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Quadratic Maps without Asymptotic Measure

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Abstract. An interval map is said to have an asymptotic measure if the time averages of the iterates of Lebesgue measure converge weakly. We construct quadratic maps which have no asymptotic measure. We also find examples of quadratic maps which have an asymptotic measure with very unexpected properties, e.g. a map with the point mass on an unstable fix point as asymptotic measure. The key to our construction is a new characterization of kneading sequences.

1. Introduction

A probability distribution ν on the phase space X of a discrete-time dynamical system $f: X \to X$ is called an asymptotic measure, if the normalized uniform measure λ on the phase space, e.g. Lebesgue measure or more generally a Riemannian volume, tends under the action of the dynamical system to the distribution ν. In mathematical terms this means that $\frac{1}{n}\sum_{k=0}^{n-1}(f^*)^k\lambda$ converges weakly to ν, where f^* is defined by $\int \psi d(f^*\lambda) = \int \psi \circ f d\lambda$ for $\psi \in C(X)$. For many hyperbolic systems asymptotic measures exist, e.g. for axiom-A systems (cf. [B]). Sometimes they are called natural measures or Bowen-Ruelle-Sinai measures.

For nonhyperbolic systems the situation is more complicated. Consider the family $f_a(x) = ax(1-x)$ with $0 < a \leqq 4$ of quadratic maps on $[0,1]$. Each f_a has either sensitive dependence to initial conditions $\left(\text{i.e. there is an } \varepsilon > 0 \text{ such that } \sup_{n>0} \text{length } (f^n J) > \varepsilon \text{ for all intervals } J \subseteq [0,1]\right)$, or there is an attractor (a stable periodic orbit or a Cantor set) which attracts Lebesgue – a.e. trajectory (cf. [G]). In the latter case, the attractor supports a unique f_a-invariant probability measure, which is an asymptotic measure of entropy zero (cf. [Ni, P]).

Work of Jakobson [Ja], Collet/Eckmann [CE 2] and others [Mi, BC, R, No, K 1, K 2, NvS] suggests that for "most" f_a with sensitive dependence there is a unique absolutely continuous invariant probability measure, which, by the ergodic theorem, is an asymptotic measure for f_a. On the other hand, Johnson [Jo] gives an example of a map f_a with sensitive dependence that has no finite absolutely continuous invariant measure. He does not investigate, however, whether his example has an asymptotic measure.

In this paper we construct parameters a, for which f_a has no asymptotic measure and others for which f_a has an asymptotic measure with unexpected properties. These f_a have necessarily sensitive dependence.

Before we state our results, we fix some notation. By a unimodal map we mean a continuous map $f:[-1,1]\to[-1,1]$ such that f is strictly increasing on $[-1,0]$ and strictly decreasing on $[0,1]$, and such that $f(-1)=f(1)=-1$. Such an f is called S-unimodal, if it is of class C^3 and if it has negative Schwarzian derivative, that is $Sf:=\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2<0$ on $[-1,1]\setminus\{0\}$.

We call $(f_t)_{0\leq t\leq 1}$ a full continuous family of unimodal maps, if

$$(t,x)\mapsto f_t(x) \text{ and } (t,x)\mapsto \frac{d}{dx}f_t(x) \text{ are continuous maps from } [0,1]\times[-1,1] \text{ to } \mathbb{R}, \tag{1.1}$$

$$f_t(0)>0 \text{ for } t\in[0,1],\ f_t^2(0)<0 \text{ for } t\in(0,1];\ f_0^2(0)=0,\ f_1(0)=1. \tag{1.2}$$

A typical example of a full continuous family of S-unimodal maps is given by $f_s(x)=s(1-x^2)-1$ for $s\in\left[\frac{\sqrt{5}+1}{2},2\right]$. By a linear change of coordinates, this family is transformed to $f_a(x)=ax(1-x)$.

For a probability measure μ on $[-1,1]$ let $\bar{\omega}_t(\mu)$ be the set of all weak accumulation points of the sequence $\left(\frac{1}{n}\sum_{k=1}^{n}(f_t^*)^k\mu\right)_{n\geq 1}$. Observe that ν is an asymptotic measure for f_t if and only if $\bar{\omega}_t(\lambda)=\{\nu\}$, where λ denotes the normalized Lebesgue measure on $[-1,1]$. Finally, denote by δ_x the unit point mass at x and by $h_\mu(f)$ the entropy of an invariant probability measure μ under f. For a full, continuous family of S-unimodal maps we prove

Theorem 1. *Let* $0\leq h_0<h_1<\log\frac{1+\sqrt{5}}{2}$. *There are uncountably many parameter values t such that*

(i) $\bar{\omega}_t(\lambda)$ *is a weakly closed, convex set, and*

$$\{h_\nu(f_t):\nu\in\bar{\omega}_t(\lambda),\ \nu \text{ ergodic}\}=[h_0,h_1],$$

(ii) $\bar{\omega}_t(\delta_x)=\bar{\omega}_t(\lambda)=\bar{\omega}_t(\delta_0)$ *for* λ-a.e. x.

Theorem 2. *Let* $0<h<\log\frac{1+\sqrt{5}}{2}$. *There are uncountably many parameter values t such that*

(i) f_t *has an asymptotic measure* ν *with entropy h which is ergodic and singular to Lebesgue measure.*

(ii) $\bar{\omega}_t(\delta_x)=\bar{\omega}_t(\lambda)=\bar{\omega}_t(\delta_0)=\{\nu\}$ *for* λ-a.e. x.

Theorem 3. *Let z_t denote the unique positive fix-point of f_t. There are uncountably many parameter values t such that*

(i) *δ_{z_t} is an asymptotic measure for f_t, but z_t is not a stable fixed point.*

(ii) *$\bar{\omega}_t(\delta_x)=\bar{\omega}_t(\lambda)=\bar{\omega}_t(\delta_0)=\{\delta_{z_t}\}$ for λ-a.e. x.*

Remarks. (a) If $t=1$ is the only parameter with $f_t(0)=1$, then in all three theorems the parameter $t=1$ is an accumulation point of the set of parameters with properties (i) and (ii). As f_1 has an absolutely continuous invariant measure equivalent to λ (see [Mi]), Theorem 3 shows that $\bar{\omega}_t(\lambda)$ depends as discontinuously as possible on t near $t=1$.

(b) In all three theorems, properties (i) and (ii) imply that f_t has sensitive dependence. For Theorems 1 and 2, this follows from the fact that f_t does not have an asymptotic measure with entropy zero, and in Theorem 3 the asymptotic measure is supported neither by a stable periodic orbit, nor by a Cantor set.

(c) In the case of Theorem 2, f_t has no ergodic absolutely continuous invariant probability measure. Hence, by Theorem A in [K 1] or by Corollary 2 in [K 2], one has

$$\limsup_{n\to\infty}\frac{1}{n}\log|(f_t^n)'(x)|\leqq 0$$

for λ-a.e. x.

On the other hand,

$$\lim_{n\to\infty}\frac{1}{n}\log|(f_t^n)'(x)|=\int\log|f_t'|dv\geqq h_v(f_t)>0$$

for v-a.e. x, although v is the weak limit of $\frac{1}{n}\sum_{k=1}^{n}\delta_{f_t^k x}$ for λ-a.e. x.

The proofs of the three theorems rely on the same basic idea, which is carried out in Sect. 3. We sketch it briefly: In a first step (Proposition 1) we construct "skeletons" of kneading sequences with the following property: For each kneading sequence $\underset{\sim}{e}$ which fits the skeleton there is at least one parameter value t such that $\underset{\sim}{e}$ is the kneading sequence of f_t and such that f_t has no finite, absolutely continuous invariant measure of positive entropy. This construction is similar to the one of Johnson [Jo]. Given such a skeleton, we construct in a second step (Proposition 2) particular kneading sequences fitting the skeleton and having prescribed $\bar{\omega}_t(\delta_0)$. Some additional care is required in order to obtain $\bar{\omega}_t(\delta_x)=\bar{\omega}_t(\lambda)=\bar{\omega}_t(\delta_0)$ for λ-a.e. x. The second step relies on a new characterization of kneading sequences, which is also of independent interest. It is given by Theorem 6 in Sect. 4. In Sect. 2 the background material is provided, which is needed for the proofs in Sects. 3 and 4.

2. Preparations

In this section we prepare the necessary tools for the construction of the examples. Let $f:[-1,1]\to[-1,1]$ be a unimodal map with critical point 0 and $f^2(0)<0<f(0)$. We restrict f to the absorbing invariant interval $[a,b]$, where $a=f^2(0)$ and $b=f(0)$. In order to apply the methods described below, we modify f. Let

$$V=\{x\in(a,b):f^k(x)=0 \text{ for some } k\geqq 0\}.$$

We substitute each $x \in V$ by two points $x-$ and $x+$ in $[a,b]$ and denote this modified interval by I. Then I is again a totally ordered set and a compact metrizable space with respect to the order topology. Define $q: I \to [a,b]$ by $q(y)=y$ if $y \notin \{x-, x+ : x \in V\}$ and $q(x+)=q(x-)=x$ for $x \in V$. Then q is continuous and surjective and every $y \in [a,b] \setminus V$ has only one inverse image. We extend f continuously from $[a,b] \setminus V$ to I and denote this map by g. Then $g: I \to I$ is continuous and $q \circ g = f \circ q$. Often we shall consider the dynamical system (I,g) instead of $([a,b],f)$. A measure on $[a,b]$ for which V is a nullset can be transferred to I by q. Hence we have the Lebesgue measure λ also on I. If the critical point 0 is not periodic, then V is a nullset of every invariant measure and hence there is a $1-1$-correspondence of invariant measures on $([a,b],f)$ and (I,g), which preserves the entropy. Furthermore, if a sequence (ν_k) of measures on I converges weakly to ν, then $q^*(\nu_k)$ converges weakly to $q^*(\nu)$, since $\psi \circ q$ is continuous on I for every continuous function ψ on $[a,b]$.

