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Ribbon Graphs and Their Invariants Derived from Quantum Groups

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Abstract. The generalization of Jones polynomial of links to the case of graphs in R^3 is presented. It is constructed as the functor from the category of graphs to the category of representations of the quantum group.

1. Introduction

The present paper is intended to generalize the Jones polynomial of links and the related Jones-Conway and Kauffman polynomials to the case of graphs in R^3.

Originally the Jones polynomial was defined for links of circles in R^3 via an astonishing use of von Neumann algebras (see [Jo]). Later on it was understood that this and related polynomials may be constructed using the quantum R-matrices (see, for instance, [Tu_1]). This approach enables one to construct similar invariants for coloured links, i.e. links each of whose components is provided with a module over a fixed algebra (see [Re_1], where the role of the algebra is played by the quantized universal enveloping algebra $U_q(G)$ of a semisimple Lie algebra G).

The Jones polynomial has been also generalized in another direction: in generalization of links of circles one considers the so-called tangles which are links of circles and segments in the 3-ball, where it is assumed that ends of segments lie on the boundary of the ball. Technically it is convenient to replace the ball by the strip $R^2 \times [0, 1]$, which enables one to distinguish the top and bottom endpoints of the tangle. The corresponding "Jones polynomial" of a coloured tanlge is a linear operator $V_1 \otimes \ldots \otimes V_k \to V^1 \otimes \ldots \otimes V^\ell$ where $V_1, \ldots, V_k$ (respectively $V^1 \otimes \ldots \otimes V^\ell$) are the modules associated with the segments incident to bottom (respectively top) endpoints of the tangle. Here the language of categories turns out to be very fruitful. The tangles considered up to isotopy are treated as morphisms of the "category of tangles." The generalized Jones polynomial is a covariant functor from this category to the category of modules (see [Tu_2, Re_2]).

Definitions of Jones-type polynomials for embedded graphs in R^3 have been given by several authors [KV, Ye] but the subject still remains open. It was clear from the very beginning that the graph should be provided with thickening, i.e.

with a surface containing the graph and contracting to it (cf. Sect. 4). Another related problem is what should be understood by colours of the vertices.

The theory of the Jones-type polynomials has proved to be closely related to the conformal field theories [KT, Wi]. In this connection one should mention Witten's paper [Wi], where it is shown that the Jones polynomial and its generalizations are related to the topological Chern-Simons action.

In the present paper we introduce the so-called coloured ribbon graphs in $R^2 \times [0,1]$ and define for them Jones-type isotopy invariants. Ribboness means that the role of vertices of graphs is played by small plane squares with two distinguished opposite bases, the role of edges is played by thin strips (or bands) whose short bases are glued either to distinguished bases of the squares or to $\partial(R^2 \times [0,1]) = R^2 \times \{0,1\}$.

Our approach to colouring is based on the Drinfel'd notion of a quasitriangular Hopf algebra (see [Dr, RS, Re$_2$]). For each quasitriangular Hopf algebra A we define A-coloured (ribbon) graphs. The colour of an edge is an A-module. The colour of a vertex is an A-linear homomorphism intertwining the modules which correspond to edges incident to this vertex. The category of an A-coloured ribbon graph happens to be a compact braided strict monoidal category in the sense of [JS].

If A satisfies a minor additional condition, then we construct a canonical covariant functor from the category of A-coloured ribbon graphs into the category of A-modules. In the case $A = U_q(sl_2)$ this functor generalizes the Jones polynomial of links.

We would like to emphasize that coloured ribbon graphs are not merely topological objects but rather mixed objects of topology and representation theory. For some algebras a purely combinatorial description of A-modules and A-homomorphisms is available (via Young tableaux, Young diagrams, etc.). Note also that for 3-valent graphs the colours of vertices are nothing else but the Clebsch-Gordan projectors and their linear combinations (cf. Sect. 7.2).

Plan of the Paper. In Sect. 2 we recall the notion of a braided monoidal category and related notions. In Sect. 3 we discuss quasitriangular and ribbon Hopf algebras. In Sect. 4 we introduce A-coloured ribbon graphs. In Sect. 5 we state and prove our main theorem on the functor from the category of graphs in the category of modules. In Sect. 6 we collect some applications and variations of the theorem. Section 7 is concerned with examples and further comments on the theorem.

2. Compact Braided Categories

2.1. Monoidal Categories (see [Ma, JS]). A monoidal category $C = (C_0, \otimes, I, a, r, \ell)$ consists of a category C_0, a covariant functor (the tensor product) $\otimes : C_0 \times C_0 \to C_0$, an object I of C_0, and natural isomorphisms

$$a = a_{UVW} : (U \otimes V) \otimes W \to U \otimes (V \otimes W),$$
$$r = r_V : V \otimes I \to V, \qquad \ell = \ell_V : I \otimes V \to V$$

(where U, V, W are arbitrary objects of C_0) so that the following two diagrams commute:

$$\begin{tikzcd}[column sep=small]
 & (U\otimes V)\otimes(W\otimes X) \arrow[dr, "a"] & \\
((U\otimes V)\otimes W)\otimes X \arrow[ur, "a"] \arrow[d, "a\otimes 1"'] & & U\otimes(V\otimes(W\otimes X)) \\
(U\otimes(V\otimes W))\otimes X \arrow[rr, "a"] & & U\otimes((V\otimes W)\otimes X) \arrow[u, "1\otimes a"']
\end{tikzcd} \tag{2.1.1}$$

$$\begin{tikzcd}
(U\otimes I)\otimes V \arrow[rr] & & U\otimes(I\otimes V) \\
 & U\otimes V \arrow[ul, "r\otimes 1"] \arrow[ur, "1\otimes\ell"'] &
\end{tikzcd} \tag{2.1.2}$$

The diagram (2.1.1) is called the pentagon for associativity and the diagram (2.1.2) is called the triangle for identities. C is called strift if each a_{UVW}, r_V, ℓ_V is an identity morphism in C.

A braiding for C consists of a natural family of isomorphisms

$$c = c_{UV} : U\otimes V \to V\otimes U$$

in C such that the following two diagrams commute:

$$\begin{tikzcd}
 & U\otimes(V\otimes W) \arrow[r, "c"] & (V\otimes W)\otimes U \arrow[dr, "a"] & \\
(U\otimes V)\otimes W \arrow[ur, "a"] \arrow[dr, "c\otimes 1"'] & & & V\otimes(W\otimes U), \\
 & (V\otimes U)\otimes W \arrow[r, "a"] & V\otimes(U\otimes W) \arrow[ur, "1\otimes c"'] &
\end{tikzcd} \tag{2.1.3}$$

$$\begin{tikzcd}
 & (U\otimes V)\otimes W \arrow[r, "c"] & W\otimes(U\otimes V) \arrow[dr, "a^{-1}"] & \\
U\otimes(V\otimes W) \arrow[ur, "a^{-1}"] \arrow[dr, "1\otimes c"'] & & & (W\otimes U)\otimes V. \\
 & U\otimes(W\otimes V) \arrow[r, "a^{-1}"] & (U\otimes W)\otimes V \arrow[ur, "c\otimes 1"'] &
\end{tikzcd} \tag{2.1.4}$$

As noted in [JS] (2.1.4) amounts to (2.1.3) with c_{UV} replaced by c_{VU}^{-1}. A monoidal category together with a braiding is called a braided monoidal category. Below (in Sects. 3 and 4) we give two fundamental examples of braided categories; for further examples see [JS].

We call a monoidal category C compact if for all objects V there exists an object V^* and arrows $e_V : V\otimes V^* \to I$, $n_V : I \to V^*\otimes V$ such that the following composites are identity morphisms:

$$V \xrightarrow{r^{-1}} V\otimes I \xrightarrow{1\otimes n} V\otimes(V^*\otimes V) \xrightarrow{a^{-1}} (V\otimes V^*)\otimes V \xrightarrow{e\otimes 1} I\otimes V \xrightarrow{\ell} V,$$

$$V^* \xrightarrow{\ell^{-1}} I\otimes V^* \xrightarrow{n\otimes 1} (V^*\otimes V)\otimes V^* \xrightarrow{a} V^*\otimes(V\otimes V^*) \xrightarrow{1\otimes e} V^*\otimes I \xrightarrow{r} V^*.$$

Remark that we do not require $V^{**} = V$.

2.2. Functors of Monoidal Categories. Let C, C' be two compact braided strict monoidal categories. Recall that a covariant functor $F: C \to C'$ associates with each object (respectively morphism) X of C an object (respectively morphism) $F(X)$ of C' so that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in Ob(C)$ and $F(f \circ g) = F(f) \circ F(g)$ for any two morphisms $g: U \to V$, $f: V \to W$ of C. We say that F preserves the tensor product if $F(I_C) = I_{C'}$ and for any two objects (respectively morphisms X, Y of C $F(X \otimes Y) = F(X) \otimes F(Y)$. We say that F preserves braiding if $F(c_{UV}) = c_{F(U)F(V)}$ for any two objects U, V of C. We say that F commutes with the compact structures if one can associate with each object V of C an isomorphism $F_V: (F(V))^* \to F(V^*)$ so that $F(n_V) = (F_V \otimes \mathrm{id}_{F(V)}) \circ n_{F(V)}$ and $e_{F(V)} = F(e_V) \circ (\mathrm{id}_{F(V)} \otimes F_V)$.

2.3. Strict Extension of Monoidal Categories. Each monoidal category C is equivalent to a strict monoidal category $C_\square$ (cf. [Ma]). The objects of $C_\square$ are finite sequences $V_1, \ldots, V_k$ (including the empty sequence $\emptyset$) of objects of C. The morphisms from $(V_1, \ldots, V_k)$ into $(W_1, \ldots, W_\ell)$ are just C-morphisms

$$(\ldots(V_1 \otimes V_2) \otimes \ldots \otimes V_{k-1}) \otimes V_k \to (\ldots(W_1 \otimes W_2) \otimes \ldots \otimes W_{\ell-1}) \otimes W_\ell .$$

(Here if $k=0$ then $V_1 \otimes \ldots \otimes V_k = I$). The tensor product in $C_\square$ is defined for objects by the rule

$$(V_1, \ldots, V_k) \otimes (W_1, \ldots, W_\ell) = (V_1, \ldots, V_k, W_1, \ldots, W_\ell)$$

and for morphisms by the evident application of the tensor product in C. It is easy to check that $C_\square$ is a strict monoidal category with $I_\square = \emptyset$.

There are two natural covariant functors relating C and $C_\square$: the inclusion $i: C \to C_\square$ and the projection $p: C_\square \to C$. The inclusion i sends $V \in Ob(C)$ into the 1-term sequence V and sends $f: V \to W$ into the same morphism f considered in the category $C_\square$. The projection p transforms each object $(V_1, \ldots, V_k)$ of $C_\square$ into $(\ldots(V_1 \otimes V_2) \otimes \ldots \otimes V_{k-1}) \otimes V_k$ and transforms each morphism of $C_\square$ in its obvious counterpart in C. Clearly, $p \circ i = \mathrm{id}_C$. Both i and p are equivalences of categories. (Recall that a covariant functor $G: C \to C'$ is called an equivalence of categories C, C' if for any objects V, W *of* C, G maps $\mathrm{Mor} f(V, W)$ bijectively onto $\mathrm{Mor} f(G(V), G(W))$ and each object of C' is isomorphic to $G(V)$ for some $V \in Ob(C)$.

