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Einstein Metrics on S^3, R^3 and R^4 Bundles

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Abstract. Starting from a $4n$-dimensional quaternionic Kähler base space, we construct metrics of cohomogeneity one in $(4n+3)$ dimensions whose level surfaces are the S^2 bundle space of almost complex structures on the base manifold. We derive the conditions on the metric functions that follow from imposing the Einstein equation, and obtain solutions both for compact and non-compact $(4n+3)$-dimensional spaces. Included in the non-compact solutions are two Ricci-flat 7-dimensional metrics with G_2 holonomy. We also discuss two other Ricci-flat solutions, one on the R^4 bundle over S^3 and the other on an R^4 bundle over S^4. These have G_2 and Spin(7) holonomy respectively.

1. Introduction

There are many examples of homogeneous Einstein metrics to be found in the literature, but inhomogeneous examples, where there is no transitively-acting isometry group, are much rarer. In this paper, we construct examples in $4n+3$ dimensions which can be described as S^3 or R^3 bundles over quaternionic Kähler base manifolds. After reviewing some relevant properties of quaternionic Kähler spaces, in this section we then discuss the notion of the twistor space Z corresponding to a quaternionic Kähler space M [1]. This space plays a central role in the rest of the paper. In Sect. 2 we give a local discussion of our construction, including details of the local calculation of the curvature of our spaces. In Sect. 3 we consider the regularity conditions on the local metrics that ensure that they can be extended to globally-defined metrics on complete manifolds, and we apply these conditions to discuss the existence of complete Einstein metrics on compact manifolds, which we have found numerically. In Sect. 4, we consider non-compact Ricci-flat spaces, and present two exact solutions in seven dimensions. These are the same as the seven-dimensional metrics with G_2 holonomy constructed recently by using different methods [2]. In Sect. 5, we consider two more exact Ricci-flat metrics, one on the manifold $R^4 \times S^3$, with G_2 holonomy, and the other on an R^4 bundle over S^4, with Spin(7) holonomy. Again, these coincide with examples constructed in [2]. In Sect. 6 we discuss the asymptotic structure of the non-compact metrics, and relate this to earlier work on gravitational instantons. Finally, in Sect. 7, we use the fact that metrics with G_2 or Spin(7) holonomy admit a covariantly-constant spinor to obtain relations between the eigenfunctions of certain differential operators on the manifold, and also to relate the space of moduli for the Ricci-flat metric to certain topological invariants of the manifold.

A quaternionic Kähler space is a Riemannian space M of real dimension $4n$ whose holonomy group is contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. It has a set of three almost complex structure tensors $J^i{}_\alpha{}^\beta$ ($i=1,2,3$; $\alpha,\beta=1,\dots,4n$) which satisfy the quaternion algebra

$$J^i{}_\alpha{}^\beta J^j{}_\beta{}^\gamma = -\delta_{ij}\delta_\alpha{}^\gamma + \varepsilon_{ijk}J^k{}_\alpha{}^\gamma. \tag{1.1}$$

The metric is quaternionic-Hermitian, which implies that $J^i{}_{\alpha\beta} = -J^i{}_{\beta\alpha}$. From the corresponding 2-forms J^i one may construct the closed 4-form $\Omega = J^i \wedge J^i$, $d\Omega = 0$. These conditions imply the existence of three local 1-forms A^i, such that

$$\nabla_\alpha J^i{}_{\beta\gamma} + \varepsilon_{ijk}A^j{}_\alpha J^k{}_{\beta\gamma} = 0. \tag{1.2}$$

A^i corresponds to the Sp(1) part of the $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ connection, and has curvature

$$F^i = dA^i + \tfrac{1}{2}\varepsilon_{ijk}A^j \wedge A^k. \tag{1.3}$$

One can show that

$$F^i{}_{\alpha\beta} = \frac{1}{2n}J^i{}_\gamma{}^\delta R^\gamma{}_{\delta\alpha\beta}. \tag{1.4}$$

All four-dimensional manifolds are quaternionic Kähler, but in $4n \geqq 8$ dimensions the quaternionic Kähler condition is more restrictive, implying in particular that for an irreducible space the metric is Einstein, $R_{\alpha\beta} = \Lambda_{4n}g_{\alpha\beta}$, and

$$F^i{}_{\alpha\beta} = \frac{\Lambda_{4n}}{n+2}J^i{}_{\alpha\beta}. \tag{1.5}$$

We shall be concerned exclusively with the case where Λ_{4n} is strictly positive, and so without loss of generality we may choose $\Lambda_{4n} = n+2$;

$$R_{\alpha\beta} = (n+2)g_{\alpha\beta}. \tag{1.6}$$

Thus if $4n \geqq 8$ and M is irreducible it follows that

$$F^i{}_{\alpha\beta} = J^i{}_{\alpha\beta}. \tag{1.7}$$

If $4n = 4$ or M is reducible, we shall impose (1.7) as a further condition. Classic examples of quaternionic Kähler spaces are provided by the quaternionic projective spaces $P_n(H)$. A more complete description of quaternionic Kähler spaces may be found in [3, 4, 5, 6].

Before describing our construction of metrics on S^3 or R^3 bundles over M, we first consider the bundle of almost complex structures on M. This has been discussed extensively in [1]. It follows from (1.1) that the tensor $J_\alpha{}^\beta$ defined by

$$J_\alpha{}^\beta = u^i J^i{}_\alpha{}^\beta \tag{1.8}$$

is an almost complex structure tensor, where the u^i are any set of three scalar fields satisfying

$$u^i u^i = 1. \tag{1.9}$$

Thus the bundle of almost complex structures on M is parametrized by points on a 2-sphere. The $(4n+2)$-dimensional total space of this S^2 bundle over M is known as the "twistor space" Z of M [1]. Defining the Sp(1)-covariant exterior derivative D of u^i by

$$\theta^i \equiv Du^i = du^i + \varepsilon^{ijk} A^j u^k, \tag{1.10}$$

one can show that $dJ = \theta^i \wedge J^i$ and that

$$u^i \theta^i = 0; \quad D\theta^i = \varepsilon^{ijk} F^j u^k, \tag{1.11}$$

where $D\theta^i = d\theta^i + \varepsilon^{ijk} A^j \wedge \theta^k$. The twister space Z may be given the metric

$$ds^2 = \lambda^2 \theta^i \theta^i + e^\alpha e^\alpha = \lambda^2 (du^i + \varepsilon^{ijk} A^j u^k)^2 + e^\alpha e^\alpha, \tag{1.12}$$

where λ is a constant, e^α is an orthonormal frame for M and the coördinates u^i on the S^2 fibres are subject to the constraint (1.9).

The isometry group $\hat{G}$ of the metric (1.12) on Z is generically equal to $SO(3) \times G$, where G is the isometry group of the quaternionic Kähler base space M and the $SO(3)$ preserves the condition (1.9) that defines the S^2 fibres. (In special case $\hat{G}$ might be larger than $SO(3) \times G$, in the same way as the $(4n+3)$-sphere, described as an $SU(2)$ principal bundle over $P_n(H)$, can have $SO(4n+4)$ rather than the generic $SU(2) \times \mathrm{Sp}(n+1)$ as isometry group in the special case that it is metrically the round sphere.)

It is straightforward to calculate the curvature of the metric (1.12) on Z. This is most easily done by first taking the three coordinates u^i to be unconstrained, and then using the Gauss–Codazzi equations to relate the curvature of this $(4n+3)$-dimensional metric to the curvature of the metric on Z defined by imposing the hypersurface condition (1.9). The non-vanishing components of the Ricci tensor on Z, in the orthonormal frame $(\lambda\theta^i, e^\alpha)$, are

$$\begin{aligned} R_{\alpha\beta} &= (n + 2 - \lambda^2)\delta_{\alpha\beta}, \\ R_{ij} &= (n\lambda^2 + \lambda^{-2}) h_{ij}, \end{aligned} \tag{1.13}$$

where

$$h_{ij} \equiv \delta_{ij} - u^i u^j \tag{1.14}$$

is the two-dimensional metric on the S^2 fibres, referred to the frame $\lambda\theta^i$.

From (1.13) we see that Z admits two Einstein metrics of the form (1.12), corresponding to taking the "squashing parameter" λ to be given by

$$\lambda^2 = 1 \quad \text{or} \quad \lambda^2 = \frac{1}{n+1}. \tag{1.15}$$

They satisfy the Einstein equation $R_{ab} = \Lambda g_{ab}$ with $\Lambda = n + 1$ or $\Lambda = (n^2 + 3n + 1)/(n+1)$ respectively. If $\lambda^2 = 1$, one can show that the metric on Z is Kähler, with the Kähler form given by

$$J' = -u^i J^i + \tfrac{1}{2}\lambda^2 \varepsilon_{ijk} u^i \theta^j \wedge \theta^k. \tag{1.16}$$

If we take $M = P_n(H)$, the corresponding twistor space Z is the complex projective space $P_{2n+1}(C)$. Taking $\lambda^2 = 1$ in (1.12) gives the usual Fubini-Study Einstein–Kähler metric on $P_{2n+1}(C)$, while $\lambda^2 = (n+1)^{-1}$ gives the second homogeneous Einstein metric discussed in [7], which is Hermitian but not Kähler. When $n = 1$, so that $M = P_1(H) = S^4$, Z is the ur twistor space $P_3(C)$.

2. Curvature and Local Calculations

In this paper, we shall construct metrics on $(4n+3)$-dimensional manifolds $\hat{M}$, for which the metrics take the local form

$$d\hat{s}^2 = \alpha^2 dr^2 + \beta^2 (du^i + \varepsilon_{ijk} A^j u^k)^2 + \gamma^2 e^\alpha e^\alpha, \tag{2.1}$$

where r is an additional coördinate and α, β and γ are functions solely of r. The level-surfaces $r =$ constant are therefore the $(4n+2)$-dimensional twistor space Z described in the previous section. The isometry group of the metric (2.1) is generically the same as that on the level surfaces, which as discussed in the previous section is itself generically $SO(3) \times G$, where G is the isometry group of the base space M.