Next we describe (I,q) using symbolic dynamics. Set

$$Z_0 = [a, 0-] \subset I \quad \text{and} \quad Z_1 = [0+, b] \subset I.$$

Then $\mathscr{Z} = \{Z_0, Z_1\}$ is a partition of I and $g|Z_0$ and $g|Z_1$ are monotone. Let $(\Omega := \{0,1\}^{\mathbb{N}}, \sigma)$ be the two-shift. We define the coding $\varphi: I \to \Omega$ by

$$\varphi(x) = \omega_1 \omega_2 \omega_3 \ldots, \quad \text{where} \quad \omega_i = j \quad \text{if} \quad g^{i-1}(x) \in Z_j. \tag{2.1}$$

We have $\varphi \circ g = \sigma \circ \varphi$. As $\mathscr{Z}$ is a partition into closed-open sets, φ is continuous, and $\varphi: I \to \varphi(I)$ is a homeomorphism, if $\mathscr{Z}$ is a generator. This happens in our examples since they contain no stable periodic orbit (see [G]).

The kneading sequence $\underline{e}$ of the one-dimensional dynamical system (I,g) is defined by $\underline{e} = \varphi(b)$. If 0 is not periodic, $\underline{e}$ is the sequence, which is usually called kneading sequence for $([a,b],f)$ and defined by $e_k = 1$, if $f^{k+1}(0) > 0$ and $e_k = 0$, if $f^{k+1}(0) < 0$ (cf. [MT, CE 2]). In order to investigate the structure of $\underline{e}$, we define a $\gamma \in \{2,3,4,\ldots\} \cup \{\infty\}$ and a sequence $(r_i)_{1 \leq i < \gamma}$ in $\mathbb{N}$ as follows. Set $r_1 = 1$. If $r_1, r_2, \ldots, r_i$ are defined, set $S_i = 1 + r_1 + \ldots + r_i$ and let $r_{i+1} \in \mathbb{N} \cup \{\infty\}$ be maximal such that $e_{S_i + j} = e_j$ for $1 \leq j < r_{i+1}$. If $r_{i+1} = \infty$ for some i, set $\gamma = i+1$, otherwise set $\gamma = \infty$.

In this way we have defined γ, $(r_i)_{1 \leq i < \gamma}$ and

$$S_0 = 1, \qquad S_k = 1 + r_1 + \ldots + r_k \quad \text{for} \quad 1 \leq k < \gamma, \tag{2.2}$$

such that

$$e_{S_{k-1}+1} e_{S_{k-1}+2} \ldots e_{S_k} = e_1 e_2 \ldots e_{r_k - 1} e'_{r_k} \quad \text{for} \quad 1 \leq k < \gamma, \tag{2.3}$$

where $x' = 1$ if $x = 0$ and $x' = 0$ if $x = 1$. [For $k=1$ this holds, since $r_1 = 1$ and $e_1 = 1$, $e_2 = 0$ in view of $g^2(0) < 0- < 0+ < g(0)$.] If $\gamma < \infty$, we have additionally

$$e_{S_{\gamma-1}+1} e_{S_{\gamma-1}+2} \ldots = e_1 e_2 \ldots. \tag{2.4}$$

We can describe the sequence $(r_i)_{1 \leq i < \gamma}$ in a different way. Set

$$\mathbb{N}_\gamma = \{l \in \mathbb{N} : 1 \leq l < \gamma\}.$$

It is shown in Lemmas 1 and 2 of [H 2] that there is a map $Q:\mathbb{N}_\gamma\to\mathbb{N}_\gamma\cup\{0\}$ such that

$$r_k=S_{Q(k)} \quad \text{for} \quad 1\leqq k<\gamma, \tag{2.5}$$

$$(r_j)_{k<j<\gamma}\geqq(r_{Q(Q(k))+j-k})_{k<j<\gamma} \quad \text{for} \quad k\in\mathbb{N}_\gamma \quad \text{with} \quad Q(k)\geqq 1. \tag{2.6}$$

Here $\geqq$ denotes the lexicographic ordering. By (2.2) and (2.5) we get

$$Q(k)<k \quad \text{for} \quad k\in\mathbb{N}_\gamma. \tag{2.7}$$

It follows from (2.5) that (2.6) is equivalent to

$$(Q(j))_{k<j<\gamma}\geqq(Q(Q(Q(k))+j-k))_{k<j<\gamma} \quad \text{for} \quad k\in\mathbb{N}_\gamma \quad \text{with} \quad Q(k)\geqq 1. \tag{2.8}$$

It is convenient, to introduce a name for these maps. A map Q is called a *kneading map*, if there is a $\gamma\in\{2,3,\ldots\}\cup\{\infty\}$ such that Q maps $\mathbb{N}_\gamma$ to $\mathbb{N}_\gamma\cup\{0\}$ and such that (2.7) and (2.8) are satisfied.

On the other hand, one can start with a kneading map Q and determine a $0-1$-sequence $\underset{\sim}{e}$ uniquely in the following way. Because of (2.7), the map Q defines uniquely a sequence $(r_i)_{1\leqq i<\gamma}$ using (2.2) and (2.5).

By (2.7) and (2.5), we get $r_i\leqq S_{i-1}$ for $1\leqq i<\gamma$. Hence setting $e_1=1$ and $e_2=0$ a $0-1$-sequence $\underset{\sim}{e}$ is defined uniquely by (2.3) and (2.4). We call $\underset{\sim}{e}$ the Q-sequence of the given kneading map Q.

We construct our examples of unimodal maps in terms of Q. To this end we need the following theorem, which is proved in Sect. 4.

Theorem 4. *Let $(f_t)_{t\in[0,1]}$ be a family of unimodal C^1-maps on $[-1,1]$ satisfying (1.1) and (1.2). Suppose that $\gamma\in\{2,3,\ldots\}\cup\{\infty\}$ and that $Q:\mathbb{N}_\gamma\to\mathbb{N}_\gamma\cup\{0\}$ is a kneading map. Let $\underset{\sim}{e}$ be its Q-sequence. Then there is a decreasing sequence of intervals $J_k=J_k(Q)$, which are open subsets of $(0,1]$, such that $J_\infty=J_\infty(Q):=\bigcap_{k=1}^{\infty}J_k$ is not empty and such that f_t has $\underset{\sim}{e}$ as its kneading sequence for all $t\in J_\infty$. J_∞ is constructed such that the critical point of f_t is nonperiodic for all $t\in J_\infty$, and hence $\underset{\sim}{e}$ is also the kneading sequence of g_t.*

If $\gamma=\infty$ and Q is not eventually periodic, then J_∞ is closed. If $\tilde{Q}$ is another kneading map with $\tilde{Q}$-sequence $\tilde{\underset{\sim}{e}}$ and if $e_i=\tilde{e}_i$ for $1\leqq i\leqq k$, then $J_i(Q)=J_i(\tilde{Q})$ for $1\leqq i\leqq k$.

For later use we state two more properties of kneading maps Q. For the sequence $(r_i)_{1\leqq i<\gamma}$ assigned to Q we have

$$S_k=1+r_1+\ldots+r_k\leqq 2^k \quad \text{for} \quad 1\leqq k<\gamma. \tag{2.9}$$

This follows by induction from (2.5) and (2.7). If $\underset{\sim}{e}$ is the Q-sequence, then we have

$$\underset{\sim}{e} \text{ eventually periodic} \Rightarrow \gamma<\infty \text{ or } Q \text{ eventually periodic}. \tag{2.10}$$

In order to prove (2.10), suppose that $\gamma=\infty$. Since $\underset{\sim}{e}$ is eventually periodic, there are k and l such that $\sigma^{k+l}\underset{\sim}{e}=\sigma^k\underset{\sim}{e}$. As $\gamma=\infty$, there is an $\alpha\in\{0,1,\ldots,l-1\}$ such that $\{\alpha+lm:m\in\mathbb{N}\}$ contains S_i for infinitely many $i\in\mathbb{N}_\gamma$. Hence there are $i\neq j$ with $S_i\geqq k$ and $S_j\geqq k$, such that $\sigma^{S_i}\underset{\sim}{e}=\sigma^{S_j}\underset{\sim}{e}$. From the definition of the r_l we get $r_{i+l}=r_{j+l}$ for $l\geqq 1$. Now (2.5) implies $Q(i+l)=Q(j+l)$ for $l\geqq 1$, and (2.10) follows.

