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On Positive Multi-Lump Bound States of Nonlinear Schrödinger Equations under Multiple Well Potential

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Abstract. In this paper, we first construct multi-lump (nonlinear) bound states of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi - \gamma|\psi|^{p-1}\psi$$

for sufficiently small $\hbar > 0$, in which sense we call them "semiclassical bound states." We assume that $1 \leqq p < \infty$ for $n = 1, 2$ and $1 \leqq p < 1 + 4/(n-2)$ for $n \geqq 3$, and that V is in the class $(V)_a$ in the sense of Kato for some a. For any finite collection $\{x_1, \ldots, x_N\}$ of nondegenerate critical points of V, we construct a solution of the form $e^{-iEt/\hbar}v(x)$ for $E < a$, where v is real and it is a small perturbation of a sum of one-lump solutions concentrated near $x_1, \ldots, x_N$ respectively. The concentration gets stronger as $\hbar \to 0$. And we also prove these solutions are positive, and unstable with respect to perturbations of initial conditions for possibly smaller $\hbar > 0$. Indeed, for each such collection of critical points we construct 2^{N-1} distinct unstable bound states which may have nodes in general, and the above positive bound state is just one of them.

1. Introduction

In [W.a] and [FW.a], the following nonlinear Schrödinger equation (abbreviated as NLS) on $\mathbb{R}^n$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + V\psi - \gamma|\psi|^{p-1}\psi \tag{1}$$

was proposed to study stabilizing linear modes concentrated near local minima for sufficiently small $\hbar > 0$ for potentials bounded below. In [FW.a], Floer and Weinstein (Alan) proved the existence of solutions of (1) for sufficiently small $\hbar > 0$ for bounded potentials, which are localized near each given nondegenerate critical point of V for all time; in fact, solutions of the form $e^{-iEt/\hbar}v(x)$. In [O1], the present author generalized their existence result to arbitrary potentials in the class $(V)_a$.

(As we pointed out in [O1], this restriction on V is needed even for bounded potentials.) These solutions have only one "lump" in the sense that they are concentrated at one point, for whose precise meaning we refer to [O1] or Theorem 4.1 in the present paper. However, as already pointed out in [FW.a] and [O1], this concentration gets stronger as $\hbar \to 0$ and so, when $\hbar$ is sufficiently small, we may try to construct N-lump solutions by adding N such one-lump solutions which are concentrated at distinct N nondegenerate critical points of V. In the present paper, we construct such N-lump solutions which are small perturbations of the sums of N one-lump solutions. We refer readers to later sections for its precise meaning. The method of the proof is again the Lyapunov–Schmidt reduction as in [FW.a, O1]. In other words, we first find nice approximate solutions whose errors can be controlled with respect to $\hbar$ and then try to perturb them to get exact solutions. The approximate trial solutions will be sums of N one-lump solutions and their slight translates. Then we estimate the norm of the Fredholm inverse of the linearized operator at trial solutions to reduce the problem to a finite dimensional one. Finally, we solve this finite dimensional problem by some simple topological means.

Although the basic line of proof is the same as in the one-lump case in [FW.a, O1], most of the estimates are more involved, depend on some judicious cut-off functions, and furthermore we need a new idea to estimate the norm of the Fredholm inverse, which was not needed for the one-lump case. In fact, estimating the Fredholm inverse is the most essential and difficult step in these kinds of problems (see [JaT, T] for a problem of this kind).

If we substitute $e^{-iEt/\hbar}v(x)$ into (1) where v is real, we get the following nonlinear eigenvalue problem.

$$-\frac{\hbar^2}{2}\Delta v + (V - E)v - |v|^{p-1}v = 0. \tag{2}$$

If we divide the equation by $\hbar^2$ and set $\lambda = 1/\hbar^2$, we get NLS without $\hbar$,

$$-\tfrac{1}{2}\Delta v + (\lambda V - \lambda E)v - \lambda |v|^{p-1}v = 0, \tag{2'}$$

and then our semiclassical result can be interpreted as a "quantum" result when the wells are deep enough, E is negative enough and $\gamma = \lambda$ is big enough.

If we change variables by $x = \hbar y$ and set $u(y) = v(\hbar y)$, $V_\hbar(y) = V(\hbar y)$, we get NLS

$$-\tfrac{1}{2}\Delta v + (V_\hbar - E)u - |u|^{p-1}u = 0, \tag{2''}$$

and then we get a quantum existence result when the distance between wells is large enough and the wells are wide enough. In this paper, we solve (2″) for $\hbar$ sufficiently small as in [FW.a, O1].

In [RW.m], Rose and Weinstein (Michael) got an existence result of different sorts of one-lump bound states which bifurcate from the bound states of linear Schrödinger equations, while the bound states obtained in [FW.a, O1] are perturbations of the well-known ground state solution of NLS,

$$-\tfrac{1}{2}\Delta u + \lambda u - |u|^{p-1}u = 0, \tag{3}$$

where $\lambda = V(x_0) - E$, x_0 is the critical point being considered. Our lump solutions will be perturbations of sums of N ground states of the latter kind whose centers locate near $x_1, \ldots, x_N$, respectively.

After we establish the existence result, we consider the stability and positivity of the solutions. We prove if $\hbar > 0$ is sufficiently small, the N-lump solutions found above which are, in some sense, sums of positive one-lump bound states, are all positive, and unstable for $N \geqq 2$. Recall that the stability of one-lump solutions depend on whether the critical point x_0 is a local maximum or a minimum (See [GrSS, O3]). The method of the proof is that we first show that the real and imaginary parts $L_\hbar^\pm$ of the linearized operator of (1) at the solutions satisfy certain spectral results, and then apply the instability criterion by Jones and Grillakis (see [Gr, Jo]). As a by-product, we show that the solutions are positive. The same line of ideas was used in [O3].

We also apply the same method of constructing the above positive solutions to construct the bound states which are now, in some sense, signed sums of N positive one-lump bound states and so have nodes in general.

We now briefly outline the organization of the contents of this paper. In Sect. 2, we give the definition of the class $(V)_a$ and some of its consequences which are needed for later estimates, and set up the problem. Sections 3, 4 and 5 deal with the problem of the one-dimensional and two-lump case for bounded potentials. Sections 3 and 4 study the existence problem using the Lyapunov–Schmidt reduction as in [FW.a, O1]. Section 3 contains main estimates for reducing the problem to a finite dimensional one and Sect. 4 finishes the proof of the existence for the two-lump problem solving the finite dimensional problem by an elementary degree theory. Section 5 deals with the positivity and stability of the solutions found in Sects. 3 and 4. Section 6 shows how we can refine the results for the one-dimensional two-lump case to generalize them to the N-lump problem for general dimensions under unbounded potentials in the class $(V)_a$. Since these generalizations are only a matter of complicating the estimates, we just indicate how we modify the proof of the one-dimensional two-lump problem for the general cases. Finally in Sect. 7, we indicate that the same proof goes through to construct 2^{N-1} distinct bound states which have nodes in general, and then give some remark.

2. Preliminaries

In this section we recall the definition of the class $(V)_a$ in [K, O1], and its consequences.

Definition 2.1. We say that a potential V defined on $\mathbb{R}^n$ is in the class $(V)_a$ for $a \in \mathbb{R}$, if either $V \equiv a$ identically or $V(x) > a$ and $(V-a)^{-1/2} \in \mathrm{Lip}(\mathbb{R}^n)$.

Remark 2.2. As already mentioned in [K, O1], most potentials that increase (eventually) monotonically as $|x| \to \infty$ belong to the class $(V) := \bigcup_{a \in \mathbb{R}} (V)_a$, but the following bounded potentials which have accelerated oscillations as $|x| \to \infty$ are not in the class (V):

$$V(x) = \sin|x|^2 \text{ or } \sin e^{|x|^2},$$

although $\sin|x|$ is in the class.

Proposition 2.3 (see [K] or Proposition 2.3 [O1, O2]). *Let $V \in (V)_a$ with $b := \|(V-a)^{-1/2}\|_{\mathrm{Lip}}$ and $H = -\frac{1}{2}\Delta + V$. If $b < 1$, then*
i) *H is self-adjoint with domain $D(H) = D(\Delta) \cap D(V)$.*
ii) *For each $u \in D(H)$*

$$\|(H-E)u\|_2 \geqq (1-b)\|(V-E)u\|_2$$

for any $E \in \mathbb{R}$ with $V - E > 0$, where such E exists by the definition of the class $(V)_a$.

Now let us define the operators

$$H_\hbar := -\tfrac{1}{2}\Delta + V_\hbar,$$

where $V_\hbar(y) := V(\hbar y)$. Then it is easy to see that $V_\hbar \in (V)_a$ for all $\hbar$ if V is in $(V)_a$ and

$$\|(V_\hbar - a)^{-1/2}\|_{\mathrm{Lip}} = \hbar \cdot \|(V-a)^{-1/2}\|_{\mathrm{Lip}}.$$

Corollary 2.4 (see Lemma 3.2 [O1] and also see [O2]). *Let $V \in (V)_a$ with $b := \|(V-a)^{-1/2}\|_{\mathrm{Lip}}$ and $E < a$. Then if $0 < \hbar, < d < 1/b$ for some d, we have*

$$\|(H_\hbar - E)u\|_2 \geqq \lambda \|u\|_\hbar$$

for all $u \in D(H_\hbar)$ and for some $\lambda > 0$ independent of $\hbar$, where $\|\cdot\|_\hbar$ is defined by

$$\|u\|_\hbar^2 = \int |\Delta u|^2 + \int (V_\hbar - E)^2 |u|^2.$$

Note that when V is bounded, all $\|\cdot\|_\hbar$ are equivalent to the H^2 norm and so we will use H^2 norm for bounded potentials. We refer to [O1, O2] for the proofs of Proposition 2.3 and Corollary 2.4. From now on until Sect. 5, for simplicity of proof, we will consider only the two-lump case where $n = 1$, V bounded and $p = 3$. In Sect. 6, we indicate how to remove these restrictions.

We also need the nondegeneracy on the linearized operator of (3) at the ground states.

Nondegeneracy. Let R_0 be the unique ground states of (3) and consider the linearized operator at R_0

$$L_0 = -\tfrac{1}{2}\Delta + \lambda - pR_0^{p-1}.$$

Then L_0 has the kernel

$$\ker L_0 = \operatorname{span}\left\{\frac{\partial R_0}{\partial x_i}\right\}_{1 \leqq i \leqq n}.$$

Proof. This is previously known to be true for $n = 1$ and for $n = 3$ and $1 < p \leqq 2$ (see [W.m, Appendix A]). The complete proof for the general case is also contained in [W.m, Appendix A] modulo the fact that any solution of the following equation:

$$\begin{cases} \left(-\dfrac{d^2}{dr^2} - \dfrac{n-1}{r}\dfrac{d}{dr} + 1 - pu_0^{p-1}\right)w = 0 \\ w'(0) = 0 \\ w(0) = 1 \end{cases}$$

is unbounded. However this is now established by Kwong [Kw] in the course of his complete proof of the uniqueness theorem of the ground state of the equation

$$-\Delta u+u-u^p=0$$

(see Sects. 4 and 5 in [Kw]). □

From now on we shall seek solutions of the form

$$\psi(x,t)=\exp\left(-\frac{iEt}{\hbar}\right)v(x)$$

of the equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2}+V\psi-\gamma|\psi|^2\psi, \tag{1'}$$

where v is real-valued and $V-E>\varepsilon>0$. Then the function v must satisfy the nonlinear eigenvalue equation

$$-\frac{\hbar^2}{2}v''+Vv-\gamma v^3=Ev$$

which we will study as $\hbar\to 0$. Without loss of any generality, we may assume $\gamma=1$ so that the above equation is reduced to

$$-\frac{\hbar^2}{2}v''+(V-E)v-v^3=0.$$

As in [FW.a, O1], we introduce a new variable $y=x/\hbar$ and define the function u by $u(y)=v(\hbar y)$. Then u satisfies Eq. (3):

$$-\tfrac{1}{2}u''+(V_\hbar-E)u-u^3=0,$$

where $V_\hbar(y):=V(\hbar y)$.