We introduce an orthonormal frame $\hat{e}^a$ for (2.1) that is defined by

$$\hat{e}^0 = \alpha dr; \quad \hat{e}^i = \beta\theta^i; \quad \hat{e}^\alpha = \gamma e^\alpha, \tag{2.2}$$

where θ^i is given by (1.10) and as before e^α is an orthonormal frame for M. The connection 1-form $\hat{\omega}_{ab}$, defined by $d\hat{e}^a = -\hat{\omega}_{ab} \wedge \hat{e}_b$ and $\hat{\omega}_{ab} = -\hat{\omega}_{ba}$, is given by

$$\hat{\omega}_{0i} = -\frac{\beta'}{\alpha\beta}\hat{e}^i, \quad \hat{\omega}_{0\alpha} = -\frac{\gamma'}{\alpha\gamma}\hat{e}^\alpha,$$

$$\hat{\omega}_{i\alpha} = \frac{\beta}{2\gamma^2}\varepsilon_{ijk} J^j{}_{\alpha\beta} u^k \hat{e}^\beta, \quad \hat{\omega}_{ij} = -\varepsilon_{ijk} A^k,$$

$$\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} - \frac{\beta}{2\gamma^2}\varepsilon_{ijk} J^j{}_{\alpha\beta} u^k \hat{e}^i, \tag{2.3}$$

where a prime denotes differentiation with respect to r and $\omega_{\alpha\beta}$ is the connection 1-form for M. Note that we have not yet imposed the hypersurface condition (1.9). The curvature 2-form defined by $\hat{\Theta}_{ab} = d\hat{w}_{ab} + \hat{\omega}_{ac} \wedge \hat{\omega}_{cb}$ has the components

$$\hat{\Theta}_{0i} = \left(-\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta}\right)\hat{e}^0 \wedge \hat{e}^i + \left(\frac{\gamma'\beta}{\alpha\gamma} - \frac{\beta'}{\alpha}\right)\varepsilon_{ijk} J^j u^k,$$

$$\hat{\Theta}_{0\alpha} = \left(-\frac{\gamma''}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma}\right)\hat{e}^0 \wedge \hat{e}^\alpha + \left(-\frac{\beta'}{2\alpha\gamma^2} + \frac{\gamma'\beta}{2\alpha\gamma^3}\right)\varepsilon_{ijk} J^j{}_{\alpha\beta} u^k \hat{e}^i \wedge \hat{e}^\beta,$$

$$\hat{\Theta}_{i\alpha} = \left(\frac{\beta'}{2\alpha\gamma^2} - \frac{\beta\gamma'}{2\alpha\gamma^3}\right)\varepsilon_{ijk} J^j{}_{\alpha\beta} u^k \hat{e}^0 \wedge \hat{e}^\beta + \left(\frac{\beta^2}{4\gamma^4}u^2 - \frac{1}{2\gamma^2}\right)\varepsilon_{ijk} J^k{}_{\alpha\beta} \hat{e}^j \wedge \hat{e}^\beta$$

$$+ \left(\frac{\beta^2}{4\gamma^4}u^2 - \frac{\beta'\gamma'}{\alpha^2\beta\gamma}\right)\hat{e}^i \wedge \hat{e}^\alpha,$$

$$\hat{\Theta}_{ij} = -\left(\frac{\beta'}{\alpha\beta}\right)^2 \hat{e}^i \wedge \hat{e}^j - \left(\frac{\beta^2}{2\gamma^2}u^2 - 1\right)\varepsilon_{ijk}J^k,$$

$$\hat{\Theta}_{\alpha\beta} = \Theta_{\alpha\beta} - \frac{\beta^2}{4\gamma^4}(\delta_{ij}u^2 - u^i u^j)(J^i{}_{\alpha\beta}J^j{}_{\gamma\delta} + J^i{}_{\alpha\gamma}J^j{}_{\beta\delta})\hat{e}^\gamma \wedge \hat{e}^\delta$$

$$-\left(\frac{\gamma'}{\alpha\gamma}\right)^2 \hat{e}^\alpha \wedge \hat{e}^\beta + \left(\frac{\beta^2}{4\gamma^4}u^2 - \frac{1}{2\gamma^2}\right)\varepsilon_{ij}\hat{e}^i \wedge \hat{e}^j, \tag{2.4}$$

where $\Theta_{\alpha\beta}$ is the curvature 2-form on M and $u^2 = u^i u^i$. The calculate the curvature of the metric (2.1) with the hypersurface condition (1.9) imposed, we use the Gauss–Codazzi equation (see, for example, [8])

$$\hat{R}_{abcd} \equiv {}^{4n+3}\hat{R}_{abcd} = {}^{4n+4}\hat{R}_{efgh}h_{ae}h_{bf}h_{cg}h_{dh} + \chi_{ac}\chi_{bd} - \chi_{ad}\chi_{bc}, \tag{2.5}$$

where

$$h_{ab} = \delta_{ab} - u^a u^b, \tag{2.6}$$

with $u^a = (u^0, u^i, u^\alpha) = (0, u^i, 0)$ the orthonormal frame components of the unit vector $N = \beta^{-1}u^i\partial/\partial u^i$ orthogonal to the hypersurface, and $\chi_{ab} \equiv h_{ac}h_{bd}\hat{\nabla}_c u_d$ the second fundamental form of the hypersurface. Since $\hat{\nabla}u_a \equiv du_a + \hat{\omega}_{ab}u_b = \hat{\nabla}_b u_a \hat{e}^b$, it follows from (2.3) and (2.6) that the only non-zero components of h_{ab} and χ_{ab} are given by

$$h_{00} = 1; \quad h_{ij} = \delta_{ij} - u_i u_j; \quad h_{\alpha\beta} = \delta_{\alpha\beta}; \quad \chi_{ij} = \frac{1}{\beta}h_{ij}. \tag{2.7}$$

The Ricci tensor $\hat{R}_{ab}$ of our $(4n+3)$-dimensional metric (2.1) with the constraint (1.9) therefore has the following non-zero orthonormal-frame components:

$$\hat{R}_{00} = -\frac{2\beta''}{\alpha^2\beta} + \frac{2\alpha'\beta'}{\alpha^3\beta} - \frac{4n\gamma''}{\alpha^2\gamma} + \frac{4n\alpha'\gamma'}{\alpha^3\gamma},$$

$$\hat{R}_{ij} = \left(-\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta} - \frac{4n\beta'\gamma'}{\alpha^2\beta\gamma} + \frac{1}{\beta^2} - \frac{\beta'^2}{\alpha^2\beta^2} + \frac{n\beta^2}{\gamma^4}\right)h_{ij},$$

$$\hat{R}_{\alpha\beta} = \left(-\frac{\gamma''}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma} - \frac{2\beta'\gamma'}{\alpha^2\beta\gamma} - \frac{(4n-1)\gamma'^2}{\alpha^2\gamma^2} + \frac{n+2}{\gamma^2} - \frac{\beta^2}{\gamma^4}\right)\delta_{\alpha\beta}. \tag{2.8}$$

In subsequent sections we shall find it useful to construct a certain harmonic 3-form on the space $\hat{M}$, and here we give the local construction of this object. There are three 3-forms on $\hat{M}$ that are invariant both under the isometry group G of the base space M and under the $SO(3)$ symmetry of the 2-spheres defined by (1.9). These 3-forms, which we denote by ω_1, ω_2 and ω_3, take the local forms

$$\omega_1 = \theta^i \wedge J^i; \quad \omega_2 = dr \wedge J; \quad \omega_3 = dr \wedge \Sigma, \tag{2.9}$$

where $J = u^i J^i$ as in (1.8) and

$$\Sigma \equiv \tfrac{1}{2}\varepsilon_{ijk}u^i\theta^j \wedge \theta^k \tag{2.10}$$

is the volume element on the S^2 fibres $u^i u^i = 1$. Using $Du^i = \theta^i, D\theta^i = \varepsilon_{ijk}J^j u^k$ and $DJ^i = 0$, where $Da^i \equiv da^i + \varepsilon_{ijk}A^j \wedge a^k$ for any Sp(1)-valued p-form a^i, it follows that the exterior derivatives of the 3-forms (2.9) are

$$d\omega_1 = 0; \quad d\omega_2 = -dr \wedge \omega_1; \quad d\omega_3 = -dr \wedge \omega_1. \tag{2.11}$$

The duals of the 3-forms (2.9), in the $(4n+3)$-dimensional metric (2.1), are

$$*\omega_1 = \tfrac{1}{4}\alpha\varepsilon_{ijk}u^i dr \wedge \theta^j \wedge J^k; \quad *\omega_2 = \frac{\beta^2}{2\alpha}\Sigma \wedge J; \quad *\omega_3 = \frac{\gamma^4}{2\alpha\beta^2}J \wedge J. \tag{2.12}$$

An harmonic 3-form, which is invariant under G and Sp(1), can be written as

$$\omega = f\omega_1 + g\omega_2 + h\omega_3, \tag{2.13}$$

where f, g and h are functions of r to be determined. From (2.10) it follows that the condition that ω be closed, $d\omega = 0$, implies that

$$f' = g + h. \tag{2.14}$$

The condition that ω be co-closed, $d*\omega = 0$, implies that

$$af = \left(\frac{\beta^2 g}{\alpha}\right)' = \left(\frac{\gamma^4 h}{\alpha\beta^2}\right)'. \tag{2.15}$$

There is therefore an harmonic form (2.13) if f, g and h satisfy

$$\alpha f = \left[\frac{\beta^2 f'}{\alpha}\left(1 - \frac{\beta^4}{\gamma^4}\right)^{-1}\right]', \tag{2.16}$$

$g = f'(1 - \beta^4/\gamma^4)^{-1}$ and $h = -g\beta^4/\gamma^4$.

Finally in this section we note that for some purposes it is advantageous to have a parametrization of the metric on $\hat{M}$ that is given in terms of $(4n+3)$ independent coördinates rather than the $(4n+4)$ coördinates subject to the hypersurface constraint (1.9) that we have been using so far. This can be done by writing the metric $d\hat{s}^2$ in the form

$$d\hat{s}^2 = \alpha^2 dr^2 + \beta^2 g_{ij}(dx^i + K^{iA}A^A)(dx^j + K^{jB}A^B) + \gamma^2 ds^2, \tag{2.17}$$

where $i, j, \ldots$ run over the values $1, 2$, g_{ij} is the standard metric on the unit two-sphere and K^{iA}, $A = 1, 2, 3$, are the three Killing vectors on the two-sphere. Thus we may introduce the orthonormal basis

$$\begin{aligned}
&\hat{e}^0 = \alpha dr; \quad \hat{e}^\alpha = \gamma e^\alpha,\\
&\hat{e}^1 = \beta(d\theta - \sin\phi A^1 + \cos\phi A^2),\\
&\hat{e}^2 = \beta\sin\theta(d\phi - \cos\phi\cot\theta A^1 - \sin\phi\cot\theta A^2 + A^3).
\end{aligned} \tag{2.18}$$