In Sect. 3 we need the following construction related to the kneading sequence. In [K 1] and [K 2] a Markov extension of the dynamical system (I, g) is constructed. Define the following subintervals D_k of I for $k \geqq 1$. If $k = S_i$ for some $i < \gamma$ set $D_k = [0+, g^{k-1}(b)]$ if $g^{k-1}(b) \geqq 0+$, and $D_k = [g^{k-1}(b), 0-]$ if $g^{k-1}(b) \leqq 0-$. For $k \notin \{S_i : 0 \leqq i < \gamma\}$ choose j maximal such that $S_j < k$ and let D_k be the closed subinterval of I with endpoints $g^{k-1}(b)$ and $g^{k-S_j-1}(b)$. In particular, $D_1 = Z_1$ and $D_2 = Z_0$. By (2.1) we get $g^{l-1}(b) \in Z_{e_l}$ for $l \geqq 1$. Since $g|Z_0$ and $g|Z_1$ are monotone, we get the following results by induction using (2.3), (2.4), and (2.5) (cf. [H 3]).

$$D_k \subset Z_{e_k} \quad \text{for} \quad k \geqq 1, \tag{2.11}$$

$$g(D_{k-1}) = D_k \quad \text{if} \quad k \notin \{S_i : 0 \leqq i < \gamma\}, \tag{2.12}$$

$$g(D_{S_i - 1}) = D_{S_i} \cup D_{r_i}, \qquad D_{S_i} \cap D_{r_i} = \emptyset \quad \text{for} \quad 0 \leqq i < \gamma. \tag{2.13}$$

Let the sets $\hat{D}_k$ be disjoint copies of the sets D_k. Set $\hat{I} = \bigcup_{k=1}^{\infty} \hat{D}_k$. Let $\pi_k : \hat{D}_k \to D_k$ be the identity and let $\pi : \hat{I} \to I$ be given by $\pi(x) = \pi_k(x)$, if $x \in \hat{D}_k$. We define $\hat{g} : \hat{I} \to \hat{I}$ as follows. Fix $x \in \hat{I}$ and let k be such that $x \in \hat{D}_k$. If $k \notin \{S_i - 1 : 0 \leqq i < \gamma\}$, then set $\hat{g}(x) = \pi_{k+1}^{-1} \circ g \circ \pi_k(x)$, which is defined by (2.12). If $k = S_i - 1$ for some $i < \gamma$, then $g \circ \pi_k(x)$ is either in D_{S_i} or in D_{r_i} by (2.13). In the first case set $\hat{g}(x) = \pi_{S_i}^{-1} \circ g \circ \pi_k(x)$, in the second case set $\hat{g}(x) = \pi_{r_i}^{-1} \circ g \circ \pi_k(x)$. Then $(\hat{I}, \hat{g})$ is a Markov map with countable Markov partition $\{\hat{D}_k : k \geqq 1\}$. It is called a Markov extension of (I, g). We have $\pi \circ \hat{g} = g \circ \pi$.

3. Proofs of Theorems 1–3

Remember that $g_t : I \to I$ $(0 \leqq t \leqq 1)$ is obtained from a full, continuous family of S-unimodal maps by doubling all preimages of the critical point, and $\varphi_t : I \to \Omega = \{0, 1\}^{\mathbb{N}}$ is the coding associated with g_t. Observe that the endpoints $a = g_t(b)$ and $b = g_t(0 \pm)$ of I depend also on t.
For $N \geqq 1$ let

$$\Omega_N := \{\omega \in \Omega : 0^N \text{ and } 01^{2i+1}0 \text{ (for } i \geqq 0) \text{ do not occur as subwords of } \omega\}.$$

Ω_N is a closed subshift of Ω. It is a strongly transitive sofic system, and therefore it is a factor of an aperiodic topological Markov chain of the same entropy (cf. [F]). In particular Ω_N has the specification property (see [DGS]). Using the results of [F] it is easy to check that

$$\lim_{N \to \infty} h_{\text{top}}(\sigma|_{\Omega_N}) = \log \frac{1 + \sqrt{5}}{2}. \tag{3.1}$$

Denote by $\mathcal{M}(I)$, $\mathcal{M}(\Omega)$, etc. the spaces of Borel probability measures on I, Ω, etc. endowed with their weak topologies. As $\varphi_t : I \to \varphi_t(I)$ is a homeomorphism (cf. Sect. 2), $\varphi_t^* : \mathcal{M}(I) \to \mathcal{M}(\varphi_t(I))$ is homeomorphic, where φ_t^* is defined by

$$\int u \, d(\varphi_t^* \nu) = \int u \circ \varphi_t \, d\nu \quad (u \in C(\varphi_t(I))).$$

Finally let

$$\mathcal{M}_\sigma(\Omega_N) := \{\nu \in \mathcal{M}(\Omega_N) \colon \nu \ \sigma\text{-invariant}\}.$$

The following theorem has the theorems from the introduction as corollaries:

Theorem 5. *Let C be a closed convex subset of $\mathcal{M}_\sigma(\Omega_N)$ for some $N \geqq 1$. There is an uncountable set $T \subseteq [0,1]$ such that for all $t \in T$ holds:*

(i) *The kneading sequence of g_t is not eventually periodic.*

(ii) *g_t has no λ-absolutely continuous invariant probability measure of positive entropy.*

(iii) $\nu \in \bar{\omega}_t(\delta_b) \Leftrightarrow \varphi_t^* \nu \in C$.

(iv) $\bar{\omega}_t(\delta_x) = \bar{\omega}_t(\lambda) = \bar{\omega}_t(\delta_b)$ *for* λ-a.e. x.

Before we turn to the proof of this theorem, we show how to deduce Theorems 1–3 from it:

Proof of Theorems 1 and 2. If $0 \leqq h_0 \leqq h_1 < \log \frac{1+\sqrt{5}}{2}$, there is $N \geqq 1$ such that $h_N := h_{\mathrm{top}}(\sigma_{|\Omega_N}) > h_1$, see (3.1). Restricting the length of permitted blocks of 1's, we find an irreducible subshift of finite type in Ω_N with entropy h', $h_1 < h' < h_N$ which can support ergodic shift-invariant probability measures of all entropies between 0 and h'. In particular we find a set $C_0 \subseteq \mathcal{M}_\sigma(\Omega_N)$ containing only ergodic measures and such that

$$\{h_\nu(f_t) \colon \varphi_t^* \nu \in C_0\} = \{h_\mu(\sigma) \colon \mu \in C_0\} = [h_0, h_1].$$

If $h_0 = h_1$, we may assume that C_0 contains only one element. Let C be the convex closure of C_0. Then Theorem 1 and 2 follow from Theorem 5, since $q^* \colon \mathcal{M}(I) \to \mathcal{M}([a,b])$ is continuous and its restriction to invariant measures is $1-1$ and preserves entropy, cf. Sect. 2.

Proof of Theorem 3. Let μ be the point-mass on $1^\infty \in \Omega_1$. Then $(\varphi_1)^{-1}\{\mu\} = \{\delta_{z_t}\}$, where z_t is the fix-point of g_t (different from a). Hence Theorem 3 follows from Theorem 5 for $C = \{\mu\}$, because if z_t were stable, then the kneading sequence of g_t would be eventually periodic.

We prepare the proof of Theorem 5 with a collection of some facts and definitions. Let $\mathcal{Q}$ be the set of all mappings $Q \colon \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying

$$Q(i) < i \quad \text{for all } i \tag{3.2}$$

and

$$(Q(i+j))_{j \geqq 1} \geqq (Q(Q(Q(i))+j))_{j \geqq 1} \quad \text{for all } i \text{ with } Q(i) > 0. \tag{3.3}$$

$\mathcal{Q}$ is the set of kneading maps with $\gamma = \infty$.

A sequence $\mathcal{F} = (0 = V_0 < U_1 < V_1 < U_2 < V_2 < \ldots)$ of integers is a *frame*, if

$$U_{k+1} \geqq k \cdot 2^{k+V_k} \quad \text{for all} \quad k \geqq 0 \tag{3.4}$$

and,

$$V_k \geqq k^2 \cdot 2^{U_k} \quad \text{for all} \quad k \geqq 1. \tag{3.5}$$

Given a frame $\mathcal{F}$ we define the *skeleton* $\mathcal{S}(\mathcal{F})$ as the set of all $Q \in \mathcal{Q}$ satisfying

$$U_k < i \leqq V_k \Rightarrow Q(i) = U_k, \tag{3.6}$$

$$Q(U_{k+1}) < U_k \tag{3.7}$$

for all $k \geqq 1$.

Let $\nu_{x,t,n} := \frac{1}{n} \sum_{k=1}^{n} \delta_{g_t^k(x)}$. We postpone the proofs of the following two propositions. The first one is inspired by Johnson's construction [Jo].

Proposition 1. *For each $N \geqq 1$ there are uncountably many different frames $\mathcal{F}$ with $U_1 = N+1$ such that for each $Q \in \mathcal{S}(\mathcal{F})$ and each $t \in J_\infty(Q)$ (cf. Theorem 4) holds:*

i) *g_t has no ergodic λ-absolutely continuous invariant probability measure of positive entropy.*

ii) *For λ-a.e. $x \in I$ and each $\psi \in C(I)$.*

$$\lim_{n \to \infty} (\nu_{x,t,S(V_n)}(\psi) - \nu_{b,t,S(U_n)}(\psi)) = 0. \tag{3.8}$$

Here $S(k) = S_k$ is determined by Q using (2.2) and (2.5).

Proposition 2. *Let C be a closed convex subset of $\mathcal{M}_\sigma(\Omega_N)$, and let $\mathcal{F}$ be a frame with $U_1 = N+1$. There is $Q \in \mathcal{S}(\mathcal{F})$ with $Q(i) = 0$ $(1 \leqq i \leqq N)$ and such that for each $t \in J_\infty(Q)$ holds*

$$\varphi_t^*(\bar{\omega}_t(\delta_b)) = \varphi_t^*(L_t) = C,$$

where L_t is the set of weak accumulation points of the sequence $(\nu_{b,t,S(U_k)})_{k \geqq 1}$.

Proof of Theorem 5. Let $\mathcal{F}$ be a frame as in Proposition 1 and $Q \in \mathcal{S}(\mathcal{F})$, $t \in J_\infty(Q)$ as in Proposition 2. We prove i)–iv) of the theorem for this t:

i) $Q \in \mathcal{S}(\mathcal{F})$ is unbounded and hence not eventually periodic because of (3.6), such that the Q-sequence $\underline{e}$ is not eventually periodic by (2.10). But $\underline{e}$ is the kneading sequence of g_t for $t \in J_\infty(Q)$ by Theorem 4.

ii) Follows from Proposition 1 and Corollary 2 of [L].

iii) Follows from Proposition 2.

iv) By Proposition 1, g_t has no ergodic absolutely continuous invariant probability measure of positive entropy, whence the same is true for f_t. Therefore, Theorem 4 of [K 2] implies $\bar{\omega}_t(\delta_x) \subseteq \bar{\omega}_t(\delta_b)$ for λ-a.e. x and $\bar{\omega}_t(\lambda) \subseteq \bar{\omega}_t(\delta_b)$. On the other hand, (3.8) of Proposition 1 implies $L_t \subseteq \bar{\omega}_t(\lambda)$ and $L_t \subseteq \bar{\omega}_t(\delta_x)$ for λ-a.e. x, such that the assertion follows from $L_t = \bar{\omega}_t(\delta_b)$, see Proposition 2.