Suppose that $\{x_1,x_2\}$ is two non-degenerate critical points of V and without loss of any generality assume that $x_1=-R$, $x_2=R$. Denote $a=V(-R)-E$, $b=V(R)-E$. By the choice of E above, we have

$$a,b>\varepsilon>0.$$

The rescaled potentials $V_\hbar$ have the corresponding nondegenerate critical points at $\pm R/\hbar$ with the same values respectively as V. Define

$$u_1(y)=\sqrt{2a}\,\mathrm{sech}\sqrt{2a}\,y,$$
$$u_2(y)=\sqrt{2b}\,\mathrm{sech}\sqrt{2b}\,y,$$

which are unique solutions (up to translation) of the equations

$$-\tfrac{1}{2}u''+au-u^3=0, \tag{4}$$
$$-\tfrac{1}{2}u''+bu-u^3=0 \tag{5}$$

respectively. Due to the translational invariance of these equations

$$u_{i,c}(y):=u_i(y-c),\quad i=1,2,\quad c\in\mathbb{R}$$

will be also solutions of (4) and (5) respectively. We define trial solutions by $u_{0,\hbar}(y) = u_1(y + R/\hbar) + u_2(y - R/\hbar)$ and

$$u_{\vec{z},\hbar}(y) = u_1\left(y + \frac{R + z_1}{\hbar}\right) + u_2\left(y - \frac{R + z_2}{\hbar}\right),$$

where $\vec{z} = (z_1, z_2)$ and $|z_i| < \frac{1}{2}$. Following [FW.a, O1], we define

$$S_\hbar(u) = -\frac{1}{2}\frac{d^2u}{dy^2} + (V_\hbar - E)u - u^3.$$

Then $S_\hbar$ is a smooth map from H^2 to L^2 and its Frêchet derivative is given by

$$S_\hbar'(u) = -\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - 3u^2.$$

We want to find a zero of $S_\hbar$, i.e. a solution of (3) of the form $u_{\vec{z},\hbar} + \phi$ for sufficiently small $\hbar > 0$ and small ϕ. We have

$$S_\hbar(u_{\vec{z},\hbar} + \phi) = S_\hbar(u_{\vec{z},\hbar}) + S_\hbar'(u_{\vec{z},\hbar})\phi + N_{\vec{z},\hbar}(\phi), \tag{6}$$

where $N_{\vec{z},\hbar}(\phi) = 3u_{\vec{z},\hbar}\phi^2 + \phi^3$.

In the following sections, all constants k_i's, K_i's and C_i's will be independent of $\hbar$. Also for a later purpose, we choose the following partitions of unity $\{\alpha_\hbar, \beta_\hbar\}$ for each $\hbar > 0$ such that $\alpha_\hbar(y) = \alpha(\hbar y)$, where

$$\alpha = \begin{cases} 1 & \text{for} \quad y < -\dfrac{R}{2} \\ 0 & \text{for} \quad y > \dfrac{R}{2} \end{cases}$$

and $\beta_\hbar = 1 - \alpha_\hbar$.

3. Reduction to Finite Dimension

3.1. *Error estimates: $S_\hbar(u_{\vec{z},\hbar})$.*

Proposition 3.1. *There exists positive constant k_1 such that for every $\rho > 0$, we have*

$$\|S_\hbar(u_{\vec{z},\hbar})\|_2^2 \leqq k_1[e^{-2\mu\rho} + e^{-\mu R/\hbar} + (V - V(-R))^2_{(\rho\hbar)}(-R - z_1) + (V - V(R))^2_{(\rho\hbar)}(R + z_2)], \tag{7}$$

where $\mu = \min\{\sqrt{2a}, \sqrt{2b}\}$, and we use the notation $W_{(r)}(z)$ to denote the minimum of the function W on the closed interval $B_r(z)$ (See [FW.a, Lemma 3.5]*). In particular,*

$$\|S_\hbar(u_{\vec{z},\hbar})\|_2^2 \to 0 \quad \text{as} \quad (|\vec{z}|, \hbar) \to 0.$$

Proof.

$$S_\hbar(u_{\vec{z},\hbar}) = S_\hbar(u_{1,-(R+z_1)/\hbar} + u_{2,(R+z_2)/\hbar})$$
$$= -\frac{1}{2}\frac{d^2}{dy^2}(u_{1,-(R+z_1)/\hbar} + u_{2,(R+z_2)/\hbar}) + (V_\hbar - E)(u_{1,-(R+z_1)/\hbar} + u_{2,(R+z_2)/\hbar})$$

$$
\begin{aligned}
&- (u_{1,-(R+z_1)/\hbar} + u_{2,(R+z_2)/\hbar})^3 \\
&= \underbrace{S_\hbar(u_{1,-(R+z_1)/\hbar})}_{\text{(I)}} + \underbrace{S_\hbar(u_{2,(R+z_2)/\hbar})}_{\text{(II)}} \\
&\underbrace{- 3u_{1,-(R+z_1)/\hbar} \cdot u_{2,(R+z_2)/\hbar} \cdot (u^2_{1,-(R+z_1)/\hbar} + u^2_{2,(R+z_2)/\hbar})}_{\text{(III)}}.
\end{aligned}
$$

We estimate (I), (II) and (III) separately. We have

$$(\mathrm{I}) = S_\hbar(u_{1,-(R+z_1)/\hbar}) = [V_\hbar(y) - V(-R)]u_{1,-(R+z_1)/\hbar}$$

using the fact that $u_{1,-(R+z_1)/\hbar}$ satisfies (4). Similarly, we have

$$(\mathrm{II}) = S_\hbar(u_{2,(R+z_2)/\hbar}) = [V_\hbar(y) - V(R)]u_{2,(R+z_2)/\hbar}.$$

Then, these two terms can be estimated to prove

$$\|(\mathrm{I})\|_2^2 \leqq K_1[e^{-2a\rho} + (V - V(-R))^2_{(\rho\hbar)}(-R - z_1)], \tag{8}$$

$$\|(\mathrm{II})\|_2^2 \leqq K_2[e^{-2b\rho} + (V - V(R))^2_{(\rho\hbar)}(R + z_2)] \tag{9}$$

for any $\rho > 0$ in exactly the same way as in the proof of Lemma 3.5 [FW.a]. To estimate (III), first note that

$$\|u_1\|_\infty \leqq \sqrt{2a}, \quad \|u_2\|_\infty \leqq \sqrt{2b}.$$

Therefore,

$$
\begin{aligned}
\|(\mathrm{III})\|_2^2 &\leqq \int_{\mathbb{R}} 3(2a + 2b)u^2_{1,-(R+z_1)/\hbar}(y) \cdot u^2_{2,(R+z_2)/\hbar}(y)\,dy \\
&= \int_{\mathbb{R}} 3(2a + 2b)u_1^2\left(y + \frac{R+z_1}{\hbar}\right)u_2^1\left(y - \frac{R+z_2}{\hbar}\right)dy \\
&= 6(a+b)\int_0^\infty u_1^2\left(y + \frac{R+z_1}{\hbar}\right)u_2^2\left(y - \frac{R+z_2}{\hbar}\right)dy \\
&\quad + 6(a+b)\int_{-\infty}^0 u_1^2\left(y + \frac{R+z_1}{\hbar}\right) \cdot u_2^2\left(y - \frac{R+z_2}{\hbar}\right)dy \\
&= 6(a+b)\int_{-(R+z_1)/\hbar}^\infty u_1^2\left(Y + \frac{R+z_1}{\hbar} + \frac{R+z_2}{\hbar}\right)u_2^2(y)\,dy \\
&\quad + 6(a+b)\int_{-\infty}^{(R+z_2)/\hbar} u_1^2(h)u_2^2\left(y - \frac{R+z_1}{\hbar} - \frac{R+z_2}{\hbar}\right)dy \\
&\leqq 6(a+b)\left\{u_1^2\left(\frac{R+z_1}{\hbar}\right)\|u_2\|_2^2 + u_2^2\left(\frac{R+z_2}{\hbar}\right)\|u_1\|_2^2\right\} \\
&\leqq K_3(e^{-2\sqrt{2a}\cdot(R+z_1)/\hbar} + e^{-2\sqrt{2b}\cdot(R+z_2)/\hbar}).
\end{aligned}
$$

Now setting $\mu = \min\{\sqrt{2a}, \sqrt{2b}\}$ and $k_1 = \max\{K_1, K_2, K_3\}$, we have finished the proof of (7). The last statement comes from setting $\rho = \hbar^{-1/2}$. □

3.2. Estimates of the Fredholm Inverse: $S'_\hbar(u_{\vec{z},\hbar})$.

Definition 3.2.

- $K_{\vec{z},\hbar} = \text{span}\,\{u'_{1,-(R+z_1)/\hbar}, u'_{2,(R+z_2)/\hbar}\}$.
- $K^\perp_{\vec{z},\hbar} = L^2$-orthogonal complement of $K_{\vec{z},\hbar}$ in H^2.
- $\pi_{\vec{z},\hbar}, \pi^\perp_{\vec{z},\hbar}$: the restrictions to H^2 of the L^2-orthogonal projections to $K_{\vec{z},\hbar}$ and $K^\perp_{\vec{z},\hbar}$ respectively.
- $L_{\vec{z},\hbar}: \pi^\perp_{\vec{z},\hbar}(S'_{\vec{z},\hbar}(u_{\vec{z},\hbar}))$.

The operator $L_{\vec{z},\hbar}$ maps H^2 to $K^\perp_{\vec{z},\hbar}$. Now we have the following analogue of Proposition 2.3 in [FW.a].

Proposition 3.3. *There exist positive real numbers $k_2, \hbar_0$ and $\alpha_0(<\frac{1}{2})$ so that for $|z_0| < \alpha_0$, $0 < \hbar < \hbar_0$ and $u \in K^\perp_{\vec{z},\hbar}$,*

$$\| L_{\vec{z},\hbar} u \|_2 \geqq k_2 \| u \|_{H^2}.$$

Proof. We use the same indirect argument as in [FW.a]. Suppose the contrary. Then there exists $\vec{z}_i = (z_{1,i}, z_{2,i})$ and $\hbar_i$ with $|z_{1,i}|, |z_{2,i}|$ and $\hbar_i \to 0$ such that there are some $\phi_i \in K^\perp_{\vec{z}_i,\hbar_i}$ with

$$L_{\vec{z}_i,\hbar_i}\phi_i \to 0 \quad \text{and} \quad \|\phi_i\|_{H^2} = 1,$$

$$\begin{aligned}
&S'_{\hbar_i}(u_{\vec{z}_i,\hbar_i})\phi_i \\
&= -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i - 3u^2_{\vec{z}_i,\hbar_i}\phi_i \\
&= -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i - 3(u^2_{1,-(R+z_{1,i})/\hbar_i} + u^2_{2,(R+z_{2,i})/\hbar_i})\phi_i \\
&\quad - 6u_{1,-(R+z_{1,i})/\hbar_i}\cdot u_{2,(R+z_{2,i})/\hbar_i}\phi_i \\
&= \underbrace{S'_{\hbar_i}(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i}_{\text{(I)}} + \underbrace{S'_{\hbar_i}(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i}_{\text{(II)}}. \\
&\quad \underbrace{+(3(\alpha_{\hbar_i} - 1)u^2_{1,-(R+z_{1,i})/\hbar_i} + 3(\beta_{\hbar_i} - 1)u^2_{2,(R+z_{2,i})/\hbar_i} - 6u_{1,-(R+z_{1,i})/\hbar_i}\cdot u_{2,(R+z_{2,i})/\hbar_i})\phi_i}_{\text{(III)}}
\end{aligned}$$

Let us first estimate (III). By the choice of $\alpha_{\hbar_i}, \beta_{\hbar_i}$, it is easy to see that

$$\begin{aligned}
\| 3(\alpha_\hbar - 1)u^2_{1,-(R+z_1)/\hbar} \|_\infty &\leqq 3\cdot 2a\,\text{sech}^2\sqrt{2a}\left(-\frac{R+z_1}{\hbar} - \frac{R}{2\hbar}\right) \\
&\leqq K_4 \cdot e^{-2\sqrt{2a}\cdot R/2\hbar}\ (\text{using that } |z_1| < \tfrac{1}{2}). \qquad (10)
\end{aligned}$$

Similarly,

$$\| 3(\beta_\hbar - 1)u^2_{2,(R+z_2)/\hbar} \|_\infty \leqq K_5 \cdot e^{-2\sqrt{2b}\cdot R/2\hbar}. \qquad (11)$$

Next,

$$6u_{1,-(R+z_1)/\hbar}\cdot u_{2,(R+z_2)/\hbar} = 12\sqrt{ab}\,\text{sech}\sqrt{2a}\left(y + \frac{R+z_1}{\hbar}\right)\cdot\text{sech}\sqrt{2b}\left(y + \frac{R+z_2}{\hbar}\right).$$

Therefore,

$$\| 6u_{1,-(R+z_1)/\hbar} \cdot u_{2,(R+z_2)/\hbar} \|_\infty \leqq K_6 \cdot e^{-2\min\{\sqrt{2a},\sqrt{2b}\}\cdot R/2\hbar}. \tag{12}$$

Combining (10), (11) and (12), we have

$$\begin{aligned} \|(\mathrm{III})\|_2 &\leqq \| 3(\alpha_{\hbar_i} - 1)u^2_{1,-(R+z_{1,i})/\hbar_i} + 3(\beta_{\hbar_i} - 1)u^2_{2,(R+z_{2,i})/\hbar_i} \\ &\quad - 6u_{1,-(R+z_{1,i})/\hbar_i} \cdot u_{2,(R+z_{2,i})/\hbar_i} \|_\infty \| \phi_i \|_2 \\ &\leqq K_7 \cdot e^{-\mu R/\hbar_i}, \end{aligned} \tag{13}$$

where we recall that $\mu = \min\{\sqrt{2a}, \sqrt{2b}\}$.