Speaking non-formally one may say that $C_\square$ has the same morphisms as C but more objects: each decomposition $V = (\ldots(V_1 \otimes V_2) \otimes \ldots \otimes V_{k-1}) \otimes V_k$ of $V \in Ob(C)$ gives rise to an object $(V_1, \ldots, V_k)$ of $C_\square$.

It is not difficult to show that each braiding in C (respectively a compact structure in C) extends naturally to a braiding (respectively a compact structure) in $C_\square$. In particular, $(V_1, \ldots, V_k)^* = (V_k^*, \ldots, V_1^*)$.

3. Quasitriangular and Ribbon Hopf Algebras

3.1. Quasitriangular Hopf Algebras (see [Dr]). Let A be a Hopf algebra with the comultiplication Δ, counit ε and antipodal mapping s. Let R be an invertible element of $A^{\otimes 2} = A \otimes A$. Denote the permutation homomorphism $a \otimes b \mapsto b \otimes a: A \otimes A \to A \otimes A$ by P. The pair (A, R) is called a quasitriangular Hopf algebra if for any $a \in A$,

$$\Delta'(a) = R\Delta(a)R^{-1}, \tag{3.1.1}$$

$$(\Delta \otimes \mathrm{id}_A)(R) = R_{13}R_{23}, \tag{3.1.2}$$

$$(\mathrm{id}_A \otimes \Delta)(R) = R_{13}R_{12}. \tag{3.1.3}$$

Here $\Delta' = P \circ \Delta$, $R_{12} = R \otimes 1 \in A^{\otimes 3}$, $R_{13} = (\mathrm{id} \otimes P)(R_{12})$, $R_{23} = 1 \otimes R$. Note that if R is a (finite) sum $\sum_i \alpha_i \otimes \beta_i$ with $\alpha_i, \beta_i \in A$, then $R_{12} = \sum_i \alpha_i \otimes \beta_i \otimes 1$, $R_{13} = \sum_i \alpha_i \otimes 1 \otimes \beta_i$, $R_{23} = \sum_i 1 \otimes \alpha_i \otimes \beta_i$.

The element R is called the universal R-matrix of A. This element satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.1.4}$$

Indeed, it follows from (3.1.1) and (3.1.2) that

$$R_{12}R_{13}R_{23} = R_{12}(\Delta \otimes \mathrm{id})(R) = (\Delta' \otimes \mathrm{id})(R)R_{12} = R_{23}R_{13}R_{12}.$$

Let us note the following important equalities:

$$(\varepsilon \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon)(R) = 1, \tag{3.1.5}$$

$$(s \otimes \mathrm{id})(R) = (\mathrm{id} \otimes s^{-1})(R) = R^{-1}, \tag{3.1.6}$$

$$(s \otimes s)(R) = R. \tag{3.1.7}$$

Indeed, from the axiom of counit we have $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$. Therefore

$$R = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = (\varepsilon \otimes \mathrm{id})(R) \cdot R,$$

$$R = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(R_{13}R_{12}) = (\mathrm{id} \otimes \varepsilon)(R) \cdot R.$$

Since R is invertible, these equalities imply (3.1.5). To prove (3.1.6) note that if $m: A \otimes A \to A$ is the algebra multiplication then the axiom of antipode states that

$$m(s \otimes \mathrm{id})(\Delta(a)) = m(\mathrm{id} \otimes s)(\Delta(a)) = \varepsilon(a)1_A \tag{3.1.8}$$

for any $a \in A$. Put $m_{12} = m \otimes \mathrm{id}: A^{\otimes 3} \to A^{\otimes 2}$ and $m_{23} = \mathrm{id} \otimes m: A^{\otimes 3} \to A^{\otimes 2}$. We have

$$m_{12}((s \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)) = (\varepsilon \otimes \mathrm{id})(R) = 1,$$

$$m_{23}((\mathrm{id} \otimes \mathrm{id} \otimes s^{-1})(\mathrm{id} \otimes \Delta')(R)) = (\mathrm{id} \otimes s^{-1})(\mathrm{id} \otimes \varepsilon)(R) = 1.$$

(Note that s^{-1} and s are antiautomorphism of A.) On the other hand (3.1.2, 3.1.3) imply that

$$m_{12}((s \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)) = (s \otimes \mathrm{id})(R) \cdot R, \tag{3.1.9}$$

$$m_{23}((\mathrm{id} \otimes \mathrm{id} \otimes s^{-1})(\mathrm{id} \otimes \Delta')(R)) = R \cdot (\mathrm{id} \otimes s^{-1})(R). \tag{3.1.10}$$

Clearly, (3.19), (3.1.10) imply (3.1.6). Also, (3.1.6) implies (3.1.7).

With each quasitriangular Hopf algebra (A, R) we associate the element $u = u_A = \sum_i s(\beta_i)\alpha_i$, where as above $R = \sum_i \alpha_i \otimes \beta_i$. This element is invertible and satisfies the following identities (see [Dr]):

$$s^2(a)=uau^{-1} \quad \text{for all} \quad a\in A, \tag{3.1.11}$$

$$us(u)\in(\text{centre of } A), \tag{3.1.12}$$

$$\Delta(u)=(R_{21}R_{12})^{-1}(u\otimes u)=(u\otimes u)(R_{21}R_{12})^{-1}, \tag{3.1.13}$$

$$u^{-1}=\sum_i \beta_i s^2(\alpha_i). \tag{3.1.14}$$

Remark. One may more generally define topological quasitriangular Hopf algebras so that the comultiplication Δ takes values not in $A\otimes A$ but rather in its certain completion $A\hat{\otimes}A$. One also has to assume that $R\in A\hat{\otimes}A$. The properties of quasitriangular Hopf algebras mentioned above may be extended to the topological case (cf. [Dr]).

3.2. Category Rep$A$. With each algebra A one associates the category Rep$A$ of its finite dimensional linear representations. The objects of Rep$A$ are left A-modules finitely generated over the ground field. The morphisms of Rep$A$ are A-linear homomorphisms. Note that the action of A in A-module V induces an algebra homomorphism $A\to\operatorname{End} V$, denoted by ϱ_V.

Let (A, R) be a quasitriangular Hopf algebra. We shall provide Rep$A$ with a structure of compact braided monoidal category. The comultiplication Δ in A induces the tensor product in Rep$A$: for objects V, W their tensor product is the ordinary tensor product $V\otimes W$ of vector spaces equipped with the (left) A-action by the formula $\varrho_{V\otimes W}(a)=(\varrho_V\otimes\varrho_W)(\Delta(a))$ for $a\in A$. The unit object I of Rep$A$ is the ground field equipped with the action of A by means of the counit of A. The homomorphisms a, r, ℓ (see Sect. 2.1) are given respectively by the formulas $(x\otimes y)\otimes z\mapsto x\otimes(y\otimes z)$, $x\otimes j\mapsto jx$, $j\otimes x\mapsto jx$. Thus Rep$A$ is a monoidal category.

It follows from (3.1.1) that for any A-modules V, W the mapping

$$c_{V,W}=P^{V,W}\circ(\varrho_V\otimes\varrho_W)(R): V\otimes W\to W\otimes V,$$

where P is the permutation homomorphism $x\otimes y\mapsto y\otimes x$, is an A-linear homomorphism, i.e. a morphism of Rep$A$. The formulas (3.1.2), (3.1.3) imply the commutativity of the diagrams (2.1.3), (2.1.4) so that Rep$A$ becomes a braided monoidal category.

For any A-module V we provide the dual linear space $V^\vee$ with the (left) action of A by the formula

$$\varrho_{V^\vee}(a)=(\varrho_V(s(a)))^*\in\operatorname{End} V^\vee \tag{3.2.1}$$

for $a\in A$. Denote by e_V the canonical pairing $(x, y)\mapsto y(x): V\otimes V^\vee\to \mathrm{I}$. Denote by n_V the homomorphism $I\to V^\vee\otimes V$ which sends 1 into $\sum_i e^i\otimes e_i$, where $\{e_i\}$ is an arbitrary basis in V and $\{e^i\}$ is the dual basis in $V^\vee$. It is easy to deduce from definitions that e_V, n_V are A-linear and that the conditions of the compact category are fulfilled.

Therefore, Rep$A$ is a compact braided monoidal category. Its strict extension $(\mathrm{Rep}A)_\square$ will be denoted by $\mathrm{Rep}_\square(A)$.

Remark. All objects of Rep$A$ are reflexive up to a canonical isomorphism. Indeed, $\varrho_{V^{\vee\vee}}(a)=\varrho_V(s^2(a))=u_V\varrho_V(a)u_V^{-1}$, where we have identified the vector spaces $V^{\vee\vee}$ and

V and where $u_V = \varrho_V(u)$. Therefore, the composition of the canonical identification $V^{**} \to V$ and the homomorphism $x \mapsto u_V^{-1} x : V \to V$ is an A-isomorphism.

3.3. Ribbon Hopf Algebras. Let (A, R) be a quasitriangular Hopf algebra with the comultiplication Δ, counit ε and antipode s. Put $u = u_A \in A$. We call the triple $(A, R$, a central element $v \in A)$ a ribbon Hopf algebra if

$$v^2 = us(u), \qquad s(v) = v, \qquad \varepsilon(v) = 1, \tag{3.3.1}$$

$$\Delta(v) = (R_{21} R_{12})^{-1} (v \otimes v), \tag{3.3.2}$$

where $R_{12} = R$ and $R_{21} = \mathrm{Perm}(R)$. Since u is invertible, v is also invertible in A.

There exist quasitriangular Hopf algebras which do not contain v as above. If v exists it may be non-unique. To describe non-uniqueness of v we define the abelian group $E(A) \subset A$ consisting of central elements $E \in A$ such that

$$E^2 = 1, \qquad s(E) = E, \qquad \varepsilon(E) = 1, \qquad \Delta(E) = E \otimes E. \tag{3.3.3}$$

It is obvious that for any ribbon Hopf algebra (A, R, v) and any $E \in A$ the triple (A, R, Ev) is a ribbon Hopf algebra iff $E \in E(A)$.

Each quasitriangular Hopf algebra (A, R) canonically extends to a ribbon Hopf algebra $(\tilde{A}, R, v)$. Namely, let $\tilde{A}$ be the module whose elements are formal expressions $a + bv$ with $a, b \in A$. We provide $\tilde{A}$ with the Hopf algebra structure by the formulas

$$(a + bv)(a' + b'v) = (aa' + bb'us(u)) + (ab' + ba')v,$$

$$\Delta(a + bv) = \Delta(a) + \Delta(b)(R_{21} R_{12})^{-1} (v \otimes v),$$

$$s(a + bv) = s(a) + s(b)v, \qquad \varepsilon(a + bv) = \varepsilon(a) + \varepsilon(b).$$

We identify A with a subset of $\tilde{A}$ by the formula $a = a + 0v$. Clearly, $R \in A \otimes A \subset \tilde{A} \otimes \tilde{A}$.