If T denotes the set of parameters with properties i)–iv), then T is uncountable by Proposition 1. If $t=1$ is the only parameter for which $f_t(0) = 1$, then $\{1\} = J_\infty(\tilde{Q})$, where $\tilde{Q} \equiv 0$. By Theorem 4, Q as in Proposition 2 satisfies $J_N(Q) = J_N(\tilde{Q})$, whence $t \in J_N(\tilde{Q})$ and $\bigcap_N J_N(\tilde{Q}) = \{1\}$. This proves Remark a) from Sect. 1.

Proof of Proposition 1. Recall from Sect. 2 that $\hat{g}_t : \hat{I} \to \hat{I}$ is the Markov extension of g_t and that also the space $\hat{I}$ varies with t. Let $\hat{I}_k := \bigcup_{i=1}^{S_k - 1} \hat{D}_i$ be the part of $\hat{I}$ "below level S_k," and denote by $\hat{\lambda}$ the Lebesgue-measure on $\hat{I}$.

We construct inductively a frame $\mathscr{F}$ which determines a sequence $(\mathscr{S}_n)_{n\geqq 0}$ of "partial skeletons" by

$$\mathscr{S}_0=\mathscr{Q} \quad \text{and, for} \quad n\geqq 1,$$

$$\mathscr{S}_n=\{Q\in\mathscr{Q}\colon (3.6) \text{ and } (3.7) \text{ hold for } k=1,\dots,n\}. \tag{3.9}$$

Obviously $\mathscr{S}(\mathscr{F})=\bigcap_{n\geqq 0}\mathscr{S}_n$. Let $V_0=0$, $U_1=N+1>1$. Suppose that

$$V_0<U_1<V_1<\dots<U_{n-1}<V_{n-1}<U_n$$

are determined. Then $\mathscr{S}_{n-1}$ is well defined and to each $Q\in\mathscr{S}_{n-1}$ we consider $\bar{Q}$ defined by

$$\bar{Q}(i)=Q(i) \quad (i\leqq U_n), \qquad \bar{Q}(i)=U_n \quad (i>U_n). \tag{3.10}$$

We check that $\bar{Q}\in\mathscr{S}_{n-1}:\bar{Q}$ satisfies (3.2) and it satisfies (3.6), (3.7) for $k\leqq n-1$ because Q does. It satisfies (3.3) for $i<U_n$ because Q does and $\bar{Q}(U_n+1)=U_n$, and for $i\geqq U_n$ because

$$\begin{aligned}\bar{Q}(\bar{Q}(\bar{Q}(i))+1)\leqq\bar{Q}(\bar{Q}(i))<\bar{Q}(i) \quad &\text{by (3.2)}\\ \leqq U_n=\bar{Q}(i+1).&\end{aligned}$$

Let $\bar{\mathscr{S}}_{n-1}=\{\bar{Q}:Q\in\mathscr{S}_{n-1}\}$. $\bar{\mathscr{S}}_{n-1}$ is finite, whence also

$$T_{n-1}:=\{t\in[0,1]\colon t \text{ is endpoint of an interval } J_\infty(\bar{Q}),\ \bar{Q}\in\bar{\mathscr{S}}_{n-1}\}$$

is finite. As $\bar{Q}(i)=U_n$ for all $\bar{Q}\in\bar{\mathscr{S}}_{n-1}$ and all $i>U_n$, we see from (2.12), (2.13), and (2.5) that $\hat{g}_t(\hat{I}\setminus\hat{I}_{U_n})\subseteqq\hat{I}\setminus\hat{I}_{U_n}$ for all $t\in T_{n-1}$. Hence, by assertion (3.14) in [K2] or Proposition 2.1 of [Mi], there is $p_n>2^{U_n}$ such that

$$\hat{\lambda}\{\hat{x}\in\hat{I}_{U_n}:\hat{g}_t^{p_n}(\hat{x})\in\hat{I}_{U_n}\}<\frac{1}{n^2} \tag{3.11}$$

for all $t\in T_{n-1}$. As T_{n-1} is finite it follows from Theorem 4 that (3.11) holds also for all $t\in J_m(\bar{Q})\setminus J_\infty(\bar{Q})$, $\bar{Q}\in\bar{\mathscr{S}}_{n-1}$, if m is large enough. As $J_m(Q')=J_m(\bar{Q})$ for $m<S_l$ if Q' is any map in $\mathscr{Q}$ which coincides with $\bar{Q}$ on $\{1,\dots,l\}$, there is $V_n\geqq n^2\cdot p_n$ such that (3.11) holds for all $t\in J_\infty(Q')$ if $Q'\neq\bar{Q}$ but $Q'(i)=\bar{Q}(i)$ for $i=1,\dots,V_n$.

Finally choose $U_{n+1}\geqq n\cdot 2^{n+V_n}$. By definition of $\mathscr{S}_n$

$$t\in\bigcup_{Q\in\mathscr{S}_n}J_\infty(Q)\Rightarrow t \text{ has property (3.11)}. \tag{3.12}$$

This finishes the recursive construction of $\mathscr{F}$. $\mathscr{F}$ is a frame, as $U_{n+1}\geqq n\cdot 2^{n+V_n}$ and $V_n\geqq n^2\cdot 2^{U_n}$ by choice of U_n and V_n. As we have much freedom in choosing U_n and V_n, the construction produces uncountably many different frames.

We prove that if $t\in J_\infty(Q)$, $Q\in\mathscr{S}(\mathscr{F})$, then g_t has no ergodic, absolutely continuous invariant probability measure of positive entropy: Suppose for a contradiction that μ is such a measure. By Theorem 3 of [K 3] (cf. also Theorem 3 of [H 1]) there is an ergodic $\hat{g}_t$ invariant probability measure $\hat{\mu}$ on $\hat{I}$ of the same entropy and such that $\mu=\pi^*\hat{\mu}$, $\hat{\mu}\ll\hat{\lambda}$.

As $\hat{I}_{U_n}\nearrow\hat{I}$ with $n\to\infty$, we have

$$\lim_{n\to\infty}\hat{\mu}(\hat{g}_t^{-p_n}\hat{I}_{U_n})=\lim_{n\to\infty}\hat{\mu}(\hat{I}_{U_n})=1,$$

whence
$$\lim_{n\to\infty} \hat{\mu}(\hat{I}_{U_n} \cap \hat{g}_t^{-p_n} \hat{I}_{U_n}) = 1,$$
which together with (3.11) is in contradiction to $\hat{\mu} \ll \hat{\lambda}$. This proves i) of Proposition 1.

Now we prove (3.8): By (3.12) and the Borel-Contelli lemma, the set $\{n \in \mathbb{N}: \hat{g}_t^{p_n}(\hat{x}) \in \hat{I}_{U_n}\}$ is finite for $\hat{\lambda}$-a.e. $\hat{x} \in \hat{I}$ if $Q \in \mathcal{S}(\mathcal{F}) = \bigcap_{n \geq 0} \mathcal{S}_n$ and $t \in J_\infty(Q)$. For such an $\hat{x}$ let n be so large that $\hat{g}_t^{p_n}(\hat{x}) \notin \hat{I}_{U_n}$, i.e. $\hat{g}_t^{p_n}(\hat{x})$ is "above level S_{U_n}." As $Q(i) = U_n$ for $U_n < i \leq V_n$, we have $\hat{g}_t^k(\hat{x}) \notin \hat{I}_{U_n}$ for $p_n \leq k < S(V_n)$ by (2.12) and (2.13), and $((\varphi_t(\pi\hat{x}))_k)_{k=p_n,\ldots,S(V_n)}$, the itinerary of $\pi\hat{x}$ from time p_n to time $S(V_n)$, is composed of segments $e_1, \ldots, e_{S(U_n)-1}, *$ [except for an initial- and end segment of length $\leq S(U_n)$], where $*$ can be either 0 or 1, cf. (2.3). Now (3.8) follows from the facts that $S(U_n) \to \infty$ as $n \to \infty$ and $S(V_n) - p_n \geq V_n - p_n \geq (n^2-1)2^{U_n} \geq (n^2-1)S(U_n)$ by (2.9) and by the choice of U_n and V_n.

Proof of Proposition 2. Let $\mathcal{F}$ be a frame with $U_1 = N+1$. As $\mathcal{M}(\Omega_N)$ is separable, there is an at most countable set $C' = \{\mu_i : i \in \mathbb{N}\} \subseteq C$ which is dense in C. As $\mu_i(\Omega_N) = 1$ for all i and as Ω_N has the specification property, each μ_i has a generic point $\omega(i) \in \Omega_N$ (see [DGS, Corollary 21.15]). Without loss of generality $\omega_1(i) = 0$ for all i.