This yields the connection form

$$\hat{\omega}_{01} = -\frac{\beta'}{\alpha\beta}\hat{e}^1, \quad \hat{\omega}_{02} = -\frac{\beta'}{\alpha\beta}\hat{e}^2, \quad \hat{\omega}_{0\alpha} = -\frac{\gamma'}{\alpha\gamma}\hat{e}^\alpha,$$

$$\hat{\omega}_{12} = -\frac{1}{\beta}\cot\theta\hat{e}^2 - \operatorname{cosec}\theta(\cos\phi A^1 + \sin\phi A^2),$$

$$\hat{\omega}_{1\alpha} = \frac{\beta}{2\gamma^2}(-\sin\phi F^1{}_{\alpha\beta} + \cos\phi F^2{}_{\alpha\beta})\hat{e}^\beta,$$

$$\hat{\omega}_{2\alpha} = \frac{\beta}{2\gamma^2}(-\cos\phi\cos\theta F^1{}_{\alpha\beta} - \sin\phi\cos\theta F^2{}_{\alpha\beta} + \sin\theta F^3{}_{\alpha\beta})\hat{e}^\beta,$$

$$\hat{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + \frac{\beta}{2\gamma^2}(\sin\phi F^1{}_{\alpha\beta} - \cos\phi F^2{}_{\alpha\beta})\hat{e}^1$$
$$+ \frac{\beta}{2\gamma^2}(\cos\phi\cos\theta F^1{}_{\alpha\beta} + \sin\phi\cos\theta F^2{}_{\alpha\beta} - \sin\theta F^3{}_{\alpha\beta})\hat{e}^2, \tag{2.19}$$

where $\omega_{\alpha\beta}$ is the connection form for the base space M. From this, it follows that the curvature 2-form is given by

$$\hat{\Theta}_{01} = \left(-\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta}\right)\hat{e}^0 \wedge \hat{e}^1 + \left(\frac{\beta\gamma'}{\alpha\gamma} - \frac{\beta'}{\alpha}\right)(-\sin\phi F^1 + \cos\phi F^2),$$

$$\hat{\Theta}_{02} = \left(-\frac{\beta''}{\alpha^2\beta} + \frac{\alpha'\beta'}{\alpha^3\beta}\right)\hat{e}^0 \wedge \hat{e}^2 + \left(\frac{\beta\gamma'}{\alpha\gamma} - \frac{\beta'}{\alpha}\right)(-\cos\phi\cos\theta F^1 - \sin\phi\cos\theta F^2 + \sin\theta F^3),$$

$$\hat{\Theta}_{0\alpha} = \left(-\frac{\gamma'}{\alpha^2\gamma} + \frac{\alpha'\gamma'}{\alpha^3\gamma}\right)\hat{e}^0 \wedge \hat{e}^\alpha + \left(\frac{\beta\gamma'}{2\alpha\gamma^3} - \frac{\beta'}{2\alpha\gamma^2}\right)(-\sin\phi F^1{}_{\alpha\beta} + \cos\phi F^2{}_{\alpha\beta})\hat{e}^1 \wedge \hat{e}^\beta$$
$$\cdot\left(\frac{\beta\gamma'}{2\alpha\gamma^3} - \frac{\beta'}{2\alpha\gamma^2}\right)(-\cos\phi\cos\theta F^1{}_{\alpha\beta} - \sin\phi\cos\theta F^2{}_{\alpha\beta} + \sin\theta F^3{}_{\alpha\beta})\hat{e}^2 \wedge \hat{e}^\beta,$$

$$\hat{\Theta}_{12} = \left(\frac{1}{\beta^2} - \frac{\beta'^2}{\alpha^2\beta^2}\right)\hat{e}^1 \wedge \hat{e}^2 + \left(\frac{\beta^2}{2\gamma^2} - 1\right)(\cos\phi\sin\theta F^1 + \sin\phi\sin\theta F^2 + \cos\theta F^3),$$

$$\hat{\Theta}_{1\alpha} = \left(\frac{\beta'}{2\alpha\gamma^2} - \frac{\beta\gamma'}{2\alpha\gamma^3}\right)\left(-\sin\phi F_{\alpha\beta} + \cos\phi F^2{}_{\alpha\beta}\right)\hat{e}^0 \wedge \hat{e}^\beta + \left(\frac{\beta^2}{4\gamma^4} - \frac{\beta'\gamma'}{\alpha^2\beta\gamma}\right)\hat{e}^1 \wedge \hat{e}^\alpha$$
$$+ \frac{\beta}{2\gamma^3}(-\sin\phi D_\gamma F^1{}_{\alpha\beta} + \cos\phi D_\gamma F^2{}_{\alpha\beta})\hat{e}^\gamma \wedge \hat{e}^\beta$$
$$+ \left(\frac{\beta^2}{4\gamma^4} - \frac{1}{2\gamma^2}\right)(\cos\phi\sin\theta F^1{}_{\alpha\beta} + \sin\phi\sin\theta F^2{}_{\alpha\beta} + \cos\theta F^3{}_{\alpha\beta})\hat{e}^2 \wedge \hat{e}^\beta,$$

$$\hat{\Theta}_{2\alpha} = \left(\frac{\beta'}{2\alpha\gamma^2} - \frac{\beta\gamma'}{2\alpha\gamma^3}\right)(-\cos\phi\cos\theta F^1{}_{\alpha\beta} - \sin\phi\cos\theta F^2{}_{\alpha\beta} + \sin\theta F^3{}_{\alpha\beta})\hat{e}^0 \wedge \hat{e}^\beta$$
$$+ \left(\frac{\beta^2}{4\gamma^4} - \frac{\beta'\gamma'}{\alpha^2\beta\gamma}\right)\hat{e}^2 \wedge \hat{e}^\alpha$$
$$+ \frac{\beta}{2\gamma^3}(-\cos\phi\cos\theta D_\gamma F^1{}_{\alpha\beta} - \sin\phi\cos\theta D_\gamma F^2{}_{\alpha\beta} + \sin\theta D_\gamma F^3{}_{\alpha\beta})\hat{e}^\gamma \wedge \hat{e}^\beta$$
$$- \left(\frac{\beta^2}{4\gamma^4} - \frac{1}{2\gamma^2}\right)(\cos\phi\sin\theta F^1{}_{\alpha\beta} + \sin\phi\sin\theta F^2{}_{\alpha\beta} + \cos\theta F^3{}_{\alpha\beta})\hat{e}^1 \wedge \hat{e}^\beta,$$

$$\hat{\Theta}_{\alpha\beta} = \Theta_{\alpha\beta} + \left(\frac{\beta'}{\alpha\gamma^2} - \frac{\beta\gamma'}{\alpha\gamma^3}\right)(\sin\phi F^1{}_{\alpha\beta} - \cos\phi F^2{}_{\alpha\beta})\hat{e}^0 \wedge \hat{e}^1$$
$$+ \left(\frac{\beta'}{\alpha\gamma^2} - \frac{\beta\gamma'}{\alpha\gamma^3}\right)(\cos\phi\cos\theta F^1{}_{\alpha\beta} + \sin\phi\cos\theta F^2{}_{\alpha\beta} - \sin\theta F^3{}_{\alpha\beta})\hat{e}^0 \wedge \hat{e}^2$$

$$+\frac{\beta}{2\gamma^3}(\sin\phi D_\gamma F^1{}_{\alpha\beta}-\cos\phi D_\gamma F^2{}_{\alpha\beta})\hat{e}^\gamma\wedge\hat{e}^1$$

$$+\frac{\beta}{2\gamma^3}(\cos\phi\cos\theta D_\gamma F^1{}_{\alpha\beta}+\sin\phi\cos\theta D_\gamma F^2{}_{\alpha\beta}-\sin\theta D_\gamma F^3{}_{\alpha\beta})\hat{e}^\gamma\wedge\hat{e}^2. \tag{2.20}$$

With the Ricci-tensor components given by (2.8), the orthonormal components of the Einstein equation

$$\hat{R}_{ab}=\Lambda\hat{g}_{ab}=\Lambda\delta_{ab} \tag{2.21}$$

are the extremal equations of the ADM-type action

$$I=\int\left(\frac{1}{\alpha}T-\alpha V\right)\beta^2\gamma^{4n}dr, \tag{2.22}$$

where

$$T=2\frac{\beta'^2}{\beta^2}+16n\frac{\beta'\gamma'}{\beta\gamma}+4n(4n-1)\frac{\gamma'^2}{\gamma^2}, \tag{2.23}$$

$$V=(4n+1)\Lambda-\frac{2}{\beta^2}-\frac{4n(n+2)}{\gamma^2}+2n\frac{\beta^2}{\gamma^2}. \tag{2.24}$$

The variation of I with respect to the non-dynamical lapse function $\alpha(r)$ gives the first-order scalar constraint equation

$$\alpha^{-2}T+V=2\hat{G}_{00}+(4n+1)\Lambda=0. \tag{2.25}$$

So long as this constraint is maintained, one may reparametrise the coördinate r so that $\alpha\beta^2\gamma^{4n}V$ is held constant, and then [13, 14, 6] varying I gives the equation for a timelike geodesic with affine parameter r in the two-dimensional minisuperspace metric

$$d\omega^2=\beta^4\gamma^{8n}V\left(2\frac{d\beta^2}{\beta^2}+16n\frac{d\beta\, d\gamma}{\beta\gamma}+4n(4n-1)\frac{d\gamma^2}{\gamma^2}\right)$$
$$=-\beta^4\gamma^{8n}V(-dt^2+dz^2), \tag{2.26}$$

where

$$t=\left(\frac{4n+1}{4n+2}\right)^{1/2}\log(\beta^2\gamma^{4n}),\quad z=\left(\frac{8n}{4n+2}\right)^{1/2}\log\left(\frac{\beta}{\gamma}\right). \tag{2.27}$$

This geodesic of (2.26) may alternatively be viewed as the trajectory of a particle of variable mass-squared,

$$\mu^2=-\beta^4\gamma^{8n}V, \tag{2.28}$$

in the conformally-related flat metric $-dt^2+dz^2$.