3.4. Theorem. *$(\tilde{A}, R, v)$ is a ribbon Hopf algebra containing A as a Hopf subalgebra.*

Proof. The equalities (3.1.1–3.1.3, 3.3.1, 3.3.2) follow directly from definitions. Thus we have to check only the axioms of Hopf algebra for $\tilde{A}$. The associativity of $\tilde{A}$ follows from (3.1.12). Clearly, $v \in$ (centre of $\tilde{A}$). Let us show that $\Delta : \tilde{A} \to \tilde{A} \otimes \tilde{A}$ is an algebra homomorphism. It follows from (3.1.1) that for any $a \in A$,

$$(R_{21} R_{12})^{-1} \Delta(a) = \Delta(a)(R_{21} R_{12})^{-1}.$$

Since s is a coalgebra antihomomorphism,

$$\begin{aligned}\Delta(s(u)) &= \mathrm{Perm}((s \otimes s)(\Delta(u))) = (s(u) \otimes s(u))(s \otimes s)(R_{21} R_{12})^{-1} \\ &= (s(u) \otimes s(u))(R_{21} R_{12})^{-1}.\end{aligned}$$

[The last equality follows from (3.1.7).] Therefore,

$$\Delta(us(u)) = (R_{21} R_{12})^{-1} (us(u) \otimes us(u))(R_{21} R_{12})^{-1} = (R_{21} R_{12})^{-2} (us(u) \otimes us(u)).$$

Now we can prove that for any a, b, a', $b' \in A$,

$$\Delta((a + bv)(a' + b'v)) = \Delta(a + bv)\Delta(a' + b'v).$$

Indeed

$$\begin{aligned}
&\Delta((a+bv)(a'+b'v))\\
&\quad=\Delta(aa'+bb'us(u))+\Delta(ab'+ba')(R_{21}R_{12})^{-1}(v\otimes v)\\
&\quad=\Delta(aa')+\Delta(bb')(R_{21}R_{12})^{-2}(us(u)\otimes us(u))\\
&\qquad+\Delta(ab')(R_{21}R_{12})^{-1}(v\otimes v)+\Delta(b)(R_{21}R_{12})^{-1}(v\otimes v)\Delta(a')\\
&\quad=\Delta(a)\Delta(a')+\Delta(b)(R_{21}R_{12})^{-1}(v\otimes v)\Delta(b')(R_{21}R_{12})^{-1}(v\otimes v)\\
&\qquad+\Delta(a)\Delta(b'v)+\Delta(bv)\Delta(a')=\Delta(a+bv)\Delta(a'+b'v).
\end{aligned}$$

The counit axiom for $\tilde{A}$ follows from (3.1.5). It is easy to verify (using 3.1.11) and the fact that the antipode of A is an antiautomorphism of the algebra structure) that s is an antiautomorphism of $\tilde{A}$. Therefore, to check the axiom of antipode (3.1.8) it suffices to consider the case $a=v$. Let $R=\sum_i \alpha_i\otimes\beta_i$. In view of (3.1.6) $R_{12}^{-1}=\sum_i s(\alpha_i)\otimes\beta_i$ and $R_{21}^{-1}=\sum_j \beta_j\otimes s(\alpha_j)$. Therefore,

$$\begin{aligned}
&m(\mathrm{id}\otimes s)(\Delta(v))=m(\mathrm{id}\otimes s)((R_{12}^{-1}R_{21}^{-1})(v\otimes v)=v^2\sum_{i,j} s(\alpha_i)\beta_j s^2(\alpha_j)s(\beta_i)\\
&\quad=v^2\sum_i s(\alpha_i)u^{-1}s(\beta_i)=v^2\sum_i u^{-1}s^3(\alpha_i)s(\beta_i)\\
&\quad=v^2u^{-1}s(u^{-1})=v^2(s(u)u)^{-1}=v^2(us(u))^{-1}=1.
\end{aligned}$$

Here we have used (3.1.11) and (3.1.14). A similar argument shows that $m(s\otimes\mathrm{id})(\Delta(v))=1$. Here one should use the equalities $\sum s^2(\alpha_i)\beta_i=s(u^{-1})$ (which follows from 3.1.7 and 3.1.14) and $v^2\sum_i s(\alpha_i u^{-1}\beta_i)=1$ (which follows from 3.1.11).

3.5. Remark. If (A, R, w) is a ribbon Hopf algebra then $w\in A\subset\tilde{A}$, so that $v=Ew$ for some $E\in\tilde{A}$, $E\neq 1$. In view of (3.3.3) E generates a two-dimensional Hopf subalgebra, say, $\bar{E}$ of $\tilde{A}$ with the basis $\{1, E\}$. Clearly, $\tilde{A}=A\otimes\bar{E}$ in the class of Hopf algebras.

4. Coloured Ribbon Graphs

4.1. Bands and Annuli. A band is the image of the square $[0,1]\times[0,1]$ under its (smooth) imbedding in R^3. The images of the segments $[0,1]\times 0$ and $[0,1]\times 1$ under this imbedding are called bases of the band. The image of the segment $(1/2)\times[0,1]$ is called the core of the band.

An annulus is the image of the cylinder $S^1\times[0,1]$ under an imbedding in R^3. The image of the circle $S^1\times(1/2)$ under this imbedding is called the core of the annulus.

4.2. Ribbon Graphs. Let k, ℓ be non-negative integers. A ribbon (k,ℓ)-graph is an oriented surface S imbedded in $R^2\times[0,1]$ and decomposed as the union of finite collection of bands and annuli, each band being provided with a "type" 1 or 2, so that the following conditions hold:

(a) annuli do not meet each other and do not meet bands;
(b) bands of the same type never meet each other;
(c) bands of different types may meet only in the points of their bases;

(d) S meets $R^2 \times \{0,1\}$ exactly in bases of certain type 2 bands and the collection of these bases is the collection of segments

$$\{[i-\tfrac{1}{4}, i+\tfrac{1}{4}]\times 0\times 0 \mid i=1,\dots,k\}\cup\{[j-\tfrac{1}{4}, j+\tfrac{1}{4}]\times 0\times 1 \mid j=1,\dots,\ell\}, \tag{4.2.1}$$

(e) the remaining bases of type 2 bands are contained in the bases of type 1 bands.

The surface S is called the surface of the graph. The type 1 bands are called coupons, the type 2 bands are called ribbons. Those ribbons which are incident to the segments (4.2.1) are called border ribbons. Some examples of ribbon graphs are given in Figs. 1–3.

Note that the choice of orientation for S is equivalent to a choice of one side of S. The chosen side will be depicted white, the opposite side will be shaded. Note that when we rotate an annulus around its core to the angle 180° we get the same annulus with the opposite orientation. Thus, orientations of annuli are insignificant when one considers ribbon graphs up to isotopy (see Sect. 4.3).

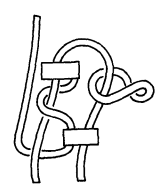


Fig. 1

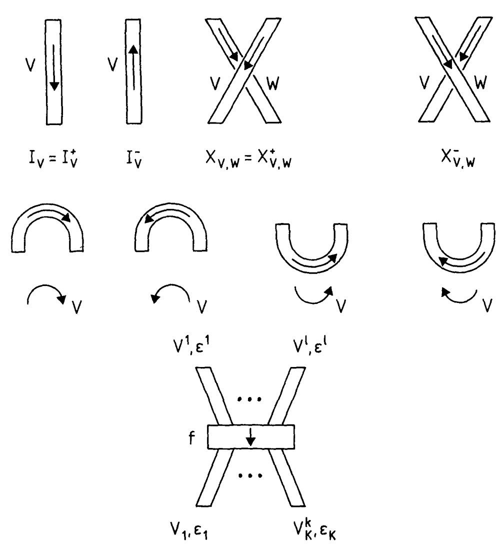


Fig. 2

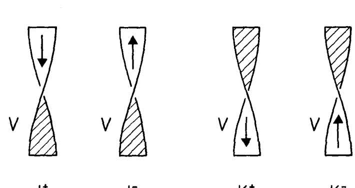


Fig. 3

A ribbon graph is called homogeneous if in a neighbor hood of the segments (4.2.1) the white side of the graph surface is turned upwards (i.e. to the reader). For instance, the graphs in Figs. 1 and 2 are homogeneous, and those in Fig. 3 are not homogeneous.

4.3. Directed Graphs. A ribbon graph is called directed if the cores of its bands and annuli are provided with directions. Note that the directed core of a band leads from one base to another one. The former base of the band is called initial, the latter one is called final.

Two directed ribbon graphs Γ, Γ' are called isotopic if there exists a (smooth) isotopy $h_t: R^2 \times [0,1] \to R^2 \times [0,1]$ of the identity mapping $h_0 = \mathrm{id}$ so that each h_t is a self-diffeomorphism of $R^2 \times [0,1]$ fixing the boundary pointwise, and h_1 transforms Γ onto Γ' preserving the decomposition into coupons, ribbons and annuli, and preserving the directions of cores and the orientation of the graph surface. Isotopy is clearly an equivalence relation.

With each directed ribbon (k,ℓ)-graph Γ we associate two sequences $\varepsilon_1(\Gamma), \ldots, \varepsilon_k(\Gamma)$ and $\varepsilon^1(\Gamma), \ldots, \varepsilon^\ell(\Gamma)$ consisting of ± 1. Here $\varepsilon_i(\Gamma) = -1$ if the segment $[i-(1/4), i+(1/4)] \times 0 \times 0$ is the initial base of the incident ribbon, and $\varepsilon_i(\Gamma) = +1$ otherwise. Similarly, $\varepsilon^j(\Gamma) = -1$ if the segment $[j-(1/4), j+(1/4)] \times 0 \times 1$ is the final base of the incident ribbon and $\varepsilon^j(\Gamma) = 1$ otherwise.

Let Q be a coupon of a directed ribbon graph Γ. Denote by $a = a(Q)$ (respectively by $b = b(Q)$) the number of ribbons of Γ incident to the initial (respectively final) base of Q. A small neighborhood of Q in $R^2 \times (0,1)$ looks as in Fig. 4 where we assue that the white side of Q is turned upwards. Denote the ribbons of Γ incident to Q by $B^1(Q), \ldots, B^a(Q)$ and $B_1(Q), \ldots, B_b(Q)$ in the order shown in Fig. 4. The directions of B^i, B_j are characterized by numbers $\varepsilon^i(Q)$, $\varepsilon_j(Q) \in \{1, -1\}$. Namely, $\varepsilon^i(Q) = -1$ if the base of B^i depicted in Fig. 4 is its initial base; otherwise $\varepsilon^i(Q) = 1$. Similarly, $\varepsilon_j(Q) = -1$ if the base of B_j depicted in Fig. 4 is its final base; otherwise $\varepsilon_j(Q) = 1$.