Let d be a metric for the weak topology on $\mathcal{M}(\Omega_N)$. For each $i \in \mathbb{N}$ there is $l(i) \in \mathbb{N}$ such that for all $l \geq l(i)$,
$$d\left(\mu_i, \frac{1}{l}\sum_{j=1}^{l} \delta_{\sigma^j \omega(i)}\right) < \frac{1}{i}. \tag{3.13}$$

Let $n_1, n_2, n_3, \ldots$ be a sequence in $\mathbb{N}$ such that each $i \in \mathbb{N}$ occurs infinitely often among the n_k and for sufficiently large k,
$$S(V_k) \geq k \cdot l(n_k), \qquad S(U_{k+1}) - S(V_k + k) \geq l(n_k). \tag{3.14}$$
Such a sequence exists as $S(U_{k+1}) - S(V_k + k) \geq U_{k+1} - 2^{V_k + k} \to \infty$ by (3.4) and $S(V_k) \geq V_k \geq k^2$ by (3.5).
We shall construct $Q \in \mathcal{S}(\mathcal{F})$ such that
$$e_{S(V_k+k)+1}, \ldots, e_{S(U_{k+1})} = \omega_1(n_k), \ldots, \omega_{S(U_{k+1})-S(V_k+k)}(n_k) \quad \text{for all} \quad k \geq 1. \tag{3.15}$$

Fix $k \geq 1$ and write $\omega(n_k) = v_1(n_k) v_2(n_k) \ldots$ where $v_j(n_k) = 0$ or 11 for all $j \geq 1$. This is possible because $\omega(n_k) \in \Omega_N$ and $\omega_1(n_k) = 0$. Define $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by
$$Q(j) = 0 \qquad (j = 1, \ldots, U_1 = N+1) \tag{3.16}$$
$$Q(U_k + j) = U_k \qquad (k \geq 1, j = 1, \ldots, V_k - U_k) \tag{3.17}$$
$$Q(V_k + j) = U_{k-j} \qquad (k \geq 1, j = 1, \ldots, k-1) \tag{3.18}$$
$$Q(V_k + k) = 1 \qquad (k \geq 1), \tag{3.19}$$
$$Q(V_k + k + j) = \begin{cases} 0 & \text{if} \quad v_j(n_k) = 0 \\ 1 & \text{if} \quad v_j(n_k) = 11 \end{cases},$$
$$(k \geq 1, j = 1, \ldots, U_{k+1} - V_k - k). \tag{3.20}$$

The choice in (3.20) is made such that

$$v_j(n_k)=e'_1=0\,,\quad \text{if}\quad r(V_k+k+j)=1$$

and (3.21)

$$v_j(n_k)=e_1e'_2=11\,,\quad \text{if}\quad r(V_k+k+j)=2$$

because $e_1=1$, $e_2=0$. [Observe

$$r(i)=1\Leftrightarrow Q(i)=0 \quad \text{and} \quad r(i)=2\Leftrightarrow Q(i)=1$$

by (2.5).]

We show that $Q\in\mathscr{S}(\mathscr{F})$, i.e. (3.2), (3.3), (3.6), and (3.7).

ad(3.2): $Q(i)<i$ for all i by definition of Q.

ad(3.3): By definition of Q, $Q(i)>0$ implies $Q(Q(i))=0$ or 1. Hence if $1\leqq j\leqq N$ and $Q(i)>0$, then $Q(Q(Q(i))+j)=0$. Suppose (3.3) is wrong. Then $Q(i+j)=0$ for $1\leqq j\leqq N$. As $Q(i)>0$ implies $i>U_1$, the $Q(i+j)$ are defined in (3.20), and it follows that there are N consecutive 0's in $\omega(n_k)$, a contradiction to $\omega(n_k)\in\Omega_N$.

ad(3.6): This is (3.17).

ad(3.7): Follows from (3.20) and the assumption $U_1=N+1>1$.

Finally (3.15) follows from (2.3) and (3.21).

We proceed to prove $\varphi_t^*(L_t)=C$, where L_t is the set of weak accumulation points of the sequence $(v_{b,t,S(U_k)})_{k\geqq 1}$. Let $\mu_i\in C'$, $i=n_k$ for infinitely many k. Observing

$$S(V_k+k)/S(U_{k+1})\leqq 2^{V_k+k}/U_{k+1}\to 0 \qquad (k\to\infty) \tag{3.22}$$

by (2.9) and (3.4) and that $\omega(i)$ is generic for μ_i, it follows from (3.15) that $(\varphi_t^*)^{-1}\mu_i\in L_t$. As L_t is closed and C' is dense in C, we have $(\varphi_t^*)^{-1}C\subseteqq L_t$.

Now let $v\in L_t$ be the weak limit of a subsequence $(m_{k_j})_{j\geqq 1}$ of $(v_{b,t,S(U_k)})_{k\geqq 1}$. If there is $i\in\mathbb{N}$ such that $\tilde{n}_j:=n_{k_j}=i$ for infinitely many j, then $\varphi_t^*v=\mu_i\in C$ by (3.15) and (3.22). Otherwise $\tilde{n}_j\to\infty$ as $j\to\infty$. In that case let $d_j:=S(U_{k_j+1})-S(V_{k_j}+k_j)$ and $\mu_{i,n}:=\frac{1}{n}\sum_{k=1}^{n}\delta_{\sigma^k\omega(i)}$. Then

$$d(\varphi_t^*v,\mu_{\tilde{n}_j})\leqq d(\varphi_t^*v,\varphi_t^*m_{k_j})+d(\varphi_t^*m_{k_j},\mu_{\tilde{n}_j,d_j})+d(\mu_{\tilde{n}_j,d_j},\mu_{\tilde{n}_j})\,.$$

The first term tends to zero by choice of $(m_{k_j})_{j\geqq 1}$, the second one by (3.15) and (3.22), and the third one is less than $\frac{1}{\tilde{n}_j}$ by (3.13) and (3.14). But $\tilde{n}_j\to\infty$ as $j\to\infty$ such that $d(\varphi_t^*v,\mu_{\tilde{n}_j})\to 0$, i.e. $\varphi_t^*v\in cl(C')=C$. This finishes the proof of $\varphi_t^*(L_t)=C$. In particular, L_t is convex.

Now we prove $\bar{\omega}_t(\delta_b)=L_t$. The "$\supseteqq$"-inclusion is trivial. So let $v\in\bar{\omega}_t(\delta_b)$ be the weak limit of a sequence $(v_{b,t,l_j})_{j\geqq 1}$. Define k_j by $S(V_{k_j})\leqq l_j<S(V_{k_j+1})$,

$$a_j:=S(V_{k_j})\,,\qquad b_j:=\min\{l_j,S(V_{k_j}+k_j)\}\,.$$

As $b_j-a_j\leqq k_j\cdot S(U_{k_j-1})\leqq k_j\cdot 2^{U_{k_j-1}}$ by (2.2), (2.5), (3.18), and (2.9), we have in view of (3.5),

$$\frac{b_j-a_j}{l_j}\leqq\frac{k_j2^{U_{k_j-1}}}{V_{k_j}}\leqq\frac{1}{k_j}\to 0 \text{ as } j\to\infty\,. \tag{3.23}$$

For $k<l$ let $\nu[k,l]:=\frac{1}{l-k}\sum_{i=l+1}^{k}\delta_{g_t^i(b)}$, and define $\nu[k,k]:=0$. Then

$$\nu_{b,t,l_j}=\nu[0,l_j]=\frac{a_j}{l_j}\nu[0,a_j]+\frac{b_j-a_j}{l_j}\nu[a_j,b_j]+\frac{l_j-b_j}{l_j}\nu[b_j,l_j].$$

Passing to a subsequence, if necessary, we may assume that

$$\frac{a_j}{l_j}\to\varrho,\ \frac{b_j-a_j}{l_j}\to 0,\ \frac{l_j-b_j}{l_j}\to 1-\varrho \qquad [\text{see } (3.23)],$$

$$\varphi_t^*\nu[0,a_j]\to\nu'\in\mathcal{M}_\sigma(\Omega_N), \qquad \varphi_t^*\nu[b_j,l_j]\to\nu''\in\mathcal{M}_\sigma(\Omega_N).$$

We shall show that $\nu'\in\varphi_t^*L_t$ and that $\nu''\in C$ if $\varrho<1$. Then

$$\varphi_t^*\nu=\varrho\cdot\nu'+(1-\varrho)\nu''\in\varphi_t^*L_t=C,$$

because C is convex, and the proof is finished.

$\underline{\nu'\in\varphi_t^*L_t}$: As $Q(U_k+j)=U_k$ for $j=1,\dots,V_k-U_k$, and as

$$S(U_k)/S(V_k)\leqq 2^{U_k}/V_k\leqq k^{-2}$$

by (2.9) and (3.5), we have

$$d(\varphi_t^*\nu[0,a_j],\quad \varphi_t^*\nu[0,S(U_{k_j})])\to 0 \quad \text{as} \quad j\to\infty,$$

and the set of weak accumulation points of $(\varphi_t^*\nu[0,S(U_{k_j})])$ is contained in $\varphi_t^*L_t$ by definition of L_t.

$\underline{\nu''\in C \text{ if } \varrho<1}$: Let $0<\varepsilon<1-\varrho$. For large j we have $\varepsilon\cdot k_j\geqq 1$ and hence

$$(l_j-b_j)\geqq\varepsilon\cdot l_j\geqq\varepsilon\cdot S(V_{k_j})\geqq\varepsilon\cdot k_j\cdot l(\tilde{n}_j)\geqq l(\tilde{n}_j) \tag{3.24}$$

by (3.14), where $\tilde{n}_j:=n_{k_j}$. Hence

$$d(\varphi_t^*\nu[b_j,l_j],\mu_{\tilde{n}_j,l_j-b_j})\to 0 \quad \text{as} \quad j\to\infty$$

by (3.15) and

$$\Delta_j:=d(\mu_{\tilde{n}_j,l_j-b_j},\mu_{\tilde{n}_j})\leqq\frac{1}{\tilde{n}_j}$$

by (3.13). If the $\tilde{n}_j$ $(j\geqq 1)$ are unbounded, there is a subsequence along which $d(\varphi_t^*\nu[b_j,l_j],\mu_{\tilde{n}_j})$ tends to zero, such that $\nu''\in C$. Otherwise there is some $i\in\mathbb{N}$ such that $i=\tilde{n}_j$ for a subsequence of $(\tilde{n}_j)$, and along this subsequence Δ_j tends to zero, because $l_j-b_j\to\infty$ by (3.24) and $\omega(i)$ is generic for μ_i. Again we conclude that $d(\varphi_t^*\nu[b_j,l_j],\mu_i)$ tends to zero along a subsequence, whence $\nu''=\mu_i\in C$.

