient Z^2/B^2, describes the collection of possible inequivalent multipliers [4]. If G is locally compact there do exist irreducible representations with each multiplier, since these can be obtained from the true representations of an extension of G by $U(1)$.

If two representations of a group belong to inequivalent multipliers, then the states of one representation space cannot be coherently mixed with the states of the other – a superselection rule exists between the spaces. Examples are the univalence superselection rule which prevents the mixing of states of integer and non-integer spin, and Bargmann's superselection rule, which prevents the mixing of states of different mass in a Galileo-invariant theory. Without Wigner's idea of allowing multipliers, and superselection rules, we could not have particles of non-integral spin, or the Heisenberg commutation relations between position and momentum in a Galilean invariant theory [5].

Wightman's hypothesis [6], that the superselection operators commute among themselves, leads to a slightly more general analysis. In Sect. 2 we consider a further generalization, non-abelian superselection rules. We show that in the type I case we can reduce the non-abelian cocycles to abelian cocycles with left-action.

Wigner's approach is not designed to cope with spontaneously broken symmetry. In the BCS model [7] the Lagrangian (or Hamiltonian) is invariant under a group G of transformations (the gauge group in this case), but the ground state is not invariant under the action of G. This idea was used by Nambu and Jona-Lasinio, and Goldstone [8] in theories of elementary particles, and is now known as a spontaneously broken symmetry. This is not a very precise criterion, since it leaves open the question as to whether or not the broken symmetry is given by a unitary (or anti-unitary) operator.

This question can be formulated more exactly in the C^*-algebra approach to quantum mechanics: a spontaneously broken symmetry is an automorphism of the C^*-algebra $\mathscr{A}$ of observables, commuting with the dynamics, but which is not a spatial automorphism in the representation in question [9]. Apart from its more precise formulation, avoiding Lagrangians, this allows us to consider representations without ground states.

Let $\{\pi_q\}$ be a collection of inequivalent irreducible representations of the algebra of observables, $\mathscr{A}$, on Hilbert spaces $\mathscr{H}_q$; the vectors of the $\mathscr{H}_q$ are possible states of the system. We can form the direct sum $\pi = \bigoplus_q \pi_q$ acting on $\mathscr{H} = \bigoplus_q \mathscr{H}_q$, with superselection rules operating between various $\mathscr{H}_q$. As constructed, π is multiplicitly free. This is mathematically equivalent to the fact that the commutant $\pi(\mathscr{A})'$ is abelian – Wightman's hypothesis. Thus, this hypothesis can be "justified" by asserting that, to get all possible states in $\mathscr{H}$, we only need to include the vector states of each $\mathscr{H}_q$ once each; there is no possible physical measurement that can distinguish between repeated copies.

In spite of this devastating argument, non-abelian gauge theories remain popular: these theories use repeated states (of different "colour") all having the same physics. It is therefore worthwhile to rethink the Wigner analysis in representations of $\mathscr{A}$ in which $\pi(\mathscr{A})'$ is not abelian, and in which the multipliers $\omega(g, h)$ [unitary elements in $\pi(\mathscr{A})'$] do not commute with the unitary operators $\{U_g, g \in G\}$ implementing the symmetries. This leads us to "non-abelian cohomology," outlined in Sect. 2. The main result is Theorem 1: if $\pi(\mathscr{A})$ is of type I, then there exists a family of gauge group elements, $\{\mu(g)\}_{g\in G} \in \pi(\mathscr{A})'$, such that $U'(g) = \mu(g)U(g)$ [cf. (1.6)] has multipliers $\omega'(g,h) = U'_g U'_h U'^{-1}_{gh}$ lying in $\mathscr{Z} = \pi(\mathscr{A})'' \cap \pi(\mathscr{A})'$, the centre of $\pi(\mathscr{A})''$, on which G acts by left action; in short, ω' is an abelian cocycle [4]. Moreover, equivalent non-abelian cocycles correspond to equivalent abelian ones.

Our general analysis can give rise to a new mechanism for breaking a symmetry, called "fact violation" in [10]. When a non-trivial multiplier occurs between a 1-parameter symmetry group and the time-evolution group, then the generator of the symmetry does not commute with the Hamiltonian H, and is thus not a conserved quantity, *even though Wigner's isometric maps exist.* This phenomenon is usually called an anomaly. In [10], the unitary operators representing space-translation do not commute. Indeed, $\mathbb{R}^4$ as a group has many multipliers; these are not used in relativistic theory [2], since they cannot be extended to multipliers of $\mathbb{P}^\uparrow_+$. This argument has no force in a non-relativistic model, or in a representation of a relativistic model in which the Lorentz group is broken (such as the charged sectors of QED). In Sect. 3 we give an example of a spin system in which the space-time automorphisms of $\mathscr{A}$ commute, but their implementing unitaries do not – a classic case of anomaly.

In Sect. 4 we give a continuous version of this model; we find that the multiplier is an operator on which the symmetry group acts non-trivially. That is, we get an operator anomaly. We also show, by this example, that it is possible for a gauge group, which by definition acts *trivially* on observables, to acquire an anomaly and be non-trivially implemented!