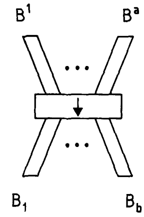


Fig. 4

4.4. Colourings of Ribbon Graphs. Fix a Hopf algebra A. Denote by $\Lambda = \Lambda(A)$ the class of A-modules of finite dimension over the ground field. Denote by $N = N(A)$ the class of finite sequences $(V_1, \varepsilon_1), \ldots, (V_k, \varepsilon_k)$ where $V_1, \ldots, V_k \in \Lambda$ and $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$. With such a sequence $\eta = ((V_1, \varepsilon_1), \ldots, (V_k, \varepsilon_k))$ we associate the A-module $V(\eta) = V_1^{\varepsilon_1} \otimes \ldots \otimes V_k^{\varepsilon_k}$, where $V^1 = V$, $V^{-1} = V^\vee$, the tensor product is taken over the ground field, and the A-module structure in $V(\eta)$ is determined by the comultiplication in A.

A colouring (or an A-colouring) of a directed ribbon graph Γ is a mapping λ which associates with each ribbon B of Γ its "colour" $\lambda(B) \in \Lambda$ and associates with each coupon Q of Γ its "colour" $\lambda(Q)\in \mathrm{Hom}_A(V(\eta),\ V(\eta'))$, where

$$\eta=((\lambda(B_1(Q)),\varepsilon_1(Q)),\ldots,(\lambda(B_b(Q)),\varepsilon_b(Q)))\in N\,,$$
$$\eta'=((\lambda(B^1(Q)),\varepsilon^1(Q)),\ldots,(\lambda(B^a(Q)),\varepsilon^a(Q)))\in N\,.$$

There is a natural isotopy relation in the class of coloured directed ribbon graphs, defined as in Sect. 4.3 with the additional condition of preservation of colouring. By a CDR-graph we shall mean a coloured directed ribbon graph considered up to isotopy. By a HCDR-graph we shall mean a homogeneous CDR-graph (see Sect. 4.2).

Some examples of CDR-graphs and the notation for these graphs are presented in Figs. 2 and 3. The homogeneous (k,ℓ)-graph with one coupon of colour f presented in Fig. 2 will be denoted by $\Gamma(f,\eta,\eta')$; here $\eta=((V_1,\varepsilon_1),\ldots,(V_k,\varepsilon_k))\in N$, $\eta'=((V^1,\varepsilon^1),\ldots,(V^\ell,\varepsilon^\ell))\in N$, where V_i, V^j are the colours of the corresponding ribbons and ε_i, ε^j are directions of their cores: $+1$ means down, -1 means up.

4.5. Category HCDR(A). Let A be a Hopf algebra. We define the category of homogeneous A-coloured directed ribbon graphs $\mathscr{H}=\mathrm{HCDR}(A)$. Its objects are elements of $N(A)$ (see Sect. 4.4). If η, $\eta'\in N(A)$ then a morphism $\eta\to\eta'$ is a HCDR-graph such that the sequence of colours λ and directions ε of the top (respectively bottom) border ribbons is equal to η' (respectively η). The composition $\Gamma'\circ\Gamma$ of two such morphisms $\Gamma:\eta\to\eta'$, $\Gamma':\eta'\to\eta''$ is defined as follows: shift Γ' by the vector $(0,0,1)$ into $R^2\times[1,2]$; glue the bottom ends of Γ' with the top ends of Γ (this glueing preserves colours and directions of ribbons since they are determined by η'); reduce twice the size along the line $0\times 0\times R$. This gives a well-defined composition law for morphisms. The identity morphism $\eta\to\eta$ is the HCDR-graph consisting of plane rectangular ribbons whose directions and colouring are determined by η.

We provide $\mathscr{H}$ with the tensor product $\otimes:\mathscr{H}\times\mathscr{H}\to\mathscr{H}$ as follows. The tensor product of objects $((V_1,\varepsilon_1),\ldots,(V_k,\varepsilon_k))$ and $((W_1,\nu_1),\ldots,(W_\ell,\nu_\ell))$ is their conjuction

$$((V_1,\varepsilon_1),\ldots,(V_k,\varepsilon_k),(W_1,\nu_1),\ldots,(W_\ell,\nu_\ell))\,.$$

The tensor product of two morphisms (i.e. graphs) Γ, Γ' is obtained by positioning Γ' to the right of Γ so that there is no mutual linking or intersection (see Fig. 5). It is obvious that $\mathscr{H}$ is strict monoidal. [Here $I\in N(A)$ is the empty sequence, id_I is the empty graph.]

Fig. 5

$$\boxed{\Gamma\otimes\Gamma'}\ =\ \boxed{\Gamma}\ \ \boxed{\Gamma'}$$

The category $\mathscr{H}$ is provided with a natural braiding

$$\{c_{\eta,\theta}:\eta\otimes\theta\to\theta\otimes\eta\mid \eta,\,\theta\in Ob(\mathscr{H})=N(A)\}$$

(see Sect. 2.1). We take the HCDR-graphs representing id_η, id_θ and form the HCDR-graph $c_{\eta,\theta}$ as in Fig. 6. (This idea is borrowed from [JS].) It is a pleasant exercise for the reader to check that we do have a braiding.

Finally, we provide $\mathscr{H}$ with a structure of a compact category. For an object $\eta=((V_1,\varepsilon_1),\ldots,(V_k,\varepsilon_k))$ put $\eta^*=((V_k,-\varepsilon_k),\ (V_{k-1},-\varepsilon_{k-1}),\ \ldots,(V_1,-\varepsilon_1))$. The construction of the homomorphisms e_η, n_η should be clear from Fig. 7.

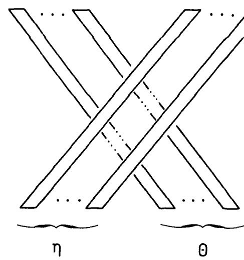


Fig. 6

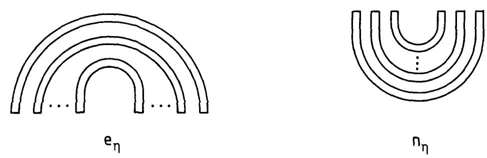


Fig. 7

4.6. Theorem. *$\mathscr{H}=\mathrm{HCDR}(A)$ is a compact braided strict monoidal category.*

Proof. Obvious.

We may similarly introduce the category CDR(A) of CDR-graphs. In contrast to HCDR(A) the objects of CDR(A) are sequences $(V_1, \varepsilon_1, v_1), \ldots, (V_k, \varepsilon_k, v_k)$, where $V_1, \ldots, V_k \in \Lambda(A)$ and $\varepsilon_i, v_i = \pm 1$. The numbers $\{v_i\}$ are responsible for the up/down position of the white side of the border ribbons near the border segments. Theorem 4.6 may be easily extended to the case of CDR(A).

5. The Main Theorem

5.1. Theorem. *Let (A, R, v) be a ribbon Hopf algebra over the field I. There exists a unique covariant functor $F=F_A: \mathrm{HCDR}(A) \to \mathrm{Rep}_\square A$ which has the following properties:*

(i) *F preserves the tensor product;*

(ii) *For any A-module V of finite dimension over I the functor F transforms the object (V, ε) into V^ε, where $V^1 = V$ and $V^{-1} = V^\vee$;*

(iii) *F transforms the* CDR*-graphs $\curvearrowright_V$, $\curvearrowleft_V$, $X_{V,W}$ (see Fig. 2) respectively in the canonical pairing*

$$(x, y) \mapsto \langle x, y\rangle = x(y): V^\vee \times V \to I, \tag{5.1.1}$$

in the pairing

$$(y, x) \mapsto \langle x, v^{-1} u y\rangle : V \otimes V^\vee \to I, \tag{5.1.2}$$

and in the homomorphism

$$x \otimes y \mapsto \sum_i \beta_i y \otimes \alpha_i x : V \otimes W \to W \otimes V, \tag{5.1.3}$$

where

$$R = \sum_i \alpha_i \otimes \beta_i \in A \otimes A;$$

(iv) *F transforms each* CDR-*graph* $\Gamma(f,\eta,\eta')$ *(see Fig. 2) into the homomorphism* $f: V(\eta)\to V(\eta')$ *(cf. Sect. 4.4.).*

Note that according to (i, ii) F transforms each object $(V_1,\varepsilon_1),\dots,(V_k,\varepsilon_k)$ of HCDR(A) into the object $V_1^{\varepsilon_1},\dots,V_k^{\varepsilon_k}$ of $\mathrm{Rep}_{\square}A$. The action of F on morphisms of HCDR(A) (i.e. on HCDR-graphs) is more subtle. Using the identities $F(\Gamma\circ\Gamma')=F(\Gamma)\circ F(\Gamma')$ and $F(\Gamma\otimes\Gamma')=F(\Gamma)\otimes F(\Gamma')$ one reduces the calculation of $F(\Gamma)$ to "building blocks" of Γ. Roughly speaking, the functor F associates with local maximums of ribbons the homomorphisms 5.1.1, 5.1.2; with positive crossings of ribbons where the ribbons are directed downwards F associates the action 5.1.3 of R, and with coupons F associates their colours. Theorem 5.1 implies that these assignments are consistent and give a well-defined isotopy invariant for each HCDR-graph Γ.

Let us show how to compute the value of F for the graphs $\smile\!\nearrow_V$, $\nwarrow\!\smile_V$. Since I_V, I_V^- are identity morphisms, $F(I_V)=\mathrm{id}_V$, $F(I_V^-)=\mathrm{id}_{V^\vee}$. It is easy to deduce from the equality $(\curvearrowright_V\otimes I_V)\circ(I_V\otimes\smile\!\nearrow_V)=I_V$ that $F(\smile\!\nearrow_V)$ is the homomorphism $I\to V\otimes V^\vee$ which transforms 1 into $\sum_i e_i\otimes e^i$, where $\{e_i\}$ is a basis in V and $\{e^i\}$ is the dual basis in $V^\vee$. Similarly, $F(\nwarrow\!\smile_V)$ is the homomorphism $I\to V^\vee\otimes V$ which transforms 1 into $\sum_i e^i\otimes u^{-1}ve_i$.

It follows easily from Theorem 5.1 and definitions that the functor F preserves braiding. The composition of F and the functor of forgetting the A-linear structure may be shown to commute with the compact structure (here for $X=((V_1,\varepsilon_1),\dots,(V_k,\varepsilon_k))$ the isomorphism $F_V:(F(X))^*\to F(X^*)$ is the tensor product of isomorphisms $(V^\varepsilon)^\vee\to V^{-\varepsilon}$, where for $\varepsilon=-1$ $V^{\vee\vee}\to V$ is the (non A-linear) canonical identification, and for $\varepsilon=1$ $V^\vee\to V^\vee$ is the multiplication by $v^{-1}u$).

To prove Theorem 5.1 we need the notion of generators of HCDR(A). We say that certain HCDR-graphs $\{X_i\}$ generate HCDR(A) if each HCDR-graph may be obtained from $\{X_i\}$ and $\{I_V, I_V^-\}_{V\in\Lambda(A)}$ using composition $\circ$ and tensor product $\otimes$. This notion is very similar to the notion of generators of a group.

5.2. Lemma. *The graphs* $\{\curvearrowright_V, \smile\!\nearrow_V, \curvearrowleft_V, \nwarrow\!\smile_V, X_{V,W}, X_{V,W}^-\mid V, W\in\Lambda(A)\}$ *and* $\{\Gamma(f,\eta,\eta')\mid\eta,\eta'\in N(A), f\in\mathrm{Hom}_A(V(\eta),V(\eta'))\}$ *generate* HCDR(A).