4. Proof of Theorem 4

We start with a $\gamma\in\{2,3,\dots\}\cup\{\infty\}$ and a kneading map

$$Q:\mathbb{N}_\gamma\to\mathbb{N}_\gamma\cup\{0\}, \quad \text{where} \quad \mathbb{N}_\gamma=\{l\in\mathbb{N}:1\leqq l<\gamma\}.$$

This means that (2.7) and (2.8) are satisfied. Let $(r_i)_{1\leqq i<\gamma}$ and $(S_i)_{0\leqq i<\gamma}$ be the sequences defined by (2.2) and (2.5). Finally let $\underset{\sim}{e}$ be the Q-sequence, which is uniquely determined by $e_1=1$, $e_2=0$, by (2.3) and by (2.4).

In order to prove Theorem 4, for a given $0-1$-sequence $\underset{\sim}{e}$ with $e_1=1$, $e_2=0$ we set

$$n=\min\{l\geqq 2: e_l=1\}$$

and define maps

$$a:\{1,2,\dots\}\to\{1,2,\dots\} \quad \text{and} \quad b:\{n,n+1,\dots\}\to\{n,n+1,\dots\}$$

as follows:

$$a(1)=1, \qquad a(k+1)=\begin{cases} a(k), & \text{if} \quad e_{k+1}=e_{k+1-a(k)} \\ k+1 & \text{otherwise} \end{cases}, \tag{4.1}$$

$$b(n)=n, \qquad b(k+1)=\begin{cases} b(k), & \text{if} \quad e_{k+1}=e_{k+1-b(k)} \\ k+1 & \text{otherwise} \end{cases}. \tag{4.2}$$

Lemma 1. *Let $\underset{\sim}{e}$ be the Q-sequence of some kneading map Q.*

(i) *$a(k)\neq b(k)$ for $k\geqq n$.*

(ii) *The existence of a k_0 with $a(k)=a(k_0)$ for all $k\geqq k_0$ is equivalent to $\gamma<\infty$. If $\gamma=\infty$ and if there is a k_0 with $b(k)=b(k_0)$ for all $k\geqq k_0$, then $(Q(j+m))_{m\geqq 1}=(Q(Q(Q(j))+m))_{m\geqq 1}$ for some j.*

(iii) *If $k\geqq n$ and $e_{k+1-a(k)}=e_{k+1-b(k)}$, then $e_{k+1}=e_{k+1-a(k)}$.*

Proof. For later use we state the following. If $i+1<\gamma$ and $l<Q(i+1)$ or $i+1=\gamma<\infty$ and $l<\gamma$, then by (2.3) for $k=i+1$ or by (2.4) we get

$$e_{S_i+S_{l-1}+1}\dots e_{S_i+S_l}=e_{S_{l-1}+1}\dots e_{S_l}.$$

Applying again (2.3) we have, if $i+1<\gamma$ and $l<Q(i+1)$ or if $i+1=\gamma<\infty$ and $l<\gamma$,

$$e_{S_i+S_{l-1}+1}\dots e_{S_i+S_l}=e_1\dots e_{r_l-1}e'_{r_l}. \tag{4.3}$$

Similarly we get for $l=Q(i+1)$ and $i+1<\gamma$ that

$$e_{S_i+S_{l-1}+1}\dots e_{S_i+S_l}=e_1\dots e_{r_l-1}e_{r_l}, \tag{4.4}$$

and for $l=\gamma<\infty$ and $i+1=\gamma$ we get

$$e_{S_i+S_{l-1}+1}e_{S_i+S_{l-1}+2}\dots=e_1e_2\dots. \tag{4.5}$$

For a given k let $p=p(k)$ be maximal such that $S_p\leqq k$. We consider two further assertions

(iv) $a(k)=S_p$.

(v) If $b(k)>a(k)$, then there is a maximal $q=q(k)\geqq 0$ with $S_p+S_q\leqq k$ and $b(k)=S_p+S_q$; if $b(k)<a(k)$, then there is a $j=j(k)\leqq p$ with

$$r_{Q(Q(j))+m}=r_{j+m} \text{ for } 1\leqq m\leqq p-j, \quad r_{Q(Q(j))+p-j+1}>k-S_p, \quad \text{and} \quad b(k)=S_j-r_{Q(j)}.$$

We prove (i), (iii), (iv), and (v) by induction on k. We begin with $k=n$. The definition of n and (2.3) imply that $r_m=1$ for $1\leqq m\leqq n-2$ and $r_{n-1}>1$. This gives $a(k)=k=S_{k-1}$ for $1\leqq k\leqq n-1$ and $a(n)=a(n-1)=n-1=S_{n-2}$. By definition, we have $b(n)=n=S_{n-2}+1=S_{n-2}+S_0$. This implies $b(n)>a(n)$, in particular (i) holds.

Because of $e_1 \neq e_2$, also (iii) is satisfied. (iv) follows with $p(n)=n-2$, and (v) holds with $q(n)=0$.

Now we suppose that (i), (iii), (iv), and (v) hold for k. We prove them for $k+1$. Write p for $p(k)$, q for $q(k)$ and j for $j(k)$. We consider six cases:

Case 1. $b(k)>a(k)$, $p(k+1)=p$, and $q(k+1)=q$. By (iv) we have $a(k)=S_p$. Because of $p(k+1)=p$ we get by (2.3) or (2.4) that $e_{k+1}=e_{k+1-a(k)}$. Hence $a(k+1)=a(k)$ by (4.1). This gives $a(k+1)=S_p$, which is (iv) for $k+1$. By (v) we have $b(k)=S_p+S_q$. Because of $p(k+1)=p$ and $q(k+1)=q$ we get by (4.3), (4.4) or (4.5) with $i=p$ and $l=q+1$ that $e_{k+1}=e_{k+1-b(k)}$. Hence $b(k+1)=b(k)$ by (4.2). This implies $b(k+1)>a(k+1)$ giving (i) for $k+1$. Since $b(k+1)=S_p+S_q$, we get (v) for $k+1$. As $e_{k+1-a(k)}=e_{k+1-b(k)}=e_{k+1}$, we get also (iii) for $k+1$.

Case 2. $b(k)>a(k)$, $p(k+1)=p$, $q(k+1)>q$. In the same way as in case 1 we get that $e_{k+1-a(k)}=e_{k+1}$ and that $a(k+1)=S_p$, implying (iv) for $k+1$. By (v) we have $b(k)=S_p+S_q$. Because of $p(k+1)=p$ and $q(k+1)>q$, which implies $q(k+1)=q+1$ and $k+1=S_p+S_{q+1}$, we get by (4.3) with $i=p$ and $l=q+1$ that $e_{k+1}\neq e_{k+1-b(k)}$. Hence $b(k+1)=k+1$ by (4.2). This implies $b(k+1)>a(k+1)$ giving (i) for $k+1$. As $b(k+1)=S_p+S_{q+1}$ we get (v) for $k+1$. Since $e_{k+1-a(k)}=e_{k+1}\neq e_{k+1-b(k)}$, we get also (iii) for $k+1$.

Case 3. $b(k)>a(k)$, $p(k+1)>p$. By (iv) we have $a(k)=S_p$. Because of $p(k+1)>p$, which implies $p(k+1)=p+1$ and $k+1=S_{p+1}$, we get by (2.3) that $e_{k+1}\neq e_{k+1-a(k)}$. Hence $a(k+1)=k+1$ by (4.1). This gives $a(k+1)=S_{p+1}$, which is (iv) for $k+1$.

By (v) we have $b(k)=S_p+S_q$. As $p(k+1)=p+1$, we get that $S_p+S_{q+1}=S_{p+1}$, i.e. $Q(p+1)=q+1$. Because of $k+1=S_{p+1}$, we get by (4.4) with $i=p$, $l=q+1$ that $e_{k+1}=e_{k+1-b(k)}$. Hence $b(k+1)=b(k)$ by (4.2). This implies $a(k+1)>b(k+1)$, giving (i) for $k+1$. Since

$$b(k+1)=S_p+S_q=S_{p+1}-r_{q+1}=S_{p+1}-r_{Q(p+1)}$$

and

$$r_{Q(Q(p+1))+1}>0=k+1-S_{p+1},$$

we get (v) for $k+1$ with $j(k+1)=p+1=p(k+1)$. As

$$e_{k+1-a(k)}\neq e_{k+1}=e_{k+1-b(k)}$$

we get also (iii) for $k+1$.

Case 4. $b(k)<a(k)$, $k+1<S_p+r_{Q(Q(j))+p-j+1}$. By (v) and (2.6) we get

$$r_{Q(Q(j))+p-j+1}\leqq r_{p+1},$$

hence $k+1<S_{p+1}$, which means $p(k+1)=p$. In the same way as in case 1 we get that $e_{k+1-a(k)}=e_{k+1}$ and that $a(k+1)=a(k)=S_p$, which implies (iv) for $k+1$.