2. Non-Abelian Cocycles

Let $\mathscr{A}$ be a C^*-algebra with unit, represented on the Hilbert space $\mathscr{H}$ by a representation $\pi:\mathscr{A}\to\mathbb{B}(\mathscr{H})=C^*$-algebra of all bounded operators on $\mathscr{H}$. Let $M=\pi(\mathscr{A})'$, a possibly non-abelian W^*-algebra. Let G be a group of automorphisms $\{\tau_g:g\in G\}$ of $\mathscr{A}$, each being spatial in π. That is, τ_g is implemented by $U_g\in\operatorname{Aut}\mathscr{H}$, thus:

$$\pi(\tau_g A)=U_g\pi(A)U_g^{-1}, \qquad g\in G,\ A\in\mathscr{A}. \tag{2.1}$$

For any $V\in\operatorname{Aut}\mathscr{H}$, denote the adjoint action of V on $\mathbb{B}(\mathscr{H})$ by $\operatorname{Ad}V$:

$$\operatorname{Ad}VB=VBV^{-1}, \qquad B\in\mathbb{B}(\mathscr{H}).$$

Thus, (2.1) can be written

$$\operatorname{Ad}U_g\pi(A)=\pi(\tau_g A), \qquad g\in G,\ A\in\mathscr{A}. \tag{2.2}$$

The map $g\to U_g$ need not be a group homomorphism, $G\to\operatorname{Aut}\mathscr{H}$; all we can say is that

$$\omega(g,h)\overset{\text{def}}{=}U_gU_hU_{gh}^{-1}\in M. \tag{2.3}$$

Indeed, for $A\in\mathscr{A}$,

$$\begin{aligned}\omega(g,h)\pi(A)-\pi(A)\omega(g,h)&=(\operatorname{Ad}\omega(g,h)\pi(A)-\pi(A))\omega(g,h)\\&=(\operatorname{Ad}U_g\circ\operatorname{Ad}U_h\circ\operatorname{Ad}U_{gh^{-1}}\pi(A)-\pi(A))\omega(g,h)\\&=(\pi(\tau_g\circ\tau_h\circ\tau_{gh}^{-1}A)-\pi(A))\omega(g,h)\\&=0,\end{aligned}$$

showing that $\omega(g,h)\in M$. Hence we get the non-abelian multiplier representation

$$U_gU_h=\omega(g,h)U_{gh}, \qquad \omega\in M,\ g,h\in G. \tag{2.4}$$

Now we show that $\operatorname{Ad}U_g$ maps M to itself; indeed if $B\in M$ and $A\in\mathscr{A}$,

$$\begin{aligned}\operatorname{Ad}U_gB\pi(A)-\pi(A)\operatorname{Ad}U_gB&=U_g\{BU_g^{-1}\pi(A)U_g-U_g^{-1}\pi(A)U_gB\}U_g^{-1}\\&=\operatorname{Ad}U_g[B,\pi(\tau_{g^{-1}}A)]=0 \quad\text{for all}\quad A\in\mathscr{A},\end{aligned}$$

showing $\operatorname{Ad}U_gB\in M$ if $B\in M$.

A similar argument shows that $\operatorname{Ad}U_g$ maps $M'=\pi(\mathscr{A})''$ to itself; or, $\operatorname{Ad}U_g|\pi(\mathscr{A})$, being (ultra)strongly continuous, has a unique extension to the closure, $\pi(\mathscr{A})''$, and this extension must be $\operatorname{Ad}U_g|\pi(\mathscr{A})''$. It follows from these results that $\operatorname{Ad}U_g$ maps $\mathscr{Z}=M\cap M'$ to itself.

Whereas the map $g\to\operatorname{Ad}U_g|M'$ is a group homomorphism, $G\to\operatorname{Aut}M'$, it is not true in general that the map $g\to\operatorname{Ad}U_g|M$ is a group homomorphism, $G\to\operatorname{Aut}M$.

Indeed, if $B\in\mathbb{B}(\mathscr{H})$,

$$\begin{aligned}\operatorname{Ad}U_g\circ\operatorname{Ad}U_hB&=U_gU_hBU_h^{-1}U_g^{-1}\\&=\omega(g,h)U_{gh}BU_{gh}^{-1}\omega(g,h)^{-1}\\&=\operatorname{Ad}\omega(g,h)\circ\operatorname{Ad}U_{gh}B\end{aligned}$$

so

$$\operatorname{Ad} U_g \circ \operatorname{Ad} U_h = \operatorname{Ad} \omega(g,h) \circ \operatorname{Ad} U_{gh}. \tag{2.5}$$

It follows that $g \to \operatorname{Ad} U_g|\mathscr{Z}$ and $g \to \operatorname{Ad} U_g|M'$ are homomorphisms and, if $\omega \in M \cap M' = \mathscr{Z}$, $g \to \operatorname{Ad} U_g|M$ is as well; but in general, we have (2.5), with a "multiplier" $\operatorname{Ad}\omega$.

The Cocycle Condition

The operators U_g are associative, and as usual this leads to a "cocycle" relation:

$$(U_g U_h)U_k = \omega(g,h)U_{gh}U_k = \omega(g,h)\omega(gh,k)U_{ghk},$$

$$U_g(U_h U_k) = U_g\omega(h,k)U_{hk} = [\operatorname{Ad} U_g\omega(h,k)]U_g U_{hk} = [\operatorname{Ad} U_g\omega(h,k)]\omega(g,hk)U_{ghk}.$$

It follows that, for all $g, h, k \in G$,

$$\omega(g,h)\omega(gh,k) = [\operatorname{Ad} U_g\omega(h,k)]\omega(g,hk). \tag{2.6}$$

This is similar to the abelian cocycle condition with left "action" $\operatorname{Ad} U_g$, except that the order is important, and $\operatorname{Ad} U_g$ is not an action of G on M but obeys (2.5). We say that ω is an $\operatorname{Ad} U$ cocycle if (2.5), (2.6) hold, and ω is a map $G \times G \to M$. ω unitary.

It is not clear that, given ω obeying (2.6) with respect to some automorphisms of M, "$\operatorname{Ad} U_g$" obeying (2.5), then there exists $U_g \in \operatorname{Aut}\mathscr{H}$ obeying (2.3).

Equivalent Multipliers

If U_g and V_g implement the automorphism τ of $\mathscr{A}$ in π, then there exists $\mu(g) \in M$ such that $V_g = \mu(g)U_g$. Conversely, if U and V are thus related, then $\operatorname{Ad} U|\pi(\mathscr{A}) = \operatorname{Ad} V|\pi(\mathscr{A})$ if either maps $\pi(\mathscr{A})$ to itself.