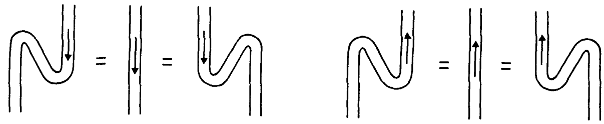
This lemma follows from the obvious fact that each graph (considered up to isotopy) may be drawn so that it lies in a standard position with respect to some square lattice in the plane of the picture. The words "standard position" mean that in each square of the lattice the picture looks as in Fig. 2.

Following the lines of group theory we may consider relations between generators. The following lemma provides us with a "generating set" of relations which meansn that all possible relations between our generators follow from the given ones. We shall present the relations in a pictorial form.

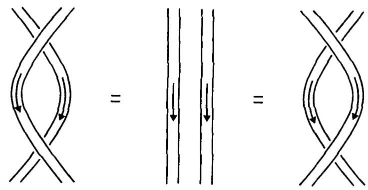
5.3. Lemma. *Figure 8 presents a generating set of relations for the generators of* HCDR(A) *given by Lemma 5.2.*

Note that all the relations Rel_1–Rel_{13} of Fig. 8 may be written purely algebraically. For example, the first relation Rel_1 means that

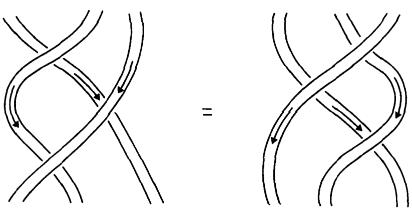
$$(\curvearrowleft_V\otimes I_V)\circ(I_V\otimes\nwarrow\!\smile_V)=I_V\,.$$



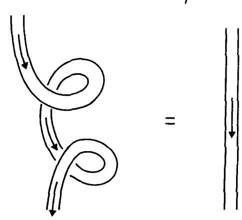
Rel_1 Rel_2 Rel_3 Rel_4



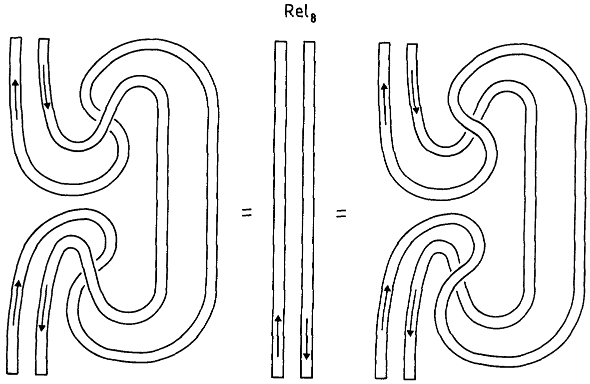
Rel_5 Rel_6



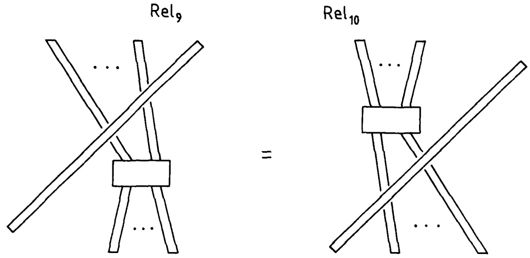
Rel_7



Rel_8



Rel_9 Rel_{10}



Rel_{11}

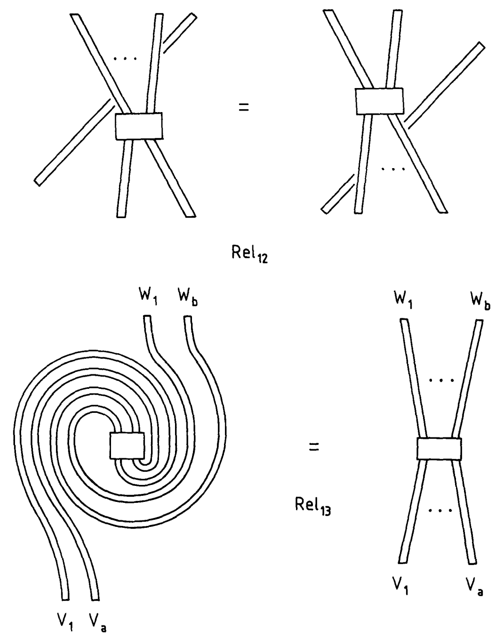


Fig. 8

The main point in the proof of Lemma 5.3 is to show that if two isotropic graphs are expressed via our generators then these two expressions are equivalent modulo our relations. In the case of graphs with no coupons this is known (see [FY, Tu$_2$]). To handle coupons note that each graph may be positioned so that all coupons are parallel to the plane of the page. This condition may be violated in the course of isotopy. However we may always assume that in the course of isotopy each coupon moves as a solid rectangle. Thus isotopy of a coupon gives rise to a path in $SO(3)$ starting in 1 and finishing in $SO(2)$ (since in the end all coupons are again parallel to the fixed plane). All such paths may be deformed to $SO(2)$. This enables one to change isotopy so that in its course all coupons are always parallel to the plane of the page. Now it is easy to see that in a neighbourhood of a coupon such isotopy changes the picture as in Fig. 8, Rel$_{11}$–Rel$_{13}$. This implies Lemma 5.3.

5.4. Proof of Theorem 5.1. Uniqueness of F follows from Lemma 5.2. Indeed, the values of F on $\curvearrowright_V$, $\curvearrowleft_V$, $C_{V,W}$, $\Gamma(f,\eta,\eta')$ are given in (iii, iv); the values of F on $\circlearrowleft_V$, $\circlearrowright_V$ are computable from the relations Rel$_1$–Rel$_4$ (cf. the remark in Sect. 5.1); finally, because of relation Rel$_5$, Rel$_6$ $F(X^-_{V,W})=(F(X_{W,V}))^{-1}$. Thus we know F for generators. This implies uniqueness.

To prove existence it suffices to check that our operators F for generators satisfy all the relations of Lemma 5.3. Rel$_1$–Rel$_6$ are straightforward. Rel$_7$ follows from (3.1.4). Let us prove Rel$_8$. Let

$$R=\sum_i \alpha_i \otimes \beta_i, \qquad u=\sum_i s(\beta_i)\alpha_i .$$

Set

$$\varphi^{\pm}=(\mathrm{id}_V \otimes F(\curvearrowleft_V))\circ(F(X^{\pm}_{V,V})\otimes \mathrm{id}_{V^*})\circ(\mathrm{id}_V\otimes F(\circlearrowright_V)): V\to V .$$

The operator counterpart of Rel_8 states that $\varphi^- \circ \varphi^+ = \mathrm{id}_V$. We shall prove that $\varphi^\pm$ is the multiplication by $v^{\pm 1}$. This would imply Rel_8. Let $\{e_j\}$ be a basis of V. A direct computation shows that

$$F(\varphi^+)(e_j) = \Big(\sum_i \beta_i v^{-1} u \alpha_i\Big) e_j$$

and

$$F(\varphi^-)(e_j) = \Big(\sum_i s(\alpha_i) v^{-1} u \beta_i\Big) e_j .$$

We have

$$\sum_i \beta_i v^{-1} u \alpha_i = \sum_i \beta_i u \alpha_i u^{-1} u v^{-1} = \sum_i \beta_i s^2(\alpha_i) u v^{-1} = v^{-1},$$

$$\sum_i s(\alpha_i) v^{-1} u \beta_i = v^{-1} \sum_i s(\alpha_i) s^2(\beta_i) u = v^{-1} s(u) u = v .$$

To prove the equalities corresponding to Rel_9, Rel_{10} fix bases $\{e_i\}$, $\{f_j\}$ respectively in V, W and dual bases $\{e^i\}$, $\{f^j\}$ respectively in $V^\vee$ and $W^\vee$. The left-hand side of Rel_9 gives rise to an operator $\psi: V^\vee \otimes W \to V^\vee \otimes W$ which is computed to act as follows:

$$\begin{aligned}\psi(e^i \otimes f_j) &= \sum_{k,\ell,p,q} e^\ell \otimes e^i(s(\alpha_p) e_k) \cdot e^k(u \alpha_q u^{-1} e_\ell) \cdot (\beta_q \beta_p)(f_j) \\ &= \sum_{k,\ell,p,q} e^k(s^2(\alpha_q) e_\ell) \cdot e^i(s(\alpha_p) e_k) e^\ell \otimes (\beta_q \beta_p)(f_j) \\ &= \sum_{p,q,\ell} e^i(s(\alpha_p) s^2(\alpha_q) e_\ell) e^\ell \otimes (\beta_q \beta_p)(f_j) = e^i \otimes f_j .\end{aligned}$$

The last equality follows from the fact that

$$\sum_{p,q} s(\alpha_p) s^2(\alpha_q) \otimes \beta_q \beta_p = (s \otimes \mathrm{id})((s \otimes \mathrm{id})(R) \cdot R) = 1 .$$

Thus $\psi = \mathrm{id}$ which proves the operator version of Rel_9. Similarly, the operator $\psi': V \otimes W^\vee \to V \otimes W^\vee$ corresponding to the right-hand side of Rel_{10} is the identity:

$$\begin{aligned}\psi'(e_i \otimes f^j) &= \sum_{k,\ell,p,q} f^k(s(\alpha_q) f_\ell) \cdot f^j(s^2(\alpha_p) f_k)(\beta_q \beta_p)(e_i) \otimes f^\ell \\ &= \sum_{p,q,\ell} f^j(s^2(\alpha_p) s(\alpha_q) f_\ell)(\beta_q \beta_p)(e_i) \otimes f^\ell = e_i \otimes f^j .\end{aligned}$$

The operator counterparts of Rel_{11}, Rel_{12} follow directly from the assumption that the colour of the coupon is A-linear.

Let us check the operator equality corresponding to Rel_{13}. Fix a bases $\{e_i\}$ in $V_1 \otimes \ldots \otimes V_b$ and a bases $\{f_j\}$ in $W_1 \otimes \ldots \otimes W_a$. The operator

$$\theta: V_1 \otimes \ldots \otimes V_b \to W_1 \otimes \ldots \otimes W_a$$

corresponding to the left-hand side of Rel_{13} acts as follows:

$$\begin{aligned}\theta(e_j) &= \sum_{j,k} e^k(u^{-1} v e_i) \cdot f^j(f(e_k))(uv^{-1})(f_j) \\ &= (uv^{-1})(f(u^{-1} v)(e_i)) = f(e_i) .\end{aligned}$$

Here we have used our assumption that the colour f of the coupon is A-linear.

6. Applications and Generalizations of Theorem 5.1

6.1. Isotopy Invariants of Coloured Links. A link in R^3 is a finite set of disjoint oriented smooth circles imbedded in R^3. The circles are called the components of the link. Fix a ribbon Hopf algebra A. An A-colouring of the link is a function associating with each component an A-module of finite dimension over the ground field. A link is called framed if each component is provided with an integer. As usual, one introduces the isotopy relation for coloured framed links [Re_2].