By (v) and $r_{Q(j)}=S_{Q(Q(j))}$, we get $r_{Q(j)}+r_{j+1}+\ldots+r_p=S_{Q(Q(j))+p-j}$. This and (2.3) imply

$$e_l=e_{l+(r_{Q(j)}+r_{j+1}+\ldots+r_p)} \quad \text{for} \quad 1\leqq l<r_{Q(Q(j))+p-j+1},$$

in particular for $l=k+1-S_p$. By (2.3) and $S_p<k+1<S_{p+1}$ we get $e_{k+1}=e_{k+1-S_p}$. Hence we get for $l=k+1-S_p$ that

$$e_{k+1}=e_l=e_{l+(r_{Q(j)}+r_{j+1}+\ldots+r_p)}=e_{k+1-S_j+r_{Q(j)}}.$$

Since $b(k)=S_j-r_{Q(j)}$, this says that $e_{k+1}=e_{k+1-b(k)}$. Hence $b(k+1)=b(k)$ by (4.2). This implies $a(k+1)>b(k+1)$ giving (i) for $k+1$. Since $b(k+1)=S_j-r_{Q(j)}$, $p(k+1)=p$, and

$$r_{Q(Q(j))+p-j+1}>k+1-S_p,$$

we get (v) for $k+1$ with $j(k+1)=j$.

As $e_{k+1-a(k)}=e_{k+1}=e_{k+1-b(k)}$, we get also (iii) for $k+1$.

Case 5. $b(k)<a(k)$, $p(k+1)=p$, $k+1=S_p+r_{Q(Q(j))+p-j+1}$. In the same way as in case 1 we get that $e_{k+1-a(k)}=e_{k+1}$ and that $a(k+1)=a(k)=S_p$, which implies (iv) for $k+1$. In the same way as in case 4 we get by (2.3) that

$$e_l \neq e_{l+(r_{Q(j)}+r_{j+1}+\ldots+r_p)} \quad \text{for} \quad l=k+1-S_p=r_{Q(Q(j))+p-j+1}\,.$$

Again as in case 4 we get $e_{k+1}=e_{k+1-S_p}$ and $e_{k+1}\neq e_{k+1-S_j+r_{Q(j)}}$. Since $b(k)=S_j-r_{Q(j)}$, this says that $e_{k+1}\neq e_{k+1-b(k)}$. Hence $b(k+1)=k+1$ by (4.2). This implies $b(k+1)>a(k+1)$ giving (i) for $k+1$. As $b(k+1)=k+1=S_p+S_q$ with $q=Q(Q(Q(j))+p-j+1)$, we get (v) for $k+1$. Since

$$e_{k+1-a(k)}=e_{k+1}\neq e_{k+1-b(k)}$$

we get also (iii) for $k+1$.

Case 6. $b(k)<a(k)$, $p(k+1)>p$, $k+1=S_p+r_{Q(Q(j))+p-j+1}$. In the same way as in case 3 we get $p(k+1)=p+1$, $e_{k+1}\neq e_{k+1-a(k)}$, and $a(k+1)=k+1=S_{p+1}$, which implies (iv) for $k+1$.

In the same way as in case 5 we get $e_l\neq e_{l+(r_{Q(j)}+r_{j+1}+\ldots+r_p)}$ for $l=k+1-S_p$. By $k+1=S_{p+1}$ we get $l=r_{p+1}$, and (2.3) implies $e_l\neq e_{l+S_p}$. Hence

$$e_{k+1}=e_{l+S_p}=e_{l+(r_{Q(j)}+r_{j+1}+\ldots+r_p)}=e_{k+1-S_j+r_{Q(j)}}\,.$$

As $b(k)=S_j-r_{Q(j)}$, this means $e_{k+1}=e_{k+1-b(k)}$. Hence $b(k+1)=b(k)$ by (4.2). This implies $a(k+1)>b(k+1)$ giving (i) for $k+1$. Since $b(k+1)=S_j-r_{Q(j)}$, $p(k+1)=p+1$, $r_{Q(Q(j))+p-j+1}=r_{p+1}$, which follows from $k+1=S_{p+1}$, and

$$r_{Q(Q(j))+p-j+2}>0=k+1-S_{p+1},$$

we get (v) for $k+1$. As

$$e_{k+1-a(k)}\neq e_{k+1}=e_{k+1-b(k)}$$

we get (iii) for $k+1$.

This finishes the induction. In particular, (i) and (iii) are shown.

We show (ii). It follows from (iv) that $a(k)=S_m$ for some fixed m and all $k\geqq k_0$ happens if and only if $r_{m+1}=\infty$, which means $\gamma=m+1<\infty$. This shows the first assertion of (ii). If $\gamma=\infty$, we have $a(k)\to\infty$ for $k\to\infty$, and hence $p(k)\to\infty$. If $b(k)=b(k_0)$ for all $k>k_0$, then $b(k)<a(k)$ happens for all k except finitely many. Choosing a larger k_0, we suppose that $b(k)<a(k)$ for all $k\geqq k_0$. As $S_j-r_{Q(j)}>S_i>S_i-r_{Q(i)}$ for $j>i$, we get $j(k)=j(k_0)=:j$ for $k\geqq k_0$. This implies $r_{Q(Q(j))+m}=r_{j+m}$ for $1\leqq m\leqq p(k)-j$ for all k and hence for all $m\geqq 1$. This implies the second assertion of (ii).

Now let $(f_t)_{t\in[0,1]}$ be a family of unimodal C^1-maps on $[-1,1]$ satisfying (1.1) and (1.2). We define maps $P_k:[0,1]\to[-1,1]$ by $P_0\equiv 0$ and

$$P_k(t)=f_t(P_{k-1}(t)) \quad \text{for} \quad k\geqq 1\,. \tag{4.6}$$

The maps P_k are continuous.

Let $\varepsilon(t)$ be the kneading sequence of f_t and $\varepsilon_k(t)$ its k^{th} coordinate, such that $\varepsilon(t) = \varepsilon_1(t)\varepsilon_2(t)\ldots$. Since $P_k(t) = f_t^k(0)$, we have

$$\varepsilon_k(t) = \begin{cases} 1 & \text{if} \quad P_k(t) > 0 \\ 0 & \text{if} \quad P_k(t) < 0 \end{cases}. \tag{4.7}$$

Furthermore (4.6) implies

$$P_m(t) = 0 \Rightarrow P_{m+k}(t) = P_k(t) \quad \text{for} \quad k \geqq 1. \tag{4.8}$$

As $f_1(0) = 1$, we get

$$P_k(1) = -1 \quad \text{for} \quad k \geqq 2. \tag{4.9}$$

Suppose that J is an open subinterval of $[0,1]$ such that $P_k(t) \neq 0$ for $t \in J$ and $1 \leqq k \leqq m$ and that s is an endpoint of J with $P_m(s) = 0$. This means that $f_s^m(0) = 0$ and the critical point 0 is a stable periodic point of f_s of period m. By (1.1) there is an open interval $U \subset J$ with endpoint s, such that f_t has a stable periodic point x of period m near 0, whose orbit attracts the orbit of 0 but does not contain 0, since $f_t^m(0) = P_m(t) \neq 0$ for $t \in J$. Hence $\operatorname{sign} f_t^k(0) = \operatorname{sign} f_t^k(x)$ for $k \geqq 1$ and $\varepsilon(t)$ is periodic for $t \in U$ by (4.7), as $P_k(t) = f_t^k(0)$. This implies also that $P_k(t) = f_t^k(0) \neq 0$ for all $k \geqq 1$ and $t \in U$. By (4.7) we get

$$\operatorname{sign} P_{m+k}(t) = \operatorname{sign} P_k(t) \quad \text{for} \quad k \geqq 1, \quad \text{if} \quad t \in U. \tag{4.10}$$

After these preparations we can show

Lemma 2. *Let $\underset{\sim}{e} = e_1 e_2 \ldots$ be a $0-1$-sequence with $e_1 = 1$ and $e_2 = 0$. Set*

$$n = \min\{l \geqq 2 : e_l = 1\}.$$

If this set is empty set $n = \infty$. Define $a : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$ and $b : \{n, n+1, \ldots\} \to \{n, n+1, \ldots\}$ by (4.1) and (4.2). If $n = \infty$, then there is no b. Suppose that $e_{k-a(k-1)} = e_{k-b(k-1)}$ implies $e_k = e_{k-a(k-1)}$ if $k > n$. Then there is a sequence $(J_k)_{k \geqq 1}$ of intervals, open as subsets of $(0, 1]$, and such that

(i) *For $k \geqq 2$ we have $J_k \subset J_{k-1}$. For $k \geqq 3$ these two intervals have a common left endpoint if and only if $a(k-1) = a(k)$. For $k > n$ they have a common right endpoint if and only if $b(k-1) = b(k)$.*

(ii) *The left endpoint of J_k is a zero of $P_{a(k)}$ for $k \geqq 2$, the right endpoint of J_k is 1 if $k < n$ and a zero of $P_{b(k)}$ otherwise.*

(iii) *For $t \in J_k$, $P_k(t) > 0$, if $e_k = 1$, and $P_k(t) < 0$, if $e_k = 0$.*

(iv) *J_k depends only on $e_1, \ldots, e_k$.*

Proof. We prove the existence of the intervals J_k and their properties (i)–(iv) by induction on k.

Set $J_1 = J_2 = (0, 1]$. Because of $f_t^2(0) < 0 < f_t(0)$ for $t \in (0, 1]$, we get $P_2 < 0 < P_1$ on $(0, 1]$. This implies (iii) for $k = 1$ and 2. By $P_2(0) = f_0^2(0) = 0$ we get (ii) for $k = 2$. (Observe that $a(2) = 2$, as $e_1 = 1$, $e_2 = 0$.) The other assertions are trivial.