To estimate $\pi^\perp_{\bar{z},\hbar}((\mathrm{I}) + (\mathrm{II}))$, we introduce more definitions.

Definition 3.4.

- $K_{z_1,\hbar} =$ span of $\{u'_{1,-(R+z_1)/\hbar}\}$
- $K_{z_2,\hbar} =$ span of $\{u'_{2,(R+z_2)/\hbar}\}$
- $K^\perp_{z_i,\hbar} = L^2$-orthogonal complement of $K_{z_i,\hbar}$ in H^2, $i = 1, 2$
- $\pi_{i,\hbar}, \pi^\perp_{i,\hbar}$: the projections to each of the above spaces respectively, $i = 1, 2$
- $L_{z_1,\hbar} := \pi^\perp_{z_1,\hbar} \cdot S'_\hbar(u_{1,-(R+z_1)/\hbar})$
- $L_{z_2,\hbar} := \pi^\perp_{z_2,\hbar} \cdot S'_\hbar(u_{2,(R+z_2)/\hbar})$.

Also from now on, we will use the notation $O(g(h))$ to mean that

$$|O(g(h))| \leqq C \cdot g(h),$$

where C is a constant independent of $\hbar$. Now, note that

$$\langle u_{1,-(R+z_1)/\hbar}, u_{2,(R+z_2)/\hbar} \rangle = O(e^{-\mu R/\hbar}), \tag{14}$$

where $\mu = \min\{\sqrt{2a}, \sqrt{2b}\}$.

Lemma 3.5 *There exist positive constants $\tilde{k}_3, \tilde{k}_4, k_3$ and k_4 such that for any $\phi \in H^2$ with $\|\phi\|_{H^2} = 1$,*

$$\| \pi_{z_2,\hbar} \cdot S'_\hbar(u_{1,-(R+z_1)/\hbar})\alpha_\hbar \phi \|_2 \leqq \tilde{k}_3 \cdot e^{-\sqrt{2b}R/2\hbar}, \tag{15}$$

$$\| \pi_{z_1,\hbar} \cdot S'_\hbar(u_{2,(R+z_2)/\hbar})\beta_\hbar \phi \|_2 \leqq \tilde{k}_4 \cdot e^{-\sqrt{2a}R/2\hbar} \tag{16}$$

and so

$$\| \pi^\perp_{\bar{z},\hbar} S'_\hbar(u_{1,-(R+z_1)/\hbar})\alpha_\hbar \phi - L_{z_1,\hbar}(\alpha_\hbar \phi) \|_2 \leqq k_3 e^{-\mu R/2\hbar}, \tag{17}$$

$$\| \pi^\perp_{\bar{z},\hbar} S'_\hbar(u_{2,(R+z_2)/\hbar})\beta_\hbar \phi - L_{z_2,\hbar}(\beta_\hbar \phi) \|_2 \leqq k_4 e^{-\mu R/2\hbar}. \tag{18}$$

Proof. The inequalities (15) and (16) can be easily proved using the fact that $\alpha_\hbar$ (respectively $\beta_\hbar$) has support $(-\infty, -R/2\hbar]$ (respectively $[R/2\hbar, \infty)$), that $u_{2,(R+z_2)/\hbar}$ (respectively $u_{1,-(R+z_1)/\hbar}$) has the maximum $\sim e^{-(\sqrt{2b}/2\hbar)/R}$ (respectively $e^{-(\sqrt{2a}/2\hbar)R}$) there and that $\|\phi\|_{H^2} = 1$. The inequalities (17) and (18) come from (14), (15) and (16). □

Now, let us continue the proof of Proposition 3.3. By the definition of $L_{\bar{z},\hbar}$ and the assumption in the beginning of the proof, we have

$$\| \pi^{\perp}_{\bar z,\hbar_i} S'_{\hbar_i}(u_{\bar z_i,\hbar_i})\phi_i \|_2^2 = \| L_{\bar z_i,\hbar_i}(\phi_i) \|_2^2 \to 0.$$

However,

$$\| \pi^{\perp}_{\bar z,\hbar_i} S'_{\hbar_i}(u_{\bar z_i,\hbar_i})\phi_i \|_2^2 = \| \pi^{\perp}_{\bar z,\hbar_i}(\mathrm{I}) + \| \pi^{\perp}_{\bar z,\hbar_i}(\mathrm{II}) + \pi^{\perp}_{\bar z,\hbar_i}(\mathrm{III}) \|_2^2$$
$$= \| L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i + L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \|_2^2 + O(e^{-\mu R/2\hbar})$$

by (13), (17), (18) and the fact that $\pi^{\perp}_{\bar z,\hbar_i}(\mathrm{I})$ and $\pi^{\perp}_{\bar z,\hbar_i}(\mathrm{II})$ are uniformly L^2-bounded since $\| \phi_i \|_{H^2} = 1$. Therefore,

$$\| L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i + L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \|_2^2 \to \quad \text{as} \quad i \to \infty. \tag{19}$$

Note that

$$L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i = S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i - \lambda_{1,i}u'_{1,-(R+z_{1,i})/\hbar_i},$$
$$L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i = S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar}{}^{i}\phi_i - \lambda_{2,i}u'_{2,-(R+z_{2,i})/\hbar_i},$$

where

$$\lambda_{1,i} = \frac{\langle S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i, u'_{1,-(R+z_{1,i})/\hbar_i}\rangle}{\| u'_{1,-(R+z_{1,i})/\hbar_i} \|_2^2},$$

and similarly for $\lambda_{2,i}$. Therefore,

$$\begin{aligned}
&\langle L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i, L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \rangle \\
&= \langle S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i, S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i \rangle \\
&\quad - \lambda_{2,i}\langle S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i, u'_{2,(R+z_{2,i})/\hbar_i} \rangle \\
&\quad - \lambda_{1,i}\langle u'_{1,-(R+z_{1,i})/\hbar_i} S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i - \lambda_{2,i}u'_{2,(R+z_{2,i})/\hbar_i} \rangle.
\end{aligned}$$

Since $\| \phi_i \|_{H^2} = 1$, and so $|\lambda_{1,i}|$, $|\lambda_{2,i}|$ are uniformly bounded over i, we can estimate the second and third terms above in the same way as before to prove that they are of order $O(e^{-\mu R/2\hbar_i})$, using the fact that $\alpha_{\hbar_i}$ (respectively $\beta_{\hbar_i}$) has support $(-\infty, -R/2\hbar_i]$ (respectively $[R/2\hbar_i, \infty)$). In particular, they converge to zero as $i \to \infty$. Now, let us deal with the first term, which is easier to deal with than $\langle L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i, L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \rangle$ because it involves only "local" operators, while $L_{z_{1,i},\hbar_i}$, $L_{z_{2,i},\hbar_i}$ are "nonlocal" operators as they involve projection operators. Now,

$$\begin{aligned}
&\langle S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i, S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i \rangle \\
&= \int_{-(R/2\hbar_i)}^{R/2\hbar_i} S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i \cdot S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i dy
\end{aligned} \tag{20}$$

from the locality of the operators $S'(u_{1,-(R+z_{1,i})/\hbar_i})$ and $S'(u_{2,(R+z_{2,i})/\hbar_i})$. Here,

$$\begin{aligned}
&S'(u_{1,-(R+z_{1,i})/\hbar_i})\alpha_{\hbar_i}\phi_i \\
&= -\frac{1}{2}\frac{d^2}{dy^2}(\alpha_{\hbar_i}\phi_i) + (V_{\hbar_i} - E)(\alpha_{\hbar_i}\phi_i) - 3u^2_{1,-(R+z_{1,i})/\hbar_i} \cdot \alpha_{\hbar_i}\phi_i \\
&= \alpha_{\hbar_i}\left(-\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right) - 3u^2_{1,-(R+z_{1,i})/\hbar_i}\alpha_{\hbar_i}\phi_i + \frac{1}{2}\left[\frac{d^2}{dy^2}, \alpha_{\hbar_i} \right]\phi_i,
\end{aligned}$$

$$S'(u_{2,(R+z_{2,i})/\hbar_i})\beta_{\hbar_i}\phi_i$$
$$= \beta_{\hbar_i}\left(-\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i\right) - 3u^2_{2,(R+z_{2,i})/\hbar_i}\beta_{\hbar_i}\phi_i + \frac{1}{2}\left[\frac{d^2}{dy^2}, \beta_{\hbar_i}\right]\phi_i,$$

where $[\cdot,\cdot]$ is the commutator. Note that

$$\| u^2_{1,-(R+z_{1,i})/\hbar_i}\alpha_{\hbar_i} \|_{\infty, -(R/2\hbar_i)<y<(R/2\hbar_i)} \leqq K_4 \cdot e^{-\sqrt{2a}R/\hbar_i}, \tag{21}$$

$$\| u^2_{2,(R+z_{2,i})/\hbar_i}\beta_{\hbar_i} \|_{\infty, -(R/2\hbar_i)<y<(R/2\hbar_i)} \leqq K_5 \cdot e^{-2\sqrt{2b}R/\hbar_i}, \tag{22}$$

where $\| f \|_{\infty, a<y<b} := \sup_{a<y<b} |f(y)|$. On the other hand, we have

$$\left\| \left[\frac{d^2}{dy^2}, \alpha_{\hbar_i}\right]\pi_i \right\|_2, \quad \left\| \left[\frac{d^2}{dy^2}, \beta_{\hbar_i}\right]\pi_i \right\|_2 < K_8 \hbar_i. \tag{23}$$

In fact,

$$\left[\frac{d^2}{dy^2}, \alpha_{\hbar_i}\right]\phi_i = 2\frac{d\alpha_{\hbar_i}}{dy}\frac{d\phi_i}{dy} + \frac{d^2\alpha_{\hbar_i}}{dy^2}\phi_i$$
$$= 2\hbar_i\alpha'(\hbar y)\frac{d\phi_i}{dy} + \hbar_i\alpha''(\hbar_i y)\phi_i$$
$$= \hbar_i(2\alpha'(\hbar y)\frac{d\phi_i}{dy} + \hbar_i\alpha''(\hbar_i y)\phi_i),$$

and so

$$\left\| \left[\frac{d^2}{dy^2}, \alpha_{\hbar_i}\right]\phi_i \right\|_2 \leqq \hbar_i \| 2\alpha'(\hbar y) \|_\infty \left\| \frac{d\phi_i}{dy} \right\|_2 + \hbar_i^2 \| \alpha''(\hbar_i y) \|_\infty \| \phi_i \|_2$$
$$\leqq K_8 \hbar_i$$

as $\| \phi_i \|_{H^2} = 1$, and α' and α'' have compact support by the definition of α. Combining the above discussions, we have

$$\langle L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i, L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \rangle = \int_{-(R/2\hbar_i)}^{R/2\hbar_i} \alpha^{\hbar_i}\beta_{\hbar_i} \left| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right|^2 dy + O(\hbar_i).$$

Hence, we have proved the following

Lemma 3.6.

$$\| L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i + L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \|_2^2$$
$$= \| L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i \|_2^2 + \| L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \|_2^2 + \langle L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i, L_{z_{2,i},\hbar_i}\beta_{\hbar_i}\phi_i \rangle$$
$$= \| L^1_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i \|_2^2 + \| L^2_{z_{1,i},\hbar_i}\beta_{\hbar_i}\phi_i \|_2^2$$
$$+ \int_{-(R/2\hbar_i)}^{R/2\hbar_i} \alpha_{\hbar_i}\beta_{\hbar_i} \left| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right|^2 dy + O(\hbar_i).$$

In particular, the right-hand side is a sum of positive terms modulo $O(\hbar_i)$ and so each term goes to zero separately as $i \to \infty$ since the left-hand side $\to 0$ from (19).

This will give a contradiction to our hypothesis in the beginning of the proof. Since $\|\phi_i\|_{H^2}=1$, we may assume (by passing to a subsequence) that one of the following holds:
Case I). $\|\phi_i\|_{H^2,(-\infty,-R/2\hbar_i]}\geqq \varepsilon_1>0$ for all sufficiently large i for some $\varepsilon_1>0$.
Case II). $\|\phi_i\|_{H^2,[R/2\hbar_i,\infty)}\geqq \varepsilon>0$ for all sufficiently large i for some $\varepsilon_2>0$.
Case III). $\|\phi_i\|_{H^2,(-R/2\hbar_i,R/2\hbar_i)}\to 1$ as $i\to\infty$.