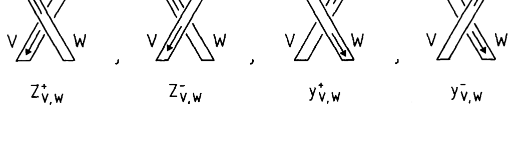
Each framed link L determines a ribbon $(0,0)$-graph Γ_L, consisting of several annuli. Namely, let $L_1, \ldots, L_k$ be the components of L and let $m_1, \ldots, m_k$ be the associated integers. Let A_i be the annulus in R^3 with the oriented core L_i and such that the linking number of two circles making ∂A_i is equal to m_i. Assume that the annuli $A_1, \ldots, A_k$ are disjoint. Compressing $\bigcup_i A_i$ into $R^2 \times (0,1)$ we get the desired ribbon $(0,0)$-graph Γ_L. Each colouring of L gives rise to a colouring of Γ_L in the obvious fashion. The formula $L \mapsto \Gamma_L$ defines a bijective correspondence between the isotopy types of coloured framed links and the isotopy types of HCDR-graphs which consist of annuli.

Note that for an arbitrary HCDR-graph Γ which has zero number of border ribbons the operator $F(\Gamma)$ is a linear endomorphism of the ground field, i.e. the multiplication by an element of this field. This element is an isotopy invariant of Γ.

In particular, with each framed coloured link L we may associate its isotopy invariant $F(\Gamma_L)$ which is an element of the ground field. For instance, if $A = U_h(sl_n(\mathbb{C}))$, (cf. Sect. 7), if all integers which determine the framing of L are chosen to be zero, and if the colours of all components of L are chosen to be the vector representation of A in C^n then the invariant $F(\Gamma_L)$ depends on $q = e^h$ as a polynomial. Up to a reparametrization this polynomial equals $P_L(q^n, q - q^{-1})$, where P_L is the 2-variable Jones-Conway polynomial of L. In the case $n = 2$ we get the Jones polynomial of L. For $A = U_h G$, $G = so(n)$, $sp(2k)$ a similar construction gives an infinite set of 1-variable reductions of the 2-variable Kauffman polynomial of L (see [Tu_1]). The invariants corresponding to the exceptional Lie algebras and to the spinor representation of $U_h so(n)$ seem to be new (see [Re_1]).

6.2. Even Graphs. A HCDR-graph is called even if it may be obtained from graphs $\cup^{\rightarrow}_V$, $\cap^{\rightarrow}_V$, $X^{\pm}_{V,W}$, $\Gamma(f, \eta, \eta')$ and the graphs $Z^{\pm}_{V,W}$ shown in Fig. 9 by applications of the operations $\otimes$ and $\circ$.

There is a simple necessary and sufficient criteria which enables one to decide if a given HCDR-graph is even. Draw a picture of this graph so that all coupons are represented by rectangles with horizontal bases (as in Figs. 1, 2 and 4). The bases of ribbons will be automatically horizontal. Assume also that white sides of the coupons are turned up and that all ribbons lie "parallel" to the plane of the picture, i.e. have no twists (cf. Fig. 1). Now, moving along the directed core of a ribbon or an annulus one meets a certain number of local extremums $\cap^{\rightarrow}$, $\cap^{\leftarrow}$, $\cup^{\rightarrow}$, $\cup^{\leftarrow}$ of the height function. Compute the number of left oriented extremums $\cup^{\leftarrow}$, $\cap^{\leftarrow}$ lying on the

Fig. 9


ribbon (respectively the annulus). It is a simple topological exercise to show that a HCDR-graph is even if and only if these numbers are even for all ribbons and annuli of the graph. For example, the 1-ribbon graphs $\curvearrowleft$, $\curvearrowright$ do not satisfy this criteria and therefore they are not even. The HCDR-graph Γ_L corresponding to a framed coloured link L is even if and only if all framing numbers of L are odd.

Let (A, R, v) and (A, R, v') be two ribbon Hopf algebras with the same underlying quasitriangular Hopf algebras. Let F, F' be the corresponding functors $\mathrm{HCDR}(A) \to \mathrm{Rep}_{\square} A$. It follows directly from definitions that the values of F, F' on $\cup_V$, $\cap_V$, $X^{\pm}_{V,W}$, $\Gamma(f, \eta, \eta')$ are the same. Using the assumption that v, v' lie in the centre of A one easily shows that $F(Z^{\pm}_{V,W}) = F'(Z^{\pm}_{V,W})$. This implies that on the set of even graphs we have $F = F'$. In other words, for even graphs the functor F does not depend on the choice of v.

Moreover, for even A-coloured HCDR-graphs we can define operators F without any use of v, i.e. for such graphs we may use an arbitrary quasitriangular Hopf algebra (A, R). Here is a sketch of the construction. First, extent the ground field to an algebraically closed field. Note that each representation of A may be extended to a representation of $\tilde{A}$ (see Sect. 3.3); we use that for any finite-dimensional operator $\mathscr{D}$ over an algebraically closed field there exists an operator d acting in the same vector space such that $d^2 = \mathscr{D}$ and d commutes with all operators commuting with $\mathscr{D}$). Thus A-colouring giver rise (non-uniquely) to an $\tilde{A}$-colouring. Apply now the functor $F = F_{\tilde{A}}$ and the reduction homomorphism $\mathrm{Rep}_{\square} \tilde{A} \to \mathrm{Rep}_{\square} A$. The composition restricted to even graphs does not depend on the choice of square roots.

6.3. Theorem. (An extension of Theorem 5.1). *Let (A, R, v) be a ribbon Hopf algebra. Assume that for each A-module V we have fixed two invertible A-linear operators θ^+_V, $\theta^-_V : V \to V$. There exists a unique covariant functor $F_\theta : \mathrm{CDR}(A) \to \mathrm{Rep}_{\square} A$ which has the following properties:* (i) *on the subcategory* HCDR(A) *of* CDR(A) F_θ equals F; (ii) F_θ *preserves the tensor product;* (iii) *for any A-module V*

$$F_\theta(J^+_V) = \theta^+_V \quad and \quad F_\theta(J^-_V) = \theta^-_V .$$

We give here a construction of F_θ. Note that an arbitrary CDR-graph Γ with k bottom ends and ℓ top ends may be uniquely represented in the form $(I^1 \otimes \ldots \otimes I^\ell) \circ \bar{\Gamma} \circ (I_1 \otimes \ldots \otimes I_k)$, where $\bar{\Gamma}$ is a homogeneous CDR-graph and each I^i is either I^ε_V or J^ε_V, $\varepsilon = \pm 1$, each I_i is either I^ε_V or K^ε_V, $\varepsilon = \pm 1$ (see Figs. 2 and 3). Put $F_\theta(J^\varepsilon_V) = \theta^\varepsilon_V$, $F_\theta(K^\varepsilon_V) = (\theta^\varepsilon_V)^{-1}$, $F_\theta(I^\varepsilon_V) = \mathrm{id}_{V^\varepsilon}$, where $V^+ = V$, $V^- = V^*$. Put

$$F(\Gamma) = (F_\theta(I^1) \otimes \ldots \otimes F_\theta(I^\ell)) \circ F(\bar{\Gamma}) \circ (F_\theta(I_1) \otimes \ldots \otimes F_\theta(I_k)) .$$

All the properties of F_θ easily follows.

6.4. Remarks. 1. It is obvious that the operator invariant $F(\Gamma, \lambda)$ of a directed ribbon graph Γ with a colouring λ is additive with respect to the colours of the coupons of Γ. This invariant is also additive with respect to the colours of those ribbons which connect 2 distinct coupons. More exactly, if the colour of such a ribbon B is a direct sum of A-modules, V, W then we split the colours of the coupons incident to B and obtain thus two A-colourings, say, λ' and λ'' of Γ. Here, λ' and λ'' coincide outside B and the incident coupons and $\lambda'(B) = V$, $\lambda''(B) = W$.

Then $F(\Gamma,\lambda)=F(\Gamma,\lambda')+F(\Gamma,\lambda'')$. This follows from the isotopy invariance of $F(\Gamma,\lambda)$, since B is isotopic to a standard plane vertical ribbon for which the assertion is obvious. Similarly, the invariant $F(\Gamma,\lambda)$ is additive with respect to the colours of annuli and of those ribbons which connect a coupon with one of the border segments (4.2.1). Note also that each annulus of Γ may be replaced by two coupons and two ribbons without changing $F(\Gamma,\lambda)$ (see Fig. 10).

2. Let Γ be a HCDR-graph and let the colour of a ribbon B of Γ is the tensor product $V\otimes W$ of two A-modules, V, W. Cutting out B along its core we get two new ribbons which make together with $\Gamma\backslash B$ another DR-graph, say, Γ'. The colouring of Γ obviously induces a colouring of Γ'. Note that looking on B from its white side we see one of the new ribbons to lie to the left of the directed core of B, the other one lying to the right; we colour them respectively by V and W. It turns out that $F(\Gamma')=F(\Gamma)$. This follows from 3.1.2, 3.1.3; these equalities directly imply equalities presented in Fig. 11. A similar assertion holds for each annulus coloured by a tensor product $V\otimes W$: We may replace such an annulus by two subannuli, obtained by cutting along the core, and coloured V and W. Such a replacement does not change $F(\Gamma)$.

3. The functor $F:\mathrm{HCDR}(A)\to\mathrm{Rep}_\square A$ is surjective since each morphism $f:\eta\to\eta'$ of $\mathrm{Rep}_\square A$ is the image of the graph $\Gamma(f,\eta,\eta')$. Moreover, f may be always presented by a 3-valent HCDR-graph, i.e. a graph whose coupons are all incident to $\leqq 3$ ribbons. This follows from the fact that the local operation on the HCDR-graphs depicted in Fig. 12 preserves the image of the graph in $\mathrm{Rep}_\square A$ and decreases the valence of the coupon by 1 by virtue of introducing an additional coupon of valence 3. Note that in Fig. 12d is the identical endomorphism of $V_{k-1}\otimes V_k$ and the colours f, f' are equal.

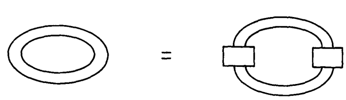


Fig. 10

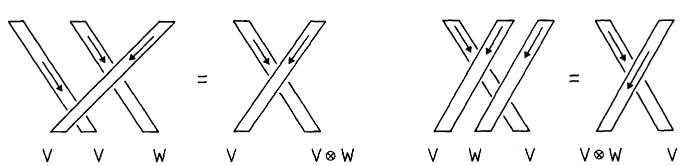


Fig. 11

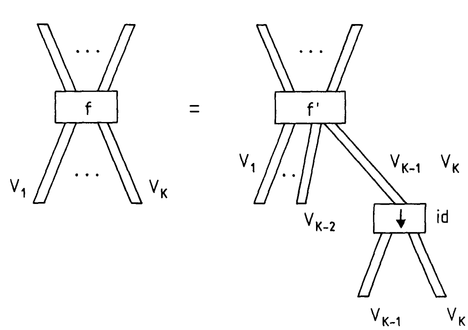


Fig. 12

4. To colour a directed ribbon graph is a problem in itself. It is natural to colour such a graph Γ in two steps: first ribbons and then coupons. The second step is somewhat more complicated as it is not easy to describe the A-homomorphisms in a combinatorial language. The following construction gives a useful method of colouring coupons.