If so, and U has multiplier ω, then

$$\begin{aligned} V_g V_h &= \mu(g)U_g\mu(h)U_h = \mu(g)(\operatorname{Ad} U_g\mu(h))\omega(g,h)U_{gh} \\ &= \mu(g)(\operatorname{Ad} U_g\mu(h))\omega(g,h)\mu(gh)^{-1}V_{gh}. \end{aligned}$$

Hence V_g has multiplier

$$\omega'(g,h) = \mu(g)(\operatorname{Ad} U_g\mu(h))\omega(g,h)\mu(gh)^{-1}. \tag{2.7}$$

Obviously, μ_g is unitary.

The multiplier $\omega'(g,h)$ obeys the cocycle relation (2.6) with $\operatorname{Ad} U_g$ replaced by $\operatorname{Ad}(\mu(g)U_g)$. We say ω' is equivalent to ω, $\omega' \sim \omega$, via μ. This sets up an equivalence relation: $\omega \sim \omega$ via 1; if $\omega' \sim \omega$ via μ, then $\omega \sim \omega'$ via μ^{-1}; and if $\omega' \sim \omega$ via μ, and $\omega'' \sim \omega'$ via μ', then $\omega'' \sim \omega$ via $\mu'\mu$. Also, $\omega \equiv 1$ is a multiplier and any $\omega \sim 1$ via μ is said to be *trivial*. Thus,

$$\omega(g,h) = \mu(g)(\operatorname{Ad} U_g\mu(h))\mu(gh)^{-1} \tag{2.8}$$

is an $\operatorname{Ad}(\mu U)$-multiplier equivalent to 1 via μ. One easily shows that if (2.8) holds, and ω' is equivalent to ω via μ', then $\omega' \sim 1$ via $\mu'\mu$. Thus, the set of multipliers of the form (2.8) for some μ are all equivalent to each other. Any equivalence class contains an ω obeying $\omega(g,e) = \omega(e,g) = 1$; we assume this relation from now on.

We can formulate these concepts without the assumed existence of U_g; we only need an "action" γ_g, obeying $\gamma_g \in \operatorname{Aut} M$,

$$\gamma_g \circ \gamma_h = \operatorname{Ad}\omega(g,h)\gamma_{gh}, \tag{2.5'}$$

and a cocycle ω obeying

$$\omega(g,h)\omega(gh,k) = [\gamma_g\omega(h,k)]\omega(g,hk). \tag{2.6'}$$

Then equivalent multipliers, and trivial multipliers can be defined, leading to similar considerations.

If $M = \mathscr{Z}$, there is a natural multiplication between multipliers belonging to the same action γ (which is then a homomorphism, $G \to \operatorname{Aut} M$), namely, the pointwise multiplication of ω as functions $G \times G \to M$. In general we can compose two projective representations as $U_1 \otimes U_2$, leading to a form of product for cocycles.

Our cohomology is an example of [4, 11], as we show in Theorem 2. Indeed, it is equivalent (Theorem 3). When M is of type I, a gauge transformation can be used to reduce the multiplier to a central one, as we now show.

First, if $\pi(\mathscr{A})$ is of type I with separable predual, then so is $\pi(\mathscr{A})' = M$. Then M is $*$-algebraically isomorphic to a W^*-algebra M_0 with abelian commutant which is then its centre, $\mathscr{Z}_0$ [13, Theorem 5.5.11]. By the central decomposition theorem [13, Corollary 4.12.5], there exists a measure space (Z, μ) and a field of Hilbert spaces $\mathscr{H}(z)$ such that M_0 is spatially isomorphic to

$$M_1 = \int_Z^{\oplus} \mathbb{B}(\mathscr{H}(z))d\mu(z)$$

acting on the direct integral

$$\mathscr{H} = \int_Z^{\oplus} \mathscr{H}(z)d\mu(z).$$

The isomorphism maps $\mathscr{Z}_0$ onto the algebra $\mathscr{Z}_1$ of diagonal operators on $\mathscr{H}$. An automorphism γ of M is transferred to an automorphism γ_1 of M_1 leaving $\mathscr{Z}_1$ globally invariant. By [12, p. 253, cor.], γ_1 is a spatial automorphism.

It is known that $\dim \mathscr{H}(z)$ is a measurable function of z [12, p. 143, Proposition 1(i)]. Thus

$$E_n = \{z \in Z : \dim \mathscr{H}(z) = n\}$$

defines the diagonalized projection

$$P_n = \int^{\oplus} \chi_n(z)1(z)d\mu(z) \in \mathscr{Z}_1,$$

where χ_n is the indicator function of E_n and $1(z)$ is the identity operator on $\mathscr{H}(z)$. The algebras $P_n M_1 P_n$ are homogeneous, $n \in \{1, 2, \ldots \infty\}$ and are spatially isomorphic to $\mathbb{B}(\mathscr{H}_n) \otimes L^\infty(E_n, \mu)$; here $\dim \mathscr{H}_n = n$ and the term is missing if $\mu(E_n) = 0$.