First, let us suppose the Case I) holds. Define

$$L_{1,i}:=\pi_1^\perp\left(-\frac{1}{2}\frac{d^2}{dy^2}+(V_i^- -E)-3u_1^2\right)\alpha_{\hbar_i}\left(\cdot-\frac{R}{\hbar_i}\right),$$

where $\pi_1^\perp$ is the L^2-projection onto $\{u_1'\}^\perp$,

$$V_i^-(y):=V_{\hbar_i}\left(y-\frac{R+z_{1,i}}{\hbar_i}\right)=V(\hbar_i y-R-z_{1,i})$$

and $\psi_i(y):=\phi_i(y-(R+z_{1,i}/\hbar_i)$. Then

$$L_{1,i}\psi_i\to 0 \tag{24}$$

because $L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i\to 0$ from Lemma 3.6 and $L_{1,i}\psi_i$ is nothing but the translation by $R/\hbar_i$ of $L_{z_{1,i},\hbar_i}\alpha_{\hbar_i}\phi_i$. Since $\|\psi_i\|_{H^2}=\|\phi_i\|_{H^2}=1$, we may assume (by passing to a subsequence) that ψ_i converges weakly to some ψ_∞ in H^2. It is easy to see that

$$\langle\psi_\infty,u_1'\rangle=0 \tag{25}$$

as $\langle\psi_i,u_1'\rangle=0$ from $\langle\phi_i,u_1',-(R+z_{1,i}/\hbar_i\rangle=0$ by the hypothesis. Defining

$$L_{1,0}:=-\frac{1}{2}\frac{d^2}{dy^2}+\alpha-3u_1^2,\quad a=V(-R)-E,$$

we show in the same way as in p. 401 [FW.a] that

$$L_{1,0}\psi_\infty=0. \tag{26}$$

In fact, for any given bounded interval $\Omega\subset\mathbb{R}$,

$$\begin{aligned}\|L_{1,0}\psi_i\|_{2,\Omega}&=\|\pi_1^\perp L_{1,0}\psi_i\|_{2,\Omega}\quad\text{from}\quad\langle\psi_i,u_1'\rangle=0\\&=\|[L_{1,i}-\pi_1^\perp(V_i^- -V(-R))]\psi_i\|_{2,\Omega}+O(e^{-\sqrt{2a}R/\hbar_i})\\&\qquad\text{for sufficiently large } i \text{ such that } \alpha_{\hbar_i}\left(\cdot-\frac{R}{\hbar_i}\right)\equiv 1\quad\text{on}\quad\Omega\\&\leqq\|L_{1,i}\psi_i\|_2+\max_{y\in\Omega}|V_i(y)-V(-R)|\cdot\|\psi_i\|_2+O(e^{-\sqrt{2a}R/\hbar_i}).\end{aligned}$$

Since $\lim_{i\to\infty}\max_{y\in\Omega}|V_i(y)-V(-R)|=0$ on any bounded interval Ω and from (24), we have

$$\lim_{i\to\infty}\|L_{1,0}\psi_i\|_{2,\Omega}=0 \tag{27}$$

on any bounded interval. The weak convergence of ψ_i to ψ_∞ in H^2 implies the weak convergence of $L_{1,0}\psi_i$ to $L_{1,0}\psi_\infty$ in L^2, and hence the weak convergence of the restrictions to Ω. From (27), the restriction of $L_{1,0}\psi_\infty$ to each bounded Ω is 0 and thus $L_{1,0}\psi_\infty = 0$. From the nondegeneracy of $L_{1,0}$, (25) and (26),

$$\psi_\infty = 0.$$

Since $\psi_i \to \psi_\infty$ weakly on H^2, $\psi_i \to \psi_\infty$ on the bounded interval in the L^2 sense. Since u_1^2 has the exponential decay, we have

$$3u_1^2\psi_i \to 0 \tag{28}$$

in the L^2 sense. Combining (24) and (28), we have

$$\| \pi_1^\perp H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i \|_2 \to 0, \tag{29}$$

where

$$H_i^- := -\frac{1}{2}\frac{d^2}{dy^2} + (V_i^- - E).$$

Since we assume that Case I) holds,

$$\left\|\alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\|_{H^2} = \|\alpha_{\hbar_i}\phi_i\|_{H^2} \geqq \|\phi_i\|_{H^2,(-\infty,-R/2\hbar_i]} \geqq \varepsilon_1. \tag{30}$$

On the other hand, we have

$$\left\|H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\|_2 \geqq \lambda \left\|\alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\|_{H^2}$$

by Corollary 2.4 and so

$$\left\|H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\|_2 \geqq \lambda\varepsilon_1 \tag{31}$$

for all i by (30). From (29) and (31), we have

$$\lim_{i\to\infty} \frac{\left\langle u_1', H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\rangle}{\|u_1'\|_2 \left\|H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\|_2} = 1. \tag{32}$$

Since u_1' is annihilated by $L_{1,0}$,

$$H_i^- u_1' = [V_i - V(-R) + 3u_1^2]u_1', \tag{33}$$

and so by the self-adjointness of H_i^-, we have

$$\left\langle u_1', H_i^- \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\rangle = \left\langle [V_i - V(-R)]u_1', \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\rangle + 3\left\langle u_1^2 u_1', \alpha_{\hbar_i}\left(\cdot - \frac{R}{\hbar_i}\right)\psi_i\right\rangle.$$

Then first term goes to 0 since $[V_i - V(-R)]u_1' \to 0$ in L^2 and $\alpha_{\hbar_i}(\cdot - (R/\hbar_i)\psi_i$ is L^2-bounded. The second term goes to 0 since $\psi_i \to 0$ weakly, $\|\psi_i\|_2 \leqq \|\psi\|_{H^2} = 1$ and $\alpha_{\hbar_i}(\cdot, -(R/\hbar_i))u_1^2 u_1' \to 0$ in L^2. This contradicts to (32). So we have taken care of Case I). Case II) can be taken care of in the same way. Therefore, we suppose Case III) holds now. Then

$$\lim_{i\to\infty} \| \phi_i \|_{H^2,(-\infty,-R/2\hbar_i]} = \lim_{i\to\infty} \| \phi_i \|_{H^2,[R/2\hbar_i,\infty)} = 0, \tag{34}$$

$$\begin{aligned} \| L_{\bar{z}_i,\hbar_i}\phi_i \|_2^2 &= \| \pi^{\perp}_{\bar{z}_i,\hbar_i} S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i \|_2^2 \\ &= \| S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i \|_2^2 - \left| \frac{\langle S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i, u_{1,-(R+z_{1,i})/\hbar_i} \rangle}{\| u'_{1,-(R+z_{1,i})/\hbar_i} \|_2^2} \right|^2 \\ &\quad - \left| \frac{\langle S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i, u'_{2,(R+z_{2,i})/\hbar_i} \rangle}{\| u'_{2,(R+z_{2,i})/\hbar_i} \|_2^2} \right|^2 + O(e^{-\mu R/2\hbar_i}) \end{aligned} \tag{35}$$

from (14). (Note that if $u_{1,-(R+z_{1,i})/\hbar_i}$ and $u_{2,(R+z_{2,i})/\hbar_i}$ were orthogonal, the equality would be exact.) However,

$$\begin{aligned} \langle S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i, u_{1,-(R+z_{1,i})/\hbar_i} \rangle &= \langle \phi_i, S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i}) u'_{1,-(R+z_{1,i})/\hbar_i} \rangle \\ &= \int_{-R/2\hbar_i}^{R/2\hbar_i} \phi_i \cdot S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i}) u'_{1,-(R+z_{1,i})/\hbar_i}\, dy \\ &\quad + \left(\int_{(-\infty,-R/2\hbar_i]} + \int_{[R/2\hbar_i,\infty)} \right) \phi_i \cdot S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i}) u'_{1,-(R+z_{1,i})/\hbar_i}\, dy. \end{aligned} \tag{36}$$

Since $S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})u_{1,-(R+z_{1,i})/\hbar_i}$ is uniformly L^2-bounded, the second term goes to 0 by (34). Moreover, $\| S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})u_{1,-(R+z_{1,i})/\hbar_i} \|_{\infty,[-R/2\hbar_i,R/2\hbar_i]} = O(e^{-\sqrt{2a}R/2\hbar_i})$, and so the first term in (36) also goes to zero. Therefore the second term in (35) goes to zero.

In the same way, we prove that the third term also goes to zero in (35). Since we assume $L_{\bar{z},\hbar_i}\phi_i \to 0$ in L^2 by the hypothesis, we have

$$\| S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i \|_2 \to 0. \tag{37}$$

And it is easy to see by the same way as before that

$$\begin{aligned} \| S'_{\hbar_i}(u_{\bar{z}_i,\hbar_i})\phi_i \|_2^2 &= \left\| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i - 3u^2_{\bar{z}_i,\hbar_i}\phi_i \right\|_2^2 \\ &= \left\| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right\|_2^2 + O(e^{-\mu R/2\hbar_i}) \\ &\quad + O(\| \phi_i \|^2_{H^2,(-\infty,-R/2\hbar_i]}) + O(\| \phi_i \|^2_{H^2,[R/2\hbar_i,\infty)}). \end{aligned}$$

Hence,

$$\left\| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right\|_2 \to 0 \tag{38}$$

by (34) and (37). On the other hand, we have

$$\left\| -\frac{1}{2}\frac{d^2\phi_i}{dy^2} + (V_{\hbar_i} - E)\phi_i \right\|_2 \geqq \lambda \| \phi_i \|_{H^2} = \lambda > 0$$

by Corollary 2.4, where λ is independent of i. This contradicts to (38) and so we finally finish the proof of Proposition 3.3. □

3.3. Reduction. Now we are ready to find a solution of the equation

$$-\tfrac{1}{2}u'' + (V_\hbar - E)u - u^3 = 0$$

modulo $K_{\vec{z},\hbar}$ for sufficiently small $\hbar > 0$.

Proposition 3.6. *There exist positive constants k_5, α_2 and $\hbar_1$ so that for every $\vec{z} = (z_1, z_2)$ and $\hbar$ with $|z_i| < \alpha_2$, $i = 1, 2$ and $0 < \hbar < \hbar_2$, there exists a unique element $\phi_{\vec{z},\hbar} \in K^\perp_{\vec{z},\hbar}$ such that*

$$S_\hbar(u_{\vec{z},\hbar} + \phi_{\vec{z},\hbar}) \in K_{\vec{z},\hbar}$$

and

$$\|\phi_{\vec{z},\hbar}\|_{H^2} \leqq k_5 \|S_\hbar(u_{\vec{z},\hbar})\|_2. \tag{39}$$

Proof. Once we have Proposition 3.1 and 3.3, the proof is the same as the one of Proposition 3.7 [FW.a]. We invite the readers to provide their own proof or refer them to [FW.a] for details. □

4. The Reduced Problem

In this section, we will prove our main existence theorem for the one-dimensional two-lump case with a bounded V.

Theorem 4.1. *Let $V \in (V)_a$ for some $a \in \mathbb{R}$ and $E < a$, and V be bounded. Then for each pair (x_1, x_2) of nondegenerate critical points of V, there is an $\hbar_3 > 0$ such that for all $\hbar$ with $0 < \hbar < \hbar_3$, the equation*

$$-\tfrac{1}{2}\hbar^2 u'' + (V - E)v - u^2 = 0$$

has a nonzero solution with the following concentration phenomena: For each given $\varepsilon, \delta > 0$, there exists some $\bar{\hbar} > 0$ such that if $0 < \hbar < \bar{\hbar}$,

$$\sup_{x \in B_\delta(x_1)} |u(x)| > k_6 \sim \sqrt{2(V(x_1) - E)},$$

$$\sup_{x \in B_\delta(x_2)} |u(x)| > k_7 \sim \sqrt{2(V(x_2) - E)}$$

and

$$\sup_{x \in \mathbb{R} \setminus B_\delta(x_1) \cup B_\delta(x_2)} |v(x)| < \varepsilon,$$

where k_6, k_7 are independent of $\hbar$.

Here we would like to note that u implicitly depends on $\hbar$.

Proof. Let α_2 and $\hbar_2$ be the constants from Proposition 3.7 and suppose $\hbar < \hbar_2$. We project $S_\hbar(u_{\vec{z},\hbar} + \phi_{\vec{z},\hbar})$ onto the space $K_{\vec{z},\hbar}$ to define a reduced "vector field" $s_\hbar : (-\alpha_2, \alpha_2) \times (-\alpha_2, \alpha_2) \to \mathbb{R}^2$ by $s_\hbar(\vec{z}) = (s_{\hbar,1}(\vec{z}), s_{\hbar,2}(\vec{z}))$, where

$$s_{\hbar,j}(\vec{z}) = \frac{1}{\hbar}\langle S_\hbar(u_{\vec{z},\hbar} + \phi_{\vec{z},\hbar}), u'_{j,z_j,\hbar}\rangle,$$

where $u'_{j,z_j,\hbar}(y)=u'_j(y-(x_j+z_j)/\hbar)$ for $j=1,2$. Consider the "linear vector field" $v_0(\vec{z})=(v_{0,1}(\vec{z}),v_{0,2}(\vec{z}))$, where

$$v_{0,1}(\vec{z})=-\tfrac{1}{2}|u_1|_2^2V''(x_1)z_1,$$
$$v_{0,2}(\vec{z})=-\tfrac{1}{2}|u_2|_2^2V''(x_2)z_2$$

together with the family of vector fields $v_\hbar$ defined on the square $[-1,1]^2$ by

$$v_\hbar(\vec{z})=\hbar^{-\nu}s_\hbar(\hbar^\nu\vec{z}),$$

where ν is a fixed number chosen $1<\nu<2$ and $\hbar\leqq\min(\hbar_2,\alpha_2^{1/2})$. Then we have the following proposition. (See Proposition 4.4 [FW.a].)