An alternative way to formulate the Einstein equation (2.21) is to use the logarithmic expansion rate, the trace of the second fundamental form of the twistor-space hypersurfaces of constant r,

$$\chi=\frac{2\beta'}{\alpha\beta}+\frac{4n\gamma'}{\alpha\gamma}. \tag{2.38}$$

The (00) component of (2.21) implies that the proper radial derivative of χ,

$$\frac{\chi'}{\alpha} = -2\left(\frac{\beta'}{\alpha\beta}\right)^2 - 4n\left(\frac{\gamma'}{\alpha\gamma}\right)^2 - \Lambda, \tag{2.39}$$

is negative semi-definite for $\Lambda \geqq 0$, so in this case χ varies monotonically and hence may be taken as the independent variable. A convenient choice of dependent variables is the twistor-space shape parameter

$$\lambda = \frac{\beta}{\gamma} \tag{2.40}$$

and its logarithmic rate of change with proper radius, the shear

$$\sigma = \frac{\lambda'}{\alpha\lambda} = \frac{\beta'}{\alpha\beta} - \frac{\gamma'}{\alpha\gamma}. \tag{2.41}$$

One can now use the definitions of χ and σ in (2.39) to write the radial proper derivative as

$$\frac{1}{\alpha}\frac{d}{dr} = \frac{\chi'}{\alpha}\frac{d}{d\chi} = -\frac{\chi^2 + 8n\sigma^2 + (4n+2)\Lambda}{4n+2}\frac{d}{d\chi}. \tag{2.42}$$

In terms of the variables χ, λ and σ, the constraint equation (2.25) gives

$$\beta = \left\{\frac{4(2n+1)[1 + 2n(n+2)\lambda^2 - n\lambda^4]}{(4n+1)\chi^2 - 8n\sigma^2 + (4n+1)(4n+2)\Lambda}\right\}^{1/2}, \quad \gamma = \frac{\beta}{\lambda}, \tag{2.43}$$

and then (2.41–2.43) and the remaining independent component of the Einstein equation (2.21) give

$$\frac{d\lambda}{d\chi} = -\frac{(4n+2)\lambda\sigma}{\chi^2 + 8n\sigma^2 + \Lambda}, \tag{2.44}$$

$$\frac{d\sigma}{d\chi} = \frac{2(4n+2)\chi\sigma - g(\lambda)[(4n+1)\chi^2 - 8n\sigma^2 + (4n+1)(4n+2)\Lambda]}{2[\chi^2 + 8n\sigma^2 + \Lambda]}, \tag{2.45}$$

where

$$\begin{aligned} g(\lambda) &= 1 - \frac{2n+1}{4n}\lambda\frac{d}{d\lambda}\log[1 + 2n(n+2)\lambda^2 - n\lambda^4] \\ &= \frac{1 - (n+2)\lambda^2 + (n+1)\lambda^4}{1 + 2n(n+2)\lambda^2 - n\lambda^4} = \frac{(1-\lambda^2)[1 - (n+1)\lambda^2]}{1 + 2n(n+2)\lambda^2 - n\lambda^4}. \end{aligned} \tag{2.46}$$

3. Complete Compact Einstein Spaces

Having given the local construction of the metric (2.1), we now turn to the consideration of the global topological structure of the manifold $\hat{M}$ on which (2.1) is to be defined. The discussion is very similar to that given in [6, 9]. The range of the radial coördinate r in (2.1) is determined by the nature of the zeroes or infinities of the r-dependent functions α, β and γ. These define the "endpoints" of the radial variable. There are essentially two kinds of possible endpoint, corresponding to one or both of β and γ going either to zero or to infinity. If $\hat{M}$ is to be a compact manifold, we require that β should vanish at the endpoints r_1 and r_2 of the range of the r coördinate, $r_1 \leqq r \leqq r_2$. This is necessary in order that the 3-dimensional fibres parametrised by r and u^i (subject to (1.9)) can be compact. Provided that β approaches zero appropriately at r_1 and r_2, the 3-dimensional fibres will have the topology of a 3-sphere. The required behaviour for β at the endpoints is that

$$\left.\frac{\beta'}{\alpha}\right|_{r_1} = +1; \quad \left.\frac{\beta'}{\alpha}\right|_{r_2} = -1, \tag{3.1}$$

together with the requirement that $(1/\alpha\beta)(\beta'/\alpha)'$ remain finite at the endpoints. The 3-sphere fibres are foliated by 2-spheres $u^i u^i = 1$, growing from zero radius at $r = r_1$ and collapsing down to zero radius again at $r = r_2$. The regularity conditions (3.1) ensure that the 3-sphere does not have conical singularities at the north and south poles r_1 and r_2. The requirement that $(1/\alpha\beta)(\beta'/\alpha)'$ be finite at the poles ensures that the curvature does not diverge there (see (2.4)).

Since a level surface $r = \text{constant}$ in $\hat{M}$ is the twistor space Z, it follows that γ must remain non-zero for all r in the interval $r_1 \leqq r \leqq r_2$. This is because unless the level surfaces are round spheres, it is impossible for them to "nest" down to zero radius without the occurrence of a conical singularity. Thus we cannot have β and γ going to zero simultaneously. To see that Z can never be a sphere, we observe that it is a Kähler space of dimension $4n + 2 \geqq 6$. Now the only sphere that is Kähler is S^2, whence Z cannot be a sphere. Not only must γ be non-zero everywhere in the interval $r_1 \leqq r \leqq r_2$, but also we see from (2.4) that in order to avoid curvature-singularities at the endpoints, $\gamma'/\alpha\beta$ must remain finite at r_1 and r_2. In particular, this implies that γ'/α must go to zero there. For $\hat{M}$ to be compact, γ must also remain finite in the interval $r_1 \leqq r \leqq r_2$.

In terms of the geodesics of the minisuperspace metric (2.26), a complete, compact, non-singular Einstein space requires $\Lambda > 0$, and corresponds to a geodesic coming from $t = -\infty, z = -\infty$ at $r = r_1$, where the regularity requirement (3.1) is that as $r \to r_1$,

$$\frac{z}{t} \to \left(\frac{2n}{4n+1}\right)^{1/2}. \tag{3.2}$$

The geodesic has t increasing until it reaches the region $V \geqq 0$, where it can turn around. Then it goes back to $t = -\infty$ at $r = r_2$, where it must again satisfy the regularity requirement (3.2). (A generic singular solution has instead $z/t \to \pm 1$.) The only known non-singular solutions are symmetric about the midpoint of r, in which case the geodesic turns around precisely on the line $V = 0$ (where it has $\beta' = \gamma' = 0$) and reverses itself to go back down the same path in the (t, z) or (β, γ) space that it came up.

In terms of the variables χ, λ and σ, and their equations (2.44) and (2.45), a complete compact Einstein space must have the shape parameter λ be zero at the two endpoints but positive in between, and the shear σ vary from $+\infty$ to $-\infty$ (not necessarily monotonically), as the expansion χ, taken as the independent variable, varies monotonically from $+\infty$ to $-\infty$. The regularity requirement (3.1) is that

$$\frac{\sigma}{\chi} \to \tfrac{1}{2} \tag{3.3}$$

at the endpoints. A solution symmetric about the midpoint would have $\sigma = 0$, where $\chi = 0$.

One can look for solutions numerically by integrating the geodesic equation in the metric (2.26), starting from the regularity condition (3.2) at $r = r_1$ or $t = -\infty$, with the initial value, γ_1, of

$$\gamma = \exp\left[(4n+1)^{-1/2}(4n+2)^{-1/2}t - (2n)^{1/2}(4n+2)^{-1/2}z\right] \tag{3.4}$$

chosen by trial and error to give a geodesic that obeys the regularity condition again when it returns to $t = -\infty$. Alternatively, one can integrate (2.44) and (2.45), with the regularity condition (3.3) at $\chi = \infty$, and look for a trial initial value of γ, given by (2.43), that yields a solution regular also at $\chi = -\infty$. In actual calculations, one should replace χ and σ, which have infinite ranges, by invertible monotonic functions of these variables that have finite ranges.

A preliminary numerical analysis indicates that complete compact solutions do exist, at least for small n. For large n, one may readily construct approximate solutions analogous to those in [6]. Hence it appears likely that solutions exist for all natural numbers n. All of the solutions found are symmetric about the midpoint. Further details of the solutions will be given in a future publication.

Having ensured that the above regularity conditions are satisfied, the metric (2.1) is now seen to be globally extendible on a compact manifold $\hat{M}$. The topology of $\hat{M}$ is that of a certain S^3 bundle over the quaternionic Kähler base manifold M. Although S^3 is topologically $SU(2)$, the S^3 bundle here is *not* a principal $SU(2)$ bundle; rather, it is an associated bundle with structure group $SO(4)$. In the case that the base space M is S^4 it is shown in [10] that such bundles are characterised by giving the transition functions on an equatorial 3-sphere on S^4. Thinking of S^3 as $SU(2)$ or the unit quarternions, the transition functions are homotopic to the map $q' \to q^{m+n}q'q^{-m}$. Steenrod called these bundles $B_{m,n}$, where $B_{m,n}$ and $B_{m+n,-n}$ are equivalent. The case $B_{0,1}$ is the standard one-instanton $SU(2)$ bundle, which as a manifold is diffeomorphic to S^7. The bundle on which we have constructed our metric is $B_{1,0}$. In fact this is reducible to an $SO(3)$ bundle where the action of $SO(3)$ on S^3 is just rotations that leave fixed the north and south poles of S^3. Because they are left fixed the bundle $B_{1,0}$, and more generally $B_{m,0}$, admits a global cross section whereas $B_{0,1}$ does not. In terms of our construction the $SO(3)$ rotates the u^i and leaves r fixed. The global sections are just $r = r_1$ and $r = r_2$. It is not difficult to find two charts on the S^4 base and to calculate the coördinate transformation between them. The A^i transform as the $B_{0,1}$ case but the induced action on the u^i is just an $SO(3)$ rotation of the u^i. Although the integers m and n distinguish the bundles as $SO(4)$ bundles, they do not distinguish the bundles as manifolds. In fact the bundles $B_{m,0}$ have the same homology and homotopy groups as the product $S^3 \times S^4$, i.e. as $B_{0,0}$. It has been shown however [11] that $B_{m,0}$ and $B_{n,0}$ have the same homotopy type if and only if $m = \pm n \bmod 12$. Thus $\hat{M}$ cannot be homeomorphic to $S^3 \times S^4$. It clearly cannot be homeomorphic to S^7 either, because its homology groups are different. The reader will appreciate the need for care here if he recalls Milnor's famous result [12] that if $k^2 \neq 1 \bmod 7$ then $B_{(k-1)/2,1}$ is homeomorphic but not diffeomorphic to S^7. It follows from Hodge theory that $\hat{M}$ has just one harmonic 3-form and just one harmonic 4-form. Since it is unique the 3-form must be invariant under G and Sp(1). It is given by (2.13), where f satisfies (2.16) and g and h are obtained from f as indicated.

By taking the base space M to be $P_2(C)$, the complex projective plane, we can obtain another example of a seven-dimensional compact Einstein space $\hat{M}$. We do not know of any discussion of the topology and differentiability of the three-sphere bundle over M in this case.

4. Non-compact Ricci-flat Metrics

In order that $\hat{M}$ be non-compact, while the base manifold M is compact, at least one limit of the range of r must be an infinite endpoint, i.e. it must correspond to a region in $\hat{M}$ that is at infinite proper distance from all points in $\hat{M}$ corresponding to the other values of r within its range. We shall consider only the case where there is just one infinite endpoint, which without loss of generality may be taken to be located at $r = \infty$, so that without loss of generality we may take the range of r to be $r_1 \leqq r < \infty = r_2$. This assumption that there is just one infinite endpoint is in fact not a restriction at all for the situation of interest to us in this section, where the Ricci tensor vanishes. This is a consequence of a theorem by Cheeger and Gromoll [15], which asserts that for any complete manifold with metric satisfying $R_{ab} \geqq 0$, there can be at most one infinite endpoint (excluding the special case of $R \times T^n$). This conclusion also follows from (2.39). Since $r = \infty$ lies at infinite proper distance from all points in the manifold, there are no regularity conditions to be imposed on the functions α, β and γ there. At $r = r_1$, which we are taking to be a finite endpoint, the condition (3.1) must hold, together with the additional requirements discussed in Sect. 3 that ensure boundedness of the curvature. Over a given point in the base manifold M, the 3-dimensional fibre has the topology R^3, since it consists of a nested sequence of 2-spheres that collapse down to a regular origin at $r = r_1$. Thus the topology of $\hat{M}$ in this case will be that of an R^3 bundle over M.