For $2 \leqq k < n$ we shall choose $J_k = (x_k, 1]$ such that $x_{k-1} < x_k$ (if $k > 2$), $P_k(x_k) = 0$ and $P_k(t) < 0$ for $t \in J_k$. As $e_k = 0$ for $2 \leqq k < n$ and hence $a(k) = k$ for $2 \leqq k < n$, this implies (i)–(iii). For $k = 2$ such a choice was made above. Hence suppose that $k \geqq 3$ and $J_2, \ldots, J_{k-1}$ are chosen. By $P_{k-1}(x_{k-1}) = 0$, (4.8) and (1.2) we get

$$P_k(x_{k-1}) = P_1(x_{k-1}) = f_{x_{k-1}}(0) > 0,$$

and (4.9) implies $P_k(1)<0$. Let x_k be maximal with $P_k(x_k)=0$. Then $x_k\in(x_{k-1},1)$ and $J_k=(x_k,1]$ has the desired properties. As $e_2=\ldots=e_k=0$ implies $n>k$, (iv) follows immediately.

If $n=\infty$, the proof is finished. Hence suppose that $n<\infty$. We have $J_{n-1}=(x_{n-1},1]$. As above we have $P_n(x_{n-1})>0$ and $P_n(1)<0$. Let y_n be the smallest zero of P_n in $(x_{n-1},1)$. Set $x_n=x_{n-1}$ and $J_n=(x_n,y_n)$. Since $e_n=1=e_1$ and $a(n-1)=n-1$, we get $a(n)=a(n-1)$, and (i) follows for $k=n$. As $b(n)=n$ we have also (ii) for $k=n$. (iii) holds for $k=n$ because $e_n=1$ and $P_n>0$ on J_n, and (iv) for $k=n$ follows from the definition of n.

Note also that $P_0(t)=0<P_n(t)=f_t^n(0)\leqq P_1(t)=f_t(0)$ holds for all $t\in J_n$, since $[f_t^2(0),f_t(0)]$ is invariant under f_t. Hence the following assertion, too, holds for $k=n$:

(v) Either $P_{k-a(k)}(t)\leqq P_k(t)\leqq P_{k-b(k)}(t)$ for all $t\in J_k$ or $P_{k-b(k)}(t)\leqq P_k(t)\leqq P_{k-a(k)}(t)$ for all $t\in J_k$, and $P_{k-a(k)}$ and $P_{k-b(k)}$ are both $\geqq 0$ or $\leqq 0$ on J_k.

We proceed to prove (i)–(v) for $k>n$ by induction. So let $k>n$ and suppose that (i)–(iv) are shown for $1,2,\ldots,k-1$ and that (v) is shown for $k-1$. We consider two cases.

Case 1. $e_{k-a(k-1)}=e_{k-b(k-1)}=:d$. By (v) for $k-1$ we have that P_{k-1} is between $P_{k-1-a(k-1)}$ and $P_{k-1-b(k-1)}$ on J_{k-1} and that they all are either $\geqq 0$ or $\leqq 0$ on J_{k-1}. As f_t is monotone on $[-1,0]$ and on $[0,1]$, this implies that P_k is between $P_{k-a(k-1)}$ and $P_{k-b(k-1)}$ on J_{k-1}. Because of $1\leqq a(k-1),\ b(k-1)\leqq k-1$, we get by (iii) for $k-a(k-1)$ and $k-b(k-1)$ that $P_{k-a(k-1)}$ and $P_{k-b(k-1)}$ are both >0 on J_{k-1}, if $d=1$, and that both are <0 on J_{k-1}, if $d=0$. Set $J_k=J_{k-1}$. By our assumption on $\underset{\sim}{e}$, we have $e_k=d$. By the above, $P_k(t)>0$ for $t\in J_k$ if $d=1$ and <0 if $d=0$. This gives (iii). Furthermore, (4.1) and (4.2) imply $a(k)=a(k-1)$ and $b(k)=b(k-1)$. This completes the proofs of (v) and (i), and the validity of (ii) for k follows from that for $k-1$. Finally (iv) holds, because the definitions of $a(i)$ and $b(i)$ for $i\leqq k$ depend only on $e_1,\ldots,e_k$.

Case 2. $e_{k-a(k-1)}\neq e_{k-b(k-1)}$. Write $J_{k-1}=(x,y)$. By (ii) for $k-1$ we have $P_{a(k-1)}(x)=0$ and $P_{b(k-1)}(y)=0$. In view of (4.10) there is $\delta>0$ with $\operatorname{sign}P_k(t)=\operatorname{sign}P_{k-a(k-1)}(t)$ for $t\in(x,x+\delta)$ and $\operatorname{sign}P_k(t)=\operatorname{sign}P_{k-b(k-1)}(t)$ for $t\in(y-\delta,y)$. These signs are different as $e_{k-a(k-1)}\neq e_{k-b(k-1)}$; observe (iii) for $k-a(k-1)$ and $k-b(k-1)$ and (4.7).

Hence the set of zeros of P_k in (x,y) is a non-empty subset of $[x+\delta,y-\delta]$. Suppose first that $e_k=e_{k-a(k-1)}$. Let $z\in[x+\delta,y-\delta]$ be minimal with $P_k(z)=0$ and choose $J_k=(x,z)$. As $P_k(t)\neq 0$ for $t\in J_k$, we get $\operatorname{sign}P_k(t)=\operatorname{sign}P_{k-a(k-1)}(t)$ for $t\in J_k$, and (iii) follows from $e_k=e_{k-a(k-1)}$. By (4.1) and (4.2) we get $a(k)=a(k-1)$ and $b(k)=k$. This implies (i) and (ii). In the same way as in Case 1 we see that P_k is between $P_{k-a(k-1)}$ and $P_{k-b(k-1)}$ on J_{k-1}. Because $\operatorname{sign}P_k=\operatorname{sign}P_{k-a(k-1)}$ on J_k, $P_k(z)=0$ and $P_{k-a(k-1)}\neq 0$ on J_{k-1}, P_k is between $P_{k-a(k-1)}$ and $P_0\equiv 0$, which is (v) since $a(k)=a(k-1)$ and $b(k)=k$. Finally (iv) follows, because $a(i)$, $b(i)$ for $i\leqq k$ depend only on $e_1,\ldots,e_k$.

Suppose now that $e_k=e_{k-b(k-1)}$. Let $z\in[x+\delta,y-\delta]$ be maximal with $P_k(z)=0$, and choose $J_k=(z,y)$. The proof is now analogous to that for $e_k=e_{k-a(k-1)}$.

Proof of Theorem 4. Lemma 1 (iii) shows that the assumptions of Lemma 2 are satisfied. $e_1=1$ and $e_2=0$ are part of the definition of Q-sequences. The existence of the intervals J_k follows from Lemma 2. If neither a nor b is eventually constant, it follows from (i) of Lemma 2 that $J=\bigcap_{k=1}^{\infty} J_k$ equals $\bigcap_{k=1}^{\infty} \bar{J}_k$ and is hence nonempty and closed. By (ii) of Lemma 1 this happens always, if $\gamma=\infty$ and Q is not eventually periodic. Now suppose that there is a k_0 with $a(k)=a(k_0)$ for $k \geqq k_0$. By (i) of Lemma 2 the intervals J_k for $k \geqq k_0$ have a common left endpoint x and $P_{a(k_0)}(x)=0$ by (ii) of Lemma 2. By (4.10), there is a $\delta>0$ such that $P_l(t) \neq 0$ for all l and all $t \in (x, x+\delta)$. By (ii) of Lemma 2 we get $(x, x+\delta) \subset J_k$ for all k proving $J_\infty \neq \emptyset$. If b is eventually constant, the proof of $J_\infty \neq \emptyset$ is similar. By (4.7) and (iii) of Lemma 2, we get that f_t has $\underset{\sim}{e}$ as its kneading sequence, and the critical point 0 is nonperiodic for all $t \in J_\infty$. The last assertion of Theorem 4 follows from (iv) of Lemma 2.

It might be useful to collect the different characterizations of kneading sequences. To this end we introduce an order relation $\lhd$ on $\Omega=\{0,1\}^{\mathbb{N}}$. If $\underset{\sim}{x} \neq \underset{\sim}{y}$ are in Ω, let j be minimal such that $x_j \neq y_j$. Then $\underset{\sim}{x} < \underset{\sim}{y}$, if $x_1, \ldots, x_{j-1}$ contains an even number of 1 and $x_j < y_j$ or if $x_1 \ldots x_{j-1}$ contains an odd number of 1 and $x_j > y_j$. Furthermore, for an $\underset{\sim}{e} \in \Omega$ with $e_1=1$ and $e_2=0$, set $n_{\underset{\sim}{e}}=\min\{l \geqq 2 : e_l=1\}$ and define

$$a_{\underset{\sim}{e}}:\{1,2,\ldots\} \to \{1,2,\ldots\} \quad \text{and} \quad b_{\underset{\sim}{e}}:\{n_{\underset{\sim}{e}}, n_{\underset{\sim}{e}}+1,\ldots\} \to \{n_{\underset{\sim}{e}}, n_{\underset{\sim}{e}}+1,\ldots\}$$

by (4.1) and (4.2). We have

Theorem 6. *For a* $0-1$*-sequence, the following are equivalent:*

(i) $\underset{\sim}{e}$ *is the kneading sequence of a unimodal map* f *with non-periodic critical point* 0 *and with* $f^2(0)<0<f(0)$.

(ii) $e_1=1$, $e_2=0$, *and* $e_k e_{k+1} \ldots \lhd e_1 e_2 \ldots$ *for all* $k \geqq 2$.

(iii) $\underset{\sim}{e}$ *is the* Q*-sequence of a kneading map* Q.

(iv) $e_1=1$, $e_2=0$, *and* $e_{k-a(k-1)}=e_{k-b(k-1)} \Rightarrow e_k=e_{k-a(k-1)}$ *for* $k>n_{\underset{\sim}{e}}$, *where* $a=a_{\underset{\sim}{e}}$ *and* $b=b_{\underset{\sim}{e}}$.

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are shown in [H 2]. (iii) ⇒ (iv) is (iii) of Lemma 1 and (iv) ⇒ (i) follows from Lemma 2 and the arguments in the proof of Theorem 4 above.

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