Proposition 4.2. *The vector fields $v_\hbar$ converges uniformly to v_0 on $[-1,1]^2$.*

Assuming this proposition for the moment, let us proceed with the proof of Theorem 4.1. We have only to prove that $v_\hbar(\vec{z})$ has a zero in $(-1,1)^2$ and so $s_\hbar(\vec{z})$ has a zero in $(-\alpha_2,\alpha_2)^2$ because this together with that

$$S_\hbar(u_{\vec{z},\hbar}+\phi_{\vec{z},\hbar})\in K_{\vec{z},\hbar}=\operatorname{span}\{u'_{1,z_1,\hbar},u'_{2,z_2,\hbar}\}$$

implies

$$S_\hbar(u_{\vec{z},\hbar}+\phi_{\vec{z},\hbar})=0$$

for some z with $-\hbar^\nu<z<\hbar^\nu$.

Note that the vector field $u_0(\vec{z})$ has degree $+1$ or -1 depending on the signs of $V''(x_1)$ and $V''(x_2)$. In any case, the degree is nonzero. Moreover, $v_0(\vec{z})$ never vanishes on the boundary of $[-1,1]^2$. Now by Proposition 4.2, $v_\hbar$ is homotopic to v_0 under the homotopy which never vanishes on $\partial[-1,1]^2$. Since the degree is invariant under such a homotopy of vector fields, $v_\hbar$ will have nonzero degree and so must have a zero in $(-1,1)^2$.

For the last statement in Theorem 4.1, recall the solutions of (2) corresponding to the solution u of (3) is

$$u(x)=u\left(\frac{x}{\hbar}\right)=u_1\left(\frac{x-x_1-z_1}{\hbar}\right)+u_2\left(\frac{s-x_2-z_2}{\hbar}\right)+\phi_{\vec{z},\hbar}\left(\frac{x}{\hbar}\right)$$

and

$$u_1(y)=\sqrt{2a}\operatorname{sech}\sqrt{2a}y,\quad a=V(x_1)-E,$$
$$u_2(y)=\sqrt{2b}\operatorname{sech}\sqrt{2b}y,\quad b=V(x_2)-E.$$

Here as $\hbar\to0$, $z/\hbar\to0$ since we choose z so that $|z|<\hbar^\nu$ and $\nu>1$, and $\phi_{\vec{z},\hbar}(\cdot/\hbar)$ converges to 0 uniformly by Proposition 3.3, 3.7 and the Sobolev inequality. Moreover, $u_1((x-x_1-z_1)/\hbar)$ and $u_2((x-x_2-z_2)/\hbar)$ become more and more concentrated at x_1,x_2 and have maximum values $\sqrt{2(V(x_1)-E)}$ and $\sqrt{2(V(x_2)-E)}$ respectively. This finishes the proof of Theorem 4.1. □

Now we have only to prove Proposition 4.2.

Proof of Proposition 4.2. We will closely follow the proof of Proposition 4.4 [FW.a] but we have to take care of two-lumps, which complicates the estimates. The expansion (6) gives

$$\hbar s'_{\hbar,j}(\vec{z}) = \langle u'_{j,z_j,\hbar}, S_\hbar(u_{\vec{z},\hbar} + \phi_{\vec{z},\hbar})\rangle$$
$$= \underbrace{\langle u'_{j,z_j,\hbar}, S_\hbar(u_{\vec{z},\hbar})\rangle}_{(I)} + \underbrace{\langle u'_{j,z_j,\hbar}, S'_\hbar(u_{\vec{z},\hbar})\phi_{\vec{z},\hbar}\rangle}_{(II)}$$
$$+ \underbrace{\langle u'_{j,z_j,\hbar}, N_{\vec{z},\hbar}(\phi_{\vec{z},\hbar})\rangle}_{(III)}$$

for $j = 1, 2$.

For the simplicity, we assume $j = 1$, and $j = 2$ can be dealt with in the same way as $j = 1$,

$$(I) = \langle u'_{1,z_1,\hbar}, S_\hbar(u_{\vec{z},\hbar})\rangle$$
$$= \left\langle u'_{1,z_1,\hbar}, -\frac{1}{2}\frac{d^2}{dy^2}(u_{1,z_1,\hbar} + u_{2,z_2,\hbar}) + (V_\hbar - E)(u_{1,z_1,\hbar} + u_{2,z_2,\hbar}) - (u_{1,z_1,z\hbar} + u_{2,z_2,\hbar})^3\right\rangle$$
$$= \langle u'_{1,z_1,\hbar}, S_\hbar(u_{1,z_1,\hbar})\rangle$$
$$+ \left\langle u'_{1,z_1,\hbar}, -\frac{1}{2}\frac{d^2}{dy^2}u_{2,z_2,\hbar} + (V_\hbar - E)u_{2,z_2,\hbar} - (u_{1,z_1,\hbar} + u_{2,z_2,\hbar})^3 - u^3_{1,z_1,\hbar}\right\rangle.$$

The second term can be estimated to prove that it is of order $O(e^{-\mu R/2\hbar})$, $R > 0$ fixed as before by noting that it involves products of u_1 and u_2. (Here, we again assume that $x_1 = -R$ $x_2 = +R$.) For the first term, the same argument as in [FW] works to prove that

$$\langle u'_{1,z_1,\hbar}, S_\hbar(u_{1,z_1,\hbar})\rangle = -\frac{1}{2}\left\langle u_{1,z_1}, V'_\hbar\left(\cdot - \frac{R}{\hbar}\right)u_{1,z_1}\right\rangle.$$

Therefore

$$(I) = -\frac{1}{2}\left\langle u_{1,z_1}, V'_\hbar\left(\cdot - \frac{R}{\hbar}\right)u_{1,z_1}\right\rangle + O(e^{-\mu R/2\hbar}).$$

Then by exactly the same argument as in p. 406 in [FW.a], we have

$$\left|\frac{(I)}{\hbar} - u'_0(z)\right| \leqq C_1[(|z_1| + \rho\hbar)^2 + e^{-\mu\rho}] \tag{40}$$

for any $\rho > 0$.

Next, it is easily to see that

$$|(III)| \leqq C_2\|\phi_{\vec{z},\hbar}\|^2_{H^2} \leqq C_3\|S_\hbar(u_{\vec{z},\hbar})\|^2_2 \tag{41}$$

by Proposition 3.1 for the second inequality and by the fact that $\|N^{(\phi)}_{\vec{z},\hbar}\|_2 \leqq C\|\phi\|^2_{H^2}$ for the first inequality, which can be easily proven. (See Lemma 3.2 [FW.a].)

Now, let us estimate the term (II). Note that

$$S'_\hbar(u_{\vec{z},\hbar}) = -\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - 3|u_{1,z_1,\hbar} + u_{2,z_2,\hbar}|^2$$
$$= S'_\hbar(u_{1,z_1,\hbar}) - 3(2u_{1,z_1,\hbar}\cdot u_{2,z_2,\hbar} + u^2_{2,z_2,\hbar}).$$

Therefore,

$$(\mathrm{II}) = \langle u'_{1,z_1,\hbar}, S'_\hbar(u_{\vec{z},\hbar})\phi_{\vec{z},\hbar}\rangle$$
$$= \langle u'_{1,z_1,\hbar}, S'_\hbar(u_{1,z_1,\hbar})\phi_{\vec{z},\hbar}\rangle - \langle u'_{1,z_1,\hbar}, 3(2u_{1,z_1,\hbar}\cdot u_{2,z_1,\hbar} + u^2_{2,z_2,\hbar})\phi_{\vec{z},\hbar}\rangle.$$

In the same way as estimating the second term in (I), we prove that the second term in (II) is of order $O(e^{-\mu R/2\hbar})$. For the first term,

$$|\langle u'_{1,z_1,\hbar}, S'_\hbar(u_{1,z_1,\hbar})\phi_{\vec{z},\hbar}\rangle| = |\langle S'_\hbar(u_{1,z_1,\hbar})u'_{1,z_1,\hbar}, \phi_{\vec{z},\hbar}\rangle|$$
$$= \left|\left\langle\left(V_\hbar - V_\hbar\left(-\frac{R}{\hbar}\right)\right)u'_{1,z_1,\hbar}, \phi_{\vec{z},\hbar}\right\rangle\right|$$
$$\leqq C_5\left\|\left(V_\hbar - V_\hbar\left(-\frac{R}{\hbar}\right)\right)u'_{1,z_1,\hbar}\right\| \|S_\hbar(u_{\vec{z},\hbar})\|_2. \tag{42}$$

Once again, the factor $(V_\hbar - V_\hbar(-R/\hbar))u'_{1,z_1,\hbar}$ can be estimated in the same way as $S_\hbar(u_{\vec{z},\hbar})$ in Proposition 3.1, which this time becomes easier.

Together with Proposition 3.1, (41) and (42) yield

$$|(\mathrm{II}) + (\mathrm{III})| \leqq C_6\left[e^{-\mu\rho} + \left(V - V\left(-\frac{R}{\hbar}\right)\right)^2_{(\rho\hbar)}(-R - z_1)\right].$$

Therefore,

$$|v_{\hbar,1}(\vec{z}) - v_{0,1}(\vec{z})| = |\hbar^{-\nu}s_\hbar(\hbar^\nu\vec{z}) - v_{0,1}(\hbar^\nu\vec{z})| = |\hbar^{-\nu}(s_\hbar(\hbar^\nu\vec{z}) - v_{0,1}(\hbar^\nu\vec{z}))|$$
$$\leqq \hbar^{-\nu}\{C_2(\hbar^\nu|z_1| + \rho\hbar)^2 + e^{-\mu\rho} + C_6(\hbar^\nu|z_1| + \rho\hbar)^4 + e^{-\mu\rho}/\hbar)\}.$$

Choosing $\rho = \hbar^{-\varepsilon}$ with $\varepsilon > 0$ which will be chosen later, and recalling that $|z_1| \leqq 1$, we have

$$|v_{\hbar,1}(\vec{z}) - v_{0,1}(\vec{z})| \leqq C_2\hbar^{-\nu}(\hbar^\nu + \hbar^{1-\varepsilon})^2 + C_6\hbar^{-(1+\nu)}(\hbar^\nu + \hbar^{1-\varepsilon})^4$$
$$+ C_7\hbar^{-\nu}\exp(-\mu\hbar^{-\varepsilon}).$$

Since we assume that $\nu < 2$, we can choose $\varepsilon > 0$ small enough so that all terms go to zero as $\hbar \to 0$. Hence we have proved that

$$|v_{\hbar,1}(\vec{z}) - v_{0,1}(\vec{z})| \to 0$$

uniformly over $\vec{z} \in [-1, 1]^2$. Similarly, we can prove that

$$|v_{\hbar,2}(\vec{z}) - v_{0,2}(\vec{z})| \to 0$$

uniformly over $\vec{z} \in [-1, 1]^2$. Hence Proposition 4.2. □

5. Instability and Positivity

In the previous sections, we have constructed a solution $u_\hbar$ of the form $u_\hbar = u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar$ of the equation

$$-\frac{1}{2}\frac{d^2u}{dy^2}+(V_\hbar-E)u-u^3=0,$$

where $u_{1,\hbar}(y)=u_1(y-(x_1+z_2)/\hbar)$, $u_{2,\hbar}(y)=u_2(y-(x_2+z_2)/\hbar)$ and $\phi_\hbar$ satisfies the following estimates:

$$\begin{aligned}\|\phi_\hbar\|_{H^2}^2 \leqq k_1k_5[e^{-2\mu\rho}+e^{-\mu R/\hbar}+(V-V(-R))^2_{(\rho\hbar)}(-R+z_1)\\ +(V-V(R))^2_{(\rho\hbar)}(R+z_2)],\end{aligned} \tag{43}$$

where we assume $x_1=-R, x_2=R$ and $|z_i|<\hbar^\nu, \nu$ being chosen any $1<\nu<2$.

In this section, we study the stability of these solutions with respect to the perturbations of initial condition. We consider the rescaled version of (1),

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial y^2}+V_\hbar\psi-|\psi|^2\psi \tag{44}$$

as in [O3]. First, we introduce the definition of Lyapunov stability.

Definition 5.1.

$$\begin{aligned}\mathcal{O}_{u_\hbar}&:=\{u_\hbar e^{i\theta}\in Q(H_\hbar)\,|\,\theta\in\mathbb{R}\},\\ \rho^2_{\mathcal{O}_{u_\hbar}}(\phi)&:=\inf_{\theta\in\mathbb{R}}(\langle(H_\hbar-E)(e^{i\theta}\phi-u_\hbar),e^{i\theta}\phi-u_\hbar\rangle)\\ &=\inf_{\theta\in\mathbb{R}}(\tfrac{1}{2}\langle e^{i\theta}\nabla\phi-\nabla u_\hbar,e^{i\theta}\nabla\phi-\nabla u_\hbar\rangle)\\ &\quad+\langle(V_\hbar-E)(e^{i\theta}\phi-u_\hbar),e^{i\theta}\phi-u_\hbar\rangle),\end{aligned}$$

where $H_\hbar=-\frac{1}{2}d^2/dy^2+V_\hbar$. Of course if we assume that $V_\hbar$ is bounded, then $Q(H_\hbar)=H^1(\mathbb{R})$ and we may use H^1-norm instead of the form norm used above.