In general, we have been unable to find explicit solutions for α, β and γ such that the Ricci tensor $\hat{R}_{ab}$ given by (2.8) is zero. However, in the case that $n = 1$, so that $\hat{M}$ is a seven-dimensional manifold, we have found an explicit solution. The radial functions take the form

$$\alpha^2 = (1 - r^{-4})^{-1}; \quad \beta^2 = \tfrac{1}{4} r^2 (1 - r^{-4}); \quad \gamma^2 = \tfrac{1}{2} r^2. \tag{4.1}$$

One can verify by substituting (4.1) into (2.8) that the metric is indeed Ricci-flat. The range of the radial coördinate is $1 \leqq r < \infty$, and it is easy to see that the regularity condition (3.1) is indeed satisfied at $r = r_1 = 1$.

Since the quaternionic Kähler base manifold M is four-dimensional in this example, the conditions (1.6) and (1.7) are not automatic, but must be imposed as further restrictions on M. We know of two cases for which these conditions are satisfied, namely $M = S^4$ and $M = P_2(C)$. When $M = S^4$, with its standard Einstein metric normalised so that

$$R_{\alpha\beta} = 3g_{\alpha\beta}, \tag{4.2}$$

$\Theta_{\alpha\beta}$ in (2.4) is given by

$$\Theta_{\alpha\beta} = e^\alpha \wedge e^\beta. \tag{4.3}$$

The almost complex structures $J^i{}_{\alpha\beta}$ take the form $J^i{}_{0j} = -\delta_{ij}$, $J^l{}_{jk} = -\varepsilon_{ijk}$, and A^i is a potential for the single-instanton bundle over S^4.

We are now in a position to calculate the holonomy group of this 7-dimensional metric. We recall that the holonomy group of an m-dimensional manifold is defined as that subgroup of the tangent space group $SO(m)$ which describes the rotation of a spinor or tensor field under parallel transport around all possible closed curves in the manifold. The rotation of a field ψ under such an infinitesimal transformation is given by

$$\delta\psi = G_{ab}\delta A^{ab}\psi, \tag{4.4}$$

where

$$G_{ab} \equiv \tfrac{1}{2}R_{abcd}\Gamma^{cd}, \tag{4.5}$$

Γ_{ab} are the generators of $SO(m)$ in the representation of the field ψ and δA^{ab} is an infinitesimal area element spanned by the closed curve. In many cases, including the metrics that we are considering in this paper, the group generated by the G_{ab}'s is the holonomy group. In general, however, the holonomy group may be larger, corresponding to the fact that parallel transport around non-infinitesimal curves may enable one to reach parts of the tangent-space group that infinitesimal curves cannot reach. This is true even for the restricted holonomy group, which is generated by contractible, but not necessarily infinitesimal, curves. Since parallel propagation around non-infinitesimal curves involves the components of the Riemann tensor away from the starting and finishing point of the curve, it follows that in general the restricted holonomy group is determined by the Riemann tensor and all its covariant derivatives. Using (2.4–2.7), we find that the generators G_{ab} are given by

$$\begin{aligned}
G_{0i} &= 4Ah_{ij}\Gamma_{0j} - A\varepsilon_{ijk}J^j{}_{\alpha\beta}u^k\Gamma_{\alpha\beta},\\
G_{0\alpha} &= -2A\Gamma_{0\alpha} - A\varepsilon_{ijk}J^j{}_{\alpha\beta}u^k\Gamma_{i\beta},\\
G_{i\alpha} &= -(A+B)h_{ij}\Gamma_{j\alpha} - Bh_{ik}h_{jl}\varepsilon_{klm}J^m{}_{\alpha\beta}\Gamma_{j\beta} + A\varepsilon_{ijk}J^j{}_{\alpha\beta}u^k\Gamma_{0\beta},\\
G_{ij} &= h_{ik}h_{jl}\Gamma_{kl} - Bh_{ik}h_{jl}\varepsilon_{klm}J^m{}_{\alpha\beta}\Gamma_{\alpha\beta},\\
G_{\alpha\beta} &= \tfrac{1}{3}(5A+B)\Gamma_{\alpha\beta} - \tfrac{1}{3}(A-B)u^iu^j[J^i{}_{\alpha\gamma}J^j{}_{\beta\delta} + J^i{}_{\alpha\beta}J^j{}_{\gamma\delta}]\Gamma_{\gamma\delta},\\
&\quad - Bh_{ik}h_{jl}\varepsilon_{klm}J^m{}_{\alpha\beta}\Gamma_{ij} - 2A\varepsilon_{ijk}J^j{}_{\alpha\beta}u^k\Gamma_{0i},
\end{aligned} \tag{4.6}$$

where A and B are the following functions of r:

$$A \equiv \frac{1}{r^6}; \quad B \equiv \frac{1}{4r^2}(3 + r^{-4}). \tag{4.7}$$

Ostensibly, there are 21 quantities G_{ab}, generating the tangent space group

$SO(7)$. However, not all 21 generators are independent. In fact, we find a total of 7 relations among the G_{ab}, which take the form

$$\begin{aligned} J^i{}_{\alpha\beta}u^i G_{0\alpha} + J^i{}_{\alpha\beta}G_{i\alpha} &= 0, \\ h_{ij}J^j{}_{\alpha\beta}G_{\alpha\beta} + 2\varepsilon_{ijk}u^j G_{0k} &= 0, \\ u^i J^i{}_{\alpha\beta}G_{\alpha\beta} + u^i\varepsilon_{ijk}G_{jk} &= 0. \end{aligned} \tag{4.8}$$

The first of these equations gives four relations amongst the G_{ab}, the second gives two relations (since although i runs over three values, the expression is orthogonal to u^i) and the final equation gives one further relation. Thus there are fourteen independent generators G_{ab} of the restricted holonomy group. In particular, it follows from (4.8) that G_{0i}, $G_{0\alpha}$ and G_{ij}, which comprise $2+4+1=7$ combinations, can be expressed in terms of the remaining combinations, and thus we may choose the $8+6=14$ combinations $G_{i\alpha}$ and $G_{\alpha\beta}$ as the independent generators. It is a straightforward exercise to show that $G_{i\alpha}$ and $G_{\alpha\beta}$ may be expressed as linear combinations of the 14 generators H_{ab} of the exceptional group G_2, which can be written in the form [16]

$$\begin{aligned} H_{0i} &= \Gamma_{0i} + \tfrac{1}{2}\varepsilon_{ijk}\Gamma_{\hat{j}\hat{k}}, \\ H_{ij} &= \Gamma_{ij} + \Gamma_{\hat{i}\hat{j}}, \\ H_{i\hat{j}} &= 2\Gamma_{i\hat{j}} + \Gamma_{j\hat{i}} - \delta_{ij}\Gamma_{k\hat{k}} + \varepsilon_{ijk}\Gamma_{o\hat{k}}, \end{aligned} \tag{4.9}$$

where $i, j = 1,2,3$ and $\hat{i}, \hat{j} = 4,5,6 = \hat{1},\hat{2},\hat{3}$. Since it turns out that the $G_{i\alpha}$ and $G_{\alpha\beta}$ generate the entire holonomy group, the Ricci-flat metric in the R^3 bundle over S^4 has G_2 holonomy, which is the exceptional possibility for 7-manifolds included in Berger's classification of holonomy groups for Riemannian manifolds [17].

The level surfaces $r = \text{constant}$ in the R^3 bundle over S^4 are the twistor space Z discussed in Sect. 1 with $M = S^4$; i.e. $Z = P_3(C)$. From (4.1), we see that the ratio of β^2 to γ^2 tends to 1/2 as r tends to infinity. Thus from (1.15) it follows that the metric on the $P_3(C)$ level surfaces is tending asymptotically to the "squashed" Einstein metric on $P_3(C)$.

There is another Ricci-flat non-compact solution corresponding to the case where M is taken to be the complex projective plane $P_2(C)$ rather than S^4. The only difference in the calculation of the curvature in this case is that the curvature 2-form $\Theta_{\alpha\beta}$ on the base space M now takes the form

$$\Theta_{\alpha\beta} = \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + K_{\alpha\gamma}K_{\beta\delta} + K_{\alpha\beta}K_{\gamma\delta})e^\gamma \wedge e^\delta \tag{4.10}$$

rather than simply (4.3), where $K_{\alpha\beta}$ is the Kähler form on $P_2(C)$. The generators G_{ab} of the holonomy group are given by (4.6) except for $G_{\alpha\beta}$, which now takes the form

$$\begin{aligned} G_{\alpha\beta} = {} & (2A - B)\Gamma_{\alpha\beta} - \tfrac{1}{3}(A - 4B)[K_{\alpha\gamma}K_{\beta\delta} + K_{\alpha\beta}K_{\gamma\delta}]\Gamma_{\gamma\delta} \\ & - \tfrac{1}{3}(A - B)u^i u^j [J^i{}_{\alpha\gamma}J^j{}_{\beta\delta} + J^i{}_{\alpha\beta}J^j{}_{\gamma\delta}]\Gamma_{\gamma\delta} \\ & - Bh_{ik}h_{jl}\varepsilon_{klm}J^m{}_{\alpha\beta}\Gamma_{ij} - 2A\varepsilon_{ijk}J^j{}_{\alpha\beta}u^k\Gamma_{0i}. \end{aligned} \tag{4.11}$$

One can show that again there are 7 relations among the 21 G_{ab}'s, and that the 14 independent ones can be written in terms of the H_{ab}'s of (4.9). Thus the

Ricci-flat metric on the R^3 bundle over $P_2(C)$ also has holonomy group G_2. As in the case where $M = S^4$, the metric on the twistor space Z that comprises the level surfaces $r = \text{constant}$ tends asymptotically to the "squashed" Einstein metric ($\lambda^2 = 1/2$ in (1.12)) as r tends to infinity. In this case, the twistor space Z for $M = P_2(C)$ is the flag manifold $SU(3)/T^2$, where T^2 is a maximal torus in $SU(3)$.