We now show that each $P_n M_1 P_n$ is mapped to itself by any automorphism γ_1 of M_1. It is enough to show that each P_n is invariant. Clearly, $\gamma_1(P_n)$ is a projection in $\mathscr{Z}_1$; let $\gamma_{1*}(E_n)$ be the corresponding measurable set in $\mathscr{Z}_1$, where γ_{1*} is the pull-back of $\gamma_1 | \mathscr{Z}_1$ to Z. By the uniqueness theorem [12, p. 222, Theorem 4], μ and $\gamma_{1*}\mu$ are equivalent, so $\mu(\gamma_{1*}E_n) \neq 0$. Also if $m \neq n$,

$$\mu(E_m \cap \gamma_{1*} E_n) = 0$$

since if not, $\mu(\gamma_{1*}^{-1} E_m \cap E_n) \neq 0$ too, and a non-trivial part of the direct integral of dimension n is mapped spatially (as γ_1 is spatial) onto a non-trivial part of dimension m. As μ is countably additive, we thus get

$$\mu((Z - E_n) \cap \gamma_{1*} E_n) = 0,$$

i.e. up to sets of measure zero,

$$\gamma_{1*} E_n \subseteqq E_n, \qquad n \in \{1, 2, \ldots\}, \text{ showing } \gamma_1(P_n) \leqq P_n.$$

Thus M_1 is the direct sum of homogeneous algebras invariant under γ_1. Reversing the isomorphism we see that M is the direct sum of homogeneous algebras each invariant under automorphisms, in particular, under $\{\mathrm{Ad}\, U_g,\ g \in G\}$. It is therefore enough to consider M to be a homogeneous algebra of type I_n, $1 \leqq n \leqq \infty$.

Theorem 1. *Suppose M is a homogeneous W^*-algebra of type I, G a group and $g \to \gamma_g$ a map $G \to \mathrm{Aut}\, M$. Let $\omega : G \times G \to M$ satisfy* (2.5′) *and* (2.6′). *Then there exists a map $\mu : G \to M$ such that*

$$\omega'(g, h) = \mu^{-1}(g) \gamma_g(\mu^{-1}(h)) \omega(g, h) \mu(gh) \tag{2.7′}$$

lies in $\mathscr{Z} = M \cap M'$, and obeys the cocycle relation

$$\omega'(g, h) \omega'(gh, k) = (\lambda_g \omega'(h, k)) \omega'(g, hk), \tag{2.6′}$$

where $g \to \lambda_g = \gamma_g|_{\mathscr{Z}}$ is a homomorphism, $G \to \mathrm{Aut}\, \mathscr{Z}$. Moreover, if ω_1 and ω_2 are equivalent, then the corresponding ω_1', ω_2' are equivalent in the abelian cohomology $H^2(G, \mathscr{Z}, \lambda)$.

Proof. Like all automorphisms, γ_g maps $\mathscr{Z}$ to itself. Since M is homogeneous of type I, there exists a $*$-algebraic isomorphism $\sigma : M \to \mathscr{Z} \otimes \mathscr{B}(\mathscr{H})$ [12, p. 251]. Then $\sigma \circ \gamma_g \circ \sigma^{-1}$ is an automorphism of $\mathscr{Z} \otimes \mathscr{B}(\mathscr{H})$ leaving the first factor invariant. Let $\lambda_g' = (\sigma \circ \gamma_g \circ \sigma^{-1}) | \mathscr{Z} \otimes 1$ and put $\lambda_g = \sigma^{-1} \circ \lambda_g' \circ \sigma$. Then $\gamma_g \circ \lambda_g^{-1} = \lambda_g^{-1} \circ \gamma_g \in \mathrm{Aut}\, M$, and leaves $\mathscr{Z}$ elementwise invariant. So, by [12], p. 255, or [13, p. 346], $\gamma_g \circ \lambda_g^{-1}$ is inner. So there exists $\mu_g \in M$ (a gauge transformation) such that $\gamma_g \circ \lambda_g^{-1} = \mathrm{Ad}\, \mu_g$. Then

$$\begin{aligned} \mathrm{Ad}\, \omega(g, h) &= \gamma_g \gamma_h \gamma_{gh}^{-1} \\ &= (\mathrm{Ad}\, \mu_g \lambda_g)(\mathrm{Ad}\, \mu_h \lambda_h)(\lambda_{gh}^{-1} \mathrm{Ad}\, \mu_{gh}^{-1}) \\ &= \mathrm{Ag}\, \mu_g\, \mathrm{Ad}(\lambda_g \mu_h)(\lambda_g \lambda_h \lambda_{gh}^{-1})\, \mathrm{Ad}\, \mu_{gh}^{-1} \\ &= \mathrm{Ad}(\mu_g (\lambda_g \mu_h) \mu_{gh}^{-1}), \end{aligned}$$

since $\lambda_g \lambda_h = \lambda_{gh}$.

Therefore,

$$\omega(g, h) = \mu_g (\lambda_g \mu_h) \omega'(g, h) \mu_{gh}^{-1}$$

for some $\omega'(g, h) \in \mathscr{Z}$. Then ω' obeys (2.7′). Since ω is an $\mathrm{Ad}\, \mu_g \circ \lambda_g$ cocycle, the general theory ensures that $\omega'(g, h)$ is a λ_g-cocycle. Thus ω is equivalent, via μ to $\omega' \in Z^2(G, \mathscr{Z}, \lambda)$. Now suppose ω_1 is equivalent to ω_2 in the sense of non-abelian cohomology, so $\omega_2 \sim \omega_1$ via μ. Let $\omega_1 \sim \omega'_1$ via μ_1 and $\omega'_2 \sim \omega_2$ via μ_2, where ω'_1 and $\omega'_2 \in \mathscr{Z}^2(G, \mathscr{Z}, \lambda)$. Then $\omega'_2 \sim \omega'_1$ via $\mu_2\mu\mu_1$. Hence the actions of ω'_1 and ω'_2 are related by $\mathrm{Ad}(\mu_2\mu\mu_1)$. But both have the same action, λ: so $\lambda = \mathrm{Ad}(\mu_2\mu\mu_1) \circ \lambda$. Hence

$$\mu_2\mu\mu_1 \in \mathscr{Z}\,.$$

Thus ω'_2 and ω'_1 are equivalent via a central element, so

$$\omega'_2\omega'^{-1}_1 \in B^2(G, \mathscr{Z}, \lambda)\,.$$

Remark. Theorem 1 shows that there is nothing new in non-abelian cohomology in the type I case. It enables us to set up cohomology groups consistently by multiplying cocycle representatives in $Z^2(G, \mathscr{Z}, \lambda)$. This provides an isomorphism between non-abelian cohomology with $\tau_g|\mathscr{Z} = \lambda_g$ given, and $H^2(G, \mathscr{Z}, \lambda)$. This does not mean that there is nothing new in the physics: it might not be possible to squeeze a non-abelian gauge quantum field theory into a multiplicity-free representation of the observables.