Definition 5.2. The solution $u_\hbar$ is (Lyapunov) stable if for any given $\varepsilon>0$, there exists a $\delta>0$ such that if $\rho_{\mathcal{O}_{u_\hbar}}(\psi(0))<\delta$, then $\rho_{\mathcal{O}_{u_\hbar}}(\psi(t))<\varepsilon$ for all $t\in\mathbb{R}$, where $\psi(t)$ satisfies the time-independent equation (44).

Now, we are ready to state the main theorem in this section.

Theorem 5.3. *The solutions $u_\hbar$ we found in the previous sections are all unstable for $N\geqq 2$ if $\hbar$ is sufficiently small.*

Remark. In [O3], we proved that for the one-lump case, the solution is stable if it is localized near a local minimum and unstable if it is localized near a local maximum (see also [GrSS, Gr]). The above theorem says that once the solution has more than two lumps, the lumps begin to interact with each other to give the instability. It would be interesting to study how they interact.

If we linearize (44) at $u_\hbar$, we have the following real and imaginary parts of the linearized operator,

$$L_\hbar^+=-\frac{1}{2}\frac{d^2}{dy^2}+(V_\hbar-E)-3u_\hbar^2,$$

$$L_\hbar^-=-\frac{1}{2}\frac{d^2}{dy^2}+(V_\hbar-E)-u_\hbar^2.$$

We will use the following instability criterion by Jones and Grillakis to get the instability result (See e.g. [Gr, Jo]).

Instability criterion) Define $N^{\pm} = \#$ of strictly negative eigenvalues of $L_{\hbar}^{\pm}$ respectively. If $|N^+ - N^-| \geqq 2$, then the linearized operator at $u_\hbar$ of (44) $\begin{pmatrix} 0 & L_\hbar^- \\ -L_\hbar^+ & 0 \end{pmatrix}$ written in the Hamiltonian form has real positive eigenvalues and so $u_\hbar$ is unstable.

Having this criterion in mind, we will study the spectral properties of $L_\hbar^+$ and $L_\hbar^-$. For this, we need the following lemma. (See Lemma 3.1 [O3].)

Lemma 5.4.
i) *The operator* $L_+(\lambda) = -\frac{1}{2}d^2/dy^2 + \lambda - 3u_0^2$ *has just one negative eigenvalue* -3λ *and one dimensional kernel in* H^1, *where* $u_0 = \sqrt{2\lambda}\,\mathrm{sech}\sqrt{2\lambda}y$. *Moreover, the corresponding eigenspaces are spanned by* u_0^2 *and* u_0' *respectively.*
ii) *The operator* $L_-(\lambda) = -\frac{1}{2}d^2/dy^2 + \lambda - u_0^2$ *has one dimensional kernel in* H^1 *which is spanned by* u_0.
iii) *Neither operator has positive eigenvalues and*

$$\inf \mathrm{ess}(L_+(\lambda)) = \inf \mathrm{ess}(L_-(\lambda)) = \lambda.$$

Proof. See Lemma 3.1 in [O3]. □

Corollary 5.5.
i) *Let* v *be orthogonal to the eigenspaces of* L_+^0, i.e., $v \perp \mathrm{span}\{u_0', u_0^2\}$. *Then,*

$$\langle L_+(\lambda)v, v\rangle \geqq \lambda\langle v, v\rangle.$$

ii) *Let* v *be orthogonal to the eigenspace of* $L_-(\lambda)$ i.e. *the* span $\{u_0\}$. *Then,*

$$\langle L_-(\lambda)v, v\rangle \geqq \lambda\langle v, v\rangle.$$

Proof. We leave the proof to the reader or refer to Corollary 3.2 [O3]. □

Using this, we prove the following propositions, which will imply Theorem 5.3 by the instability criterion.

Proposition 5.6. *There exists* $\hbar_4 > 0$ *such that if* $0 < \hbar < \hbar_4$, *then* $L_\hbar^-$ *has no negative spectrum and one dimensional kernel spanned by* $u_\hbar$.

Proposition 5.7. *There exists* $\hbar_5 > 0$ *such that if* $0 < \hbar < \hbar_5$, *then* $L_\hbar^+$ *has at least two negative eigenvalues.*

Before proving proposition, we give the proof of the fact that $u_\hbar$ is positive if $0 < \hbar < \hbar_4$ as a corollary of Proposition 5.6. We first need the following well-known fact on Schrödinger operators (See [ReS] or [GlJa]).

Lemma 5.8. *Consider the Schrödinger operator on* $\mathbb{R}^n$,

$$H = -\Delta + V,$$

and suppose that H *has a ground state,* i.e. *the bound state of the lowest eigenvalue. Then the ground state is nodeless and may be chosen strictly positive on* $\mathbb{R}^n$. *Moreover, the positive ground state is unique up to positive scalar multiple.*

Proof. See Theorem 3.3.2 and Corollary 3.3.4 in [GlJa]. □

Theorem 5.9. *The solution $u_\hbar$ found in the previous sections is positive, if $0 < \hbar < \hbar_4$ so that $L^{\hbar}_{-}$ satisfies the properties in Proposition 5.6.*

Proof. This is immediate from Proposition 5.6 and Lemma 5.8. □

Now, we go back to the proofs of Propositions 5.6, 5.7. We first need the following lemma.

Lemma 5.10. *Let $\phi_\hbar$ be as in (43). Then we have for any fixed $0 < \varepsilon < 1$.*

$$\|\phi_\hbar\|^2_{H^2} \leqq k_8^2 \hbar^{4(1-\varepsilon)}$$

if $0 < \hbar < \hbar_6$ for some $\hbar_6 > 0$, where k_8 depends only on ε.

Proof. See Lemma 2.2 [O3]. □

Proof of Proposition 5.6. We know that $u_\hbar$ satisfies the equation

$$L_\hbar^- u_\hbar = -\frac{1}{2}\frac{d^2 u_\hbar}{dy^2} + (V_\hbar - E)u_\hbar - u_\hbar^2 u_\hbar = 0$$

by the definition of $u_\hbar$, i.e., $u_\hbar \in \ker L_\hbar^-$. We will prove

$$\inf_{v \perp \{u_{1,\hbar}, u_{2,\hbar}\}} \frac{\langle L_\hbar^- v, v\rangle}{\langle v, v\rangle} > c > 0 \tag{45}$$

in Lemma 5.12 where c does not depend on $\hbar$ if $\hbar$ is sufficiently small. Assuming this for the moment, let us proceed with the proof. Equation (45) means that the restriction of $L_\hbar^-$ to $\{u_{1,\hbar}, u_{2,\hbar}\}^\perp$ is positive definite and so $L_\hbar^-$ has at most one negative eigenvalue because we already know that $L_\hbar^-$ has one zero eigenvalue with the eigenfunction $u_\hbar = u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar$. Suppose $L_\hbar^-$ has one negative eigenvalue and the corresponding eigenfunction w with $\|w\|_2 = 1$. Decompose

$$w = p u_{1,\hbar} + q u_{2,\hbar} + w_\perp,$$

where $w_\perp$ is the orthogonal projection to $\{u_{1,\hbar}, u_{2,\hbar}\}^\perp$,

$$\begin{aligned}\langle u_\hbar, w\rangle &= \langle u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar, w\rangle \\ &= \langle u_{1,\hbar} + u_{2,\hbar}, w\rangle + \langle \phi_\hbar, w\rangle \\ &= p\|u_{1,\hbar}\|^2 + q\|u_{2\hbar}\|^2 + (p+q)\langle u_{1,\hbar}, u_{2,\hbar}\rangle + \langle \phi_\hbar, w\rangle.\end{aligned}$$

Recall that $\langle u_{1,\hbar}, u_{2,\hbar}\rangle = O(e^{-\mu R/\hbar})$ from (14). And from Lemma 5.10 and $\|w\|_2 = 1$,

$$\langle \phi_\hbar, w\rangle = O(\hbar^{3/2})$$

by choosing $\varepsilon = \frac{1}{4}$. Therefore,

$$0 = \langle u_\hbar, w\rangle = p\|u_{1,\hbar}\|^2 + q\|u_{2,\hbar}\|^2 + O(\hbar^{3/2}).$$

Hence, if $\hbar$ is sufficiently small, then p and q have different signs. On the other hand, we will prove

$$\langle L_\hbar^-(p u_{1,\hbar} + q u_{2,\hbar}), v\rangle = O(\hbar^{3/2}) \tag{46}$$

for any v with $\|v\|_2 = 1$ in Lemma 5.11. Then,

$$\begin{aligned} 0 > \langle L_\hbar^- w, w\rangle &= \langle L_\hbar^- w_\perp, w_\perp\rangle + L_\hbar^-(pu_{1,\hbar} + qu_{2,\hbar}), w\rangle \\ &= \langle L_\hbar^- w_\perp, w_\perp\rangle + O(\hbar^{3/2}) \\ &= c\langle w_\perp, w_\perp\rangle + O(\hbar^{3/2}) \end{aligned}$$

by (45). Therefore we have

$$0 < \|w_\perp\|_2^2 = \langle w_\perp, w_\perp\rangle \leqq O(\hbar^{3/2}). \tag{47}$$

Moreover, we also have

$$\begin{aligned} |L_\hbar^- w_\perp, w_\perp\rangle| &= \left|\left\langle \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u_\hbar^2\right) w_\perp, w_\perp \right\rangle\right| \\ &= \left|\frac{1}{2}\left\|\frac{dw_\perp}{dx}\right\|_2^2 + \langle (V_\hbar - E - u_\hbar^2) w_\perp, w_\perp\rangle\right| \leqq O(\hbar^{3/2}). \end{aligned}$$

Since $V_\hbar$, $u_\hbar^2$ are bounded, we have

$$\left\|\frac{dw_\perp}{dx}\right\|_2^2 = O(\hbar^{3/2}). \tag{48}$$

Combining (47) and (48), we proved that

$$\|w_\perp\|_{H^1} \to 0 \quad \text{as} \quad \hbar \to 0.$$

(Here, note that $w_\perp$ implicitly on $\hbar$.) By the Sobolev inequality $H^1 \hookrightarrow L_\infty$, $w_\perp$ uniformly converges to zero. Also, p and q in

$$w = pu_{1,\hbar} + qu_{2,\hbar} + w_\perp$$

satisfy

$$\begin{aligned} p^2\|u_{1,\hbar}\|^2 + q^2\|u_{2,\hbar}\|^2 + O(\hbar^{3/2}) = \|w\|_2^2 = 1, \\ p\|u_{1,\hbar}\|^2 + q\|u_{2,\hbar}\|^2 + O(\hbar^{3/2}) = 0, \end{aligned}$$

and so

$$\begin{aligned} p &\cong \frac{\|u_{2,\hbar}\|_2}{\|u_{1,\hbar}\|_2}\sqrt{\|u_{1,\hbar}\|_2^2 + \|u_{2,\hbar}\|_2^2} = 2\left(\frac{b}{a}\right)^{1/4}(\sqrt{2a} + \sqrt{2b})^{1/2}, \\ q &\cong -\frac{\|u_{1,\hbar}\|_2}{\|u_{2,\hbar}\|_2}\sqrt{\|u_{1,\hbar}\|_2^2 + \|u_{2,\hbar}\|_2^2} = -2\left(\frac{a}{b}\right)^{1/4}(\sqrt{2a} + \sqrt{2b})^{1/2}, \end{aligned}$$

which are uniformly away from zero over $\hbar$. Therefore, if $\hbar$ is sufficiently small, w must change sign as the signs of p and q are different, and so w cannot be the ground state of $L_\hbar^-$ by Lemma 5.8, which contradicts to the hypothesis that w has the (unique) negative eigenvalue and so it is the ground state. Therefore, $L_\hbar^-$ has no negative eigenvalue. Then Proposition 5.6 comes from the uniqueness of the ground state by Lemma 5.8. □

Now, it remains to prove (45) and (46). We give the proofs of them in the form of lemmas.