5. Other Ricci-flat Metrics with Exceptional Holonomy

There are other examples of Ricci-flat metrics with exceptional holonomy groups that may be constructed by methods similar to those of the previous sections. The first of these is another seven-dimensional example, which has the topology of an R^4 bundle over S^3. Since S^3 is parallelizable, the bundle is trivial, and so in this case the seven-dimensional manifold has the product topology $R^4 \times S^3$. The metric takes the form

$$d\hat{s}^2 = \alpha^2 dr^2 + \beta^2(\sigma^i - A^i)^2 + \gamma^2 \Sigma^i \Sigma^i, \tag{5.1}$$

where Σ^i are a set of left-invariant one-forms on the S^3 base manifold, satisfying $d\Sigma^1 = -\Sigma^2 \wedge \Sigma^3$, etc., σ^i are a set of left-invariant one-forms on the fibres of a principal $SU(2)$ bundle over S^3, with connection A^i given by

$$A^i = \tfrac{1}{2}\Sigma^i, \tag{5.2}$$

and as usual α, β and γ functions of the seventh coördinate, r. The level surfaces $r = \text{constant}$ have the topology $S^3 \times S^3$. In fact the metrics $ds^2 = \lambda^2(\sigma^i - (1/2)\Sigma^i)^2 + \Sigma^i\Sigma^i$, where $\lambda = \text{constant}$, give a family of homogeneous metrics on $S^3 \times S^3$. The standard Einstein metric corresponds to taking $\lambda^2 = 4$, and there is a second Einstein metric given by taking $\lambda^2 = 4/3$.

We introduce the orthonormal basis

$$\hat{e}^0 = \alpha dr; \quad \hat{e}^{\hat{\imath}} = \beta(\sigma^i - A^i); \quad \hat{e}^i = \gamma\Sigma^i, \tag{5.3}$$

where $i, j = 1, 2, 3$, and $\hat{\imath}, \hat{\jmath} = \hat{1}, \hat{2}, \hat{3} = 4, 5, 6$. The curvature 2-form for (5.3) can be calculated by standard methods. It may in fact be read off from the results in [6], by making minor modifications to take into account the fact that the base space is three-dimensional rather than four-dimensional. One can then verify that a Ricci-flat metric is obtained by choosing the functions α, β and γ as follows:

$$\alpha^2 = (1 - r^{-3})^{-1}; \quad \beta^2 = \tfrac{1}{9}r^2(1 - r^{-3}); \quad \gamma^2 = \tfrac{1}{12}r^2. \tag{5.4}$$

As r tends to infinity, the level surfaces therefore approach the geometry of the second Einstein metric on $S^3 \times S^3$ discussed above. Defining the functions A and B as

$$A = \frac{3}{4r^5}; \quad B = \frac{1}{4r^2}(5 + r^{-3}), \tag{5.5}$$

one can show that the curvature components G_{ab}, defined by (4.5), take the form

$$G_{0\hat{\imath}} = 2A\Gamma_{0\hat{\imath}} - A\varepsilon_{ijk}\Gamma_{jk},$$
$$G_{0i} = -2A\Gamma_{0i} + A\varepsilon_{ijk}\Gamma_{\hat{\jmath}\hat{k}},$$

$$
\begin{aligned}
G_{\hat{i}j} &= -\tfrac{1}{5}(4A+3B)\Gamma_{\hat{i}j} + \tfrac{1}{5}(A-3B)\Gamma_{\hat{j}i} + \tfrac{1}{5}(A+2B)\delta_{ij}\Gamma_{\hat{k}k} + A\varepsilon_{ijk}\Gamma_{0k}, \\
G_{\hat{i}\hat{j}} &= B\Gamma_{\hat{i}\hat{j}} + B\Gamma_{ij}, \\
G_{ij} &= (2A+B)\Gamma_{ij} + B\Gamma_{\hat{i}\hat{j}} - 2A\varepsilon_{ijk}\Gamma_{0\hat{k}}.
\end{aligned}
\tag{5.6}
$$

One can easily show that these quantities generate the group G_2. For example, by interchanging the hatted and unhatted indices and reversing the sign of the epsilon tensor in (4.9), one can show that all the 21 components G_{ab} can be expressed in terms of the 14 G_2 generators H_{ab} given by (4.9).

As discussed earlier, strictly speaking it does not necessarily follow that the group generated by the curvature components (4.5) is *precisely* the restricted holonomy group of the space in question: one should really consider the (possibly larger) group generated by the Riemann tensor and all its covariant derivatives. However, in this case the following argument enables us to demonstrate that the holonomy group is indeed just G_2. We note that in a seven-dimensional space, one can define Majorana spinors that transform as the 8-dimensional representation of the Spin(7) double covering of the tangent space group. If the holonomy group is G_2, then there should exist a covariantly-constant spinor η, satisfying

$$
\hat{D}\eta \equiv d\eta + \tfrac{1}{4}\hat{\omega}_{ab}\Gamma^{ab}\eta = 0,
\tag{5.7}
$$

since under the embedding of G_2 in Spin(7) the 8 of Spin(7) decomposes to the $7+1$ of G_2. The singlet in this decomposition corresponds to the covariantly-constant spinor η. The integrability condition that follows from acting on (5.7) with another covariant exterior derivative $\hat{D}$ is precisely the condition that η be annihilated by G_{ab}, i.e. $G_{ab}\eta = 0$. But since we have noted that in this case the G_{ab}'s generate the group G_2, this *necessary* condition on η in fact determines it entirely, except for an overall factor. This factor may be fixed (up to a sign) by requiring that η be real, and be normalised to $\bar{\eta}\eta = 1$. (Since $\hat{D}\eta = 0$, it follows that η has constant norm.) It is now straightforward to calculate $\hat{D}$, defined by (5.7),

$$
\begin{aligned}
\hat{D} = d &- \frac{\beta'}{2\alpha\beta}\hat{e}^{\hat{i}}\Gamma_{0\hat{i}} - \frac{\gamma'}{2\alpha\gamma}\hat{e}^{i}\Gamma_{0i} - \frac{1}{8\beta}\varepsilon_{ijk}\hat{e}^{\hat{k}}\Gamma_{\hat{i}\hat{j}} - \tfrac{1}{4}\varepsilon_{ijk}A^{k}\Gamma_{\hat{i}\hat{j}} \\
&+ \frac{\beta}{16\gamma^2}\varepsilon_{ijk}\hat{e}^{k}\Gamma_{i\hat{j}} + \tfrac{1}{4}\omega_{ij}\Gamma_{ij} - \frac{\beta}{32\gamma^2}\varepsilon_{ijk}\hat{e}^{\hat{k}}\Gamma_{ij},
\end{aligned}
\tag{5.8}
$$

where ω_{ij} is the connection one-form for the base space with dreibein Σ^i. From (5.5), one can then show that the unit-norm real spinor annihilated by (5.6) does indeed satisfy $D\eta = 0$. Thus the holonomy group for the Ricci-flat metric on $R^4 \times S^3$ is indeed G_2.

In eight dimensions, where the holonomy group is generically $SO(8)$, there is another exceptional possibility that appears in Berger's classification [17], for which the holonomy group is Spin(7). An example of an 8-manifold with Spin(7) holonomy may also be constructed using the techniques of this paper. In fact this example is a special case of the general class of quaternionic line bundles over quaternionic projective spaces that were discussed in [6]. The metric takes the form

$$
d\hat{s}^2 = \alpha^2 dr^2 + \beta^2(\sigma^i - A^i)^2 + \gamma^2 ds^2,
\tag{5.9}
$$

where ds^2 is the metric on the base space, which we take to be $S^4 = P_1(H)$. The σ^i are left-invariant one-forms on the $SU(2)$ fibres of the single-instanton bundle over S^4, and as usual α, β and γ are functions of the remaining coördinate r. We introduce the orthonormal basis

$$\hat{e}^0 = \alpha dr; \quad \hat{e}^i = \beta(\sigma^i - A^i); \quad \hat{e}^\alpha = \gamma e^\alpha, \tag{5.10}$$

where e^α is an orthonormal basis for S^4. The curvature 2-form for (5.10) is given in [6]. One can then verify that if one takes α, β and γ to be given by

$$\alpha^2 = (1 - r^{-10/3})^{-1}; \quad \beta^2 = \frac{9}{100} r^2 (1 - r^{-10/3}); \quad \gamma^2 = \frac{9}{20} r^2, \tag{5.11}$$

then the metric (5.9) is Ricci flat. Topologically, the manifold $\hat{M}$ is an R^4 bundle over S^4. As r tends to infinity, the metric on the level surfaces $r = \text{constant}$ tends to the homogeneous "squashed" Einstein metric on the seven-sphere.

From [6], one can show that the curvature components G_{ab} defined by (4.5) are given by

$$\begin{aligned} G_{0i} &= 4A\Gamma_{0i} + AJ^i{}_{\alpha\beta}\Gamma_{\alpha\beta}, \\ G_{0\alpha} &= -3A\Gamma_{0\alpha} + AJ^i{}_{\alpha\beta}\Gamma_{i\beta}, \\ G_{i\alpha} &= -(A + 2B)\Gamma_{i\alpha} - B\varepsilon_{ijk}\Gamma_{j\beta} + AJ^i{}_{\alpha\beta}\Gamma_{0\beta}, \\ G_{ij} &= 4B\Gamma_{ij} - B\varepsilon_{ijk}J^k{}_{\alpha\beta}\Gamma_{\alpha\beta}, \\ G_{\alpha\beta} &= 2(A + B)\Gamma_{\alpha\beta} + 2AJ^i{}_{\alpha\beta}\Gamma_{0i} - B\varepsilon_{ijk}J^k{}_{\alpha\beta}\Gamma_{ij}, \end{aligned} \tag{5.12}$$

where A and B are given here by

$$A = \tfrac{5}{9} r^{-16/3}; \quad B = \frac{1}{9r^2}(4 + r^{-10/3}). \tag{5.13}$$

After some algebra, one can show that all the G_{ab}'s in (5.12) may be expressed as linear combinations of the 21 generators of the Spin(7) subgroup of the 28 generators Γ_{ab} of $SO(8)$, as given, for example, in [18].

As in the case of the Ricci-flat metric on $R^4 \times S^3$ discussed above, we can easily show explicitly that there exists a covariantly-constant spinor for our Ricci-flat metric on the R^4 bundle over S^4. In this case, the irreducible spinor representations of the Spin(8) double covering of the $SO(8)$ tangent space group are the 8_+ and 8_-, which correspond to left-handed and right-handed Majorana–Weyl spinors respectively. Under the embedding of Spin(7) in Spin(8), one of the spinors decomposes irreducibly, say $8_- \to 8$, while the other decomposes as $8_+ \to 7 + 1$. The singlet in this decomposition corresponds to the covariantly-constant spinor η. Again, this means that $G_{ab}\eta = 0$ is an integrability condition for the existence of η, which determines η uniquely up to an overall factor. As before, we fix the factor (up to a sign) by demanding that η be real and have unit norm. The exterior derivative defined by (5.7) takes the form

$$\begin{aligned} \hat{D} = d &- \frac{\beta'}{2\alpha\beta}\hat{e}^i\Gamma_{0i} - \frac{\gamma'}{2\alpha\gamma}\hat{e}^\alpha\Gamma_{0\alpha} - \frac{\beta}{4\gamma^2}J^i{}_{\alpha\beta}\hat{e}^\beta\Gamma_{i\alpha} - \frac{1}{8\beta}\varepsilon_{ijk}\hat{e}^k\Gamma_{ij} \\ &- \tfrac{1}{4}\varepsilon_{ijk}A^k\Gamma_{ij} + \tfrac{1}{4}\omega_{\alpha\beta}\Gamma_{\alpha\beta} + \frac{\beta}{8\gamma^2}J^i{}_{\alpha\beta}\hat{e}^i\Gamma_{\alpha\beta}, \end{aligned} \tag{5.14}$$

where $\omega_{\alpha\beta}$ is the connection one-form for the vierbein on the S^4 base space. Thus using (5.11), one can show that the unit-norm spinor η that is annihilated by G_{ab} given by (5.12) is indeed covariantly constant. Hence it follows that the Ricci-flat metric on the R^4 bundle over S^4 has precisely Spin(7) holonomy.