There is a connection between cocycles ω with ω-action γ and extensions of the group G. Let $\mathscr{U}$ denote the unitaries in M.

Theorem 2. *The multiplication law*

$$(g, u) \circ (h, v) = (gh, u(\gamma_g v)\omega(g, h)) \tag{2.9}$$

makes $G \times \mathscr{U}$ *into a group* $E = G \underset{\gamma}{\rtimes} \mathscr{U}$. *Then the sequence*

$$1 \rightarrow \mathscr{U} \xrightarrow{i} E \xrightarrow{\Phi} E/\mathscr{U} \rightarrow 1\,, \tag{2.10}$$

where $E/\mathscr{U} \approx G$, *is exact. Moreover, for any* ω-*representation* $g \rightarrow U_g$ *of* G,

$$V(g, \mu) = \mu U_g \tag{2.11}$$

is a true representation of E. *If* $\omega' \sim \omega$ *via* μ, *then the groups* $G \underset{\gamma}{\rtimes} \mathscr{U}$ *and* $G \underset{\gamma'}{\rtimes} \mathscr{U}$ *are isomorphic, where* $\gamma'_g = \mathrm{Ad}\,\mu_g \circ \gamma_g$, *and conversely. The isomorphism is*

$$(g, u) \rightarrow (g, u\mu_g^{-1})\,.$$

Then the corresponding true representations of (2.11) *are equal.*

Proof. Elementary.

The product law (2.9) is a generalization of the semi-direct product (when $\omega = 1$) and the central extension (when $\gamma = 1d$, $\mathscr{M} = \mathscr{Z}$).

We call (2.9) the canonical form of the extension E.

Theorem. *Let* (2.10) *be an exact sequence of groups. Then there exists a pair* (ω, γ), *where* ω *is a 2-cocycle:* $G \times G \rightarrow \mathscr{U}$ *and* γ *is an* $\mathrm{Ad}\,\omega$-*homomorphism* $\gamma : G \rightarrow \mathrm{Aut}\,\mathscr{U}$, *such that*

$$E \approx G \underset{\gamma}{\rtimes} \mathscr{U}$$

with product in canonical form (2.9).

Proof. Identify G with $E/\mathcal{U}$ and to each $g \in G$ choose $e_g \in \phi^{-1}(g)$. Since ϕ is a homomorphism, $e_g e_h \in \phi^{-1}(gh)$, so there exists $\omega(g,h) \in \mathcal{U}$ such that $e_g e_h = \omega(g,h) e_{gh}$. Let $\gamma_g = \mathrm{Ad}\, e_g \in \mathrm{Aut}\,\mathcal{U}$. Then since

$$\mathrm{Ad}\, e_g \,\mathrm{Ad}\, e_h = \mathrm{Ad}\,\omega(g,h) \circ \mathrm{Ad}\, e_{gh}$$

we obtain (2.5′). Also, associativity in G, namely $e_{(gh)k} = e_{g(hk)}$ gives the cocycle relation (2.6′).
Finally, any element of E has the unique expression $e = u e_g$. The product law is

$$e_1 e_2 = u_1 e_{g_1} u_2 e_{g_2} = u_1 \gamma_{g_1}(u_2)\omega(g_1, g_2) e_{g_1 g_2}$$

so the map

$$u e_g \to (g, u)$$

is an isomorphism in canonical form.

We see that our freedom to choose various U_g to implement the symmetry is just choosing a representative e_g in the coset in E.

3. Energy-Momentum Anomaly

We now give an example, the spin chain in a magnetic field, in which the space and time automorphisms commute and possess Wigner isometries, but these isometries do not commute. The anomalous commutator, which is of course a cocycle, arises in a way that is typical of anomalies the world over.

Let $\mathcal{A}$ be the spin $\frac{1}{2}$ C^*-algebra for the linear chain, i.e. $\mathcal{A}$ is the inductive limit over finite subsets $\Lambda \subseteqq Z$ of the algebras [14],

$$\mathcal{A}(\Lambda) = \bigotimes_{j \in \Lambda} M_{2(j)}, \qquad |\Lambda| < \infty, \tag{3.1}$$

where $M_{2(j)}$ is a copy of $M_2 = \mathbb{B}(C^2)$. The group Z acts on $\mathcal{A}$ by translations:

$$T(n) A_j = A_{j+n}; \qquad n, j \in Z,$$

where A_j is the copy of A in $M_{2(j)}$. Suppose the system lies in a magnetic field $\vec{M} = (0, 0, M)$, so that the Hamiltonian is $H = \vec{M} \cdot \vec{S}$

$$H = M \sum_{j \in Z} \left(\tfrac{1}{2}\sigma_j^3\right). \tag{3.2}$$

This formal infinite sum does not converge, but it defines a unique one-parameter group of automorphisms of $\mathcal{A}$, namely, those that reduce to

$$\tau_t(A) = \mathrm{Ad}\exp(iM\sigma^3 t/2) A \tag{3.3}$$

on each $M_{2(j)}$. This action is periodic of period $4\pi/M$, and commutes with space-translation. So we would say that space-translation is a symmetry in the algebraic formalism.