Assume that the ribbons of Γ are coloured. Assume that with each coupon Q of Γ we have associated a DR-graph Γ_Q with coloured ribbons so that the border ribbons of Γ_Q have the same directions, colours, and orientations (white/black) as the ribbons of Γ incident to Q. Gluing Γ_Q instead of Q we get a "simpler" directed ribbon graph with coloured ribbons. This in principle enables us to reduce the problem of colouring coupons of Γ to the same problem for simpler graphs $\{\Gamma_Q\}$. For instance, in many cases we may take Γ_Q without any coupons at all. (Ribbon graphs without coupons are called ribbon tangles.) This construction suggests a specific point of view on coupons: one may think that coupons are black boxes hiding inside some blocks $\{\Gamma_Q\}$.

7. Examples

7.1. Algebra $U_h\mathfrak{G}$. The theory of quantum groups developed in [Dr1, J] gives nontrivial examples of (topological) ribbon Hopf algebras. We mean the quantized universal enveloping algebras of simple Lie algebras, called also briefly quantized Lie algebras.

Let $\mathfrak{G}$ be a simple Lie algebra. The quantized Lie algebra $U_h\mathfrak{G}$ is the associative algebra over the ring of formal power series $\mathbb{C}[[h]]$ generated (as an algebra complete in the h-adic topology) by elements H_i, X_i^+, X_i^-, $i=1,\ldots,r=\operatorname{rank}\mathfrak{G}$ subject to the following relations:

$$[H_i,H_j]=0, \qquad [H_i,X_j^\pm]=\pm(\alpha_i\alpha_j)X_j^\pm,$$

$$[X_i^+,X_j^-]=\delta_{ij}\frac{sh\left(\frac{h}{2}H_i\right)}{sh\left(\frac{h}{2}\right)},$$

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}(X_i^\pm)^kX_j^\pm(X_i^\pm)^{n-k}=0, \qquad i\neq j, \qquad n=1-A_{ij},$$

where $\{\alpha_i\}_{i=1}^r$ are the simple roots of $\mathfrak{G}$; (α_i,α_j) is the value of the canonical bilinear form in the root space and

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{sh\left(\frac{hn}{2}\right)\ldots sh\left(\frac{h(k+1)}{2}\right)}{sh\left(\frac{h(n-k)}{2}\right)\ldots sh\left(\frac{h}{2}\right)}.$$

Put $U=U_h\mathfrak{G}$. The algebra U is provided with the structure of (topological) ribbon Hopf algebra as follows. The antipodal mapping $S:U\to U$ and the counit $\varepsilon:U\to\mathbb{C}$ are defined by the formulas

$$S(X_i^\pm)=-e^{\mp h\frac{(\alpha_i\alpha_i)}{4}}X_i^\pm, \qquad S(H_i)=-H_i,$$

$$\varepsilon(X_i^\pm)=\varepsilon(H_i)=0.$$

Denote by $U\hat{\otimes}U$ the completion of $U\otimes_{\mathbb{C}[\![h]\!]}U$ in the h-adic topology. The comultiplication $\Delta: U\to U\otimes U$ is given on generators by the formulas

$$\Delta(X_i^{\pm})=X_i^{\pm}\otimes\exp\left(\frac{h}{4}H_i\right)+\exp\left(-\frac{h}{4}H_i\right)\otimes X_i^{\pm},$$

$$\Delta(H_i)=H_i\otimes 1+1\otimes H_i.$$

The universal R-matrix $R\in U\hat{\otimes}U$ is described in [Dr1] as an infinite sum $\sum_i \alpha_i\otimes\beta_i$, where α_i, $\beta_i\in U$ are determined via a recurrent procedure. It is not difficult to show that the infinite sum $u=\sum_i s(\beta_i)\alpha_i$ gives a well-defined element of $\mathbb{C}[\![h]\!]$.

Let ϱ be the half-sum $(1/2)\sum_{\alpha\in\Delta_+}\alpha$ of positive roots of $\mathfrak{G}$. Let ϱ_i be the coordinates of ϱ in the basis $\alpha_1,\dots,\alpha_r$ so that $\varrho=\sum_i\varrho_i\alpha_i$. Put

$$v=u\exp\left(-h\sum_{i=1}^{r}\varrho_i H_i\right). \tag{7.1.1}$$

It is straightforward to show that v satisfies relations 3.3.1, 3.3.2 so that the triple (U,R,v) is a (topological) ribbon Hopf algebra.

7.2. Representations of U and U-Colourings of Graphs. There exists an isomorphism of $\mathbb{C}[\![h]\!]$-algebras $U=U_h\mathfrak{G}\to(U\mathfrak{G})[\![h]\!]$ identical an the Cartan subalgebra f generated by $H_1,\dots,H_r$ (see [Dr1]). This implies that i.h.w. (irreducible highest weight finite dimensional) U-modules have the same structure as the corresponding $\mathfrak{G}$-modules, i.e. they are identical as linear spaces and have the same weight decompositions with respect to f (see [L, Ro]). Finite dimensional representations of U are completely reducible. In particular for any i.h.w. U-modules V^λ, V^μ we have

$$V^\lambda\otimes V^\mu=\bigoplus_\nu(V^\nu\otimes W_\nu^{\lambda\mu}), \tag{7.2.1}$$

where $W_\nu^{\lambda\mu}$ is a vector space over $\mathbb{C}$, $V^\nu\otimes W_\nu^{\lambda\mu}$ is the primary component of V^ν in $V^\lambda\otimes V^\mu$ with multiplicity $\dim W_\nu^{\lambda\mu}$.

For the algebra $U=U_h\mathfrak{G}$ we have a more or less satisfactory combinatorial description of U-colourings of ribbon graphs. The i.h.w. U-modules are parametrized by the highest weights of $\mathfrak{G}$. Thus we can colour ribbons by associating with them some highest weights $\lambda,\mu,\nu,\dots$ of $\mathfrak{G}$. Remark 6.4.3 show that the process of colouring coupons may be reduced to the case of 3-valent coupons. Consider the

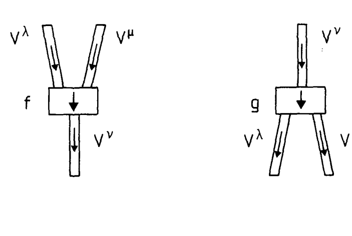


Fig. 13

3-valent coupons with coloured ribbons depicted in Fig. 13. The colours of such coupons are linear mappings respectively.

$$f: V^{\nu} \to V^{\lambda} \otimes V^{\mu}, \qquad g: V^{\lambda} \otimes V^{\mu} \to V^{\nu}.$$

Such homomorphisms bijectively correspond to elements of $W_{\nu}^{\lambda\mu}$ (respectively $(W_{\nu}^{\lambda\mu})^*$). There are rather explicit parametrizations of the elements of $W_{\nu}^{\lambda\mu}$. In the case $\mathfrak{G} = sl_n$ the simplest parametrization is given by the Young tableaux.

7.3. Homomorphism w_{λ}. Let V^{λ} be the finite dimensional i.h.w. representation of $U = U_h\mathfrak{G}$ with the highest weight. Let $(V^{\lambda})^{\vee}$ be the dual representation (3.2.1). Put $\lambda^* = -w_0(\lambda)$, where w_0 is the element of the Weil group of maximal length with respect to the Bruhat order. There is an U-linear isomorphism

$$w_{\lambda}: (V^{\lambda})^{\vee} \to V^{\lambda^*}.$$

Thus for any $a \in U$,

$$w_{\lambda}^{-1} \varrho_{\lambda^*}(a) w_{\lambda} = (\varrho_{\lambda}(S(a)))^*.$$

Since $w_0^2 = 1$ we have $(\lambda^*)^* = 1$.

For representations V^{λ} with multiplicity free weights acts in the weight basis by the formula [Re1],

$$w_{\lambda} e_{\mu}^{\lambda} = e^{-\frac{hc_{\lambda}}{4}} e^{\frac{h}{2}(\varrho, \mu)} \varepsilon_{\mu}^{\lambda} e_{w \circ \mu}^{\lambda}, \qquad (\varepsilon_{\mu}^{\lambda})^2 = 1.$$
$$c_{\lambda} = (\lambda, \lambda) + 2(\varrho, \lambda).$$

The action of w_{λ} in arbitrary V^{λ} may be defined using imbeddings of V^{λ} in the tensor powers of the vector representation of $\mathfrak{G}$. (For the case $\mathfrak{G} = sl_2$ see [KR]).

Assume that we have a HCDR-graph Γ without coupons and that Γ' is the same graph in which the directions of several ribbons and annuli have been reversed and their colours have been changed by the rule $\lambda \mapsto \lambda^* = -w_0(\lambda)$. Then one may use w to compute from $F(\Gamma)$. In particular, for generators of the subcategory of HCDR(U) consisting of graphs without coupons we have

$$F(\nwarrow\!\!\smile_{\lambda^*}) = (w_{\lambda^*}^{-1} \otimes w_{\lambda}) \circ F(\smile\!\!\nearrow_{\lambda}),$$
$$F(\swarrow\!\!\frown^{\lambda^*}) = F(\frown\!\!\searrow^{\lambda}) \circ (w_{\lambda}^{-1} \otimes w_{\lambda^*}),$$
$$F(Y_{\lambda,\mu}^{\pm}) = (1 \otimes w_{\lambda}^{-1}) F(X_{\lambda^*,\mu}^{\mp})(w_{\lambda} \otimes 1),$$
$$F(Z_{\lambda,\mu}^{\pm}) = (w_{\mu}^{-1} \otimes 1) F(X_{\lambda,\mu^*}^{\mp})(1 \otimes w_{\mu}).$$

It is easy to extend these formulas to all generators of HCDR(U) if one agrees to change not only the colours of ribbons and annuli but also conjugate by w_{λ} the colours of coupons.

Remark. The homomorphisms $\{w_{\lambda}\}$ suggest the extension $\tilde{\tilde{U}}_h\mathfrak{G}$ of $U_h\mathfrak{G}$, generated by a certain element w subject to relations

$$H_i w = -w H_{\theta(i)},$$

$$w X_i^{\pm} w^{-1} = X_{\theta(i)}^{\mp} q^{\mp\frac{(\alpha_i \alpha_i)}{4}}, \qquad q = e^h,$$

where θ is the non-trivial involution of the Dynkin diagram. The formulas

$$\Delta(w)=R^{-1}(w\otimes w), \qquad \varepsilon(w)=1, \qquad S(w)=uw^{-1}$$

make $\tilde{\tilde{U}}$ a Hopf algebra containing U as a Hopf subalgebra. One may show that

$$(w\otimes w)R=R_{21}(w\otimes w), \qquad wu=S(u)w.$$

Using these formulas it is not difficult to show that $(\tilde{\tilde{U}},R,w^2)$ is a ribbon Hopf algebra. According to results of Sect. 3.3 the element v given by (7.1.1) and w^2 are related by the formula $w^2=vE$, where E satisfies (3.3.3).