Lemma 5.11. *For any given $p, q \in \mathbb{R}$ and v with $\|v\|_2 = 1$, we have*

$$\langle L_\hbar^-(pu_{1,\hbar} + qu_{2,\hbar}), v\rangle = O(\hbar^{3/2}).$$

Proof.

$$L_\hbar^-(pu_{1,\hbar} + qu_{2,\hbar}) = pL_\hbar^- u_{1,\hbar} + qL_\hbar^- u_{2,\hbar},$$

$$\begin{aligned} L_\hbar^- u_{1,\hbar} &= \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u_\hbar^2\right)u_{1,\hbar} \\ &= \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - (u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar)^2\right)u_{1,\hbar} \\ &= \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u_{1,\hbar}^2 u_{1,\hbar}\right) - u_\hbar(u_{1,\hbar} + \phi_\hbar)u_{2,\hbar} \\ &= (V_\hbar - V(-R))u_{1,\hbar} - u_\hbar(u_{2,\hbar} + \phi_\hbar)u_{1,\hbar}. \end{aligned} \tag{49}$$

Therefore,

$$\langle L_\hbar^- u_{1,\hbar}, v\rangle = \langle (V_\hbar - V(-R))u_{1,\hbar}, v\rangle - \langle u_{1,\hbar}(u_{2,\hbar} + \phi_\hbar)u_{1,\hbar}, v\rangle.$$

We can prove

$$\langle (V_\hbar - V(-R))u_{1,\hbar}, v\rangle = O(\hbar^{3/2}) \tag{50}$$

in the same way as in (8) or (9). And,

$$\langle u_\hbar(u_{2,\hbar} + \phi_\hbar)u_{1,\hbar}, v\rangle = O(\hbar^{3/2}) \tag{51}$$

by Lemma 5.10 and the fact that

$$\|u_{2,\hbar}\cdot u_{1,\hbar}\|_\infty = O(e^{-\mu R/2\hbar}).$$

By (50), (51) we have proved

$$\langle L_\hbar^- u_{1,\hbar}, v\rangle = O(\hbar^{3/2}).$$

In the same way, we have

$$\langle L_\hbar^- u_{2,\hbar}, v\rangle = O(\hbar^{3/2}),$$

and hence the proof. □

Lemma 5.12. *If $\hbar$ is sufficiently small, there exists some $c > 0$ which is independent of $\hbar$ such that*

$$\inf_{v \perp \{u_{1,\hbar}, u_{2,\hbar}\}} \frac{\langle L_\hbar^- v, v\rangle}{\langle v, v\rangle} > c > 0.$$

Proof. We have

$$\begin{aligned} L_\hbar^- v &= \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - (u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar)^2\right)v \\ &= \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u_{1,\hbar}^2\right)\alpha_\hbar v + \left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u_{2,\hbar}^2\right)\beta_\hbar v \\ &\quad \underbrace{-(2u_{1,\hbar} - u_{2,\hbar} + (1-\alpha_\hbar)u_{1,\hbar}^2 + (1-\beta_\hbar)u_{2,\hbar}^2 + \phi_\hbar, u_\hbar)v.}_{\text{(A)}} \end{aligned}$$

Combining (10), (11), Lemma 5.9, (47) and the fact that $\|u_\hbar\|_\infty$ is bounded uniformly over $\hbar$, we have

$$\|(A)\|_2 = O(\hbar^{3/2}) \tag{52}$$

if we assume that $\|v\|_2 = 1$. If we denote

$$L^-_{\hbar,1} = -\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u^2_{1,\hbar},$$

$$L^-_{\hbar,2} = -\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - u^2_{2,\hbar},$$

then

$$\begin{aligned}\langle L^-_\hbar v, v\rangle &= \langle (L^-_{\hbar,1}\alpha_\hbar + L^-_{\hbar,2}\beta_\hbar)v, v\rangle + O(\hbar^{3/2})\\ &= \langle L^-_{\hbar,1}\alpha_\hbar v, v\rangle + \langle L^-_{\hbar,2}\beta_\hbar v, v\rangle + O(\hbar^{3/2}).\end{aligned} \tag{53}$$

Now, suppose that $v \perp \{u_{1,\hbar}, u_{2,\hbar}\}$ and $\|v\|_2 = 1$. To estimate (53), we may assume without loss of any generalities that in Sect. 3, when we choose α and β,

$$\alpha = \gamma^2 \quad \text{and} \quad \beta = \delta^2,$$

which is possible because $\alpha, \beta \geqq 0$ and are smooth. It follows that $\alpha_\hbar = \gamma_\hbar^2$ and $\beta_\hbar = \delta_\hbar^2$,

$$\begin{aligned}\langle L^-_{\hbar,1}\alpha_\hbar v, v\rangle &= \langle L^-_{\hbar,1}\gamma_\hbar^2 v, v\rangle\\ &= \langle \gamma_\hbar L^-_{\hbar,1}\gamma_\hbar v, v\rangle + \langle [L^\hbar_-, \gamma_\hbar]\gamma_\hbar v, v\rangle\\ &= \langle L^-_{\hbar,1}\gamma_\hbar v, \gamma_\hbar v\rangle + \langle [L^\hbar_-, \gamma_\hbar]\gamma_\hbar v, v\rangle.\end{aligned} \tag{54}$$

Here,

$$\langle [L^-_{\hbar,1}, \gamma_\hbar]\gamma_\hbar v, v\rangle = O(\hbar) \tag{55}$$

due to the same reason as (23). In other words, apply

$$\begin{aligned}[L^-_{\hbar,1}, \gamma_\hbar] &= \left[-\frac{1}{2}\frac{d^2}{dy^2}, V_\hbar\right]\\ &= -\frac{\hbar}{2}\left(\gamma'(\hbar y)\frac{d}{dy} + \hbar\gamma''(\hbar y)\right)\end{aligned}$$

to (55). For the first term of the right-hand side in (54), first note that

$$\langle \gamma_\hbar v, u_{1,\hbar}\rangle = O(e^{-\mu R/2\hbar}). \tag{56}$$

i.e. $\gamma_\hbar v$ is "almost orthogonal" to $u_{1,\hbar}$. In fact,

$$\begin{aligned}\langle \gamma_\hbar v, u_{1,\hbar}\rangle &= \langle v, u_{1,\hbar}\rangle + \langle (1-\gamma_\hbar)v, u_{1,\hbar}\rangle\\ &= \langle v, (1-\gamma_\hbar)u_{1,\hbar}\rangle\end{aligned}$$

as $\langle v, u_{1,\hbar}\rangle = 0$ by the hypotheses. Then it follows as before that

$$\begin{aligned}|\langle v, (1-\gamma_\hbar)u_{1,\hbar}\rangle| &\leqq \|v\|_2 \|(1-\gamma_\hbar)u_{1,\hbar}\|_2\\ &= O(e^{-\mu R/2\hbar}).\end{aligned}$$

Now,

$$\langle L_{\hbar,1}^{-}\gamma_\hbar v, \gamma_\hbar v\rangle = \langle L_{0,1}^{-}\gamma_\hbar v, \gamma_\hbar v\rangle + \langle (V_\hbar - V(-R))\gamma_\hbar v, \gamma_\hbar v\rangle, \tag{57}$$

where

$$L_{0,1}^{-} := -\frac{1}{2}\frac{d^2}{dy} + (V(-R) - E) - u_{1,\hbar}^2.$$

Using Corollary 5.5 ii) and (56), we have

$$\langle L_{0,1}^{-}\gamma_\hbar v, \gamma_\hbar v\rangle \geqq (V(-R) - E)\langle \gamma_\hbar v, \gamma_\hbar v\rangle + O(e^{-\mu R/2\hbar}).$$

Substituting this into (57), we have

$$\langle L_{\hbar,1}^{-}\gamma_\hbar v, \gamma_\hbar v\rangle \geqq \langle (V_\hbar - E)\gamma_\hbar v, \gamma_\hbar v\rangle + O(e^{-\mu R/2\hbar}).$$

Since we assume in the beginning of this paper that

$$V_\hbar - E > \varepsilon > 0,$$

we have

$$\begin{aligned}\langle L_{\hbar,1}^{-}\gamma_\hbar v, \gamma_\hbar v\rangle &\geqq \varepsilon\langle \gamma_\hbar v, \gamma_\hbar v\rangle + O(e^{-\mu R/2\hbar})\\ &= \varepsilon\langle \gamma_\hbar^2 v, v\rangle + O(e^{-\mu R/2\hbar})\\ &= \varepsilon\langle \alpha_\hbar v, v\rangle + O(e^{-\mu R/2\hbar}).\end{aligned} \tag{58}$$

Substituting (55) and (58) into (54), we have

$$\langle L_{\hbar,1}^{-}\alpha_\hbar v, v\rangle \geqq \varepsilon\langle \alpha_\hbar v, v\rangle + O(\hbar). \tag{59}$$

Similarly, we have

$$\langle L_{\hbar,2}^{-}\beta_\hbar v, v\rangle \geqq \varepsilon\langle \beta_\hbar v, v\rangle + O(\hbar). \tag{60}$$

Substituting (59) and (60) into (53), we have

$$\langle L_\hbar^{-} v, v\rangle \geqq \varepsilon(v, v\rangle + O(\hbar)$$

since $\alpha_\hbar + \beta_\hbar \equiv 1$. By choosing a smaller ε, we have proved that

$$\langle L_\hbar^{-} v, v\rangle \geqq \varepsilon\langle v, v\rangle$$

if $v \perp \{u_{1,\hbar}, u_{2,\hbar}\}$. Hence Lemma 5.12. Finally, we have finished the proof of Proposition 5.6. □

Proof of Proposition 5.7. By Lemma 5.4, the negative eigenvalues of

$$L_{0,1}^{+} = -\frac{1}{2}\frac{d^2}{dy^2} + (V(-R) - E) - 3u_{1,h}^2,$$

$$L_{0,2}^{+} = -\frac{1}{2}\frac{d^2}{dy^2} + (V(R) - E) - 3u_{2,\hbar}^2$$

are $-3a$ and $-3b$ respectively, where

$$a = V(-R) - E \quad \text{and} \quad b = V(R) - E.$$

We will show that if $\hbar$ is sufficiently small, then these eigenvalues survive as ones of

$$L_\hbar^+ = -\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - 3u_\hbar^2.$$

And we also know from Lemma 5.4 that the eigenfunctions of $L_{0,1}^+$ and $L_{0,2}^+$ corresponding to the negative eigenvalues are $u_{1,\hbar}^2$ and $u_{2,\hbar}^2$ respectively. First, note that $u_{1,\hbar}^2$ and $u_{2,\hbar}^2$ are "almost orthogonal", i.e.

$$\langle u_{1,\hbar}^2, u_{2,\hbar}^2\rangle = O(e^{-\mu R/\hbar}) \tag{61}$$

which can be proven easily. We will prove that if $\hbar$ is sufficiently small, the restriction of $L_\hbar^+$ to the span $\{u_{1,\hbar}^2, u_{2,\hbar}^2\}$ is negative definite, which will conclude by the mini-max principle (see [ReS]) that $L_\hbar^+$ has at least two negative eigenvalues. It is now easy to show that

$$\langle L_\hbar^+ u_{1,\hbar}^2, u_{2,\hbar}^2\rangle = O(e^{-\mu R/\hbar}). \tag{62}$$

On the other hand,

$$\begin{aligned}\langle L_\hbar^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle &= \left\langle\left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - 3u_\hbar^2\right)u_{1,\hbar}^2, u_{1,\hbar}^2\right\rangle\\ &= \left\langle\left(-\frac{1}{2}\frac{d^2}{dy^2} + (V_\hbar - E) - 3(u_{1,\hbar} + u_{2,\hbar} + \phi_\hbar)^2\right)u_{1,\hbar}^2, u_{1,\hbar}^2\right\rangle\\ &= \langle L_{\hbar,1}^+ u_{1,\hbar}^2 u_{1,\hbar}^2\rangle - 3\langle u_\hbar u_{2,\hbar} u_{1,\hbar}^2, u_{1,\hbar}^2\rangle - 3\langle u_\hbar \phi_\hbar u_{1,\hbar}^2, u_{1,\hbar}^2\rangle,\end{aligned}$$

where $L_{\hbar,i}^+ = -(1/2)(d^2/dy^2) + (V_\hbar - E) - 3u_{i,\hbar}^2$, $i = 1, 2$. In the same reason as (62),

$$\langle u_\hbar u_{2,\hbar} u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = O(e^{-\mu R/2\hbar}) \tag{63}$$

and from Lemma 5.10,

$$\langle u_\hbar \phi_\hbar u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = O(\hbar^{3/2}). \tag{64}$$

And,

$$\langle L_{\hbar,1}^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = \langle L_{0,1}^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle + \langle (V_\hbar - V(-R))u_{1,\hbar}^2, u_{1,\hbar}^2\rangle. \tag{65}$$

Since $u_{1,\hbar}^2$ is the eigenfunction of $L_{0,1}^+$ with the eigenvalue $-3a$, we have

$$\langle L_{\hbar,1}^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = -3a\langle u_{1,\hbar}^2, u_{1,\hbar}^2\rangle. \tag{66}$$

Moreover, we can estimate

$$\langle (V_\hbar - V(-R))u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = O(\hbar^{3/2}) \tag{67}$$

again in the same way as in (8), (9) and Lemma 5.10. Substituting (66) and (67) into (65), we have

$$\langle L_{\hbar,1}^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = -3a\langle u_{1,\hbar}^2, u_{1,\hbar}^2\rangle + O(\hbar^{3/2}).$$

Therefore, we have

$$\langle L_\hbar^+ u_{1,\hbar}^2, u_{1,\hbar}^2\rangle = -3a\langle u_{1,\hbar}^2, u_{1,\hbar}^2\rangle + O(\hbar^{3/2}). \tag{68}$$

Similarly, we prove

$$\langle L_\hbar^+ u_{2,\hbar}^2, u_{2,\hbar}^2 \rangle = -3b \langle u_{2,\hbar}^2, u_{2,\hbar}^2 \rangle + O(\hbar^{3/2}). \tag{69}$$

Combining (61), (62), (68) and (69), we have proved that the restriction of $L_\hbar^+$ to span $\{u_{1,\hbar}^2, u_{2,\hbar}^2\}$ is negative definite, and so that by the mini-max principle, $L_\hbar^+$ has at least two negative eigenvalues. This finishes the proof of the proposition. □

Remark. In fact, in the course of the proof we have proved that $L_\hbar^+$ has one eigenvalue near each of $-3a$ and $-3b$ if $\hbar$ is sufficiently small.