Each τ_t on M_2 has two pure stationary states $u = \begin{pmatrix}1\\0\end{pmatrix}$ and $d = \begin{pmatrix}0\\1\end{pmatrix}$. By taking infinite tensor products of these stationary states we get many invariant states of τ_t for the whole algebra $\mathcal{A}$. Let us choose a state that represents a magnetic domain wall at $j=0\in Z$: construct the state [14, 15]:

$$\Omega=\bigotimes_{j<0} u_j \otimes \bigotimes_{j\geqq 0} d_j . \tag{3.4}$$

Then the functional

$$\omega(A)=\langle\Omega, A\Omega\rangle, \qquad A\in\mathscr{A}(\Lambda)$$

defines by extension to the inductive limit a stationary state on $\mathscr{A}$. In the cyclic representation, say $(\mathscr{H}_\omega, \pi_\omega, \Omega)$, given by the GNS construction [13, p. 47] time-evolution is given by a continuous one-parameter group of unitaries $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathscr{H}_\omega$. The generator of $U(t)$ is given by (3.2) in which σ_j^3 is replaced by its representation in π_ω, and a "vacuum renormalization" is made so that H is convergent on a dense set in $\mathscr{H}_\omega$:

$$H=\frac{M}{2}\left(\sum_{j<0}^{-\infty}(\sigma_j^3-1)+\sum_{j\geqq 0}^{\infty}(\sigma_j^3+1)\right). \tag{3.5}$$

The state ω is not invariant under space-translations: the dual action $T^*(n)$ takes ω to ω_n, where ω_n is determined by

$$\Omega_n=\bigotimes_{j<n} u_j \otimes \bigotimes_{j\geqq n} d_j .$$

This might signal a spontaneous breakdown of translation invariance. However, we now show that $T(n)$ is spatial in π_ω, so that each $T(n)$ is a symmetry in Wigner's sense. Nevertheless, momentum is not conserved owing to an anomaly. Define $V(n)$ on Ω by

$$\begin{aligned} V(n)\Omega=\Omega_n&=\pi_\omega\left(\bigotimes_{j=0}^{n-1}\sigma_j^1\right)\Omega, \qquad n>0,\\ &=\pi_\omega\left(\bigotimes_{j=n}^{-1}\sigma_j^1\right)\Omega, \qquad n\leqq -1,\\ &=\Omega, \qquad n=0, \end{aligned}$$

where $\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

Thus, Ω_n is the state with spin at k up, $k<n$, and spin at k down, $k\geqq n$. Define $V(n)$ on a general vector $\pi(\mathscr{A})\Omega$ by

$$V(n)\pi(A)\Omega=\pi[T(n)A]\Omega_n .$$

Then $V(n)$ is unitary, obeys $V(n)V(n')=V(n+n')$, and implements $T(n)$ in π:

$$V(n)\pi(A)V^{-1}(n)=\pi(T(n)A).$$

In spite of this ($V(n)$ does not commute with $U(t)$; indeed, if $n>0$,

$$U(t)\Omega_n=U(t)\pi_\omega\left(\sum_{k=0}^{n-1}\sigma_k^1\right)\Omega=\pi_\omega\left(\tau_t\left(\bigotimes_{k=0}^{n-1}\sigma_k^1\right)\right)\Omega$$

(as dictated by the GNS construction)

$$=\pi_\omega\left(\bigotimes_{0}^{n-1}\begin{pmatrix}0&e^{iMt}\\e^{-iMt}&0\end{pmatrix}\right)\Omega=e^{inMt}\Omega_n .$$

Similarly, if $n<0$,

$$U(t)\Omega_n = U(t)\pi\left(\bigotimes_{k=n}^{-1} \sigma_k^1\right)\Omega$$
$$= \pi\left(\bigotimes_{k=n}^{-1}\begin{pmatrix} 0 & e^{iMt} \\ e^{-iMt} & 0 \end{pmatrix}\right)\Omega = (e^{-iMt})^{-n}\Omega_n$$
$$= e^{inMt}\Omega_n .$$

Hence Ω_n is an eigenstate of H/M with eigenvalue n: H/M is the *position operator* for the boundary of the domain (the boundary is only sharp for vectors of the form Ω_n). In our model, there is a *density* for the position operator, $\frac{M}{2}(\sigma^3-1)$, $n<0$ and $\frac{M}{2}(\sigma^3+1)$ for $n \geqq 0$, in view of (3.5).

The commutation relation in Weyl form

$$U(t)V(n) = e^{inMt}V(n)U(t) \tag{3.6}$$

follows immediately, at least on $\mathrm{Span}\{\Omega_n : n \in Z\}$. Then (3.6) must hold on all vectors: π is irreducible, so the multiplier lies in $U(1)$ and is determined by $U(t)V(n)U^{-1}(t)U^{-1}(n)$ on any non-zero vector. Equation (3.6) gives a multiplier for the group $Z \times \hat{Z} = Z \times U(1)$ [16]: time-evolution is periodic and so forms the group $U(1)$, dual to Z.

It is clear that momentum is not conserved in this model. How does this "fact violation" come about? Firstly, the space-translates Ω_n of the stationary state Ω are *different* stationary states; secondly, H is not positive, but has "spectral symmetry." The eigenvalue equation $H\Omega = 0$ results from the "cancellation" of infinitely many $\frac{-M}{2}$ with the balancing $\frac{+M}{2}$, provided we take the infinite limit in a balanced way. This is violated on the states Ω_n. This is exactly how the anomaly in the Virasoro algebra is described [17, p. 290], and indeed the axial anomaly also arises because the Dirac spectrum is shifted up and down by the axial gauge transformations, giving a spectral flow. In our model, the translation operator provides the spectral flow directly.

This model acts as a guide for the continuous version in Sect. 4.