7.4. The Case $\mathfrak{G}=sl_2$. The Hopf algebra $U=U_hsl_2$ is the simplest among the quantized Lie algebras. It has been extensively studied in [KR]. The finite dimensional i.h.w. representations $\{V^j\}_j$ of $U=U_hsl_2$ are parametrized by integers and half-integers $j=0, 1/2, 1, 3/2, \ldots$ where $\dim_{\mathbb{C}} V^j=2j+1$. In the weight basis $\{e^j_m\}^j_{-j}$ of V^j the action of H, $X^\pm \in U$ is given by the following formulas:

$$\varrho^j(X^\pm)e^j_m=([j\mp m]_q[j\pm m+1]_q)^{1/2}e^j_{m\pm 1},$$
$$\varrho^j(H)e^j_m=2me^j_m,$$

where

$$[n]_q=sh\left(\frac{hn}{2}\right)\Big/ sh\left(\frac{h}{2}\right), \qquad q=e^h.$$

The universal R-matrix $R\in U\hat{\otimes}U$ is computed as follows:

$$R=\exp\left(\frac{h}{4}H\otimes H\right)\sum_{n\geqq 0}\frac{(1-q^{-1})^n\left(sh\left(\frac{h}{2}\right)\right)^n}{sh\left(\frac{hn}{2}\right)\ldots sh\left(\frac{h}{2}\right)}$$
$$\times e^{\frac{h}{4}n(H\otimes 1-1\otimes H)}(X^+)^n\otimes(X^-)^n.$$

It is not difficult to compute the matrix elements of $R^{j_1j_2}=(\varrho^{j_1}\otimes\varrho^{j_2})R$ in the weight basis:

$$R^{j_1j_2}(e^{j_1}_{m_1}\otimes e^{j_2}_{m_2})=\sum_n (R^{j_1j_2})^{m_1+n,m_2-n}_{m_1m_2}e^{j_1}_{m_1+n}\otimes e^{j_1}_{m_2-n}.$$

Here the non-zero matrix elements may be expressed as follows:

$$(R^{j_1j_2})^{m_1+n,m_2-n}_{m_1m_2}$$
$$=\frac{(1-q^{-1})^n}{[n]_q!}\left(\frac{[j_1-m_1]![j_1+m_1+n]![j_2+m_2]![j_2+n-m_2]!}{[j_1-m_1-n]![j_1+m_1]![j_2+m_2-n]![j_2-m_2]!}\right)^{1/2}$$
$$\times q^{\frac{n}{2}(n_1-n_2+2n)}.$$

The tensor product $V^{j_1}\otimes V^{j_2}$ is decomposable into the direct sum of irreducible summands V^j by the formula (7.2.1) where all multiplicity spaces are 1-dimensional for $|j_1-j_j|\leqq j\leqq j_1+j_2$ and are equal to zero for other j. In other words

$$V^{j_1}\otimes V^{j_2}=\bigoplus_{\substack{|j_1-j_2|\leqq j\leqq j_1+j_2\\ 2j=2j_1+2j_2(\mathrm{mod}\,2)}} V^j. \tag{7.4.1}$$

Therefore, the $V_h sl_2$-colouring of 3-valent ribbon graphs is extremely similar. It suffices to associate with each ribbon and integer or half-integer j and with each coupon a complex number. Of course, the invariant F of the graph may be non-zero only if for every coupon the colours j_1, j_2, j of the adjacent ribbons satisfy the inequalities $|j_1-j_2|\leqq j\leqq j_1+j_2$, $2j=2j_1+2j_2(2)$.

The modules V^j are self-dual: $(V^j)^v\approx V^j$. This implies that the corresponding invariants of coloured ribbon tangles do not depend on the choice of directions of ribbon cores and annuli cores. This generalizes the fact that the Jones polynomial of a framed link L does not depend on the choice of orientation. Indeed, if all annuli of the graph Γ_L (see Sect. 6.1) are coloured by $V^{1/2}$ we get exactly the Jones polynomial. Also, if one colours all the annuli of Γ_L by V^1 one gets a one-variable reduction of the Kauffman polynomial.

Since the modules $\{V^j\}$ have the preferred bases one may study the transformation matrix of (7.4.1). For any j_1, j_2, j_3 and m_1, m_2, m_3 with $1\leqq m_i \leqq 2j_i+1$, put

$$\begin{aligned}\begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m\end{bmatrix}_q &= \delta_{m_1+m_2,m}(-1)^{j_1-m_1} q^{\frac{1}{4}(j_2(j_2+1)-j_1(j_1+1)-j(j+1))+\frac{m_1(m_1+1)}{2}}\\ &\times\left\{\frac{[j+m]!\,[j-m]!\,[j_1-m_1]!\,[j_2-m_2]!\,[j_1+j_2-j]!\,[2j+1]}{[j_1+m_1]!\,[j_2+m_2]!\,[j_1-j_2+j]!\,[j_2-j_1+j]!\,[j_2+j_2+j+1]!}\right\}^{1/2}\\ &\times\sum_{r\geqq 0}(1-)^r q^{\frac{1}{2}r(m+j+1)}\frac{[j_1+m_1+r]!\,[j_2+j-m_1-r]!}{[r]!\,[j-m-r]!\,[j_1-m_1-r]!\,[j_2-j+m_1+r]!}.\end{aligned}$$

These numbers are called $q-3j$-symbols (see [KR, K]). They constitute the transformation matrix for (7.4.1): for $|j_1-j_2|\leqq j\leqq j_1+j_2$ and $m=-j,\dots,j$, we have

$$e^j_m=\sum_{\substack{-j_i\leqq m_i\leqq j_i,\\ 2m_i=2_{ji}(2)}}\begin{bmatrix} j_1 j_2 j \\ m_1 m_2 m\end{bmatrix}_q e^{j_1}_{m_1}\otimes e^{j_2}_{m_2}.$$

For $U_h sl_2$,

$$w^j_{mm'}=e^{-\frac{h}{2}j(j+1)}(-1)^{j-m}\delta_{m,-m'}e^{-\frac{hm}{2}}.$$

7.5. *Remark.* In this paper we have constructed a functor $F:\mathrm{HCDR}(A)\to\mathrm{Rep}_\square A$. Let $p:\mathrm{Rep}_\square A\to\mathrm{Rep}A$ be the standard projection (see Sect. 2.3). Let w be the forgetting functor which transforms each A-module into the underlying vector space. Let $C(A)$ be the image of $\mathrm{Rep}(A)$ under ω. This is a "generalized Tannakian category" (cf. [D]). The commutativity and associativity morphisms in $C(A)$ are induced by the corresponding morphisms in $\mathrm{Rep}(A)$. For this it is convenient to choose in Rep$A$ a "multiplicative bases" consisting of finite dimensional irreducible A-modules $\{V^{\lambda}\}_{\lambda}$. The commutativity morphism $V^{\lambda}\otimes V^{\mu}\to V^{\mu}\otimes V^{\lambda}$ and the associativity morphism $(V^{\lambda}\otimes V^{\mu})\otimes V^{\nu}\to V^{\lambda}\otimes(V^{\mu}\otimes V^{\nu})$ are given by a calculus generalizing the Racah-Wigner calculus for groups. The composition $i\circ\omega\circ p\circ F$: HCDR$(A)\to C_{\square}(A)$ is a representation of HCDR(A) in $C_{\square}(A)$ realized via Racah-Wigner q-coefficients (the case $A=U_h sl_2$ was treated in [KR]).

A functor similar to $i\circ\omega\circ p\circ F$ was constructed in [MS] using fusion rule constants in the conformal field theory. This similarity once more suggests a close relationship between the conformal field theory and the theory of quantum groups.

References

1. [Br] Bremmer, M.R., Moody, R.V., Patera, J.: Tables of dominant weight multiplicities for representation of simple Lie groups. New York: Dekker 1984
2. [D] Deling, P., Milne, J.S.: Tannakian Categories in Hodge cicles, motives and simura varieties. Lecture Notes in Mathematics, vol. 900. Berlin, Heidelberg, New York: Springer 1982
3. [Dr1] Drinfeld, V.G.: Proceedings of the International Congress of Mathematicians, vol. 1, p. 798–820. Berkeley, California: New York: Academic Press 1986
4. [Dr2] Drinfeld, V.G.: Quasicommutative Hopf algebras. Algebra and Anal. **1**, (1989)
5. [FRT] Faddeev, L., Reshetikhin, N., Takhtajan, L.: Quantized Lie groups and Lie algebras. LOMI-preprint, E-14-87, 1987, Leningrad
6. [FY] Freyd, P.J., Yetter, D.N.: Braided compact closed categories with applications to low dimensional topology. Preprint (1987)
7. [JS] Joyal, A., Street, R.: Braided Manoidal categories. Macquarie Math. Reports. Report No. 860081 (1986)
8. [JO] Jones, V.F.R.: A polynomial invariant of Knots via von Neumann algebras. Bull. Am. Math. Soc. **12**, 103–111 (1985)
9. [KT] Tsuchiya, A., Kanlie, Y.: Vertex operators in the conformal field theory on $P1$ and monodromy representations of the braid group. In: Conformal field theory and solvable lattice models. Lett. Math. Phys. **13**, 303 (1987)
10. [KV] Kauffman, L.H., Vogel, P.: Link polynomials and a graphical calculus. Preprint (1987)
11. [K] Kirillov, A.N.: q-analog of Clebsch-Gordon coefficients for slz. Zap. Nauch. Semin. LOMI, **170**, 1988
12. [KR] Kirillov, A.N., Reshetikhin, N.Yu.: Representation of the algebra $U_q(sl_2)$, q-orthogonal polynomials and invariants of links. LOMI-preprints, E-9-88, Leningrad 1988
13. [L] Luztig, G.: Quantum deformations of certain simple modules over enveloping algebras. M.I.T. Preprint, December 1987
14. [Ma] MacLane, S.: Natural associativity and commutativity. Rice Univ. Studies **49**, 28–46 (1963)
15. [MS] Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177–254 (1989)
16. [Re1] Reshetikhin, N.Yu.: Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links. I. LOMI-preprint E-4-87 (1988), II. LOMI-preprint E-17-87 (1988)
17. [Re2] Reshetikhin, N.Yu.: Quasitriangular Hopf algebras and invariants of Tangles. Algebra and Anal. **1**, (1988) (in Russian)
18. [RS] Reshetikhin, N.Yu., Semenov:Tian-Shansky, M.A.: Quantum R-matrices and factorization problem in quantum groups. J. Geom. Phys. v. **V** (1988)

19. [RO] Rosso, M.: Représentations irréductibels de dimension finie du q-analogue de l'algébre enveloppante d'une algèbre de Lie simple. C.R. Acad. Sci. Paris, t. **305**, Se'rie 1, 587–590 (1987)
20. [Tu1] Turaev, V.G.: The Yang-Baxter equation and invariants of links. Invent. Math. **92**, 527–553 (1988)
21. [Tu2] Turaev, V.G.: The Conway and Kauffman modules of the solid torus with an appendix on the operator invariants of tangles. LOMI Preprint E-6-88. Leningrad (1988)
Turaev, V.G.: Operator invariants of tangles and R-matrices. To appear in Izv. An SSSR
22. [Ye] Yetter, D.N.: Category theoretic representations of Knotted graphs in S^3, Preprint
23. [Wi] Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys. (in press)

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