Using the covariantly-constant spinor η in the seven and eight-dimensional spaces that we have discussed in this paper, one can immediately construct certain covariantly-constant p-forms, with components given by

$$\omega_{a_1 \cdots a_p} = \bar{\eta} \Gamma_{a_1 \cdots a_p} \eta, \tag{5.15}$$

where $\Gamma_{a_1 \cdots a_p} = \Gamma_{[a_1} \cdots \Gamma_{a_p]}$, the totally antisymmetric product of p Dirac matrices. In the case that $\hat{M}$ is seven dimensional, it follows from the fact that η is Majorana (and commuting) that (5.15) is non-zero only if $p = 0, 3, 4$ or 7, owing to the antisymmetry of the Dirac-matrix products under the interchange of spinor indices when $p = 1, 2, 5$ or 6. When $p = 0$ or 7 we just get a constant scalar or its dual, but the case $p = 3$ gives the 3-form that we discussed in Sect. 2. When $p = 4$ we obtain the dual of this 3-form. When $\hat{M}$ is eight dimensional, η is Majorana and Weyl, so (5.15) is non-zero only if $p = 0, 4$ or 8. The non-trivial case $p = 4$ yields a covariantly-constant 4-form in the Ricci-flat R^4 bundle over S^4 discussed earlier in this section.

It is interesting to note that the existence of a covariantly-constant spinor η in a space automatically implies that it must be Ricci flat. To see this, consider the integrability condition

$$R_{abcd} \Gamma_{cd} \eta = 0 \tag{5.16}$$

which follows from taking the commutator of covariant derivatives on η. Multiplying on the left with Γ_b, and using the cyclic identity $R_{a[bcd]} = 0$, we obtain

$$R_{ab} \Gamma_b \eta = 0, \tag{5.17}$$

where R_{ab} is the Ricci tensor. Multiplying on the left by $\bar{\eta} \Gamma_c$ then yields the result

$$R_{ab} = 0. \tag{5.18}$$

This provides a simple alternative derivation of the result [19] that any seven-dimensional space with G_2 holonomy or any eight-dimensional space with Spin(7) holonomy must be Ricci flat. Indeed, more generally, whenever the holonomy group H of a space is such that the decomposition of the spinor representation under H includes a singlet, the space must be Ricci flat. Examples are provided by $2n$-dimensional Kähler manifolds that have $SU(n)$ holonomy.

6. The Asymptotic Structure of the Non-Compact Metrics

In this section we discuss the asymptotic behaviour of our new metrics in more detail. In all cases the metric tends to a Ricci-flat metric on a *generalised cone*, i.e. the n-dimensional metric tends to

$$ds^2 = dr^2 + r^2 d\Omega_{n-1}^2, \tag{6.1}$$

where Ω_{n-1}^2 is a metric on the "base" of the cone. The metric (6.1) will be Ricci-flat if and only if $d\Omega^2_{n-1}$ is Einstein, with the scale chosen so that

$$R_{AB} = (n-2)g_{AB}, \tag{6.2}$$

where $A, B = 1, 2, \ldots, n-1$. It seems to be a fairly general feature of cohomogeneity-one Ricci-flat matrics that they tend to such cones, where $d\Omega^2_{n-1}$ is an Einstein metric on the homogeneous space G/H, G being the isometry group and H the isotropy subgroup. The generic orbit of G is $(n-1)$ dimensional, but in the interior the orbit collapses to an orbit of lower dimension of the form G/H', $G \supset H' \supset H$. Such an orbit of lower dimension is called a "Bolt of the second kind" unless it is zero-dimensional in which case it is called a "NUT of the second kind". In earlier work the terms "NUT" and "bolt" were used to describe an isolated (NUT) or 2-dimensional (bolt) fixed-point set of a $U(1)$ subgroup (possibly proper) of the isometry group of a 4-metric [20]. Thus our non-compact metrics can be thought of as smoothed-out cones, the bolt replacing the singular vertex. In fact our examples can also be thought of as vector bundles over the bolt; the bolt in fact corresponds to the zero section of the vector bundle. The space thus retracts onto the bolt and this is useful in understanding the topology of our manifolds. The Euler number is, for example, given by the Euler number of the base.

It is also useful to think of our spaces as manifolds $\hat{M}$ with boundary $\partial\hat{M}$, the boundary being a level surface $r = \text{constant}$. By glueing two such manifolds together across the boundary we obtain a compact manifold without boundary called the "double," $2\hat{M}$. Our two compact Einstein 7-manifolds with positive Ricci scalar constructed in Sect. 3 are of course just the doubles of the smoothed-out cones with base $\partial\hat{M}$ where $\partial\hat{M}$ is $P_3(C)$ or $SU(3)/T^2$. Similarly the compact Einstein 8-manifold with positive Ricci scalar constructed in [6] is topologically the double of the non-compact 8-manifold with Spin(7) holonomy constructed in Sect. 5. There is a useful relation [21] between the Euler number of a manifold, its boundary and its double. This is

$$\chi(\partial\hat{M}) - 2\chi(\hat{M}) + \chi(2\hat{M}) = 0. \tag{6.3}$$

Compact odd-dimensional manifolds without boundary have vanishing Euler number so that we deduce that the Euler number of the boundary is twice that of the non-compact 7-manifold, that is, twice that of the bolt. On the other hand for compact 8-manifolds the first term in (6.3) vanishes and the Euler number of the double is twice that of the non-compact manifold. Clearly not all generalised cones can be smoothed out even topologically to give a compact manifold of which the base is the boundary. The base must be cobordant to zero. In particular from (6.3) it follows [21] that the Euler number of the base must be even.

Since the occurrence of asymptotic behaviour of this conical sort is common we wish to expand on it a little. If the metric $d\Omega^2_{n-1}$ is not only Einstein but of constant positive curvature, i.e.

$$R_{ABCD} = (g_{AC}g_{BD} - g_{AD}g_{BC}), \tag{6.4}$$

the associated cone is flat. If $d\Omega^2_{n-1}$ is the standard round metric on S^{n-1}/Γ with $\Gamma \subset SO(n)$ the metric (6.1) is said to be asymptotically locally Euclidean, ALE, unless $\Gamma = 1$ in which case it is asymptotically Euclidean, AE. It is known [22, 23] that there are no complete Ricci-flat AE metrics, but examples of Ricci-flat ALE metrics are known. Note that S^{n-1}/Γ need not be homogeneous. If it is not homogeneous the associated ALE metrics are not isotropic at infinity: certain directions are picked out. The condition that S^{n-1}/Γ be homogeneous restricts Γ considerably. From [24] one finds that Γ must be a finite multiplicative group of numbers chosen from the real, complex or quaternion fields. In all three cases the sphere is thought of as the set of unit vectors in a k-dimensional vector space over the three fields and the group acts by multiplication on the vectors. Thus the possibilities are:

1. $S^k/\pm 1$,
2. S^{2k-1}/C_p, $p=2,3,\ldots$,
3. S^{4k-1}/D_p^*, S^{4k-1}/T^*, S^{4k-1}/O^*, S^{4k-1}/I^*,

where C_p is the cyclic group of order p, D_p^* is the binary dihedral group of order $4p, T^*$ the binary tetrahedral group, O^* the binary octahedral group and I^* the binary icosehedral group. Familiar examples of Ricci-flat ALE metrics of this sort are generalized Eguchi–Hanson metrics [25, 26, 9] with $p=k$, for all k, and the self-dual gravitational instantans for $k=1$[20, 27, 28]. Both of these examples have special holonomy. The Eguchi–Hanson metrics are Kähler and so have holonomy $SU(k)$. Their isometry group is $U(k+1)$ acting in the standard way on the k^{th} power of the Hopf bundle over $P_k(C)$. They are cohomogeneity-one metrics and are thus similar to our present examples, being R^2 bundles over the Einstein–Kähler base manifold with bolt corresponding to $P_k(C)$. The gravitational instantons are not of cohomogeneity one but they do have special holonomy, namely Sp(1); i.e. they are hyper-Kähler. It is interesting to note that the only odd-dimensional flat cones (homogeneous or not) are those over $P_{2n}(R)$. However this has Euler number one so that there is no way of smoothing out the vertex to obtain a manifold. Thus ALE spaces must be even dimensional. As far as we know all known ALE spaces, which may be of interest as generalisations to higher dimensions of the conical metric of a cosmic string, have special holonomy.

The metrics we have constructed also have special holonomy but they are not ALE, indeed from our remark above, those with holonomy G_2, being 7-dimensional, could not possibly be ALE; rather they and the 8-dimensional example with holonomy Spin(7) can be said to be AC, that is, they tend asymptotically to Ricci-flat cones. In fact Bryant's original incomplete examples [29] were *precisely* Ricci-flat cones. The metrics we have constructed in this paper (and which were previously constructed by Bryant and Salamon [2]) are smoothed-out cones, the singular "vertex" at $r=0$ being replaced by a smoothly-embedded bolt. In this respect they are similar to Calabi's hyper-Kähler metric on $T^*(P_k(C))$ [25]. This has isometry group $U(k+1)$ acting on orbits of co-dimension one of the form $U(k+1)/(U(k-1)\times U(1))$. The bolt is $P_k(C)$, and at infinity the metric is asymptotic to a Ricci-flat cone with base an Einstein metric on $U(k+1)/(U(k-1)\times U(1))$. If $k=2$ we have an 8-dimensional metric on an R^4 bundle over $P_2(C)$ which is analogous to the example with holonomy Spin(7), which is an R^4 bundle over S^4.