4. Energy-Momentum Anomaly in the Continuum

We now give the continuous version of the model of Sect. 3. But it is not possible to define a continuous tensor product of the space C^2 carrying a spin $\frac{1}{2}$ representation of $U(2)$ [18]. The problem lies in the failure of the positive-definiteness of the continuous product

$$\mathop{\times}_{x\in\mathbb{R}} \langle \psi(x), \phi(x)\rangle = \exp \int dx \log\langle\psi(x), \phi(x)\rangle \tag{4.1}$$

even though $\langle\, ,\rangle$ is positive-definite. The solution to this problem is given in [19]: at each point $x \in \mathbb{R}$, we choose an "infinitely divisible" cyclic representation

$(\mathscr{D}_x, \Omega_x)$, instead of the spin $\frac{1}{2}$ representation $\mathscr{D}^{1/2}$. For $SU(2)$, any infinitely divisible representation is an exponential of a cyclic representation [20].

To set up the continuous analogue of the model of Sect. 3, let $\lambda \in C$ and

$$\mathrm{Exp}\,\lambda u = 1 \oplus \lambda u \oplus \lambda^2 \frac{u \otimes u}{\sqrt{2!}} \oplus \ldots \in \mathrm{Exp}\, C^2\,, \tag{4.2}$$

where $\mathrm{Exp}\,C^2$ is the Fock space over C^2. Similarly we define $\mathrm{Exp}\lambda d$. Then

$$\Omega_+ = \exp(-|\lambda|^2/2)\,\mathrm{Exp}\,\lambda u$$

and

$$\Omega_- = \exp(-|\lambda|^2/2)\,\mathrm{Exp}\,\lambda d$$

are unit vectors in $\mathrm{Exp}\,C^2$. Let $U \in U(2)$ and define its second quantization

$$\mathrm{Exp}\,U = 1 \oplus U \oplus (U \otimes U) \oplus \ldots \in \mathrm{Aut}(\mathrm{Exp}\,C^2)\,.$$

Let G denote the group of piecewise constant maps $U : \mathbb{R} \to U(2)$, equal to the identity outside a compact set, with pointwise multiplication. We will sometimes write $U = \bigotimes_x U(x)$ for the map $U : x \mapsto U(x)$.

Then we may define the continuous tensor product

$$\mathscr{H} = \bigotimes_{x \in \mathbb{R}} (\mathrm{Exp}\,C^2)$$

with cyclic vector

$$\begin{aligned} \Omega = \bigotimes_x \Omega_x,\ \Omega_x &= \Omega_+,\ x < 0 \\ &= \Omega_-,\ x > 0\,. \end{aligned} \tag{4.4}$$

Then (4.3) carries a representation of G by the pointwise left action

$$\left(\bigotimes_x U(x)\right)\left(\bigotimes_x \psi(x)\right) = \bigotimes_x (\mathrm{Exp}\,U(x) \cdot \psi(x)) \tag{4.5}$$

and by definition, $\mathscr{H}$ is spanned by G acting on Ω.

The scalar product (4.1) is then positive-definite, thus

$$\begin{aligned} \left\langle \bigotimes_x U_1(x)\Omega, \bigotimes_x U_2(x)\Omega \right\rangle &= \exp \int dx \log \langle \mathrm{Exp}\,U_1(x)\Omega_x, \mathrm{Exp}\,U_2(x)\Omega_x \rangle \\ &= \exp |\lambda|^2 \int dx (\langle U_1(x)\omega_x, U_2(x)\omega_x \rangle_{C^2} - 1)\,, \end{aligned} \tag{4.6}$$

where $\omega_x = u$, $x < 0$, and $\omega_x = d$, $x > 0$. Let $\mathscr{A}$ be the W^*-algebra generated by the operators $\bigotimes_x U(x) \in G$ on $\mathscr{H}$. Certain automorphisms of $\mathscr{A}$ are spatial. Naturally, conjugation by elements of G, being inner, are spatial. More, let $U_1 = \bigotimes_x U_1(x)$ be such that, outside a compact set K, U_1 leaves the states defined by u, d invariant, i.e. U_1 is a rotation about the third axis. Suppose U_1 is piecewise constant in K. Then the automorphism

$$U \mapsto \bigotimes_x U_1(x) U(x) U_1^{-1}(x) \tag{4.7}$$

is a spatial automorphism. Thus, let τ_t be the automorphism of $\mathscr{A}$ due to a constant magnetic field M in the third direction. On G, this reduces to (4.7) with $U_1(x)=\begin{pmatrix} e^{itM/2} & 0 \\ 0 & e^{-itM/2}\end{pmatrix}\forall x$. Then Ω defines a state on $\mathscr{A}$ invariant under τ_t, which is therefore implemented, by $V(t)$ say, given by

$$V(t)\Big(\bigotimes_x U(x)\Omega\Big)=\bigotimes_x \tau_t^0(U(x))\Omega_x, \tag{4.8}$$

where $\tau_t^0=\operatorname{Ad}U_1(x)$. Then $V(t_1)V(t_2)=V(t_1+t_2)$, and $V(t+4\pi/M)=V(t)$. The group of space translations acting on G namely $T_a\bigotimes_x U(x)=\bigotimes_x U(x-a)$ is also spatial. I claim that T_a is implemented by $W(a)$:

$$W(a)\bigotimes_x U(x)\Omega=\bigotimes_x U(x-a)\Omega_a,$$

where Ω_a has the wall of the magnetic domain at a instead of at 0:

$$\Omega_a=\bigotimes_{-\infty}^{a}\Omega_+\otimes\bigotimes_{a}^{\infty}\Omega_-\,. \tag{4.9}$$