6. Generalizations

6.1. Construction. We can remove the assumption that V is bounded as long as V is in the class $(V)_a$, by using, as in [O1], cut-off functions and the operator domain $D(H_\hbar)$ and the corresponding weighted norm instead of the space H^2. Now, we explain how we modify the previous construction to deal with multi-lump case.

Let a collection of nondegenerate critical points $\{x_1, \ldots, x_N\}$ of V be given. We again consider the rescaled equation

$$-\tfrac{1}{2}u'' + (V_\hbar - E)u - u^3 = 0,$$

and now trial solutions given by

$$u_{\vec{z},\hbar} = u_1\left(\cdot - \frac{x_1 + z_1}{\hbar}\right) + u_2\left(\cdot - \frac{x_2 + z_2}{\hbar}\right) + \cdots + u_N\left(\cdot - \frac{x_N + z_N}{\hbar}\right),$$

where $\vec{z} = (z_1, \ldots, z_N)$ and

$$u_i(y) = \sqrt{2a_i}\,\operatorname{sech}\sqrt{2a_i}\,y, \quad a_i = V(x_i) - E.$$

We can estimate the error $S_\hbar(u_{\vec{z},\hbar})$ in the similar way as in Proposition 3.1 using the fact that the N lumps $u_i(\cdot - (x_i + z_i/\hbar))$ farther and farther away as $\hbar \to 0$. To estimate the Fredholm inverse of $S'_\hbar(u_{\vec{z},\hbar})$, we introduce the partitions of unity given as follows:

Let $d :=$ the minimum of $|x_i - x_j|$, $i,j = 1, \ldots, N$, and define

$$\alpha_j(x) = \begin{cases} 1 & \text{for} \quad |x - x_j| < \dfrac{d}{3} \\[2ex] 0 & \text{for} \quad |x - x_j| > \dfrac{2d}{3} \end{cases},$$

$$\beta(x) = 1 - \sum_{j=1}^{N} \alpha_i.$$

Choose $\{a_{j,\hbar}, \beta_\hbar\}_{j=1,\ldots,N}$ as our partitions of unity where

$$\alpha_{j,\hbar}(y) = \alpha_j(\hbar y), \quad \beta_\hbar(y) = \beta(\hbar y).$$

Now, we have to take care of the following cases separately as in the proof of Proposition 3.3

Case I). $\|\phi_i\|_{H^2,[-(d/3\hbar_i)+(x_j/\hbar_i),(d/3\hbar_i)+(x_j/\hbar_i)]} \geqq \varepsilon_1 > 0$ for all sufficiently large i and for some $j = 1,\dots,N$ and $\varepsilon_1 > 0$.

Case II) $\lim_{i\to\infty} \max_{j=1,\dots,N} \|\phi_i\|_{H_2,[-(d/3\hbar_i)+(x_j/\hbar_i),(d/3\hbar_i)+(x_j/\hbar_i)]} = 0.$

Case I) can be taken care of as in the Case I) or II) in Proposition 3.3 and Case II) can be taken care of as in Case III) there. For the reduced problem, consider the reduced vector field defined by

$$s_\hbar(\vec{z}) = (s_{\hbar,1}(\vec{z}),\dots,s_{\hbar,N}(\vec{z})),$$
$$v_\hbar(\vec{z}) = \hbar^\nu s_\hbar(\hbar^{-\nu}\vec{z}),$$

where $s_{\hbar,j}(\vec{z}) = (1/\hbar)\langle u'_{j,z_j,\hbar}, S_\hbar(u_{\vec{z},\hbar})\rangle$ for $j = 1,\dots,N$. Now we prove in the same way as in the two-loop case that $v_\hbar$ uniformly converges to the linear vector field v_0 on $[-1,1]^N$ defined by

$$v_0(\vec{z}) = (v_{0,1}(\vec{z}),\dots,v_{0,N}(\vec{z}))$$
$$v_{0,j}(\vec{z}) = -\tfrac{1}{2}\|u_j\|^2 V''(x_j)z_j, \quad j = 1,\dots,N.$$

Then we prove that $v_\hbar$ has the nonzero degree since v_0 does so because

$$\deg(v_0) = \prod_{j=1}^{N} \sin V''(x_j) \neq 0$$

due to the nondegeneracy of critical points. Moreover, these arguments work even for high dimensional situations under the same hypothesis and so we have the following generalization of Theorem 4.1.

Theorem 6.1. *Let $V\in(V)_a$ for some $a\in\mathbb{R}$, $V - E > \varepsilon > 0$ and p be chosen so that it satisfies the basic hypothesis in the abstract of the present paper. Then for each collection $\{x_1,\dots,x_N\}$ of nondegenerate critical points of V, there is an $\hbar_4 > 0$ such that for all $\hbar$ with $0 < \hbar < \hbar_4$, the equation*

$$-\frac{\hbar^2}{2}\Delta v + (V - E)v - |v|^{p-1}v = 0 \tag{70}$$

has a nonzero solution with the corresponding concentration phenomena as in Theorem 4.17.

So far, we studied Eq. (1) in the semi-classical point of view, i.e. when $\hbar\to 0$. However, we can reinterpret this existence result as a genuine "quantum" result:

I) *The Case of Deep Wells.* If we divide Eq. (70) by $\hbar^2$ and set $\lambda = 1/\hbar^2$, we have the equation

$$-\tfrac{1}{2}\Delta v + \lambda(V - E)v - \lambda|v|^{p-1}v = 0. \tag{71}$$

If we consider the family of potentials V_λ, eigenvalues E_λ and γ_λ defined by

$$V_\lambda(x) = \lambda V(x), \quad E_\lambda = \lambda E, \quad \gamma_\lambda = \lambda,$$

then as $\lambda\to\infty$, each potential well of V_λ gets deeper and deeper, and $\inf V_\lambda(x) - E_\lambda$ and γ_λ become bigger and bigger.

Theorem 6.1′. *Suppose that* $V \in (V)_a$ *and* $V - E > \varepsilon > 0$. *Consider the nonlinear Schrödinger equation*

$$-\tfrac{1}{2}\Delta v + (V_\lambda - E_\lambda)v - \gamma_\lambda |v|^{p-1} v = 0,$$

where $V_\lambda = \lambda V$, $E_\lambda = \lambda E$ *and* $\gamma_\lambda = \lambda$. *Then for each collection of critical points* $\{x_1, \ldots, x_N\}$ *there exists some* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, *the equation has a nonzero solution with the corresponding concentration phenomena as in Theorem 4.1.*

Note that we do not need the separations of the wells as long as the wells are deep enough.

II) *The Case of Wide Wells with Large Separations.* On the other hand, if we look at the rescaled equation itself

$$-\tfrac{1}{2}\Delta u + (V_\hbar - E)u - |u|^{p-1} u = 0$$

as $\hbar \to 0$, the wells of $V_\hbar$ become wider and wider and the distances between the wells become larger and larger.

Theorem 6.1″. *Suppose that* $V \in (V)_a$ *and* $V - E > \varepsilon > 0$. *Consider the nonlinear Schrödinger equation*

$$-\tfrac{1}{2}\Delta u + (V_\hbar - E)u - |u|^{p-1} u = 0, \tag{72}$$

where $V_\hbar(y) = V(\hbar y)$. *Then for each collection of critical points* $\{x_1, \ldots, x_N\}$ *of* V *(and so critical points* $\{x_1/\hbar, \ldots, x_N/\hbar\}$ of $V_\hbar$), *there exists some* $\hbar_0 > 0$ *such that for any* $0 < \hbar < \hbar_0$, (72) *has a nonzero solution with the lumps concentrated with nonzero concentration near of* $x_j/\hbar$. *In particular, the lumps becomes more and more separated.*

Remark. The reason why we have chosen (72) to solve (43) and (45) is that it has the nicest limit among them as $\hbar \to 0$ that we can deal with, while (43) and (45) have singular limits as either $\lambda \to \infty$ or $\hbar \to 0$ respectively.

6.2. Instability and Positivity.

Theorem 6.2. *The solutions of the form* $u_\hbar = u_{\bar{z},\hbar} + \phi_\hbar$, *where*

$$u_{\bar{z},\hbar} = u_{1,(x_1+z_1)/\hbar} + \cdots + u_{N,(x_N+z_N)/\hbar}$$

are all positive and (Lyapunov) unstable if $\hbar$ *is sufficiently small.*

Again this theorem comes from the following two propositions and the Instability Criterion.

Proposition 6.3. *The operator*

$$L_\hbar^- = -\tfrac{1}{2}\Delta + (V_\hbar - E) - |u_\hbar|^{p-1}$$

has one-dimensional kernel, no negative eigenvalue and all the other spectra are positive.

Proposition 6.4. *The operator*

$$L_\hbar^+ = -\tfrac{1}{2}\Delta + (V_\hbar - E) - p|u_\hbar|^{p-1}$$

has at least N negative eigenvalues.

Proofs of these propositions are following essentially the same line of ideas as in the proofs of Proposition 6.6 and 6.7 using the following facts:
1. The operator $L_0^- = -\frac{1}{2}\Delta + \lambda - |u_0|^{p-1}$ has one-dimensional kernel spanned by u_0 and no negative eigenvalues, where u_0 is the "unique" ground state of the equation

$$-\frac{1}{2}\Delta u + \lambda u - |u|^{p-1}u = 0.$$

2. The ground state of a Schrödinger operator

$$H = -\frac{1}{2}\Delta + V$$

has no node and so can be chosen to be positive everywhere and such a ground state is unique (See Lemma 5.8).
3. The operator

$$L_0^+ = -\frac{1}{2}\Delta + \lambda - p|u_0|p^{-1}$$

has one negative eigenvalue.
1. and 2. will be needed to prove Proposition 6.3, and 3) will be needed to prove Proposition 6.4.

7. Final Remarks

Note that summing up one-lump solutions is not the only way of getting an approximate N-lump wave solutions. For example, we can choose any of the following

$$u_{1,(x_1+z_1)/\hbar} \pm u_{2,(x_2+z_2)/\hbar} \pm \cdots \pm u_{N,(x_N+z_N)/\hbar}$$

as an approximate solution. Then it is easy to see that exactly the same estimates hold as before and that we may even choose the same constants in the estimates as in the case $u_{1,(x_1+z_1)/\hbar} + \cdots + u_{N,(x_N+z_N)/\hbar}$.

This gives 2^{N-1} distinct N-lump bound states of NLS (1). Of course, these solutions have nodes, i.e., change their signs. Again, we can prove that all of these solutions are also unstable this time using a more refined instability criterion in [Gr, Theorem 1.2]. In fact, we can prove that the real part $L_\hbar^+$ has at least N negative eigenvalues (all of which are of order $O(1)$) while the imaginary part $L_\hbar^-$ has at most $N-1$ negative eigenvalues (all of which are of order $O(\hbar)$).

We may even choose a sum of one-lumps with different phases as approximate solutions (the above solutions correspond to the phase $e^{0\pi} = 1$ or $e^{i\pi} = -1$). However, unless the phase is real, i.e. $r^{i\theta} = \pm 1$, the corresponding standing NLS equation is not a single equation but a system of equations. For example, when $N = 2$, if we let the solution be of the form

$$e^{-iEt/\hbar}(u + e^{i\theta}v),$$

where u and v are real and $e^{i\theta} \neq \pm 1$, then u and v satisfy the equation

$$-\frac{\hbar^2}{2}\Delta(u + e^{i\theta}v) + (V - E)(u + e^{i\theta}v) - |u + e^{i\theta}v|^2(u + \varepsilon^{i\theta}v) = 0,$$

and so

$$-\frac{\hbar^2}{2}\Delta u + (V - E)u - (u^2 + v^2 + 2uv\cos\theta)u = 0,$$

$$-\frac{\hbar^2}{2}\Delta v + (V - E)v - (u^2 + v^2 + 2uv\cos\theta)v = 0,$$

since $e^{i\theta}$ is not real. (If $e^{i\theta}$ is real, then we have just one equation.) One can easily see that the pair $(u = u_{1,\hbar}, v = u_{2,\hbar})$ is an approximate solution of this system of equations whose error can be estimated as before, but this time estimating the Fredholm inverse is not as clear as before. So far now, it is not clear whether such N-lumps with different phases exist. This will be a subject of our future investigations.

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