For the reader's convenience we list here the bases, bolts, holonomy and Euler numbers of our four examples and the two generalisations of the 4-dimensional

Eguchi–Hanson metric:

1. $U(k+1)/(U(k-1)\times U(1))$: $P_k(C)$: $\mathrm{Sp}(k)$: $\chi=k+1$,
2. $U(k+1)/U(1)$: $P_k(C)$: $SU(k)$: $\chi=k+1$,
3. S^7: S^4: Spin(7): $\chi=2$,
4. $P_3(C)$: S^4: G_2: $\chi=2$,
5. $SU(3)/T^2$: $P_2(C)$: G_2: $\chi=3$,
6. $S^3\times S^3$: S^3: G_2: $\chi=0$.

The manifolds that these metrics live on are respectively 1) the tangent bundle $P_k(C)$, 2) The k^{th} power of the Hopf bundle over $P_k(C)$, 3) the bundle of positive chirality spinors over S^4, 4) the bundle of anti-self-dual 2-forms over S^4, 5) the bundle of anti-self-dual 2-forms over $P_2(C)$ and 6) the spin bundle over S^3.

Finally we should remark that not all cohomogeneity-one Ricci-flat metrics are asymptotic to Ricci-flat cones. The higher-dimensional Schwarzschild metric [30]:

$$ds^2=(1-r^{3-n})d\tau^2+(1-r^{3-n})^{-1}dr^2+r^2d\Omega^2_{n-2}, \tag{6.5}$$

where $d\Omega^2_{n-2}$ is an $n-2$-dimensional Einstein metric with scale chosen to satisfy (6.2), is asymptotic to $S^1\times$ whatever Einstein manifold we choose but with the length of the circle direction going to a constant. It can thus be said to tend to $S^1\times$ an $(n-1)$-dimensional Ricci-flat cone. A more complicated but similar example is provided by the higher dimensional version of Taub-NUT [9, 31] which is defined on R^{2k+2} and has a NUT at the origin. The isometry group is $U(k+1)$ which acts on the Hopf bundle over $P_k(C)$. The orbits of the isometry group correspond to S^{2k-1} but at infinity the length of the Hopf circles goes to a constant whereas the $P_k(C)$ base expands. The asymptotic form in similar to the associated Schwarzschild (6.5) metric with $d\Omega^2_{n-2}$ being an appropriate multiple of the Fubini-Study metric on $P_{(n/2-1)}(C)$. This can be said to be asymptotic to a circle bundle over the Ricci-flat cone with base $P_{(n/2-1)}(C)$. As far as we know neither of these two examples, unlike the closely-related ALE Ricci-flat Kähler generalised Eguchi-metric, has special holonomy except the 4-dimensional Taub-NUT metric.

7. Eigenfunction Relations and Moduli

For a general space in n dimensions, one can consider the spectrum of various differential operators acting on fields carrying certain representations of the $SO(n)$ tangent space group. These operators would include the Hodge–de Rham operator $(d+\delta)^2$ acting on p-forms, the Dirac operator $i\Gamma^a\nabla_a$ acting on spinors, and the Lichnerowicz operator Δ_L acting on symmetric tracefree tensors. Generically, there will be no relation between the spectra of these various operators. If, however, the holonomy group H of the space is a proper subgroup of the tangent space group $SO(n)$, then rather than classifying fields by their $SO(n)$ representations one can instead classify them according to their representations under H, since the differential operators under consideration commute with H. In general a field that transforms irreducibly under $SO(n)$ will be reducible under H. This means that the spectra of the various differential operators will be at least partially related. This was discussed in detail in [32] for the case of four-dimensional manifolds with self-dual Riemann tensor.

Let us begin by considering the case of an eight-manifold with Spin(7) holonomy. As we know, this implies that there exists a covariantly-constant Majorana–Weyl spinor η, which we shall take to be left-handed and normalised to unit length. Thus we see that the 8_- representation of Spin(8), the double cover of the $SO(8)$ tangent–space group, decomposes under Spin(7) as

$$8_- \to 7 + 1. \tag{7.1}$$

This defines the embedding of Spin(7) in $SO(8)$ uniquely. It follows that the 8_v vector representation and the 8_+ right-handed spinor representation decompose irreducibily as

$$8_v \to 8; \quad 8_+ \to 8. \tag{7.2}$$

Equation (7.2) shows that in a manifold with Spin(7) holonomy, right-handed spinors and vectors transform in the same way under the holonomy group, and in fact the spectrum of the square of the Dirac operator on right-handed spinors is the same as the spectrum of the Hodge–de Rham operator on 1-forms (which are equivalent to vectors). This can be seen explicitly by noting that if V^a is a vector, then we may form the right-handed spinor ψ given by

$$\psi = i V^a \Gamma_\alpha \eta. \tag{7.3}$$

Conversely, given by a right-handed spinor ψ, we can form the vector V^a given by

$$V^a = -i\bar{\eta}\Gamma^a\psi. \tag{7.4}$$

It is easy to check from the algebra of the gamma matrices that these maps are invertible, so that *any* right-handed spinor may be mapped into a vector, and *vice versa*. If V^a is an eigenfunction of the Hodge–de Rham operator Δ,

$$\Delta V_a \equiv -\nabla^b\nabla_b V_a + R_{ab}V^b = \lambda V_a, \tag{7.5}$$

then substituting (7.4) into this equation shows that ψ is an eigenfunction of the square of the Dirac operator with the same eigenvalue:

$$(i\Gamma^a\nabla_a)^2\psi = -\nabla^a\nabla_a\psi + \tfrac{1}{4}R\psi = \lambda\psi. \tag{7.6}$$

(Note that the curvature terms in (7.5) and (7.6) are zero in our case, since the eight-manifold must be Ricci flat.)

In a similar manner one can establish that there is a one–one mapping between self-dual 4-forms and symmetric traceless 2-index tensors, which transform as the 35_+ and 35_v of $SO(8)$ respectively. Under Spin(7), both of these representations decompose irreducibly:

$$35_+ \to 35; \quad 35_v \to 35. \tag{7.7}$$

The mapping can be made explicit by defining the anti-self-dual 4-form

$$\eta_{abcd} = \bar{\eta}\Gamma_{abcd}\eta, \tag{7.8}$$

which is clearly covariantly constant. Given a self-dual 4-form ω_{abcd}, one can now show that h_{ab}, defined by

$$h_{ab} = \tfrac{1}{6}\omega_{acde}\eta_b{}^{cde}, \tag{7.9}$$

is symmetric and tracefree. Conversely, given a symmetric tracefree h_{ab}, ω_{abcd} defined by

$$\omega_{abcd} = -h_{[a}{}^{e}\eta_{bcd]e} \tag{7.10}$$

can be shown to be anti-self-dual. The maps (7.9) and (7.10) are invertible, so *any* anti-self-dual 4-form is equivalent to a symmetric traceless 2-index tensor, and *vice versa*. Substituting into the Hodge–de Rham operator for 4-forms, and the Lichnerowicz operator for symmetric tracelss 2-index tensors, one finds that if ω_{abcd} is a self-dual 4-form satisfying $\Delta\omega = \lambda\omega$, then h_{ab} given by (7.9) is an eigenfunction of the Lichnerowicz operator with the same eigenvalue;

$$\Delta_L h_{ab} \equiv -\nabla^c\nabla_c h_{ab} - 2R_{acbd}h^{cd} + 2R_{(a}{}^{c}h_{b)c} = \lambda h_{ab}. \tag{7.11}$$

(The Ricci tensor term will be zero in here.) Conversely, if h_{ab} is any eigenfunction of the Lichnerowicz operator, satisfying (7.11), then ω_{abcd} given by (7.10) is a self-dual 4-form that is an eigenfunction of the Hodge–de Rham operator, with the same eigenvalue λ.

One consequence of the relation discussed above is that if ω_{abcd} is an harmonic self-dual 4-form, then h_{ab} given by (7.9) is a divergence-free zero mode of the Lichnerowicz operator; i.e. it corresponds to an infinitesimal deformation of the metric that maintains the Ricci-flatness of the eight-dimensional space. Thus such deformations are in one–one correspondence with the volume-preserving moduli of the metric. In addition, there is a trivial volume-changing modulus corresponding to scaling the metric by a constant factor. Thus for eight-manifolds with Spin(7) holonomy, we have the result that the total number of parameters for Ricci-flat metrics is $b_4^+ + 1$, where b_4^+ is the dimension of the space of self-dual harmonic 4-forms.

Similar considerations apply to the case of seven-dimensional spaces with G_2 holonomy. Here, the spinor representation of the double cover of the $SO(7)$ tangent-space group decomposes under G_2 as

$$8 \to 7 + 1. \tag{7.12}$$

Since the dimension of the space is odd, the Dirac operator here maps the space of spinors onto itself. Thus we can look at the first-order eigenvalue equation

$$i\Gamma^a\nabla_a\psi = \mu\psi \tag{7.13}$$

for spinors. Simple algebra shows that if ϕ is a scalar eigenfunction satisfying

$$-\nabla^a\nabla_a\phi = \lambda\phi, \tag{7.14}$$

then $\psi_\pm$ defined by

$$\psi_\pm = \phi\eta \pm i\lambda^{-1/2}(\nabla_a\phi)\Gamma^a\eta \tag{7.15}$$

satisfies (7.13) with eigenvalues $\mu = \pm\lambda^{1/2}$. Similarly, if V^a is a divergence-free vector eigenfunction satisfying (7.5), then $\psi_{\pm}$ defined by

$$\psi_{\pm} = iV^a\Gamma_a\eta \pm \lambda^{-1/2}(\nabla_a V_b)\Gamma^{ab}\eta \tag{7.16}$$

satisfies (7.13) with eigenvalues $\mu = \mp\lambda^{1/2}$. There are also corresponding inverse transformations from spinor eigenfunctions to scalar and vector eigenfunctions.

In manifolds with G_2 holonomy there is a relation between the moduli of Ricci-flat metrics and the space of harmonic 3-forms. To see this, we note that 3-forms, which correspond to the 35 representation of the $SO(7)$ tangent–space group, decompose under G_2 into the $27 + 7 + 1$ representations. On the other hand, symmetric traceless tensors, which are in the 27 of $SO(7)$, decompose irreducibly as the 27 of G_2. Thus together with (7.12), this shows that 3-forms are equivalent to the sum of spinors and symmetric traceless tensors. One can exhibit these relations explicitly in much the same way as those that we have discussed previously. In particular, it follows that the harmonic 3-forms are in one–one correspondence with the set of Dirac zero modes together with the Lichnerowicz zero modes. Now if the space is compact, we know from Lichnerowicz's theorem that any Dirac zero mode is in fact covariantly constant (recall that the Ricci tensor vanishes for a metric with G_2 holonomy). We know that there is just one covariantly constant spinor if the holonomy group is exactly G_2, and hence it follows that b_3, the number of harmonic 3-forms, must be equal to the number of traceless Lichnerowicz zero modes plus one. Thus the number of moduli of Ricci-flat metrics in this case is $b_3 - 1 + 1 = b_3$, where we have included the trivial scaling mode in addition to the volume-preserving traceless modes.

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