We must justify (4.9) by showing that it lies in $\mathscr{H}$. In the spin $\frac{1}{2}$ representation of $U(2)$, the element $\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ converts λu to λd and vice versa. It follows that $\operatorname{Exp}\sigma^1$ takes $\operatorname{Exp}\lambda u$ to $\operatorname{Exp}\lambda d$ in $\operatorname{Exp}C^2$, and vice versa. Therefore the element of $\mathscr{A}$, by (4.5)

$$F_{a,b}=\bigotimes_{x\in[a,b]}\sigma_x^1$$

takes $\bigotimes_{x\in[a,b]}(\operatorname{Exp}\lambda u)_x$ to $\bigotimes_{x\in[a,b]}(\operatorname{Exp}\lambda d)_x$ and vice versa. Thus

$$\begin{aligned}\Omega_a&=\mathrm{F}_{0,a}\Omega, \quad && a>0,\\ &=\mathrm{F}_{a,0}\Omega, && a<0,\end{aligned}$$

lies in $\mathscr{H}$. It is then trivial to show that $W(a)$ is unitary on $\mathscr{H}$, and $W(a_1)W(a_2)=W(a_1+a_2)$.

As in the discrete model, Sect. 3, space and time automorphisms commute: $\tau_tT_a=T_a\tau_t$; but their implementing operators $V(t)$, $W(a)$ do not commute. We can now compute the multiplier of the representation of the group $\mathbb{R}^2$ given by $U(a,t)=W(a)V(t)$. We get

$$\begin{aligned}U(a_1,t_1)U(a_2,t_2)A\Omega&=W(a_1)V(t_1)W(a_2)\tau_{t_2}A\Omega\\ &=W(a_1)V(t_1)(T_{a_2}\tau_{t_2}A)\Omega(a_2)=W(a_1)(T_{a_2}\tau_{t_1+t_2}A)V(t_1)\Omega(a_2)\\ &=W(a_1)(T_{a_2}\tau_{t_1+t_2}A)\Big(\tau_{t_1}\bigotimes_{x\in[0,a_2]}\sigma_x^1\Big)\Omega \quad\text{if}\quad a_2>0\,.\end{aligned}$$

Now, $\tau_t\bigotimes_{x\in[0,a]}\sigma_x^1=\bigotimes_{x\in[0,a]}\tau_x^0\sigma_x^1$ and $\tau_t^0\sigma^1=\begin{pmatrix}0 & e^{iMt}\\ e^{-iMt} & 0\end{pmatrix}$. This multiplies u by e^{-iMt}. In the interval $[0,a_2]$ the second quantization of this is

$$e^{iMtN[0,a_2]}=\operatorname{Exp}(e^{iMt\chi[0,a_2]}),$$

where $N[a,b]$ is the number operator in the interval $[a,b]$. Hence

$$U(a_1,t_1)U(a_2,t_2)A\Omega = W(a_1)(T_{a_2}\tau_{t_1+t_2}A)e^{-iMt_1N[0,a_2]}\Omega(a_2)$$
$$=(T_{a_1+a_2}\tau_{t_1+t_2}A)e^{-iMt_1N[a_1,a_1+a_2]}\Omega(a_1+a_2).$$

Since

$$U(a_1+a_2,t_1+t_2)A\Omega=(T_{a_1+a_2}\tau_{t_1+t_2}A)\Omega(a_1+a_2)$$

we see that the multiplier is

$$\omega(a_1,t_1;a_2,t_2)=e^{-iMt_1N[a_1,a_1+a_2]}, \qquad a_2>0.$$

For $a_2<0$ we get the same formula if we interpret $N[a,b]=-N[b,a]$. The action of the group $\mathbb{R}^2$ on the cocycle is

$$\mathrm{Ad}\,U(a,t)\omega(a_1,t_1,a_2,t_2)=\omega(a_1+a,t_1;\ a_2+a,t_2).$$

One verifies that, with this action, ω obeys the cocycle relation (2.6). In this case, $M=\mathscr{Z}$ is abelian, being the W^*-algebra generated by the number density, i.e. by $\{\exp iN[a,b],\ a<b\in\mathbb{R}\}$.

We have an operator anomaly, albeit an abelian one. Heuristically, this arises as follows. The local currents, which generate G, are given by $\vec{J}(x)=\frac{1}{2}a_i^*(x)\vec{\sigma}_{ij}a_j(x)$, where $a_i^*(x)$, $i=1,2$ create the two spin-state u,d in Fock space. The naive formula for the energy in a magnetic field in the third direction, $M\int\limits_{-\infty}^{\infty}J^3(x)dx$, diverges on Ω; we must subtract half the number density, $N(x)/2$, $x<0$ and add $N(x)/2$, $x>0$, to remove the zero point energy. Then the "renormalized" Hamiltonian

$$H=M\int\limits_{-\infty}^{0}\left(J(x)-\frac{N(x)}{2}\right)dx+\int\limits_{0}^{\infty}\left(J(x)+\frac{N(x)}{2}\right)dx$$

annihilates Ω and makes sense on the states of $\mathscr{H}$. But it fails to commute with space-translation:

$$W(a)H-HW(a)=-\int\limits_{0}^{a}MN(x)dx=-MN[0,a],$$

giving the anomaly, again caused by spectral flow.

By choosing $a=t$, we obtain a projective unitary representation of $\mathbb{R}:t\to U(t,t)$ not reducible to a true representation.

We can turn this model round, and choose $\mathscr{Z}$ as the algebra of observables, and $\mathscr{A}$ as the non-abelian gauge group. The observables then are generated by the number operator density, which is invariant under the "time-evolution" τ_t given by the external constant gauge field in the third direction, which (being the identity automorphism) commutes with space-translation. Let us look at this system in the state Ω, which has a kink in its gauge potential at $x=0$. We find that the *trivial action* of the time evolution, τ_t on $\mathscr{Z}$, is non-trivially represented in the Hilbert space $\mathscr{H}$ built on Ω, namely, it is given by $V(t)$! A shocking violation of fact.

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